Imperial College London

Imperial College London

Department of Mathematics

LMM to FMM: An Illustration of SONIA Cap Pricing

Author: Weiqing CAO (CID: 02282339)

A thesis submitted for the degree of

MSc in Mathematics and Finance, 2022-2023

Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature: Weiqing Cao
Date: 05/09/2023

Acknowledgements

First and foremost, I would like to express my gratitude to the staff at Imperial College London for opening the doors to the world of quantitative finance. I am deeply thankful to my internal supervisor, Damiano Brigo, for his wealth of knowledge and invaluable guidance during the course of my dissertation. I feel fortunate that he not only introduced me to interest rate models but also guided my dissertation through the final stages of my master's degree in this area.

I would also like to extend my appreciation to Mazars for providing me with the internship opportunity and allowing me to explore such an intriguing field. My external supervisor, Ludvig Hallman, exhibited immense patience and a keen interest in my progress. He diligently addressed every one of my inquiries with insightful responses. His kindness and profound theoretical knowledge greatly facilitated the completion of my internship project. I am equally grateful to my colleagues who offered advice and assistance with my thesis: Nicolas Cerrajero, Paul Padiou and Martin Ricouard. I would like to extend my heartfelt appreciation to all of my colleague. The team really helps me a lot.

Lastly, I wish to express my deepest gratitude to my parents for their unwavering support, which has enabled me to pursue studies. I am also thankful to my friends for their support throughout this journey.

Abstract

Interest rate benchmarks play a pivotal role within the global financial system, serving as essential reference points that guide various financial activities. However, the credibility of Interbank Offered Rates (IBORs) has been called into question due to issues such as liquidity constraints and manipulation. As a result, regulatory bodies and central banks have taken significant steps to overhaul the interest rate markets, aiming to address these concerns.

In response to these challenges, the financial industry is seeking a more dependable alternative to replace IBORs, which has led to the emergence of Risk-Free Rates (RFRs).

This research delves into the critical transition from the London Interbank Offered Rate (LIBOR) Market Model (LMM) to the Forward Market Model (FMM). By refining discount factor definitions, converting forward-looking rates into their backward-looking equivalents, and introducing decay functions to emulate the dynamic behavior of these backward-looking rates, a comprehensive framework for the transition process is established. To express these intricate shifts, the Bachelier and Black models are employed as effective frameworks.

In this study, the adaptable Bachelier model is harnessed for cap pricing. The research encompasses the thorough examination of three distinct parameter setups against market prices sourced from Bloomberg. Through this analysis, the effectiveness and accuracy of the proposed model transitions are rigorously evaluated.

Contents

0.1 The Meaning of LIBOR 6
0.2 Reasons Why Not to Use LIBOR 6
0.3 The Differences Between LIBOR and RFR 7
0.4 The Impact of LIBOR transition 7

1 Mathematical Framework **9**
1.1 Basic Definitions 9
1.2 Change of Numeraire 9
1.3 RFR Rates 11
1.4 Decay volatility 13

2 RFR Instruments Pricing Formula **16**
2.1 RFR Cap 16
2.1.1 Bachelier Model 17
2.1.2 Black Model 19
2.2 RFR swaps 20
2.3 RFR swaptions 21

3 Calibration **23**
3.1 Yield Curve Calibration 23
3.2 Boot-stripping RFR Caplet Volatility 24
3.3 Data Processing 28

4 Pricing Results **30**
4.1 Spot Starting 31
4.1.1 Linear Method on Zero Rates 31
4.1.2 Logcubic Method on Discount Factor 31
4.1.3 Monte Carlo Results 32
4.2 Start in The Future 33
4.3 Valuation Date in $[T_{j-1}, T_j]$ 33

A Technical Proofs **36**
A.1 Proof of LIBOR Swaption pricing 36

Bibliography **39**

List of Figures

1 Source: Schrimpf Andreas and Sushko Vladyslav (2019)[1, page 35], The new benchmark rates . . . 5
2 Source: BOE (2020)[2, Chart A], The low number of transactions . . . 6
3 Source: BOE (2020)[2, Chart B], LIBOR, SONIA and Bank Rate . . . 7
4 Source: ChathamFinancial (2022)[3, section 4], Proprietary credit-sensitive benchmark rates . . . 8

1.1 Dynamic of RFR Rates with Decay Volatility. . . . 14

2.1 Source: Ioannis Rigopoulos (2023)[4], Cap term . . . 16
2.2 Source: BOE (2020)[5, page 6], Look-back without Observation Shift . . . 17

3.1 Source: Ametrano (2013)[6, section 4.5, page 42], The impact of interpolation on curve shape . . . 24
3.2 Source: Olegalexandrov (2008)[7], Newton–Raphson method . . . 26
3.3 Volatility smile . . . 27
3.4 Volatility vs Maturity . . . 27
3.5 Volatility Comparison Between Par & Fwd_decay. . . . 28

4.1 Cash flow with linear method on zero rate using BBG when spot starting . . . 31
4.2 Cash flow with logcubic method on discount factor using BBG when spot starting 32
4.3 Monte Carlo of the first caplet . . . 32
4.4 Cash flow priced by Bloomberg when starting in the future . . . 33
4.5 Cash flow priced by Bloomberg when valuation date in $[T_{j-1}, T_j]$ . . . 34

List of Tables

3.1 Source: Burgess (2019)[8, section 4.4.1, page 38], OIS Curve Calibration Instruments, USDOIS 24
3.2 The par volatility surface 26
3.3 The forward decay volatility surface 27

4.1 Cap contract 30
4.2 Cash flow with linear method on zero rate using VP when spot starting 31
4.3 Cash flow with logcubic method on discount factor using VP when spot starting . 32
4.4 Cash flow priced by VP when starting in the future 33
4.5 Cash flow priced by VP when valuation date in $[T_{j-1}, T_j]$ 33

Introduction

Interest rate benchmarks are crucial linchpins in the financial world, acting as vital reference points. Unfortunately, their reliability has been shaken by liquidity constraints and rigging etc. This has led to a erosion in trust concerning the dependability and robustness of these benchmarks. Consequently, regulatory authorities and central banks are orchestrating substantial reforms within the interest rate markets. These efforts are directed at facilitating a seamless transition toward innovative alternative benchmark rates that are in alignment with updated standards, as highlighted by LBR [8, page 4].

It's noteworthy that the UK Financial Conduct Authority (FCA), the supervisory body responsible for overseeing the LIBOR benchmark, has publicly indicated its stance against coercing financial institutions to continue issuing LIBOR rates beyond 2021, as reported in the FCA2019 document [9, section 8]. This decision marks a significant turning point, given that pricing models, the intricate financial landscape, and the foundational market infrastructure have become closely interwoven with the LIBOR benchmark over the span of decades. There are several examples of RFR rates for different currencies in Figure 1.

Overview of identified alternative RFRs in selected currency areas — Table 1

	United States	**United Kingdom**	**Euro area**	**Switzerland**	**Japan**
Alternative rate	**SOFR** (secured overnight financing rate)	**SONIA** (sterling overnight index average)	**ESTER** (euro short-term rate)	**SARON** (Swiss average overnight rate)	**TONA** (Tokyo overnight average rate)
Administrator	Federal Reserve Bank of New York	Bank of England	ECB	SIX Swiss Exchange	Bank of Japan
Data source	Triparty repo, FICC GCF, FICC bilateral	Form SMMD (BoE data collection)	MMSR	CHF interbank repo	Money market brokers
Wholesale non-bank counterparties	**Yes**	**Yes**	**Yes**	**No**	**Yes**
Secured	**Yes**	**No**	**No**	**Yes**	**No**
Overnight rate	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**
Available now?	Yes	Yes	Oct 2019	Yes	Yes

FICC = Fixed Income Clearing Corporation; GCF = general collateral financing; MMSR = money market statistical reporting; SMMD = sterling money market data collection reporting.

Sources: ECB; Bank of Japan; Bank of England; Federal Reserve Bank of New York; Financial Stability Board; Bank of America Merrill Lynch; International Swaps and Derivatives Association.

Figure 1: Source: Schrimpf Andreas and Sushko Vladyslav (2019)[1, page 35], The new benchmark rates

This transition to new benchmark rates is posing one of the most monumental challenges the financial industry has faced. As the sector navigates this profound transformation, careful consideration is being given to preserve the integrity and efficiency of the financial markets, thus ushering in an era of increased transparency and renewed investor confidence.

0.1 The Meaning of LIBOR

The "London Interbank Offered Rate"(LIBOR) is a widely adopted benchmark interest rate. This rate signifies the average interest rate which is used by major global banks engage in interbank to lend within the London market. LIBOR spans various currencies, ranging from overnight to one year, and it is updated daily[10, section 3, page 1].

In the realm of finance, the London Interbank Offered Rate (LIBOR) holds a pivotal role as a critical reference indicator. It plays a vital role in determining the interest rates for different financial instruments such as loans, derivatives, and bonds. Its influence is particularly significant in the realm of pricing variable-rate loans[11, section 1].

Greek banker Minos Zombanakis carefully orchestrated syndicated loans tied to the six-month interbank rate in London during the 1960s. This initiative paved the way for the emergence of the London Interbank Offered Rate (LIBOR). Initially designed for evaluating variable-rate notes, it gained formal recognition from the British Bankers' Association in 1986. Over time, its influence expanded across major currencies including USD, GBP, EUR, JPY, and CHF, encompassing a variety of loan tenures[12].

Although LIBOR is important, we are not going to use it anymore.

0.2 Reasons Why Not to Use LIBOR

The first reason is rigging.The mounting body of evidence concerning the manipulation of LIBOR is being augmented by diverse legal actions, stemming from the initial revelation of LIBOR rigging through a settlement involving Barclays Bank and the Financial Services Authority in June 2012[13, abstract].

Following the financial crisis, the LIBOR gauged market has experienced a decline in activity. This decrease in transaction volume has eroded LIBOR's reliability as a benchmark. The market's response which is due to the Covid-19 accentuated the weaknesses of LIBOR and emphasized the necessity for financial markets to adopt alternative benchmarks.

Amid the market turbulence in 03/2020, which was triggered by the pandemic, the market activity that supports LIBOR diminished a lot. As a result, the determination of rates relied heavily on expert judgment from banks (as illustrated in Figure 2).

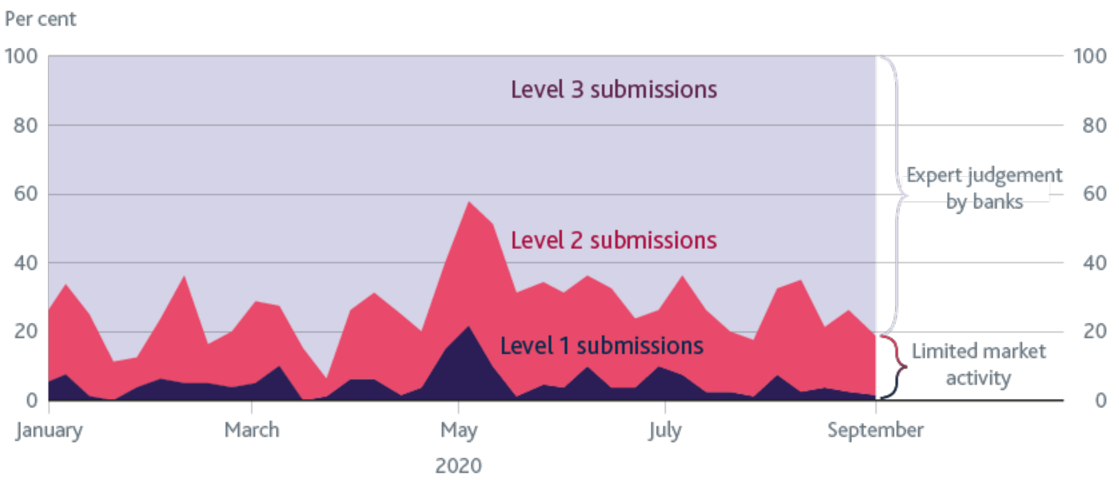


Figure 2: Source: BOE (2020)[2, Chart A], The low number of transactions

LIBOR rates, and subsequently borrowing costs, increased, while Bank Rate became some new low levels, rendering more affordable funding accessible to banks (depicted in Figure 3). However, the GBP risk-free-rate SONIA shown the same trend with Bank Rate[2].

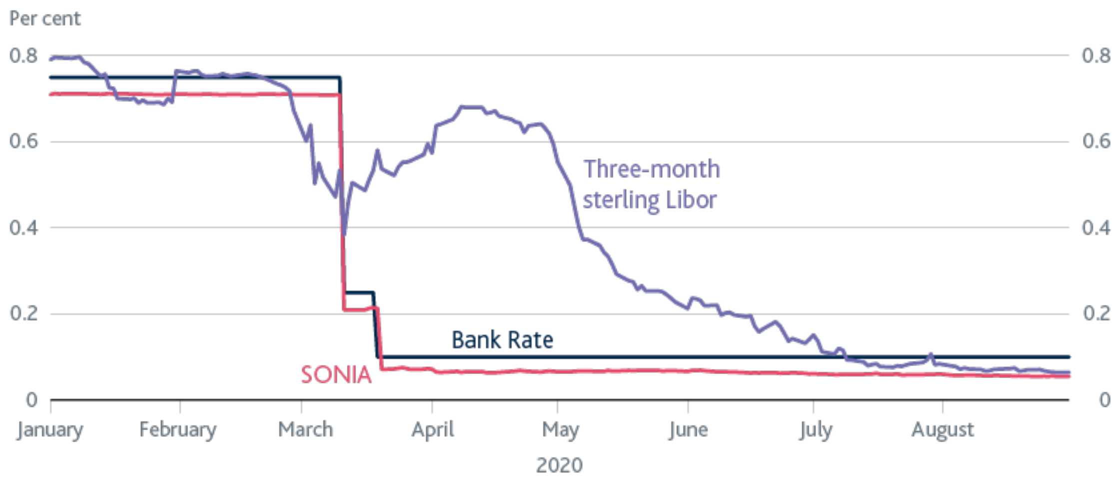


Figure 3: Source: BOE (2020)[2, Chart B], LIBOR, SONIA and Bank Rate

0.3 The Differences Between LIBOR and RFR

Initially, IBORs encompass interest rates applicable to unsecured interbank loans spanning various timeframes, encompassing an embedded credit spread. Conversely, RFRs denote overnight interest rates that inherently involve minimal to no credit-related risks. By the characteristic of their definition, the differences are as follows[14, page 2]:

- Forward-looking vs. Backward-looking
  - IBORs are forward-looking term rates. They are calculated at the accrual period start.
  - RFRs are backward-looking rates. They are calculated by the historical data and provided at the end of the day or the next business day.
- In Arrears Calculations
  - Because IBORs are forward-looking rates, the payoff at the payment date of a instrument such as cap can be known at the start of a period.
  - RFRs are backward-looking rates, so they are not known in advance. Thus, calculations for products linked to RFRs are typically done in arrears.

Due to the nature of RFR rates, it can effectively circumvent the shortcomings of IBOR rates proposed in Section 0.2:

- Because RFR rates are released at the end of a period based on historical data, it can effectively avoid the risk of fraud.
- The trading volume of derivatives based on RFR rates is also much larger in the market.

Because of these distinctions separating IBORs and RFRs, and with the intention of valuing interest rate instruments within the RFR framework, our aim is to formulate forward RFR term rates that can replace IBOR.

0.4 The Impact of LIBOR transition

As the transition from IBORs to RFRs takes place, it's essential to consider how contracts tied to IBORs will be impacted. For contracts that continue to reference LIBOR after the end of 2021, there's a possibility that the relevant LIBOR benchmark might no longer be available or appropriate.

Some banks advocate for credit-sensitive alternatives to RFRs like SOFR. The adoption of such rates can help banks manage potential situations where their cost of funds rises while their lending rates remain static or fall. Nowadays, various groups have create their own benchmark rates that consider credit sensitivity (see Figure 4). These rates are developed as alternatives to SOFR[3, section 4].

Criteria	SOFR	NY Fed Average SOFR Rates	Term SOFR	AMERIBOR	BSBY
First published	April 2018	March 2020	April 2021	2015	October 2020 (made available to trade in March 2021)
Historical data available	Historical proxy transaction data based on repo data back to August 2014	Can calculate averages from SOFR Proxy data, but Fed has only published back to March 2020	Indicative data back to January 2019	None	1/6/2016
Publisher	The Federal Reserve Bank of NY (daily)	The Federal Reserve Bank of NY	CME Group	American Financial Exchange (AFX), an electronic exchange for over 1,000 U.S. banks and financial institutions	Bloomberg
Term structure(s)	Daily	30-, 90-, and 180-day averages are compounded daily on each business day	CME began to publish SOFR term rates in May 2021	AMERIBOR Term-30, announced April 2021 AMERIBOR Term90, announced May 2021	One-, three-, six-, and twelve-month structures
Objective	Reflect overnight collateralized bank borrowing costs.	Reflect movements in interest rates over a given time period, and smooths out day-to-day volatility in market rates over that period.	Indication of the forward-looking compound average of overnight SOFR, based on SOFR Futures markets.	Reflect member banks' and financial institutions' actual unsecured borrowing costs.	Reflect the average yields at which large global banks access U.S. dollar senior unsecured marginal wholesale funding.
Methodology	Takes a weighted average of all transactions in the overnight Treasury repo markets.	Calculated by compounding daily rates on each business day. On non-business days, the SOFR value for the preceding business day applies. Act/360 day count for interest.	Calculated based on a combination of SOFR overnight indexed swaps and 1-month/3-month SOFR futures contracts.	Calculated daily based on the average interest AFX users charge one another for unsecured overnight loans.	Calculated daily based on wholesale primary market funding transactions (e.g., interbank deposits and CD, commercial paper).
Credit sensitivity	No	No	No	Yes	Yes
Underlying market volume	$1.0 trillion of daily transactions	Same as SOFR	$218 billion notional in daily SOFR futures volumes transacted on CME Group's regulated marketplace (data as of H1 2021)	$2.5 billion of daily transactions	Minimum daily thresholds: Overnight: $60 billion 1M, 3M, and 6M: $10 billion 12M: $9 billion
IOSCO compliance (self-certified)	Yes (as of June 2018)	Yes	Yes (as of April 2021)	Overnight: Yes (as of Aug 2019) 30-day: Yes (as of April 2021)	Yes (as of April 2021)
Designated benchmark by FASB	Yes (overnight)	Yes	No	No	No
Other observations	• ARRC's choice following vetting process • Discount rate for all centrally cleared trades			• Supported by regional and community banks • "Fully appropriate" for certain market participants per Fed Chair Powell	• First issuance from Tier 1 bank, BofA issued 6-month $1.0 billion note • Index tends to track LIBOR but tends to lie below it

Figure 4: Source: ChathamFinancial (2022)[3, section 4], Proprietary credit-sensitive benchmark rates

In essence, the thesis aims to introduce mathematical models for valuing interest rate instruments, particularly focusing on SONIA caps in the RFR framework, along with model calibrations, but also mentions swaps and swaptions. Additionally, it serves as theoretical support for integrating a vanilla RFR cap pricing module into the Valuation Platform, a Python-based toolkit developed for pricing interest rate instruments.

Chapter 1

Mathematical Framework

In this section, we first introduce some basic definitions for calculating interest rate instruments. Then we recall change of numeraire which is used to derive forward rates, the dynamic in FMM and finally the pricing model.

1.1 Basic Definitions

Assume the instantaneous risk-free rate at time t is $r(t)$. Then the dynamic of bank account $B(t)$ can be written as

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t$$

where $B(0) = 1$, which equals to

$$B(t) = e^{\int_0^t r(u)\mathrm{d}u}$$

The risk-neutral measure of numeraire $B(t)$ is denoted as Q. The associated expectation is written as $\mathbb{E}$ to simplify and a family of σ-fields $\{\mathcal{F}_t\}$ is defined to be a filtration. We denote the price $P(t,T)$ which represents a zero-coupon bond price with maturity T and final payoff 1, and for $t \leq T$,

$$P(t,T) = \mathbb{E}[e^{-\int_t^T r(u)\mathrm{d}u}|\mathcal{F}_t] \tag{1.1.1}$$

A.Lyashenko and F.Mercurio[15, section 2, page 4] extended equation (1.1.1) for $t > T$,

$$P(t,T) = \mathbb{E}[e^{-\int_T^t r(u)\mathrm{d}u}|\mathcal{F}_t] = e^{-\int_T^t r(u)\mathrm{d}u} = \frac{B(t)}{B(T)} \tag{1.1.2}$$

Because $e^{-\int_T^t r(u)\mathrm{d}u}$ is $\mathcal{F}_t$-measurable. Then $P(t,T)$ can be extended to be a new numeraire.

The extended T-forward measure Q^T is a blended measure that merges the traditional T-forward measure up to the maturity T with the risk-neutral measure Q after T[15, section 2.1, page 4].

1.2 Change of Numeraire

Recall the change of numeraire

Definition 1.2.1 (Equivalent measures[16, page 119])**.** Two measures $\mathbb{P}$ and $\mathbb{Q}$ on the space $(\Omega, \mathcal{F})$ are considered equivalent if they share the same assessment of events with zero probability.

Definition 1.2.2 (Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$[16, page 120])**.** This refers to a stochastic variable such that, for any other stochastic variable X, the expected value under $\mathbb{Q}$ can be expressed as:

$$\mathbb{E}^{\mathbb{Q}}[X] = \int X\mathrm{d}\mathbb{Q} = \int X\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\mathrm{d}\mathbb{P} = \mathbb{E}^{\mathbb{P}}\left[X\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]$$

Theorem 1.2.3 (Girsanov's theorem[16, page 121])**.** *If the dynamic of measure $\mathbb{P}$ is $\mathrm{d}X_t = f^{\mathbb{P}}(X_t)\mathrm{d}t + \sigma^{\mathbb{P}}(X_t)\mathrm{d}W_t^{\mathbb{P}}$, we define for all $t \in [0,T]$*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := exp[-\frac{1}{2}\int_0^t (\frac{f^{\mathbb{Q}}(X_s) - f^{\mathbb{P}}(X_s)}{\sigma^{\mathbb{P}}(X_s)})^2\mathrm{d}s + \int_0^t \frac{f^{\mathbb{Q}}(X_s) - f^{\mathbb{P}}(X_s)}{\sigma^{\mathbb{P}}(X_s)}\mathrm{d}W_s^{\mathbb{P}}]$$

then

$$\mathrm{d}X_t = f^{\mathbb{Q}}(X_t)\mathrm{d}t + \sigma^{\mathbb{P}}(X_t)\mathrm{d}W_t^{\mathbb{Q}}$$

Here, $W^{\mathbb{Q}}$ represents a Brownian Motion (BM) under $\mathbb{Q}$, while the measures $\mathbb{P}$ and $\mathbb{Q}$ are considered to be equivalent. An adequate requirement for this situation to be valid is the fulfillment of the Novikov criterion.

$$\mathbb{E}\left[exp(\frac{1}{2}\int_0^T (\frac{f^{\mathbb{Q}}(X_s) - f^{\mathbb{P}}(X_s)}{\sigma^{\mathbb{P}}(X_s)})^2 \mathrm{d}s)\right] < +\infty$$

Theorem 1.2.4 (Ito's formula[16, page 81])**.** *If the dynamic is $\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$, we have*

$$\mathrm{d}\varphi(t, X_t) = \frac{\partial \varphi}{\partial t}\mathrm{d}t + \frac{\partial \varphi}{\partial X}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 \varphi}{\partial X^2}\mathrm{d}X_t\mathrm{d}X_t$$

Here is the derivation of how to change numeraire.

Let's consider the scenario following Brigo's train of thought[17, page 292-301], where we want to express the evolution of X using the benchmark of S, but now we wish to switch to a new benchmark U. This will lead to the modified dynamics given by $\mathrm{d}X_t = \mu_t^U(X_t)\mathrm{d}t + \sigma_t(X_t)C\mathrm{d}W_t^U$. Here we denote W^U as a multi-dimensional standard Brownian motion according to the measure Q^U.

Girsanov's theorem (referred to as Theorem 1.2.3) is used to derive the Radon-Nykodim derivative (defined as in Definition 1.2.2) between Q^U and Q^S by comparing the X's dynamics under these distinct measures:

$$\zeta_T := \left.\frac{\mathrm{d}Q^S}{\mathrm{d}Q^U}\right|_{\mathcal{F}_T} = exp(-\frac{1}{2}\int_0^T |(\sigma_t(X_t)C)^{-1}[\mu_t^S(X_t) - \mu_t^U(X_t)]|^2 \mathrm{d}t$$
$$+ \int_0^T \{(\sigma_t(X_t)C)^{-1}[\mu_t^S(X_t) - \mu_t^U(X_t)]\}'\mathrm{d}W_t^U)$$

We can express ζ_t as

$$\mathrm{d}\zeta_t = \alpha_t \zeta_t \mathrm{d}W_t^U \tag{1.2.1}$$

where $\alpha_t = \{(\sigma_t(X_t)C)^{-1}[\mu_t^S(X_t) - \mu_t^U(X_t)]\}'$. Conversely, as per the definition of Q^S, for any asset whose price is Z,

$$\mathbb{E}_0^{Q^S}\left[\frac{Z_T}{S_T}\right] = \mathbb{E}_0^{Q^U}\left[\frac{U_0}{S_0}\frac{Z_T}{U_T}\right] = \frac{Z_0}{S_0} \tag{1.2.2}$$

According to the Radon-Nykodym derivative's definition (as provided in Definition 1.2.2), it is also clear that for any Z,

$$\mathbb{E}_0^{Q^S}\left[\frac{Z_T}{S_T}\right] = \mathbb{E}_0^{Q^U}\left[\frac{Z_T}{S_T}\frac{\mathrm{d}Q^S}{\mathrm{d}Q^U}\right] \tag{1.2.3}$$

Upon contrasting equations (1.2.2) and (1.2.3), we derive that

$$\zeta_T = \left.\frac{\mathrm{d}Q^S}{\mathrm{d}Q^U}\right|_{\mathcal{F}_T} = \frac{U_0}{S_0}\frac{S_T}{U_T}$$

because ζ constitutes a martingale under the Q^U measure

$$\zeta_T = \mathbb{E}_t^{Q^U}\zeta_T = \mathbb{E}_t^{Q^U}\left[\frac{U_0}{S_0}\frac{S_T}{U_T}\right] = \frac{U_0}{S_0}\frac{S_t}{U_t} \tag{1.2.4}$$

This can be deduced through differentiation, resulting in

$$\mathrm{d}\zeta_T = \frac{U_0}{S_0}\mathrm{d}\frac{S_t}{U_t} = \frac{U_0}{S_0}\sigma_t^{S/U}C\mathrm{d}W_t^U \tag{1.2.5}$$

Here, considering that the ratio S/U maintains a martingale property under the Q^U measure, we have made an assumption about the subsequent dynamics of martingales:

$$\mathrm{d}\frac{S_t}{U_t} = \sigma_t^{S/U}C\mathrm{d}W_t^U$$

Analyze equations (1.2.1) and (1.2.5) in conjunction which infer that

$$\alpha_t \zeta_t = \frac{U_0}{S_0} \sigma_t^{S/U} C$$

Considering equation (1.2.4), we arrive at the outcome

$$\frac{S_t}{U_t} \alpha_t = \sigma_t^{S/U} C$$

Recall α and substituting it into the equation,

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \frac{U_t}{S_t} \sigma_t(X_t) \rho (\sigma_t^{S/U})'$$

utilize $\rho = CC'$. It provides change into the alteration of a stochastic process's drift during the transition from the benchmark of U to that of S. Additionally, if we make the further assumption

$$\begin{aligned} \mathrm{d}S_t &= (...)\mathrm{d}t + \sigma_t^S C \mathrm{d}W_t^U \\ \mathrm{d}U_t &= (...)\mathrm{d}t + \sigma_t^U C \mathrm{d}W_t^U \end{aligned}$$

By Ito's formula (Theorem 1.2.4) we have

$$\sigma_t^{S/U} = \frac{\sigma_t^S}{U_t} - \frac{S_t}{U_t} \frac{\sigma_t^U}{U_t}$$

so that

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \sigma_t(X_t) \rho (\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t})'$$

What's more,

$$\mu_t^U(X_t)\mathrm{d}t + \sigma_t(X_t) C \mathrm{d}W_t^U = \mu_t^S(X_t)\mathrm{d}t + \sigma_t(X_t) C \mathrm{d}W_t^S$$

so we also have

$$C\mathrm{d}W_t^S = C\mathrm{d}W_t^U - \rho (\frac{\sigma_t^S}{S_t} - \frac{\sigma_t^U}{U_t})' \mathrm{d}t$$

And then we can use "DC" to represent the vector diffusion coefficient. What's more, we can use $\mathrm{d}Z = C\mathrm{d}W$ to represent the correlated Brownian motion, we can express the final equation as

$$\mathrm{d}Z_t^S = \mathrm{d}Z_t^U - \rho (\frac{DC(S)}{S_t} - \frac{DC(U)}{U_t})' \mathrm{d}t$$

and

$$\begin{aligned} \frac{DC(S)}{S_t} - \frac{DC(U)}{U_t} &= DC(\log(S)) - DC(\log(U)) \\ &= DC(\log(S) - \log(U)) \\ &= DC(\log(S/U)) \end{aligned}$$

This will be used in Section 1.4 to calculate the drift.

1.3 RFR Rates

The Risk-Free Rate (RFR) is a daily published interest rate based on the latest available data. For instance, the Bank of England releases its previous working day's SONIA rate on the following working day[18, section 2]. Our upcoming discussion will focus on how to construct the term structure of RFR rates, and display the final expression. The theoretical framework is mainly based on Andrei Lyashenko and Fabio Mercurio's article[15, page 5-7].

For the sake of clarity in our subsequent discussions, we introduce the following definitions for the symbols we will be using: Our observation points are denoted as T_i, $i = 0, 1, \ldots, M$, and the year fraction is represented by τ_j, which corresponds to the time interval of $[T_{j-1}; T_j)$. To simplify matters, we will use $P_j(t)$ to signify the bond price $P(t, T_j)$.

The actual daily-compounded setting-in-arrears rate which is calculated during $[T_{j-1}; T_j)$ is given by

$$R(T_{j-1}, T_j) = \frac{1}{\tau_j}\left[\prod_{i=1}^{n}(1 + r_i\delta_i) - 1\right]$$

Here, the multiplication encompasses the business days within the interval $[T_{j-1}, T_j)$, and r_i represents the RFR daily rate on date i, with its corresponding day-count fraction denoted as δ_i. If we converge δ_i towards zero, we arrive at the approximation captured in Equation (1.3.1).

$$R(T_{j-1}, T_j) = \frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\mathrm{d}u} - 1\right] = \frac{1}{\tau_j}\left[\frac{B(T_j)}{B(T_{j-1})} - 1\right] = \frac{1}{\tau_j}[P_{j-1}(T_j) - 1] \tag{1.3.1}$$

Next, we clarify the concepts of forward-looking and backward-looking rates:

- Backward looking rate: This pertains to the compounded rate calculated retrospectively over a term, known as the arrears term rate.
- Forward looking rate: This involves the par rate over the accrual period.

Moving forward, we can establish a forward-looking rate whose maturity is $T_j$and valuation date is T_{j-1}, which we denote as $F(T_{j-1}, T_j)$.

$$F(T_{j-1}, T_j) = \mathbb{E}^{T_j}[R(T_{j-1}, T_j)|\mathcal{F}_{T_{j-1}}] \tag{1.3.2}$$

The backward-looking forward rate $R_j(t)$ at time t can be denoted as

$$R_j(t) = \mathbb{E}^{T_j}[R(T_{j-1}, T_j)|\mathcal{F}_t] \tag{1.3.3}$$

Drawing upon Equation (1.3.2) and (1.3.3), we can establish that $F(T_{j-1}, T_j)$ is identical to $R_j(t)$ at time $t = T_{j-1}$ for each $j = 1, \ldots, M$. This relationship can be expressed as:

$$F(T_{j-1}, T_j) = R_j(T_{j-1})$$

Altering the measure to Q, we have

$$\begin{aligned} 1 + \tau_j R_j(t) &= \mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\mathrm{d}u}|\mathcal{F}_t\right] \\ &= \frac{1}{P_j(t)}\mathbb{E}\left[e^{-\int_t^{T_j} r(u)\mathrm{d}u}e^{\int_{T_{j-1}}^{T_j} r(u)\mathrm{d}u}|\mathcal{F}_t\right] \\ &= \frac{1}{P_j(t)}\mathbb{E}\left[e^{-\int_t^{T_{j-1}} r(u)\mathrm{d}u}|\mathcal{F}_t\right] \\ &= \frac{P_{j-1}(t)}{P_j(t)} \end{aligned}$$

Therefore, the forward rate is

$$R_j(t) = \frac{1}{\tau_j}\left[\frac{P_{j-1}(t)}{P_j(t)} - 1\right]$$

Finally,

$$R_j(t) = \begin{cases} \dfrac{1}{\tau_j}\left[\dfrac{P_{j-1}(t)}{P_j(t)} - 1\right], & t \leq T_{j-1}, \\ \dfrac{1}{\tau_j}\left[\dfrac{B(t)/B(T_{j-1})}{P_j(t)} - 1\right], & T_{j-1} < t < T_j, \\ \dfrac{1}{\tau_j}\left[\dfrac{B(T_j)}{B(T_{j-1})} - 1\right], & t \geq T_j. \end{cases} \tag{1.3.4}$$

- For $t \leq T_{j-1}$, $P_{j-1}(t)$, $P_j(t)$ are discount factors which the calculation method is the same as LIBOR rates and the only difference is the curve.
- For $T_{j-1} < t < T_j$, $P_{j-1}(t) = B(t)/B(T_{j-1})$ which becomes daily compounding rate.
- For $t \geq T_j$,$R_j(t) = \frac{1}{\tau_j}[P_{j-1}(T_j) - 1]$ where $P_{j-1}(T_j) = \frac{B(T_j)}{B(T_{j-1})}$ is the daily compounding rate which is already known.

This forward FMM rate is the rate we use to price RFR instruments.

1.4 Decay volatility

Following the article written by Andrei Lyashenko and Fabio Mercurio's article[15, page 9-11], due to the relationship described in Equation (1.3.3), the forward rate $R_j(t)$ maintains a martingale property within its associated T_j-forward measure, where $j = 1, \dots, M$. In this context, we consider the Q^{T_j}-dynamics as given by (1.4.1):

$$\mathrm{d}R_j(t) = \sigma_j(t)\mathbb{1}_{\{t \leq T_j\}}\mathrm{d}W_j(t) \tag{1.4.1}$$

In the context of each $j = 1, \dots, M$, $W_j(t)$ represents a standard Brownian motion, while $\sigma_j(t)$ is considered as the state of the adapted process $\sigma_j = (\sigma_j^s)_{s\geq 0}$ at time $s = t$, where $\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{i,j}\mathrm{d}t$. Because for time $t > T_j$ the rate is already known, so we use $1_{t\leq T_j}$ to express.

However, the rate $R_j(t)$ is daily compounded during the accrual period, so it will decrease towards zero. The simplest form of such decay is linear. Thus, (1.4.1) is not used anymore, and instead, a function g_j is selected as per A. Lyashenko and F. Mercurio [15, section 3, page 9]. Specifically, $g_j(t)$ satisfies the conditions:

- $g_j(t) = 1, \quad t \leq T_{j-1}$
- $g_j(t)$ monotonically decreases within the interval $[T_{j-1}, T_j]$
- $g_j(t) = 0, \quad t \geq T_j$

The currently widely used function g_j is:

$$g_j(t) = \min\left\{\frac{(T_j - t)^+}{T_j - T_{j-1}}, 1\right\}$$

Subsequently, we can express the dynamic of $R_j(t)$ as:

$$\mathrm{d}R_j(t) = \sigma_j(t)g_j(t)\mathrm{d}W_j(t)$$

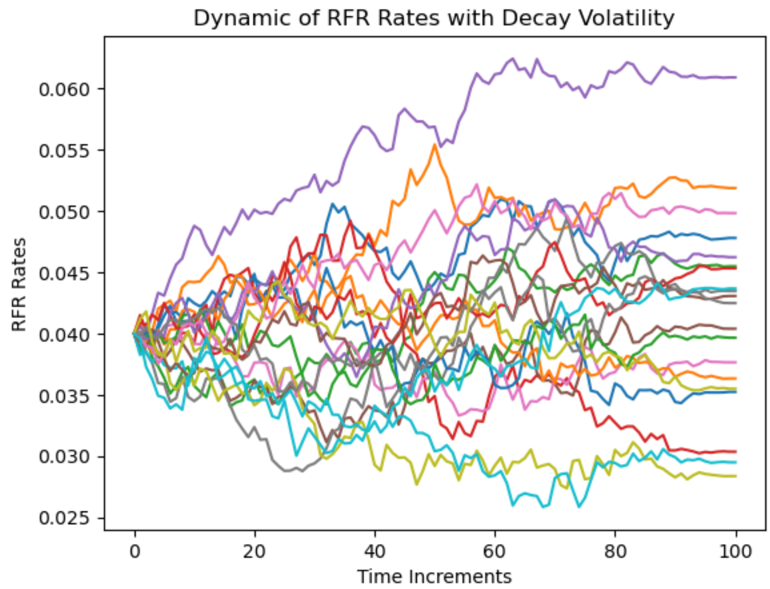
Here are images of the RFR rate simulated with normal model (see Figure 1.1(a)) and lognormal model (see Figure 1.1(b)) using Monte Carlo respectively. We set $R(0) = 0.04$, $\sigma = 0.01$, $T_{j-1} = 0.75$ (time increments=75) and $T_j = 1$ (time increments=100). It can be observed that the volatility gradually decreases after entering the interval $[T_{j-1}, T_j]$, and the curve becomes almost a horizontal line at the end, which means that the volatility finally approaches 0.

Following Andrei Lyashenko and Fabio Mercurio's guidance [15, page 10-12], we're unifying the dynamics of each $R_j(t)$ under a common measure. Using the numeraire $(B(t), Q)$, we apply the discussed numeraire change in Section 1.2 to derive:

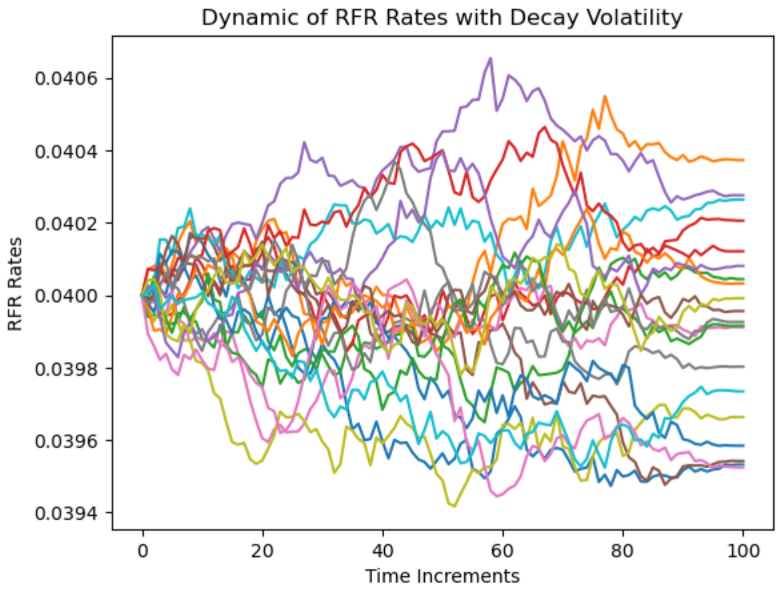
$$\mu_j^Q(t) = \frac{\mathrm{d}R_j(t)\mathrm{d}\log(B(t)/P(t,T_j))}{\mathrm{d}t}$$

By the definition of P,

$$\begin{aligned}
\log\frac{B(t)}{P(t,T_j)} &= \log\frac{P(t,0)}{P(t,T_j)} \\
&= \log\prod_{i=1}^{j}\frac{P(t,T_{i-1})}{P(t,T_i)} \\
&= \log\prod_{i=1}^{j}[1 + \tau_i R_i(t)] \\
&= \sum_{i=1}^{j}\log[1 + \tau_i R_i(t)]
\end{aligned}$$



(a) Normal Model.



(b) Lognormal Model.

Figure 1.1: Dynamic of RFR Rates with Decay Volatility.

The drift can be

$$
\begin{aligned}
\mu_j^Q(t) &= \frac{\mathrm{d}R_j(t)\mathrm{d}\log(B(t)/P(t,T_j))}{\mathrm{d}t} \\
&= \frac{\mathrm{d}R_j(t)\mathrm{d}\sum_{i=1}^{j}\log[1+\tau_i R_i(t)]}{\mathrm{d}t} \\
&= \sum_{i=1}^{j}\frac{\tau_i}{1+\tau_i R_i(t)}\frac{\mathrm{d}R_j(t)\mathrm{d}R_i(t)}{\mathrm{d}t} \\
&= \sum_{i=1}^{j}\frac{\tau_i}{1+\tau_i R_i(t)}\frac{\sigma_j(t)g_j(t)\mathrm{d}W_j(t)\sigma_i(t)g_i(t)\mathrm{d}W_i(t)}{\mathrm{d}t} \\
&= \sigma_j(t)g_j(t)\sum_{i=1}^{j}\frac{\tau_i}{1+\tau_i R_i(t)}\frac{\rho_{i,j}\sigma_i(t)g_i(t)\mathrm{d}t}{\mathrm{d}t} \\
&= \sigma_j(t)g_j(t)\sum_{i=1}^{j}\frac{\tau_i\rho_{i,j}\sigma_i(t)g_i(t)}{1+\tau_i R_i(t)}
\end{aligned}
$$

Hence, the dynamics of R_j under the Q measure become

$$
\mathrm{d}R_j(t) = \sigma_j(t)g_j(t)\sum_{i=1}^{j}\frac{\tau_i\rho_{i,j}\sigma_i(t)g_i(t)}{1+\tau_i R_i(t)}\mathrm{d}t + \sigma_j(t)g_j(t)\mathrm{d}W_j^{\mathbb{Q}}(t)
$$

This type of change of numeraire can be used to complete the generalized forward market model. We can find more details in Lyashenko Andrei and Mercurio Fabio's article[19].

Chapter 2

RFR Instruments Pricing Formula

In this section, we will begin by presenting the description of RFR caps, swaps and swaptions. Following that, we will illustrate two distinct pricing equations derived by Bachelier and Black models for caps.

2.1 RFR Cap

An interest rate cap provides the holder with the ability to set a maximum limit on payment. It is structured as a sequence of caplets with a certain payment frequency. We can envision a caplet to be a Call option on the forward rate with a designated strike price K. In a broader context, the value of a cap in the present is computed as the product of the discount factor $P(t,T)$ and the anticipated payoff at time t.

Consider a sequence of time intervals given by $0 \leq t \leq T_0 \leq \cdots \leq T_n = T$, where $(T_k)_{k=1}^n$ represents the payment dates. Define τ_j as $\frac{1}{T_j - Tj-1}$. Let K be a constant, referred to as the cap strike, with the condition $K \geq 0$.

Due to the backward-looking nature of RFRs, we can establish two types of caplets over the interval $[T_{j-1}, T_j]$ utilizing a strike value of K[15, section 6.3, page 18]. The resulting payoffs at T_j are as follows:

- $\tau_j[R_j(T_{j-1}) - K]^+$, where R_j is known at the beginning of the period T_{j-1} because of it's forward-looking nature.
- $\tau_j[R_j(T_j) - K]^+$, where R_j is known at the end of the period T_j because of it's backward-looking nature.

The cap term can be shown as Figure 2.1

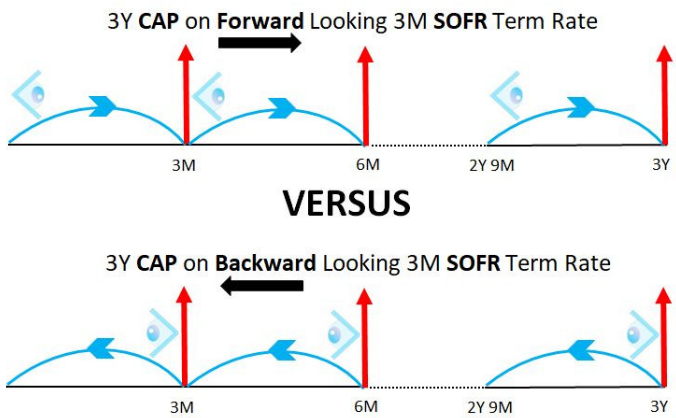


Figure 2.1: Source: Ioannis Rigopoulos (2023)[4], Cap term

The caplet price on $[T_{j-1}, T_j]$ is

$$V_{caplet}(t) = \tau_j P(t, T_j) \mathbb{E}^{T_j}[(R_j - K)^+ | \mathcal{F}_t]$$

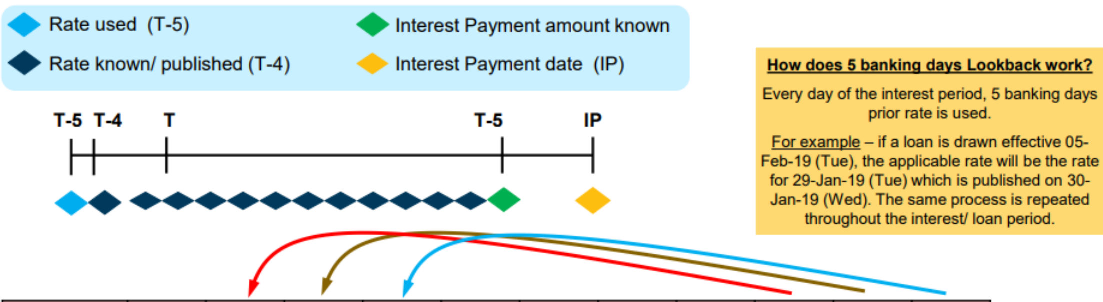
The determination of this valuation is contingent upon the approach employed to model the retrospective risk-free-rate $R_j(t)$. As articulated by Andrei Lyashenko and Fabio Mercurio (reference [15, section 6.3, page 18]), it has been observed that caplets with a backward-looking characteristic tend to possess a higher value compared to caplets with a forward-looking aspect. This phenomenon can be inferred by considering the following factors:

$$\begin{aligned} \mathbb{E}^{T_j}[(R_j(T_j) - K)^+ | \mathcal{F}_t] &= \mathbb{E}^{T_j}[\mathbb{E}^{T_j}[(R_j(T_j) - K)^+ | \mathcal{F}_{T_{j-1}}] | \mathcal{F}_t] \\ &\geq \mathbb{E}^{T_j}[(\mathbb{E}^{T_j}[R_j(T_j) | \mathcal{F}_{T_{j-1}}] - K)^+ | \mathcal{F}_t] \\ &= \mathbb{E}^{T_j}[(R_j(T_{j-1}) - K)^+ | \mathcal{F}_t] \end{aligned}$$

Because R_j is calculated in-arrears for the backward-looking case, RFR caps often have a look-back period which is 5D or 2D before the payment date to calculate the rate R_j. According to the pdf published by Bank of England [5, page 4], there are two types of look-back conventions:

- Look-back without Observation Shift
- Look-back with Observation Shift

The main difference of these two method is whether we consider holidays. The recommended method is look-back without observation shift which doesn't consider holidays. The process is explained in details as Figure 2.2.



Rate for	28-Jan Mon	29-Jan Tue	30-Jan Wed	31-Jan Thu	01-Feb Fri	02-Feb Sat	03-Feb Sun	04-Feb Mon	05-Feb Tue	06-Feb Wed	07-Feb Thu
Published on	29-Jan Tue	30-Jan Wed	31-Jan Thu	01-Feb Fri	04-Feb Mon	-	-	05-Feb Tue	06-Feb Wed	07-Feb Thu	08-Feb Fri
	0.7054	0.7036	0.7034	0.7034	0.7025	-	-	0.7051	0.7048	0.7066	0.7065

Loan Period - 05-Feb-19 to 12-Feb-19

Observation Date	Start Date	End Date	Daily RFR	Comment
Tue,29-Jan-19	Tue,05-Feb-19	Wed,06-Feb-19	0.7036	Use rate for 29-Jan published on 30-Jan
Wed,30-Jan-19	Wed,06-Feb-19	Thu,07-Feb-19	0.7034	Use rate for 30-Jan published on 31-Jan
Thu,31-Jan-19	Thu,07-Feb-19	Fri,08-Feb-19	0.7034	Use rate for 31-Jan published on 1-Feb
Fri,01-Feb-19	Fri,08-Feb-19	Mon,11-Feb-19	0.7025	Use rate for 1-Feb published on 4-Feb
Mon,04-Feb-19	Mon,11-Feb-19	Tue,12-Feb-19	0.7051	Use rate for 4-Feb published on 5-Feb

[1] Also known as 'Lag'

Figure 2.2: Source: BOE (2020)[5, page 6], Look-back without Observation Shift

Next, we are going to introduce two models used for valuation: Bachelier model (see Section 2.1.1) and Black model (see Section 2.1.2).

2.1.1 Bachelier Model

Becasue of the backward property of RFRs, we recall the Forward Market Model (FMM)

$$dR_j(t) = \sigma_j(t) g_j(t) dW_j(t)$$

And $g_j(t)$ is given as:

$$g_j(t) = \min\left\{\frac{(T_j - t)^+}{T_j - T_{j-1}}, 1\right\}$$

Assume that the dynamics of $R_j(s)$ is normal with constant volatility σ_j and decaying function g_j. Then we have

$$\mathrm{d}R_j(s) = \sigma_j g_j(s)\mathrm{d}W^{T_j}(s)$$

where $W^{T_j}(s)$ is Brownian motion under $\mathbb{Q}^{T_j}$ and R_j is a martingale. Then we are going to deduce the pricing formula following Yan's process[14, section 3.1, page 15-17]. Therefore, we have

$$R_j(T_{j-1}) = R_j(t) + \int_t^{T_{j-1}} \sigma_j g_j(s)\mathrm{d}W_s^{T_j} \quad \text{(forward rate)}$$

$$R_j(T_j) = R_j(t) + \int_t^{T_j} \sigma_j g_j(s)\mathrm{d}W_s^{T_j} \quad \text{(backward rate)}$$

Then the mean for both rates is $R_j(t)$ (deterministic) based on the property of martingale.

By Ito's formula, variances are

$$\mathbb{E}\left[\int_t^{T_{j-1}} (\sigma_j g_j(s)\mathrm{d}W_s)^2\right] = \int_t^{T_{j-1}} (\sigma_j g_j(s))^2 \mathrm{d}s \quad \text{(forward looking)}$$

$$\mathbb{E}\left[\int_t^{T_j} (\sigma_j g_j(s)\mathrm{d}W_s)^2\right] = \int_t^{T_j} (\sigma_j g_j(s))^2 \mathrm{d}s \quad \text{(backward looking)}$$

Thus,

$$R_j(T_{j-1}) \sim \mathcal{N}(R_j(t), \int_t^{T_{j-1}} (\sigma_j g_j(s))^2 \mathrm{d}s) \quad \text{(forward looking)}$$

$$R_j(T_j) \sim \mathcal{N}(R_j(t), \int_t^{T_j} (\sigma_j g_j(s))^2 \mathrm{d}s) \quad \text{(backward looking)}$$

And we can write R_j as $R_j(t) - vZ$ where Z is a standard normal distribution $\mathcal{N}(0,1)$ and

- $v^2 = \int_t^{T_{j-1}} \sigma_j^2 \mathrm{d}s = \sigma_j^2 (T_{j-1} - t)$ for forward-looking rate.
- For the backward-looking rate, when $t \le T_{j-1}$

$$\begin{aligned} v^2 &= \int_t^{T_j} \sigma_j^2 g_j(s)^2 \mathrm{d}s \\ &= \int_t^{T_j} \sigma_j^2 \min\left\{\frac{(T_j - s)^+}{T_j - T_{j-1}}, 1\right\} \mathrm{d}s \\ &= \sigma_j^2 \int_{T_{j-1}}^{T_j} \left(\frac{T_j - s}{T_j - T_{j-1}}\right)^2 \mathrm{d}s + \int_t^{T_{j-1}} \mathrm{d}s \\ &= \sigma_j^2 \left(\frac{1}{3}(T_j - T_{j-1}) + (T_{j-1} - t)\right) \end{aligned}$$

- For the backward-looking rate, when $T_{j-1} < t < T_j$

$$\begin{aligned} v^2 &= \int_t^{T_j} \sigma_j^2 g_j(s)^2 \mathrm{d}s \\ &= \int_t^{T_j} \sigma_j^2 \min\left\{\frac{(T_j - s)^+}{T_j - T_{j-1}}, 1\right\} \mathrm{d}s \\ &= \sigma_j^2 \int_t^{T_j} \left(\frac{T_j - s}{T_j - T_{j-1}}\right)^2 \mathrm{d}s \\ &= \frac{\sigma_j^2 (T_j - t)^3}{3(T_j - T_{j-1})^2} \end{aligned}$$

The expected payoff under Bachelier Model is:

$$
\begin{aligned}
\mathbb{E}^{T_j}\left[(R_j(T)-K)^+|\mathcal{F}_t\right] &= \mathbb{E}^{T_j}\left[(R_j(T)-K)\mathbf{1}_{\{R_j(T)>K\}}|\mathcal{F}_t\right] \\
&= \mathbb{E}^{T_j}\left[(R_j(t)-vZ-K)\mathbf{1}_{\{R_j(t)-vZ>K\}}|\mathcal{F}_t\right] \\
&= \mathbb{E}^{T_j}\left[(R_j(t)-vZ-K)\mathbf{1}_{\{Z<\frac{R_j(t)-K}{v}\}}|\mathcal{F}_t\right] \\
&= (R_j(t)-K)\mathbb{E}^{T_j}\left[\mathbf{1}_{\{Z<\frac{R_j(t)-K}{v}\}}|\mathcal{F}_t\right] - v\mathbb{E}^{T_j}\left[Z\mathbf{1}_{\{Z<\frac{R_j(t)-K}{v}\}}|\mathcal{F}_t\right] \\
&= (R_j(t)-K)\Phi\left(\frac{R_j(t)-K}{v}\right) - v\mathbb{E}^{T_j}\left[Z\mathbf{1}_{\{Z<\frac{R_j(t)-K}{v}\}}|\mathcal{F}_t\right] \\
&= (R_j(t)-K)\Phi\left(\frac{R_j(t)-K}{v}\right) - v\int_{-\infty}^{\frac{R_j(t)-K}{v}} z\phi(z)\mathrm{d}z \\
&= (R_j(t)-K)\Phi\left(\frac{R_j(t)-K}{v}\right) - v\phi\left(\frac{R_j(t)-K}{v}\right)
\end{aligned}
$$

where Φ is the cumulative density function(cdf) of standard normal distribution while $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp -\frac{z^2}{2}$ is the probability density function (pdf) of standard normal distribution. $\mathbf{1}_{\{z<x\}}$ is the indicator function which takes value 1 when $z < x$ and 0 otherwise.

Therefore, we have the valuation for each caplet:

- Forward-looking caplet:

$$V^F_{\text{caplet}}(t) = \tau_j P(t, T_j)((R_j(t)-K)\Phi\left(\frac{R_j(t)-K}{v}\right) + v\phi\left(\frac{R_j(t)-K}{v}\right)$$

  where $v^2 = \sigma_j^2(T_{j-1}-t)$

- Backward-looking caplet:

$$V^B_{\text{caplet}}(t) = \tau_j P(t, T_j)((R_j(t)-K)\Phi\left(\frac{R_j(t)-K}{v}\right) + v\phi\left(\frac{R_j(t)-K}{v}\right)$$

$$\text{where } v^2 = \begin{cases} \sigma_j^2\left(\frac{1}{3}(T_j - T_{j-1}) + (T_{j-1}-t)\right), & t \le T_{j-1}, \\ \frac{\sigma_j^2(T_j-t)^3}{3(T_j-T_{j-1})^2}, & T_{j-1} < t < T_j. \end{cases}$$

2.1.2 Black Model

Assume that the dynamics of $R_j(t)$ is lognormal with constant volatility. That is

$$\mathrm{d}R_j(t) = \sigma_j R_j(t) g_j(t) \mathrm{d}W^{T_j}(t)$$

Firstly, we explain the Black pricing Here we mainly forcus on Call options whose strike is K and maturity is T. S_t represents the asset price at time t. The payoff function is

$$\text{Payoff} = \max(S_T - K, 0)$$

Black-Scholes model set the dynamic of S_t to be

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Here μ and σ represents drift and volatility respectively.

The pdf of S_t at maturity T is calculated from the lognormal distribution:

$$f(S_T) = \frac{1}{S_T\sigma\sqrt{2\pi T}} \exp\left(-\frac{(\ln(S_T/S_0) - (\mu - \frac{\sigma^2}{2})T)^2}{2\sigma^2 T}\right)$$

Taking expectation, the price at time t becomes :

$$\text{BlackCallPrice} = \mathbb{E}^Q[\max(S_T - K, 0)]$$

and then we have:

$$\text{BlackCallPrice} = \int_0^\infty \max(S_T - K, 0) f(S_T) dS_T$$

Solving this integral, we arrive at the final pricing formula

$$\text{BlackCallPrice} = S_t e^{r(T-t)}\Phi(d_1) - K\Phi(d_2)$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Thus, the expected payoff for caplet under Black Model is:

$$\mathbb{E}^{T_j}\left[(R_j(T) - K)^+ | \mathcal{F}_t\right] = R_j(t)\Phi(d_1) - K\Phi(d_2)$$

where $d_{1,2} = \frac{\log(R_j(t)/K) \pm v^2/2}{v}$ and Φ is the cdf of standard normal distribution.

Then the valuation for caplet is:

- Forward-looking caplet:

$$V^F_{\text{caplet}}(t) = \tau_j P(t, T_j)[R_j(t)\Phi(d_1) - K\Phi(d_2)], \quad t \leq T_{j-1}$$

  where $v^2 = \sigma_j^2(T_{j-1} - t)$

- Backward-looking caplet:

$$V^B_{\text{caplet}}(t) = \tau_j P(t, T_j)[R_j(t)\Phi(d_1) - K\Phi(d_2)], \quad t \leq T_j$$

$$\text{where } v^2 = \begin{cases} \sigma_j^2 \left(\frac{1}{3}(T_j - T_{j-1}) + (T_{j-1} - t)\right), & t \leq T_{j-1}, \\ \frac{\sigma_j^2(T_j - t)^3}{3(T_j - T_{j-1})^2}, & T_{j-1} < t < T_j. \end{cases}$$

After we get the price of each caplet, we can add them up and we will have the value of a cap in the RFR world.

In general, the valuation of RFR caps is similar to the valuation of IBOR caps but need to consider decay volatility. We can still use Black and Bachelier model which is the special case of FMM.

2.2 RFR swaps

An interest rate swap (fixed-floating) provides the holder with the ability to exchange cash flow with fixed/floating rates. Following Andrei Lyashenko and Fabio Mercurio's guidance [15, section 6.2, page 17]

Consider a sequence of floating leg payment dates T_j, where $j = a+1, \ldots, b$. And a sequence of fixed leg payment dates T'_j, where $j = c+1, \ldots, d$. Start date $T_a = T'_c$, and end date $T_b = T'_d$. Define τ_j as $\frac{1}{T_j - T_{j-1}}$ and τ'_j as $\frac{1}{T'_j - T'_{j-1}}$. Let K be a constant, referred to as the fixed rate. $R_j(t)$

is the risk-free rate.

Looking at the payoff at $t < T_{a+1}$ from the standpoint of the fixed-rate payer

$$\sum_{j=a+1}^{b} \tau_j P(t, T_j) R_j(t) - K \sum_{j=c+1}^{d} \tau'_j P(t, T'_j)$$

When valuing swaps, our primary focus lies on assessing the cash flows associated with fixed and floating legs. These cash flows are directly derived from forward rates. It's important to note that a swap is not an option, which means there's no need to factor in decay volatility for its pricing. Instead, we solely rely on the forward rates obtained from the market. Consequently, the pricing methodology closely resembles that of LIBOR-based swaps. However, one crucial aspect requires attention. If the valuation date is on or before the accrual period's commencement, we utilize the forward rate for pricing. If the valuation date falls within the accrual period, we employ the compounded rate within the interval $[T_a, T_{a+1}]$.

The swap rate, which we select a proper K such that the payoff function to be 0 at time t, is denoted as:

$$S(t) = \frac{\sum_{j=a+1}^{b} \tau_j P(t, T_j) R_j(t)}{\sum_{j=c+1}^{d} \tau'_j P(t, T'_j)} = \frac{P(t, T_a) - P(t, T_b)}{\sum_{j=c+1}^{d} \tau'_j P(t, T'_j)} \tag{2.2.1}$$

2.3 RFR swaptions

An RFR swaption, whether in the role of the payer or the receiver, is essentially a contract that provides the choice to engage in a spot RFR swap once the maturity date of the swaption is reached.

Looking at the payoff at time T_a from the standpoint of the fixed-rate payer[15, section 6.4, page 19]

$$[S(T_a) - K]^+ \sum_{j=c+1}^{d} \tau'_j P(t, T'_j)$$

where $S(t)$ is defined as Equation (2.2.1).

We need to calculate the dynamic of S(t) to get the price. Just like a swap rate based on LIBOR, an RFR swap rate follows a martingale pattern under the forward swap measure linked to its annuity numeraire[20, section 3, page 5-6]. Consequently, we can hypothesize certain behaviors of S(t) within this framework and value swaptions accordingly. For the convenience of representation, we assume that the term sheets of the fixed leg and the floating leg are the same. The volatility is needed for the option's valuation. The dynamic of $S(t)$ in swap model (SMM) can be[17, page 403]

$$\mathrm{d}S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t) \mathrm{d}W_t^{\alpha,\beta} \tag{2.3.1}$$

In LIBOR case, we approximate the volatility $\sigma^{(\alpha,\beta)}$ in SMM by using LMM. For the technical proof details we can find it in A.1 which is provided by Brigo[17, page 403-407].

The final estimated variance calculated by LMM is

$$\begin{aligned}
(v_{\alpha,\beta}^{\mathrm{LMM}})^2 &= \int_0^{T_\alpha} (d \ln S_{\alpha,\beta}(t))(d \ln S_{\alpha,\beta}(t)) \\
&= \sum_{i,j=\alpha+1}^{\beta} w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) dt.
\end{aligned}$$

Using the previous notation, if we change the time of freezing drift from 0 to valuation date t, we have

$$(v_{\alpha,\beta}^{\mathrm{LMM}})^2 = \sum_{i,j=\alpha+1}^{\beta} w_i(t) w_j(t) F_i(t) F_j(t) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(t)^2} \int_t^{T_\alpha} \sigma_i(s) \sigma_j(s) ds.$$

This variance can be used in the lognormal dynamic of Equation (2.3.1) to calculate the final price. For RFR swaptions, same as caps in Section 2.1, we can consider the case of forward-looking interest rates and backward-looking interest rates separately.

As in Section 2.1.2, set the dynamic of forward rates to be

$$\mathrm{d}R_j(t) = \sigma_j R_j(t) g_j(t) \mathrm{d}W^{T_j}(t)$$

where

$$g_j(t) = \min\left\{ \frac{(T_j - t)^+}{T_j - T_{j-1}}, 1 \right\}$$

Then we have the variance

- Forward-looking:

$$\begin{aligned}(v_{\alpha,\beta}^{\mathrm{LMM}})^2 &= \sum_{i,j=\alpha+1}^{\beta} w_i(t) w_j(t) R_i(t) R_j(t) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(t)^2} \int_t^{T_\alpha} \sigma_i(s)\sigma_j(s) ds \\ &= \sum_{i,j=\alpha+1}^{\beta} w_i(t) w_j(t) R_i(t) R_j(t) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(0)^2} \sigma_i \sigma_j (T_\alpha - t)\end{aligned}$$

- Backward-looking:

$$\begin{aligned}(v_{\alpha,\beta}^{\mathrm{LMM}})^2 &= \sum_{i,j=\alpha+1}^{\beta} w_i(t) w_j(t) R_i(t) R_j(t) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(t)^2} \int_t^{T_{\alpha+1}} \sigma_i(s)\sigma_j(s) ds \\ &= \sum_{i,j=\alpha+1}^{\beta} w_i(t) w_j(t) R_i(t) R_j(t) \rho_{i,j} \frac{1}{S_{\alpha,\beta}(0)^2} v^2\end{aligned}$$

where $v^2 = \begin{cases} \sigma_i \sigma_j \left(\frac{1}{3}(T_{\alpha+1} - T_\alpha) + (T_\alpha - t) \right), & t \leq T_\alpha, \\ \frac{\sigma_i \sigma_j (T_{\alpha+1} - t)^3}{3(T_{\alpha+1} - T_\alpha)^2}, & T_\alpha < t < T_{\alpha+1}. \end{cases}$

Chapter 3

Calibration

In this section, we would like to focus our analysis on SONIA index and the RFR cap quoted on Bloomberg. According to [21, page 7], the normal model can more appropriately reflect historical data, and it allows negative interest rates. Thus, we mainly focus on Bachelier model.

3.1 Yield Curve Calibration

Yield curve calibration involves deducing discount factors $P(t, T)$ and then calculate forward-forward rates. These instruments can include swaps, cash instruments, and various discount factors, among others. In this context, we opt to directly utilize the discount factor $P(t, T)$, which is readily accessible from Bloomberg, for yield curve computation.

After acquiring the discount factor $P(t, T)$, the initial consideration pertains to the interpolation method. This interpolation can be executed using either the zero rate z or the discount factor $P(t, T)$. This concept can be illustrated by Equation (3.1.1):

$$P(0, t) = \exp(-zt) \tag{3.1.1}$$

The typical interpolation state variables include[8, section 4.3.4, page 31-33]

- Discount Factors
- Log of Discount Factors
- Zero Rate
- Zero Rate times Time

The outcome of the interpolation on zero rate times time is the same as it on log of discount factors because of Equation (3.1.1). The interpolation method we can choose are

- Piecewise-Constant
  In this method, forward rates maintain a constant value between two pillars, resulting in a piecewise-constant pattern.
- Linear
  The rates are interpolated by linear function between two pillars.
- Smooth
  This approach assumes that forward rates show a smooth pattern by using tension-spline interpolation method.

For curves with a mixture of futures and swaps we require a mixed linear and smooth interpolator as explained by Burgess [8, section 4.3.4, page 32], see in Table 3.1. However, here we only consider the simplest case which uses discount factor and single interpolation method to calibrate.

Ametrano [6, section 4.5, page 42] provides a visual representation of how interpolation affects the shape of the curve, as demonstrated in Figure 3.1.

Instrument	Tenor	Quote	Interpolation Style
Cash Deposit	1D	2.0248	Linear
OIS swap	6M	7.7345	Spline
OIS swap	1Y	1.5989	Spline
OIS swap	18M	1.5205	Spline
OIS swap	2Y	1.4605	Spline
OIS swap	5Y	1.369	Spltne
LIBOR-OIS Basis swap	7Y	0.26563	Spline
LIBOR-OIS Basis swap	10Y	0.26063	Spline
LIBOR-OIS Basis swap	15Y	0.255	Spline
LIBOR-OIS Basis swap	20Y	0.25375	Spline
LIBOR-OIS Basis swap	30Y	0.25375	Spline
LIBOR-OIS Basis swap	40Y	0.25375	Spline
LIBOR-OIS Basis swap	50Y	0.25375	Spline

Table 3.1: Source: Burgess (2019)[8, section 4.4.1, page 38], OIS Curve Calibration Instruments, USDOIS

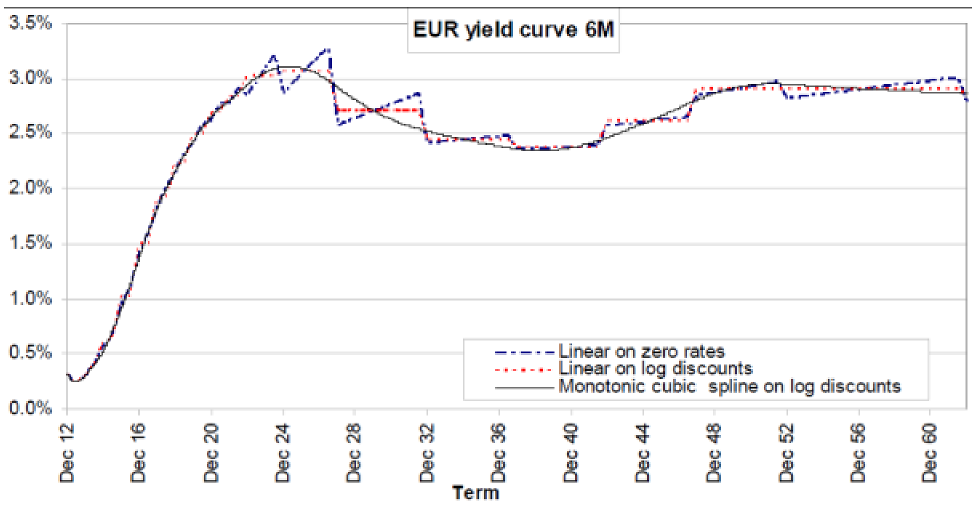


Figure 3.1: Source: Ametrano (2013)[6, section 4.5, page 42], The impact of interpolation on curve shape

We choose linear on zero rates and logcubic on discount factors to compare the results with Bloomberg.

3.2 Boot-stripping RFR Caplet Volatility

In this section we are using Bachelier model in Section 2.1.1. Thus, the caplet pricing formula becomes

$$V_{\text{caplet}}(t) = \tau_j P(t,T_j)((R_j(t) - K)\Phi\left(\frac{R_j(t) - K}{v}\right) + v\phi\left(\frac{R_j(t) - K}{v}\right)$$

where $v^2 = \begin{cases} \sigma_j^2 \left(\frac{1}{3}(T_j - T_{j-1}) + (T_{j-1} - t) \right), & t \leq T_{j-1}, \\ \frac{\sigma_j^2 (T_j - t)^3}{3(T_j - T_{j-1})^2}, & T_{j-1} < t < T_j. \end{cases}$

At present, the cap volatility matrix accessible on Bloomberg offers par volatility, calculated using the Bachelier model [22, section 4.2, page 11]. This volatility matrix is constructed based on the presumption of a 3M payment frequency. When we are using "par volatility", it means that the volatility is a constant value for each 3M caplet from valuation date to the quoted maturity.

Note that for RFR volatility matrix quoted by Bloomberg it doesn't consider the decay function when calculating the par volatility. Thus $v^2 = \sigma_j(T_j - t)$, which is different from v^2 for both forward-looking and backward-looking formula. We represent $v^2 = \sigma_j(T_j - t)$ as $\text{Caplet}_{\text{par}}$.

The relationship between par volatility and forward volatility is:

$$\text{Cap}(T_n) = \sum_{i=1}^{n} \text{Caplet}_{\text{par}}(T_i, \sigma_{\text{par}}(T_i), K_i) = \sum_{i=1}^{n} \text{Caplet}(T_i, \sigma_{\text{forward}}(T_i), K_i)$$

Our aim is to calculate the σ_{forward} from σ_{par}. We can first calculate the cap price for each maturity using par volatility σ_{par}. Then calculate the difference between two cap price with neighboring maturities[23, section 2]. The difference we get is the caplet price for a certain period, and the price is the same as the price calculated by forward volatility with a decay pattern.

The calculation is following the bootstrapping method expressed by Yan[14, section 4.2, page 20]:

- Because the volatility is the same for each caplet, we can sum $\text{Caplet}_{\text{par}}$ to get the cap price

$$\text{Cap}(T_n) = \sum_{i=1}^{n} \text{Caplet}_{\text{par}}(T_i, \sigma_{\text{par}}(T_i), K_i)$$

- For the first cap which is also the first caplet ($T_1 \approx 0.25$), the pricing formula is different when calculating σ_{par} and σ_{forward}, so we still need to use root-finding method to calculate σ_{forward}

$$\text{Cap}(T_1) = \text{Caplet}_{\text{par}}(T_1, \sigma_{\text{par}}(T_1), K_1) = \text{Caplet}(T_1, \sigma_{\text{forward}}(T_1), K_1)$$

- And then, for the index $i = 2, ..., n$
  - In real situations, we can get the cap price by adding up the price of each caplets whose volatility is σ_{forward} with decay pattern. It can be shown as:

$$\text{Cap}(T_{i-1}, K_i) = \sum_{j=1}^{i-1} \text{Caplet}(T_j, \sigma_{\text{forward}}(T_j), K_i)$$

  - The difference we get is the caplet price for a certain period:

$$\text{Caplet}(T_i, \sigma_{\text{forward}}(T_i), K_i) = \text{Cap}(T_i, K_i) - \text{Cap}(T_{i-1}, K_i)$$

  - To get the value of σ_{forward}, we can use the Newton–Raphson method (see Figure 3.2). The Newton–Raphson method can be expressed as:

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)}{f'(\sigma_n)}$$

It's important to highlight that, when pricing IBOR caps, the rate for the first caplet is already known which means that the caplet price is alreadt known as well. Thus, we don't consider the first caplet when pricing. In contrast, if we are pricing RFR caps, we can't no the rate for the first caplet because it's daily compounded and back-ward looking. So we need to consider the first

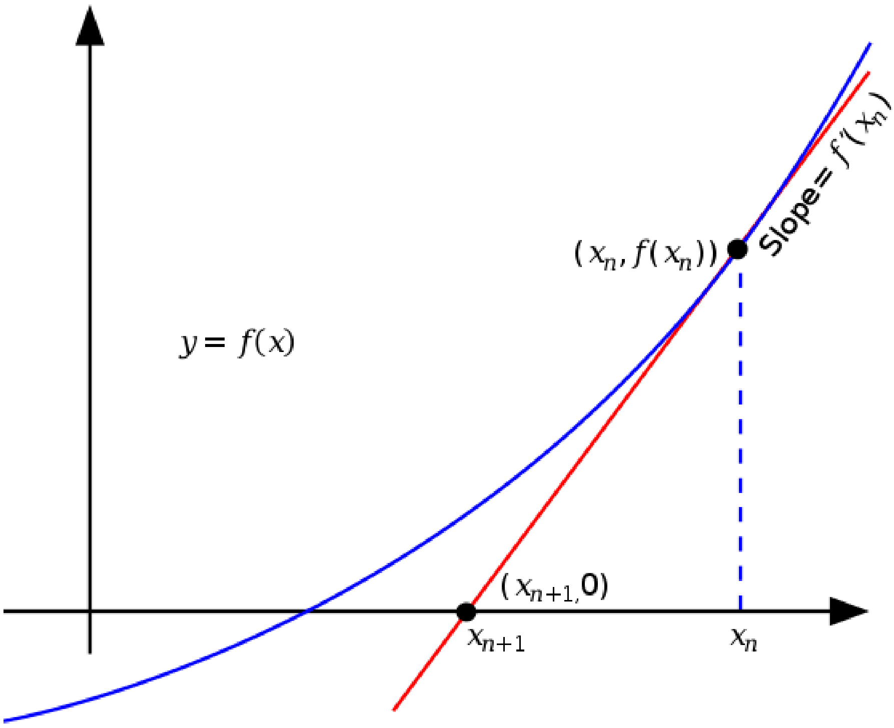


Figure 3.2: Source: Olegalexandrov (2008)[7], Newton–Raphson method

Term	0.0025	0.005	0.0075	0.01	0.0125	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.055	0.06
31/05/2024	207.27	203.24	199.08	194.79	190.39	185.87	176.46	166.5	155.76	143.88	130.72	118.64	113.1	112.66	113.13
02/06/2025	190.54	187.9	185.18	182.4	179.57	176.69	170.82	164.82	158.66	152.12	144.9	137.56	133.23	132.94	135.04
01/06/2026	175.97	173.56	171.1	168.63	166.16	163.71	158.95	154.52	150.58	147.14	143.95	141.08	140.33	142.81	147.76
01/06/2027	163.77	161.65	159.51	157.38	155.29	153.25	149.46	146.25	143.88	142.41	141.63	141.4	142.87	146.95	153.19
31/05/2028	156.07	154.21	152.36	150.53	148.76	147.07	144.06	141.76	140.43	140.15	140.73	141.96	144.68	149.63	156.43
31/05/2029	149.6	147.9	146.21	144.56	142.98	141.49	138.91	137.12	136.36	136.76	138.13	140.24	143.68	149.01	155.92
31/05/2030	142.47	141.01	139.56	138.17	136.85	135.63	133.64	132.46	132.33	133.37	135.39	138.13	142.05	147.58	154.46
02/06/2031	137.21	135.87	134.56	133.31	132.14	131.08	129.44	128.65	128.96	130.45	132.95	136.19	140.49	146.21	153.11
01/06/2032	130.66	129.65	128.67	127.77	126.94	126.23	125.26	125.08	125.87	127.72	130.47	133.89	138.22	143.76	150.3
31/05/2033	125.39	124.61	123.87	123.2	122.61	122.14	121.62	121.84	122.96	125.05	127.96	131.49	135.81	141.17	147.39
31/05/2035	117.6	117.12	116.68	116.3	116.02	115.84	115.87	116.57	118.07	120.41	123.48	127.08	131.32	136.36	142.06
01/06/2038	110.46	110.44	110.43	110.46	110.53	110.67	111.18	112.14	113.65	115.77	118.41	121.42	124.85	128.79	133.11
01/06/2043	102.54	102.88	103.22	103.56	103.94	104.34	105.31	106.57	108.2	110.27	112.72	115.41	118.36	121.62	125.09
01/06/2048	97.18	97.47	97.81	98.19	98.63	99.16	100.48	102.23	104.45	107.15	110.24	113.58	117.17	121.04	125.14
02/06/2053	93.72	94.04	94.42	94.87	95.4	96.02	97.59	99.64	102.18	105.19	108.58	112.22	116.09	120.22	124.56

Table 3.2: The par volatility surface

caplet when dealing with RFR case.

We use linear interpolation method and flat extrapolation method to strip 4 caplet volatility a year with reset period 3M. Table 3.2 shows the par volatility surface.

There are several features we can see from the volatility matrix:

- Smile and Skew
  As shown in the Figure 3.3, for a given maturity, volatility follows a U-shaped pattern across various strike prices, with lower volatility near the current interest rate and higher volatility for strikes further from the current rate.

- Maturities
  With increasing maturity, volatility may progressively approach a relatively consistent level (refer to Figure 3.4). This phenomenon can be attributed to the difficulty in precisely forecasting price expectations for distant future periods.

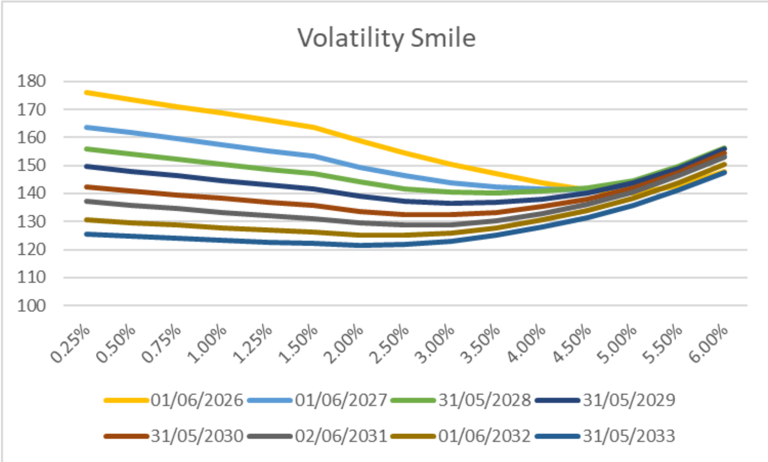


Figure 3.3: Volatility smile

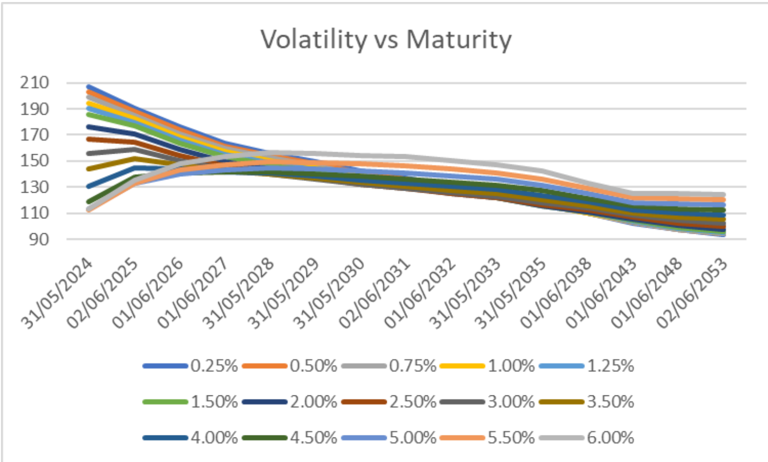


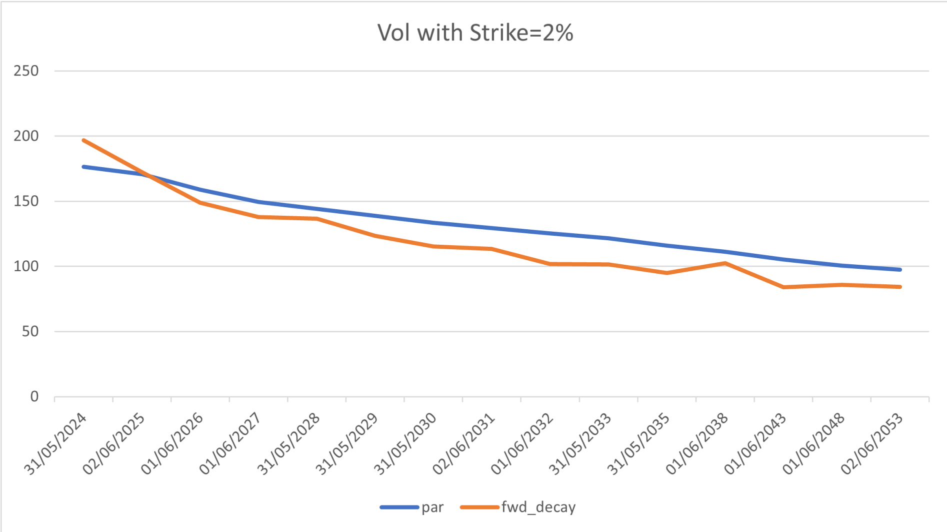
Figure 3.4: Volatility vs Maturity

The forward decay volatility is shown in Table 3.3. Because the forward decay volatility will decrease during a accrual period, the forward decay volatility is greater than par volatility when we look at the first caplet. When it comes to pricing caplets, we use linear interpolation to get the volatility for quarterly caplets.

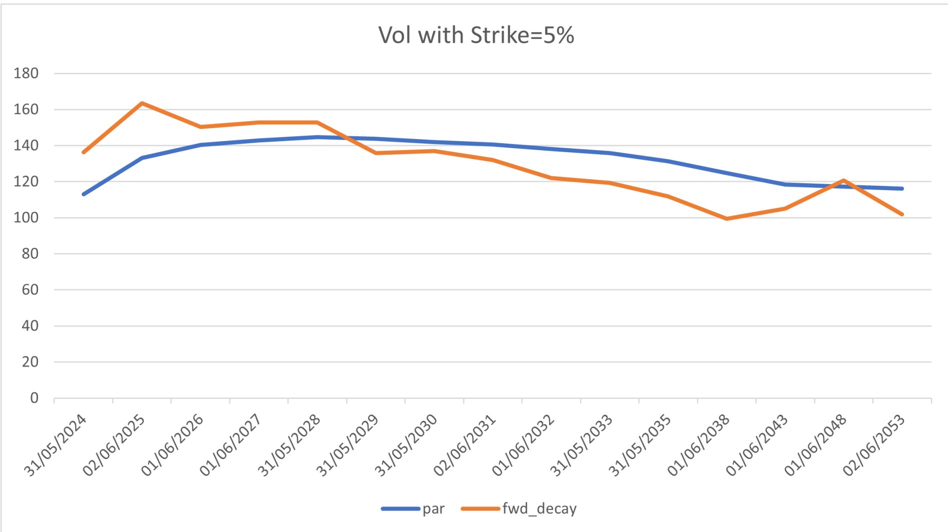
Term	0.0025	0.005	0.0075	0.01	0.0125	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.055	0.06
31/05/2024	229.12	224.83	220.42	215.88	211.24	206.50	196.74	186.66	176.31	165.76	154.99	144.47	136.33	131.02	128.90
02/06/2025	189.57	187.12	184.62	182.12	179.63	177.16	172.39	168.00	164.31	161.66	160.61	161.46	163.57	166.71	170.50
01/06/2026	168.95	166.04	163.08	160.12	157.18	154.31	148.88	144.24	140.89	139.48	140.42	144.01	150.38	159.11	169.55
01/06/2027	150.42	148.37	146.37	144.40	142.55	140.80	137.86	135.98	135.70	137.26	140.88	146.27	152.73	159.81	167.34
31/05/2028	147.26	145.43	143.67	141.96	140.33	138.88	136.55	135.33	135.51	137.41	141.02	146.15	152.76	160.60	169.09
31/05/2029	134.65	132.83	130.98	129.24	127.59	126.03	123.34	121.68	121.29	122.60	125.56	130.13	135.80	142.09	149.00
31/05/2030	120.11	119.08	118.08	117.23	116.44	115.82	115.39	115.93	117.82	121.05	125.49	130.74	136.87	143.65	150.82
02/06/2031	122.14	120.53	119.05	117.60	116.30	115.14	113.37	112.77	113.73	116.19	120.24	125.77	132.08	139.09	146.54
01/06/2032	92.43	93.62	94.70	96.02	97.25	98.67	101.76	105.20	108.54	112.01	115.35	118.66	122.08	125.55	129.16
31/05/2033	103.76	103.16	102.74	102.18	101.89	101.70	101.64	102.31	104.07	106.71	110.21	114.58	119.40	124.60	130.13
31/05/2035	90.31	90.78	91.18	91.74	92.41	93.10	94.87	97.07	99.51	102.22	105.27	108.37	111.85	115.51	119.34
01/06/2038	98.01	98.92	99.81	100.60	101.14	101.72	102.36	102.56	102.41	102.03	101.28	100.49	99.44	98.32	96.95
01/06/2043	80.46	80.69	80.95	81.21	81.83	82.33	84.07	86.37	89.14	92.44	96.40	100.54	105.14	110.07	115.36
01/06/2048	86.01	85.13	84.57	84.32	83.99	84.41	85.82	88.83	93.48	99.36	105.91	113.18	120.60	127.99	135.34
02/06/2053	77.92	78.66	79.31	80.05	81.22	81.93	84.25	86.83	89.47	92.21	95.32	98.52	101.99	105.86	109.90

Table 3.3: The forward decay volatility surface

The volatility at strike rates of 2% in Figure 3.5(a) and 5% in Figure 3.5(b) is compared. When we compare the performance of par volatility and forward decay volatility, we observe that forward decay volatility exhibits greater fluctuations, while par volatility remains relatively stable. This is because par volatility is calculated based on the collective forward decay volatility for all maturities leading up to it, mitigating the individual impact of any single maturity.



(a) Volatility with strike=2%.



(b) Volatility with strike=5%.

Figure 3.5: Volatility Comparison Between Par & Fwd_decay.

3.3 Data Processing

Once we have obtained the discount factors, volatility matrix, and historical rates from Bloomberg, we can proceed to price caps in the Valuation Platform (VP). The data processing can be roughly categorized into four main steps.

- Yield Curve
  Firstly, we address the yield curve by interpolating the discount factors and calculating the forward-forward rates.

- Volatility
  Next, we manage the volatility matrix, converting par volatility to forward decay volatility. This allows us to select suitable interpolation methods for volatility extraction during subsequent calculations.

- Contract

Subsequently, we process the cap contracts input by extracting relevant parameters and selecting appropriate interpolation techniques based on these parameters.

- Pricing
  Lastly, we move on to the pricing process. We choose a pricing model according to the specified parameters and establish the term structure. If the valuation date falls within the period of the first caplet, historical rates are utilized for the calculations. Simultaneously, we employ suitable interpolation methods to determine discount factors, forward-forward rates, and volatility. These values are then incorporated into the model to ultimately derive the cap's price.

Chapter 4

Pricing Results

To check if the pricing process we use is correct, we choose currency GBP and set the valuation date to be 31/05/2023. We price caps for the following cases:

- spot starting (valuation date is T_{j-1})
- start in the future (valuation date is before T_0)
- valuation date in $[T_{j-1}, T_j]$

The cap contract is set to be as Table 4.1. According to [21, page 7], the normal model can more appropriately reflect historical data, and it allows negative interest rates. We set model to be Bachelier for the rest. We use normalised error (4.0.1) to check the gap between VP and

Valuation currency	GBP
Valuation Date	31/05/2023
Interpolation variable for yield curves	modified following
Interpolation method	modified following
Model for cap	Normal
Cap strike in volatility surface	Absolute
Interpolation for cap volatility surface strike axis	Linear
Interpolation for cap volatility surface expiry axis	Linear
Extrapolation for cap volatility surface strike axis	Flat
Extrapolation for cap volatility surface strike axis	Flat
Discount tenor for pricing	OIS
Index tenor	OIS
Start date	modified following
End date	modified following
Period Frequency	3 M
Notional	10,000,000
Cap Rate / Strike	4.00%
lookback period	0 D
Day counter	Actual/365
Calendar	Target
Stub	short start
Payment lag	0 D

Table 4.1: Cap contract

Bloomberg.

$$\text{Normalised error(bp)} = \frac{|\text{Cap}_{\text{VP}} - \text{Cap}_{\text{BBG}}|}{\text{Notional} * \max(1, \text{Time to Maturity})} * 10000 \tag{4.0.1}$$

4.1 Spot Starting

For spot starting we set start date to be 31/05/2023 and end date to be 31/05/2025. In this case we can use the volatility matrix which is directly collected from Bloomberg without any interpolation. Therefore, we can check if the yield curve we get is correct. Then we check the reset rates and cash flow. Finally we will know if the pricing process is correct.

One of the reasons that will cause errors is the interpolation method for yield curve. Thus we use both linear method on zero rates and logcubic method to check with Bloomberg respectively.

4.1.1 Linear Method on Zero Rates

First we compare the cash flow calculated by VP (see Table 4.2) and Bloomberg (see Figure 4.1).

Start period	End period	Payment date	Notional	Strike	Volatility	Reset Rate	Discount	Price
31/05/2023	31/08/2023	31/08/2023	10000000	0.04	0.015499	0.047553	0.988156	19026.4
31/08/2023	30/11/2023	30/11/2023	10000000	0.04	0.015499	0.052259	0.975447	30674.16
30/11/2023	29/02/2024	29/02/2024	10000000	0.04	0.015499	0.05373	0.962553	34683.64
29/02/2024	31/05/2024	31/05/2024	10000000	0.04	0.015499	0.052696	0.949937	33815.78
31/05/2024	30/08/2024	30/08/2024	10000000	0.04	0.015639	0.049637	0.938324	29066.78
30/08/2024	29/11/2024	29/11/2024	10000000	0.04	0.015779	0.04882	0.927027	28933.91
29/11/2024	28/02/2025	28/02/2025	10000000	0.04	0.015919	0.046044	0.916506	25996.48
28/02/2025	30/05/2025	30/05/2025	10000000	0.04	0.016056	0.044703	0.906406	25363.33

Table 4.2: Cash flow with linear method on zero rate using VP when spot starting

Pay Date	Da...	Notional	Cap Strike	Payment	Discount	Intrinsic PV	Time PV	PV
08/31/2023	92	10,000,000.00	4.00000	19,251.52	0.988156	18,814.64	208.86	19,023.50
11/30/2023	91	10,000,000.00	4.00000	31,446.10	0.975447	29,811.98	862.02	30,674.00
02/29/2024	91	10,000,000.00	4.00000	36,044.95	0.962553	32,949.88	1,745.28	34,695.16
05/31/2024	92	10,000,000.00	4.00000	35,613.32	0.949937	30,378.09	3,452.33	33,830.42
08/30/2024	91	10,000,000.00	4.00000	31,201.11	0.938325	22,552.50	6,724.27	29,276.77
11/29/2024	91	10,000,000.00	4.00000	31,422.55	0.927027	20,526.29	8,603.26	29,129.55
02/28/2025	91	10,000,000.00	4.00000	28,324.53	0.916506	13,809.13	12,150.48	25,959.61
05/30/2025	91	10,000,000.00	4.00000	27,745.97	0.906406	10,605.59	14,543.53	25,149.13

Figure 4.1: Cash flow with linear method on zero rate using BBG when spot starting

The payment date is the same. There are only slight differences on discount and PV. Then we check the normalized error to see if it's within threshold. The cap price calculated by VP is 227, 560.48 GBP and by Bloomberg is 227, 738.14 GBP. The normalized error is 0.089 bps which means the price is good.

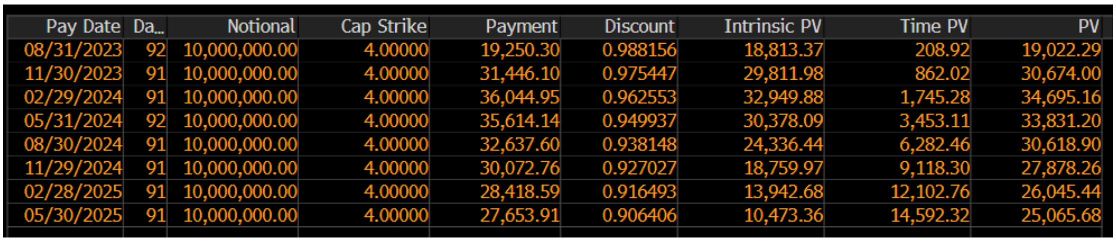
4.1.2 Logcubic Method on Discount Factor

Then we compare the cash flow calculated by VP (see Table 4.3) and Bloomberg (see Figure 4.2).

The payment date remains the same, with only minor disparities in discount and present value. We subsequently assess the normalized error to determine whether it falls within the specified threshold. The cap price calculated by VP is 227, 684.41 GBP and by Bloomberg is 227, 830.94 GBP. The normalized error is 0.073 bps which means the price is good as well. From this we can conclude that the pricing method is correct and then we can look at if the interpolation of volatility matrix is acceptable.

Start period	End period	Payment date	Notional	Strike	Volatility	Reset Rate	Discount	Price
31/05/2023	31/08/2023	31/08/2023	10000000	0.04	0.015499	0.047553	0.988156	19026.42
31/08/2023	30/11/2023	30/11/2023	10000000	0.04	0.015499	0.052259	0.975447	30674.24
30/11/2023	29/02/2024	29/02/2024	10000000	0.04	0.015499	0.05373	0.962553	34683.77
29/02/2024	31/05/2024	31/05/2024	10000000	0.04	0.015499	0.052706	0.949937	33834.79
31/05/2024	30/08/2024	30/08/2024	10000000	0.04	0.015639	0.050378	0.938147	30326.55
30/08/2024	29/11/2024	29/11/2024	10000000	0.04	0.015778	0.048088	0.927027	27783.69
29/11/2024	28/02/2025	28/02/2025	10000000	0.04	0.015917	0.046102	0.916493	26075.52
28/02/2025	30/05/2025	30/05/2025	10000000	0.04	0.016053	0.044643	0.906406	25279.43

Table 4.3: Cash flow with logcubic method on discount factor using VP when spot starting



Pay Date	Da...	Notional	Cap Strike	Payment	Discount	Intrinsic PV	Time PV	PV
08/31/2023	92	10,000,000.00	4.00000	19,250.30	0.988156	18,813.37	208.92	19,022.29
11/30/2023	91	10,000,000.00	4.00000	31,446.10	0.975447	29,811.98	862.02	30,674.00
02/29/2024	91	10,000,000.00	4.00000	36,044.95	0.962553	32,949.88	1,745.28	34,695.16
05/31/2024	92	10,000,000.00	4.00000	35,614.14	0.949937	30,378.09	3,453.11	33,831.20
08/30/2024	91	10,000,000.00	4.00000	32,637.60	0.938148	24,336.44	6,282.46	30,618.90
11/29/2024	91	10,000,000.00	4.00000	30,072.76	0.927027	18,759.97	9,118.30	27,878.26
02/28/2025	91	10,000,000.00	4.00000	28,418.59	0.916493	13,942.68	12,102.76	26,045.44
05/30/2025	91	10,000,000.00	4.00000	27,653.91	0.906406	10,473.36	14,592.32	25,065.68

Figure 4.2: Cash flow with logcubic method on discount factor using BBG when spot starting

4.1.3 Monte Carlo Results

We use Monte Carlo method to price the first caplet (See Figure 4.3). It can directly show the dynamic of R_i using the real data. And can also help us to check the outcome. We set the reset rate to be 0.047553, volatility to be 0.015499, discount factor to be 0.988156, strike to be 0.04, notional to be 10000000 and time period to be 0.25 as shown in Table 4.2 and 4.3. Because it is spot starting, we can see that the volatility decays during the process and finally becomes 0.

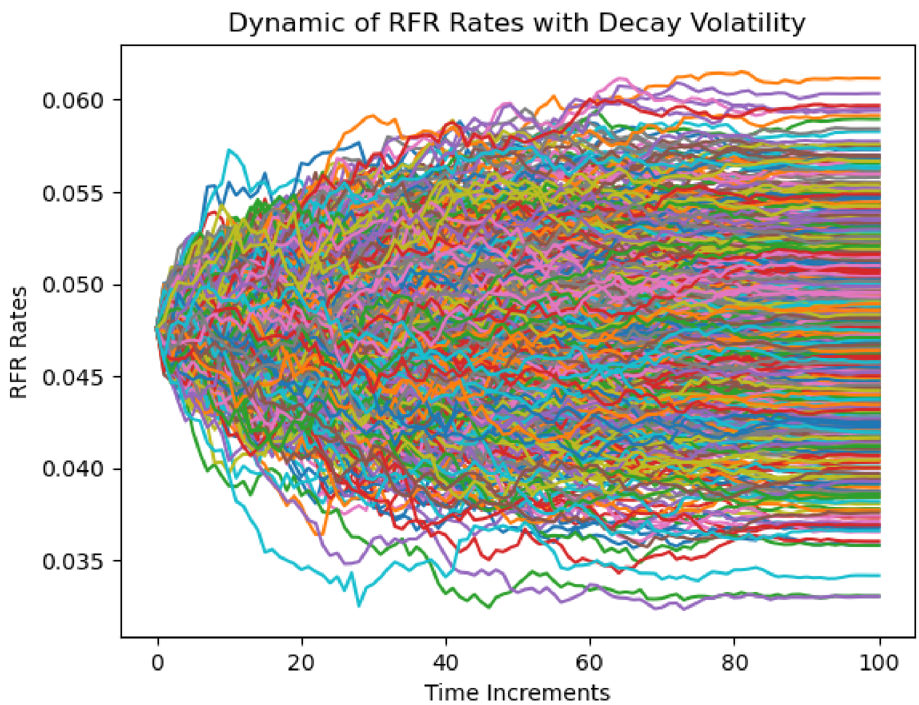


Figure 4.3: Monte Carlo of the first caplet

With a step size of 100 and 1000 paths, the caplet price computed using the Monte Carlo method stands at 18,592.59. When compared to 19,026.42 from VP and 19,022.29 from Bloomberg, the normalized errors are 0.217 bps and 0.215 bps, respectively, which is quite satisfactory.

4.2 Start in The Future

The logcubic method exhibits smoothness, as evident in Figure 3.1. For the remaining calculations, we've configured the interpolation method to be logcubic on the discount factor.

We set start date to be 31/07/2023 and end date to be 31/07/2025. In this case we need to use linear interpolation method to extract volatility from the volatility matrix. Therefore, we can check if this interpolation method is acceptable.

Start period	End period	Payment date	Notional	Strike	Volatility	Reset Rate	Discount	Price
31/07/2023	31/10/2023	31/10/2023	10000000	0.04	0.015499	0.051036	0.979727	27923.83
31/10/2023	31/01/2024	31/01/2024	10000000	0.04	0.015499	0.053792	0.966621	34946.03
31/01/2024	30/04/2024	30/04/2024	10000000	0.04	0.015499	0.053145	0.954118	33679.42
30/04/2024	31/07/2024	31/07/2024	10000000	0.04	0.015591	0.051233	0.941956	31782.54
31/07/2024	31/10/2024	31/10/2024	10000000	0.04	0.015733	0.048837	0.930502	28878.8
31/10/2024	31/01/2025	31/01/2025	10000000	0.04	0.015874	0.046688	0.91968	26818.49
31/01/2025	30/04/2025	30/04/2025	10000000	0.04	0.01601	0.04503	0.909691	24874.47
30/04/2025	31/07/2025	31/07/2025	10000000	0.04	0.015739	0.043967	0.899721	24959.58

Table 4.4: Cash flow priced by VP when starting in the future

Pay Date	Da...	Notional	Cap Strike	Payment	Discount	Intrinsic PV	Time PV	PV
10/31/2023	92	10,000,000.00	4.00000	28,512.29	0.979727	27,256.38	677.87	27,934.25
01/31/2024	92	10,000,000.00	4.00000	36,165.70	0.966621	33,600.71	1,357.82	34,958.53
04/30/2024	90	10,000,000.00	4.00000	35,301.45	0.954118	30,927.30	2,754.44	33,681.74
07/31/2024	92	10,000,000.00	4.00000	33,912.40	0.941957	26,635.04	5,308.99	31,944.04
10/31/2024	92	10,000,000.00	4.00000	31,188.36	0.930503	20,732.37	8,288.48	29,020.85
01/31/2025	92	10,000,000.00	4.00000	29,166.54	0.919680	15,507.05	11,316.82	26,823.87
04/30/2025	89	10,000,000.00	4.00000	27,157.42	0.909691	11,154.43	13,550.45	24,704.88
07/31/2025	92	10,000,000.00	4.00000	26,898.08	0.899721	8,990.15	15,210.63	24,200.77

Figure 4.4: Cash flow priced by Bloomberg when starting in the future

Then we compare the cash flow calculated by VP (see Table 4.4) and Bloomberg (see Figure 4.4). We verify the normalized error to determine if it falls within the established threshold. The cap price computed by VP is 233,863.17 GBP, while Bloomberg's calculation yields 233,268.93 GBP. The normalized error stands at 0.30 basis points (bps), indicating that the interpolation of the volatility matrix is deemed acceptable.

4.3 Valuation Date in $[T_{j-1}, T_j]$

We set start date to be 31/03/2023 and end date to be 31/03/2025. In this case we need to use compounded rate from 31/03/2023 to 31/05/2023 to calculate the price of the first caplet. Therefore, we can check if this calculation is correct.

Start period	End period	Payment date	Notional	Strike	Volatility	Reset Rate	Discount	Price
31/03/2023	30/06/2023	30/06/2023	10000000	0.04	0.015499	0.043637	0.996304	9034.616
30/06/2023	29/09/2023	29/09/2023	10000000	0.04	0.015499	0.049407	0.9842	23542.68
29/09/2023	29/12/2023	29/12/2023	10000000	0.04	0.015499	0.05322	0.971312	33058.77
29/12/2023	28/03/2024	28/03/2024	10000000	0.04	0.015499	0.05354	0.958655	34164.78
28/03/2024	28/06/2024	28/06/2024	10000000	0.04	0.015542	0.05208	0.946234	32960.52
28/06/2024	30/09/2024	30/09/2024	10000000	0.04	0.015686	0.049651	0.934287	30434.69
30/09/2024	31/12/2024	31/12/2024	10000000	0.04	0.015825	0.047384	0.923262	27425.37
31/12/2024	31/03/2025	31/03/2025	10000000	0.04	0.015964	0.045522	0.913014	25410.76

Table 4.5: Cash flow priced by VP when valuation date in $[T_{j-1}, T_j]$

Then we compare the cash flow calculated by VP (see Table 4.5) and Bloomberg (see Figure 4.5). The cap price derived from VP is 216,032.19 GBP, while Bloomberg's calculation yields 216,106.35 GBP. The normalized error is 0.037 basis points (bps), indicating that the compounding method

Pay Date	Da...	Notional	Cap Strike	Payment	Discount	Intrinsic PV	Time PV	PV
06/30/2023	91	10,000,000.00	4.00000	9,066.04	0.996304	9,032.52	0.01	9,032.53
09/29/2023	91	10,000,000.00	4.00000	23,728.06	0.984200	22,885.30	467.87	23,353.17
12/29/2023	91	10,000,000.00	4.00000	34,045.93	0.971312	32,023.55	1,045.66	33,069.21
03/28/2024	90	10,000,000.00	4.00000	35,646.07	0.958655	32,012.83	2,159.46	34,172.28
06/28/2024	92	10,000,000.00	4.00000	34,922.05	0.946235	28,804.12	4,240.33	33,044.45
09/30/2024	94	10,000,000.00	4.00000	32,826.08	0.934288	23,222.45	7,446.56	30,669.01
12/31/2024	92	10,000,000.00	4.00000	29,737.07	0.923262	17,171.25	10,283.87	27,455.11
03/31/2025	90	10,000,000.00	4.00000	27,722.01	0.913014	12,429.06	12,881.53	25,310.59

Figure 4.5: Cash flow priced by Bloomberg when valuation date in $[T_{j-1}, T_j]$

is considered acceptable.

In conclusion, the pricing difference between VP and Bloomberg is small enough which is within the threshold. The pricing method we implemented in VP is reliable.

Conclusion

We have constructed the transition from the LMM to the FMM based on the methodology described in reference [15]. This process encompasses extending the definition of discount factors, transforming forward-looking rates into backward-looking rates, and introducing decay functions to simulate the dynamics of backward-looking rates within an interval. Subsequently, we presented the expressions of this dynamic process for caps within the framework of both the Bachelier model and the Black model. Also, we introduced some ideas for pricing RFR swaps and swaptions.

Furthermore, we employed the more widely used Bachelier model for pricing. We selected three different parameter configurations and compared them with the prices from Bloomberg. The observed differences in prices were quite small and fell within an acceptable range. This indicates that the approach we employed is indeed acceptable.

Throughout this study, whether in the construction of the pricing model, the selection of yield curve and volatility interpolation methods, or the approach to bootstrap the volatility matrix, we have provided a comprehensive validation and practical implementation of these methods.

It is noteworthy that the volatility matrix provided on Bloomberg is specifically designed for caps with a payment frequency of three months. For pricing caps with a different payment frequency, such as six months, an adjustment is required using techniques like freezing drift and utilizing swap rates. The exact implementation of these adjustments requires further investigation and research.

Appendix A

Technical Proofs

A.1 Proof of LIBOR Swaption pricing

As discussed in Brigo's lecture notes[17, page 403-407], the dynamic of $S(t)$ in swap model (SMM) can be

$$\mathrm{d}S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t)S_{\alpha,\beta}(t)\mathrm{d}W_t^{\alpha,\beta}$$

Then, we have

$$\int_0^{T_\alpha} \sigma^2_{\alpha,\beta}(t)dt = \int_0^{T_\alpha} \sigma_{\alpha,\beta}(t)dW_{\alpha,\beta}(t)\sigma_{\alpha,\beta}(t)dW_{\alpha,\beta}(t) = \int_0^{T_\alpha} \left(d\ln S_{\alpha,\beta}(t)\right)\left(d\ln S_{\alpha,\beta}(t)\right)$$

$$\begin{aligned}
S_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t), \\
w_i(t) &= w_i(F_{\alpha+1}(t), F_{\alpha+2}(t), \ldots, F_\beta(t)) \\
&= \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}}
\end{aligned}$$

Freeze the values of the w_i at time 0, we have

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t)F_i(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)F_i(t).$$

Then

$$dS_{\alpha,\beta} \approx \sum_{i=\alpha+1}^{\beta} w_i(0)dF_i = (\ldots)dt + \sum_{i=\alpha+1}^{\beta} w_i(0)\sigma_i(t)F_i(t)dZ_i(t)$$

$$\begin{aligned}
dS_{\alpha,\beta}(t)dS_{\alpha,\beta}(t) &\approx \sum_{i,j=\alpha+1}^{\beta} w_i(0)\sigma_i(t)F_i(t)dZ_i w_j(0)F_j(t)\sigma_j(t)dZ_j \\
&= \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)dt.
\end{aligned}$$

$$\begin{aligned}
(d\ln S_{\alpha,\beta}(t))(d\ln S_{\alpha,\beta}(t)) &= \frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} \cdot \frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} \\
&\approx \frac{1}{S_{\alpha,\beta}(t)^2} \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)dt.
\end{aligned}$$

Here we use freezing drift method again to approximate. Set $F(t)$ to be $F(0)$

$$(d\ln S_{\alpha,\beta})(d\ln S_{\alpha,\beta}) \approx \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}\frac{1}{S_{\alpha,\beta}(0)^2}\sigma_i(t)\sigma_j(t)dt.$$

Thus, the final estimated variance calculated by LMM is

$$\begin{aligned}(v_{\alpha,\beta}^{\mathrm{LMM}})^2 &= \int_0^{T_\alpha}(d\ln S_{\alpha,\beta}(t))(d\ln S_{\alpha,\beta}(t))\\ &= \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}\frac{1}{S_{\alpha,\beta}(0)^2}\int_0^{T_\alpha}\sigma_i(t)\sigma_j(t)dt.\end{aligned}$$

Bibliography

[1] Andreas Schrimpf and Vladyslav Sushko. Beyond libor: a primer on the new benchmark rates. *BIS Quarterly Review March*, 2019.

[2] Bank of England. Why do firms need to accelerate the transition from libor benchmarks? https://www.bankofengland.co.uk/bank-overground/2020/why-do-firms-need-to-accelerate-the-transition-from-libor-benchmarks, 2020. Online; accessed 29 July 2023.

[3] Kristianna Nelson and Kevin Jones. Usd libor transition: Credit sensitive fallback rates. https://www.chathamfinancial.com/insights/usd-libor-transition-credit-sensitive-fallback-rates, 2022. Online; accessed 29 July 2023.

[4] Ioannis Rigopoulos. Using the bootstrapped market sofr caplet normal vol surface to price in excel interest rate caps/floors on backward/forward looking sofr term rates. https://blog.deriscope.com/index.php/en/interest-rate-sofr-cap-floor, 2023. Online; accessed 29 July 2023.

[5] Bank of England. Working group on sterling risk-free rates detailed loans conventions. https://www.bankofengland.co.uk/-/media/boe/files/markets/benchmarks/rfr/uk-loan-conventions-supporting-slides.pdf, 2021. Online; accessed 29 July 2023.

[6] Ferdinando M Ametrano and Marco Bianchetti. Everything you always wanted to know about multiple interest rate curve bootstrapping but were afraid to ask. *Available at SSRN 2219548*, 2013.

[7] Wikipedia. Newton's method. https://en.wikipedia.org/wiki/Newton%27s_method, 2008. Online; accessed 29 July 2023.

[8] Nicholas Burgess. Libor benchmark reform: An overview of libor changes and its impact on yield curves, pricing and risk. *Pricing and Risk (September 6, 2019)*, 2019.

[9] Financial Conduct Authority. UK Financial Conduct Authority Presentation: Feedback on the dear CEO letter on Libor Transition. https://www.fca.org.uk/news/speeches/libor-preparing-end, 2019, Online; accessed 29 July 2023.

[10] CIBC Mellon. Cibc mellon libor transition client faqs. https://www.cibcmellon.com/en/_locale-assets/pdf/our-thinking/2021/cibc-mellon-libor-transition-client-faqs-november-2021.pdf, 2021, Online; accessed 29 July 2023.

[11] Bank of England. Transition to Sterling Risk-Free Rates from LIBOR. https://www.bankofengland.co.uk/markets/transition-to-sterling-risk-free-rates-from-libor, 2023, Online; accessed 29 July 2023.

[12] Gavin Finch and Liam Vaughan. The man who invented libor. https://www.bloomberg.com/news/features/2016-11-29/the-man-who-invented-libor-iw3fpmed?in_source=embedded-checkout-banner, 2016. Online; accessed 29 July 2023.

[13] Eckart Bueren and Holger Fleischer. Die libor-manipulation zwischen kapitalmarktrecht und kartellrecht (libor manipulation: Between capital markets law and competition law). *Der Betrieb*, 65(45):2561–2568, 2012.

[14] Yan Zhou. Ibor transition: Pricing of interest rate instruments in the rfr world. 2022.

[15] Andrei Lyashenko and Fabio Mercurio. Looking forward to backward-looking rates: a modeling framework for term rates replacing libor. *Available at SSRN*, 3330240, 2019.

[16] Damiano Brigo. A crash course in probability and stochastic calculus. Imperial College London, 2022.

[17] Damiano Brigo. Interest rate models with credit risk, collateral, funding liquidity risk and multiple curves. Imperial College London, 2020.

[18] Bank of England. Sonia interest rate benchmark. `https://www.bankofengland.co.uk/markets/sonia-benchmark`, 2023. Online; accessed 29 July 2023.

[19] Andrei Lyashenko and Fabio Mercurio. Looking forward to backward-looking rates: completing the generalized forward market model. *Available at SSRN*, 2019.

[20] Nicholas Burgess. Interest rate swaptions-a review & derivation of swaption pricing formulae. *Journal of Economics and Financial Analysis*, 2(2):87–103, 2018.

[21] Daniel Hohmann, Mario Hörig, D Aktuar, and Florian Ketterer. The new normal. Technical report, Milliman Research Report, 2015.

[22] Quantitative Analytics Bloomberg L.P. Volatility cube for compounded overnight risk-free rates. 2021.

[23] MANU. Rates volatility. `http://www.murex.expert/2015/06/09/rates-volatility`, 2015. Online; accessed 29 July 2023.