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Stochastic Control Problem with Constrained Condition and Random Drift

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

In this thesis we will study the stochastic linear quadratic (SLQ) control problem in the stock market with finite continuous time horizon. We first focus on the unconstrained problem and solve it using both Hamiltonian-Jacobi-Bellman (HJB) equation method and Forward Backward Stochastic Differential Equation (FBSDE) method directly and through its dual problem, and we will discuss the relations among the 4 methods. We also plot the graphs to show the consistency of the methods numerically by Python and compare the differences among the methods. To consider more realistic situations we also introduce convex constrained condition and random drift to the SLQ control problem.

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Introduction

In this thesis we will discuss 4 methods to solve the unconstrained stochastic linear quadratic control problem in the stock market in finite time horizon in Chapter 1. We are going to minimize the functional that consists of a utility function, quadratic in the associated cash process, and an integral of the convex cost function which is quadratic in terms of the time-dependent portfolio strategy and the cash process.

In the 1950s, Bellman discovered the dynamic programming principle (DPP). We can derive a second order non-linear PDE, Hamilton–Jacobi–Bellman (HJB) equation which is also known as dynamic programming equation (DPE) from DPP. Plenty of literature and books have discussed the HJB equation, see [Bertsekas et al., 1995], [Pham, 2009a] and [Yong and Zhou, 1999]. We will state the DPP and HJB equation specifically in Section 1.1.1. If the PDE can be solve analytically, we can apply the verification theorem to validate the solution of the HJB equation, see [Soner, 2004]. However, we will not go further into this point since the problem we discussed cannot be solved explicitly by the commonly used methods like stochastic Riccati equation (SRE), instead we would solve the PDE using Runge-Kutta Method, see [Ascher and Petzold, 1998]. The main drawback of HJB equation is to assume the existence of a regular solution to the HJB equation, which is not true in some situations, see a specific example in [Pham, 2009b].

In the 1980s, in the papers [Karatzas et al., 1987] and [Cox and Huang, 1989], Karatzas, Lehoczky & Shreve and Cox & Huang found that the duality methods provides powerful insights into the solutions for the optimal investment problem. For our convex SLQ problems, we are going to use the convex duality method in [Labbé and Heunis, 2007] to formulate an associated dual optimization problem by deriving the dual process of the cash process. Assuming the product of the cash process and its dual one is a supermartingale, we define the convex conjugate of the cost and duality functions, and we can then construct the optimal portfolio strategy from the solution of the dual problem. In our case, the cost function is quadratic in the primal control process, i.e. the portfolio strategy, which means that we still need to find the dual control which is conjugate of the portfolio strategy. If the cost function is independent of the primal control process, we can see that it is much easier to get the value function by solving the dual problem rather than the primal problem, since the convex conjugate of the primal control would be 0. we We would discuss this specifically in Section 1.1.2 and 1.1.3.

In the 1990s Pardoux & Peng [Pardoux and Peng, 1990] introduced the backward stochastic differential equations (BSDEs) which is now extensively used in mathematical finance due to its connection with the stochastic control problem and partial differential equation. Combining the BSDE derived from our stochastic control problem with the forward dynamic of the associated cash process, we have the forward backward stochastic differential equation (FBSDE) system. By solving the FBSDE system we find the value function to the problem, this will be discussed specifically in Section 1.2.1. We also apply the FBSDE method to the dual control problem we derived in Section 1.2.2, and we will study the relations between the results from the primal FBSDE and dual FBSDE in Section 1.2.3.

Considering sometimes we will have some subjective judgement to the investment, we will also introduce constrained conditions which make the second entry of the portfolio process to be 0, which is kind of special convex cone. In this case, the primal problem would be even easier to formulate as it degenerate from 2-dimensional case to a 1-dimensional case, we can see this clearly in Section 2.1.1. But the dual problem with the constrained condition becomes more complicated as we cannot directly say that the second entry of the dual process of the portfolio strategy is also 0 which is actually not 0. We need to recalculate the convex conjugate of the cost function under the given condition. Even if the cost function is independent of the primal control process, we cannot directly treat the dual control as 0 as we are formulating the problem in a constrained space. We will discuss this further in Section 2.1.2. The FBSDE method does not show much difference compare with Chapter 1 but the calculation is much more complicated, see Section 2.2.

In Chapter 3 we consider a more realistic case which assumes the appreciation rate of a stock to be a random process. We simply regard it as an Ornstein–Uhlenbeck (OU) process which is also known as the Vasicek model in mathematical finance. In this situation, we can still apply the HJB method and FBSDE methods on both primal and dual problem as we mentioned above. However, we cannot get the linear PDE using HJB method, so we are going to introduce the BSDE representation and non-linear Feynman-Kac theorem to solve the semi-linear PDE [Pham, 2009b]. We will discuss this further in Section 3.2.1 and 3.2.2. In the FBSDE systems we assume that the correlation between the Brownian motion in the OU-process and the one in the associated cash process is 1 to ensure the consistency of the measure. Even we add constrained condition to the correlation, the FBSDE system cannot be analytically solved using ansatz. We need to apply reinforcement learning to the system, but we will not going further into this point in this thesis.

Chapter 1

Unconstrained Problem

Following the problem settings of [Li and Zheng, 2018]. We are considering the market model on the finite time horizon $[0,\ T]$, where $T > 0$ is a fixed terminal time,let $\{W(t), t \in [0,T]\}$ an $\mathbb{R}^2$-valued standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, by $\{\mathcal{F}_t\}$ the $\mathbb{P}$-augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \le s \le t)$ generated by W, by $\mathcal{P}\left(0, T; \mathbb{R}^2\right)$ the set of all $\mathbb{R}^2$ -valued progressively measurable processes on $[0,T] \times \Omega$, by $\mathcal{H}^2\left(0, T; \mathbb{R}^2\right)$ the set of processes x in $\mathcal{P}\left(0, T; \mathbb{R}^2\right)$ satisfying $E\left[\int_0^T |x(t)|^2 dt\right] < \infty$, and by $\mathcal{S}^2\left(0, T; \mathbb{R}^2\right)$ the set of processes x in $\mathcal{P}\left(0, T; \mathbb{R}^2\right)$ satisfying $E\left[\sup_{0 \le t \le T} \left|x_t^2\right|\right] < \infty$.

Suppose the market consists of a bank account with price $\{S_0(t)\}$ given by

$$dS_0(t) = r(t)S_0(t)dt,\ 0 \le t \le T,\ S_0(0) = 1$$

and 2 stocks with prices $\{S_n(t)\}, n = 1,\ 2$ given by

$$dS_n(t) = S_n(t)\left[b_n(t)dt + \sum_{m=1}^{2} \sigma_{nm}(t)dW_m(t)\right], 0 \le t \le T, S_n(0) > 0$$

$r \in \mathcal{P}(0,T;\mathbb{R})$ is the scalar interest rate of the market at time t, $b \in \mathcal{P}\left(0, T; \mathbb{R}^2\right)$ is the vector of appreciation rate at time t and $\sigma \in \mathcal{P}\left(0, T; \mathbb{R}^{2\times 2}\right)$ is the volatility matrix are uniformly bounded, we also assume that $\sigma(t)$ and $\sigma'(t)$ are invertible for $t \in [0,T]$, which means that there exist a constant $k \in \mathbb{R}^+$ such that

$$x'\sigma(t)\sigma'(t)x \ge k|x|^2$$

for all $(x, \omega, t) \in \mathbb{R}^N \times \Omega \times [0,T]$, where x' means the transpose of x.

Define the set of admissible investing process to be

$$\mathcal{A} := \left\{\pi \in \mathcal{H}^2\left(0, T; \mathbb{R}^2\right) : \pi(t) \in K \text{ for } t \in [0,T] \text{ a.e. } \right\},$$

Here we simply set $K = \mathbb{R}^2$.

For a given $\pi \in \mathcal{A}$, the wealth process is

$$dX^\pi(t) = \left[r(t)X^\pi(t) + \pi'(t)\sigma(t)\theta(t)\right]dt + \pi'(t)\sigma(t)dW(t),\ 0 \le t \le T,\ X^\pi(0) = x_0$$

where $\theta(t) := \sigma^{-1}(t)[b(t) - r(t)I]$ is the market price of risk at time t which is uniformly bounded and $I \in \mathbb{R}^2$ is the vector with all the entry equal to 1.

Define the functional $J : \mathcal{A} \to \mathbb{R}$ such that

$$J(\pi) := E\left[\int_0^T f\left(t, X^\pi(t), \pi(t)\right)dt + g\left(X^\pi(T)\right)\Big|X^\pi(0) = x_0\right] \tag{1.0.1}$$

where $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ has form

$$\begin{cases} f(\omega, t, x, \pi) := \frac{1}{2}\left[Q(t)x^2 + \pi' R(t)\pi\right] \\ g(\omega, x) := \frac{1}{2}\left[ax^2 + 2cx\right] \end{cases}$$

here we simply assume that $a, c \in R$ are deterministic constants, and the processes $Q \in \mathcal{P}(0,T;\mathbb{R}), R \in \mathcal{P}\left(0,T;\mathbb{R}^{2\times 2}\right)$ are uniformly bounded, $R(t)$ is a symmetric matrix, and the combined matrix

$$\begin{pmatrix} Q(t) & \mathbf{0}' \\ \mathbf{0} & R(t) \end{pmatrix}$$

is semi-positive definite for all $(\omega, t) \in \Omega \times [0,T]$, where $\mathbf{0}$ represents the 2×1 matrix with all the entries are 0, these assumptions lead $J(\pi)$ to be a convex functional of π.

Now we are considering to minimize $J(\pi)$ subject to some admissible (X, π), thus we set the value function.

$$V(0, x_0) = \inf_{\pi \in \mathcal{A}} J(\pi)$$

1.1 HJB Method

From [Soner, 2004] we have the following theorems of DPP and HJB equation

Theorem 1.1.1. ***(Dynamic Programming Principle)*** *Let the dynamic of the state variable $X^\pi(t)$ on $\mathbb{R}^n$ be*

$$dX = b(t, X^\pi(t), \pi(t))dt + \sigma(t, X^\pi(t), \pi(t))dW(t)$$

where $W(t)$ is a d-dimensional Brownian motion

Let $\mathcal{A}$ be the admissible control set, define the functional as

$$J(t, x, \pi) = E\left[\int_t^T f\left(s, X^\pi(s), \pi(s)\right) ds + g\left(X^\pi(s)\right) \mid \mathcal{F}_t\right]$$

$$V(t, x) := \inf_{\pi \in \mathcal{A}_{t,x}} J(t, x, \pi)$$

Then for any stopping time $\tau > t$, we have

$$v(t, x) = \inf_{\pi \in \mathcal{A}_{t,x}} E\left[\int_t^\tau f\left(s, X^\pi(s), \pi(s)\right) ds + g\left(X^\pi_{t,x}(\tau)\right) \mid \mathcal{F}_t\right]$$

Theorem 1.1.2. ***(HJB equation)***
Assume DPP holds, then the value function $V(t, x)$ satisfies the equation

$$\partial_t V + \inf_{\pi \in \mathcal{A}_{t,x}} \left[\mathcal{L}_\pi V(s, X^\pi(s)) + f\left(s, X^\pi(s), \pi(s)\right)\right] = 0$$

where $\mathcal{L}_\pi V = b(t, x, \pi) \cdot \nabla V + \frac{1}{2} \operatorname{tr} a(t, x, \pi) D^2 V$ is the infinitesimal generator of process $X^\pi(s)$ with terminal condition

$$V(T, X^\pi(T)) = g\left(X^\pi(t)\right)$$

1.1.1 Primal Problem

In our case, we can write the HJB equation as

$$V_t + \frac{1}{2}Q(t)X^\pi(t)^2 + r(t)X^\pi(t)V_x + \inf_{\pi \in K} L_v(t) = 0$$

with terminal condition

$$V(T, X^\pi(T)) = \frac{a}{2}x^2 + cx \tag{1.1.1}$$

where

$$L_v(t,\pi) = \pi'(t)\sigma(t)\theta(t)V_x + \frac{1}{2}\pi'(t)\left(\sigma(t)\sigma'(t)V_{xx} + R(t)\right)\pi(t)$$

To optimize $L_v(t,\pi)$ we take the partial derivative of L_v respect to π , according to the first order condition we have the feedback form of the control process

$$\hat{\pi}(t) = -\left(\sigma(t)\sigma'(t)V_{xx} + R(t)\right)^{-1}\sigma(t)\theta(t)V_x \tag{1.1.2}$$

Substituting (1.1.2) back into the HJB equation we can get the PDE

$$V_t + \frac{1}{2}Q(t)x^2 + r(t)xV_x - \frac{1}{2}\theta'(t)\sigma'(t)\left(\sigma(t)\sigma'(t)V_{xx} + R(t)\right)^{-1}\sigma(t)\theta(t)V_x^2 = 0 \tag{1.1.3}$$

Through observing the terminal condition (1.1.1) we can make the ansatz

$$V(t,x) = x^2 g_1(t) + x g_2(t) + g_3(t). \tag{1.1.4}$$

which is inspired by [Cartea et al., 2015]

Substitute our ansatz (1.1.4) into the PDE (1.1.3) we can get a the updated PDE where the time t and state x are separated

$$\begin{aligned}
&x^2\dot{g}_1 + x\dot{g}_2 + \dot{g}_3 + \frac{1}{2}Q(t)x^2 + r(t)x\left(2xg_1 + g_2\right) \\
&-\frac{1}{2}\theta'(t)\sigma'(t)\left(2\sigma(t)\sigma'(t)g_1 + R(t)\right)^{-1}\sigma(t)\theta(t)\left(4x^2g_1^2 + 4xg_1g_2 + g_2^2\right) = 0
\end{aligned}$$

By summing the terms in the power of x we can get the series of ODEs

$$\begin{cases}
\dot{g}_1 - 2\theta'(t)\sigma'(t)\left(2\sigma(t)\sigma'(t)g_1 + R(t)\right)^{-1}\sigma(t)\theta(t)g_1^2 + 2r(t)g_1 + \frac{1}{2}Q(t) = 0 \\
\dot{g}_2 + \left(r(t) - 2\theta'(t)\sigma'(t)\left(2\sigma(t)\sigma'(t)g_1 + R(t)\right)^{-1}\sigma(t)\theta(t)g_1\right)g_2 = 0 \\
\dot{g}_3 - \frac{1}{2}\theta'(t)\sigma'(t)\left(2\sigma(t)\sigma'(t)g_1 + R(t)\right)^{-1}\sigma(t)\theta(t)g_2^2 = 0
\end{cases}$$

with terminal condition

$$\begin{cases}
g_1(T) = \frac{a}{2} \\
g_2(T) = c \\
g_3(T) = 0
\end{cases}$$

1.1.2 Dual Problem

Sometimes the primal problem maybe hard to solve and difficult to ensure the existence of the regular solution, thus we can use the convex duality method to find the value function.

Following the method in [Labbé and Heunis, 2007], we define

$$\mathbb{B} := \mathbb{R} \times \mathcal{H}^2(0,T;\mathbb{R}) \times \mathcal{H}^2\left(0,T;\mathbb{R}^2\right)$$

and $X \in \mathbb{B}$ if and only if

$$X(t) = x_0 + \int_0^t \dot{X}(s)ds + \int_0^t \Lambda'_X(s)dW(s), 0 \leq t \leq T$$

for some $\left(x_0, \dot{X}, \Lambda_X\right) \in \mathbb{B}$, to formulate the dual problem over $\mathbb{B}$ we define that for any $X \in \mathbb{B}$

$$\begin{aligned} \mathcal{U}(X) := \{\pi \in \mathcal{A} \text{ such that } \dot{X}(t) = r(t)X(t) + \pi'(t)\sigma(t)\theta(t) \\ \text{and } \Lambda_X(t) = \sigma'(t)\pi(t) \text{ for } \forall t \in [0,T], \text{ P-a.e. } \} \end{aligned} \tag{1.1.5}$$

We can see that $\mathcal{U}$ contains all the admissible control $\pi \in \mathcal{A}$ and $\mathcal{U}(X) \neq \varnothing$ if and only if $(X(t), \Lambda_X(t)) \in \mathcal{S}(t, X(t))$ where $S(\omega, t, x)$ is the set-valued function

$$\mathcal{S}(\omega,t,x) := \left\{(v,\xi) : v = r(t)x + \xi'\theta(t) \text{ and } [\sigma']^{-1}(t)\xi \in K\right\}$$

Define the penalty function $L : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to [0,\infty]$ as

$$L(\omega,t,x,v,\xi) = f\left(\omega,t,x,[\sigma']^{-1}(t)\xi\right) + \Psi_{\mathcal{S}(\omega,t,x)}(v,\xi)$$

$\Psi_U(u) = 0$ if $u \in U$ and $+\infty$ otherwise.

Let

$$\Phi(X) := l_0(x_0) + E[g(X(T))] + E\left[\int_0^T L\left(s, X(s), \dot{X}(s), \Lambda_X(s)\right)ds\right] \tag{1.1.6}$$

where

$$l_0(x) = \Psi_{\{x_0\}}(x),$$

We can see that minimizing $\Phi(X)$ is equivalent to minimize (2.0.1)

The following definition is taken from [Phelps, 2009]

Definition 1.1.3. (Convex Conjugate) Suppose $f : X \to \mathbb{R}$ is a convex function defined on a convex set $X \subset \mathbb{R}^n$, then its convex conjugate $g : Y \to \mathbb{R}$ is defined by

$$g(y) := \sup_{x \in X}(\langle x,y\rangle - f(x)), \quad y \in Y$$

with its domain

$$Y = \left\{y \in \mathbb{R}^n : \sup_{x \in X}(\langle x,y\rangle - f(x)) < \infty\right\}$$

where $\langle x,y\rangle$ denotes the dot product of x and y.

Note that the convex conjugate is always well-defined if f is a convex function.

Now by taking the convex conjugate of (1.1.6) we can establish the dual problem i.e. minimize

$$\tilde{\Psi}(y,\alpha,\beta) := m_0(y) + E\left[m_T(Y(T))\right] + E\left[\int_0^T \phi(s,\alpha(s),\beta(s))ds\right]$$

for $Y \equiv \left(y, \dot{Y}, \Lambda_Y\right) \in \mathbb{B}$ where

$$m_0(y) := \sup_{x\in\mathbb{R}} \left\{xy - \Psi_{\{x_0\}}(x)\right\} = x_0 y$$

$$m_T(\omega, y) := \sup_{x\in\mathbb{R}}\{-xy - g(\omega, x)\} = \frac{(y+c)^2}{2a}$$

$$\begin{aligned}\phi(\omega, t, \alpha, \beta) : &= \sup_{x\in\mathbb{R}, \pi\in K} \{x\alpha + \pi'\beta - f(\omega, t, x, \pi)\} \\ &= \frac{1}{2}\left[\frac{\alpha^2}{Q} + \beta' R^{-1}\beta\right]\end{aligned}$$

and Y has the dynamic

$$\begin{cases} dY(t) = [\alpha(t) - r(t)Y(t)]dt + \left[\sigma^{-1}(t)\beta(t) - \theta(t)Y(t)\right]' dW(t) \\ Y(0) = y \end{cases}$$

Regard y as a constant, then we need to solve the control problem with

$$\tilde{J}(\alpha, \beta) := E\left[\int_0^T \phi(t, \alpha(t), \beta(t))dt + m_T(y)\Big| Y(0) = y\right]$$

The value function is

$$W(0, y) = \inf_{(\alpha,\beta)\in\mathcal{A}} \tilde{J}(\alpha, \beta)$$

According to DPP we can write the HJB equation as

$$W_t - r(t)yW_y + \inf_{(\alpha,\beta)\in\mathcal{B}} L_w(t) = 0 \tag{1.1.7}$$

with terminal condition

$$W(T, Y^{(y,\alpha,\beta)}(T)) = \frac{1}{2a}y^2 + \frac{c}{a}y + \frac{c^2}{2a}$$

where

$$L_w(t, \alpha, \beta) = \alpha(t)W_y + \frac{1}{2}\left|\sigma(t)^{-1}\beta(t) - \theta(t)y\right|^2 W_{yy} + \phi(t, \alpha(t), \beta(t))$$

To optimize $L_w(t, \alpha, \beta)$ we need to take the partial derivatives respect to $\alpha(t)$ and $\beta(t)$. According to the first order condition we have the feedback form of the control processes

$$\begin{cases} \hat{\alpha}(t) = -Q(t)W_y \\ \hat{\beta}(t) = \left(\sigma(t)^{-1} + \dfrac{\sigma'(t)R(t)^{-1}}{W_{yy}}\right)^{-1} \theta(t)y \end{cases} \tag{1.1.8}$$

substitute back to the HJB equation (1.1.7), we can get

$$\begin{aligned} &W_t - \frac{1}{2}Q(t)W_y^2 - r(t)yW_y + \frac{1}{2}\left|\left(I + \frac{K(t)}{W_{yy}}\right)^{-1}\theta(t) - \theta(t)\right|^2 \\ &y^2 W_{yy} + \frac{1}{2}\left|\left(I + \frac{K(t)}{W_{yy}}\right)^{-1}\theta(t)\right|^2_{K(t)} y^2 = 0 \end{aligned} \tag{1.1.9}$$

where $K(t) = \sigma'(t)R(t)^{-1}\sigma(t)$, I refers to the identity matrix, $|x|^2_A = x'Ax$.

This PDE is difficult to solve so we make the ansatz according to the terminal condition

$$W(t,y) = y^2 h_1(t) + y h_2(t) + h_3(t) \tag{1.1.10}$$

Substitute the ansatz (1.1.10) into (1.1.9) we can change the PDE to

$$\begin{aligned}
&y^2\dot{h}_1 + y\dot{h}_2 + \dot{h}_3 - 2Qh_1^2y^2 - 2Qh_1h_2y - \frac{1}{2}Qh_2^2 - 2rh_1y^2 - rh_2y \\
&+ \left|\left(I + \frac{K}{2h_1}\right)^{-1}\theta - \theta\right|^2 h_1y^2 + \frac{1}{2}\left|\left(I + \frac{K}{2h_1}\right)^{-1}\theta\right|_K^2 y^2 = 0
\end{aligned}$$

Summing the terms in the power of y, we have

$$\begin{cases}
\dot{h}_1 - 2Q(t)h_1^2 - 2r(t)h_1 + \left|\left(I + \frac{K(t)}{2h_1}\right)^{-1}\theta(t) - \theta(t)\right|^2 h_1 + \\
\frac{1}{2}\left|\left(I + \frac{K(t)}{2h_1}\right)^{-1}\theta(t)\right|_{K(t)}^2 = 0 \\
\dot{h}_2 - \left(2Q(t)h_1 + r(t)\right)h_2 = 0 \\
\dot{h}_3 - \frac{1}{2}Q(t)h_2^2 = 0
\end{cases}$$

with terminal condition

$$\begin{cases}
h_1(T) = \frac{1}{2a} \\
h_2(T) = \frac{c}{a} \\
h_3(T) = \frac{c^2}{2a}
\end{cases}$$

Substitute $W(0,y)$ back to $\tilde{\Psi}(Y(t),\alpha(t),\beta(t))$ and minimize $\tilde{\Psi}(y,\hat{\alpha}(0),\hat{\beta}(0))$ respect to y, we can get

$$\hat{y} = -\frac{h_2(0) + x_0}{2h_1(0)}$$

and

$$\begin{aligned}
V(0,x_0) &= -\tilde{\Psi}(\hat{y},\hat{\alpha},\hat{\beta}) \\
&= -x_0\hat{y} - \hat{y}^2h_1(0) - \hat{y}h_2(0) - h_3(0) \\
&= \frac{1}{4h_1(0)}x_0^2 + \frac{h_2(0)}{2h_1(0)}x_0 + \frac{h_2^2(0)}{4h_1(0)} - h_3(0)
\end{aligned}$$

Compared with the ansatz $V(t,X^\pi(t)) = x^2g_1(t) + xg_2(t) + g_3(t)$, we can get the dual relation between the primal ansatz and dual ansatz

$$\begin{cases}
g_1(t) = \frac{1}{4h_1(t)} \\
g_2(t) = \frac{h_2(t)}{2h_1(t)} \\
g_3(t) = \frac{h_2^2(t)}{4h_1(t)} - h_3(t)
\end{cases}$$

Following [Li and Zheng, 2018] we can also derive the dual problem in the following way .

To find the dual process, we need to construct the process $Y(t)$ such that $X^\pi(t)Y(t)$ is a supermartingale for all admissible control π. Suppose

$$dY(t) = \alpha_1(t)dt + \beta_1(t)dW(t)$$

we have

$$d\left(X^\pi(t)Y(t)\right) = \left(X^\pi(t)\alpha(t) + \pi'(t)\beta(t)\right)dt + local\ martingale$$

where

$$\begin{cases} \alpha(t) = \alpha_1(t) + r(t)Y(t) \\ \beta(t) = \sigma(t)\left(\beta_1(t) + \theta(t)Y(t)\right) \end{cases}$$

Thus we have

$$dY^{(y,\alpha,\beta)}(t) = [\alpha(t) - r(t)Y^{(y,\alpha,\beta)}(t)]dt + \left[\sigma^{-1}(t)\beta(t) - \theta(t)Y^{(y,\alpha,\beta)}(t)\right]' dW(t),\ Y^{(y,\alpha,\beta)}(0) = y$$

we can see that $X^\pi(t)Y^{y,\alpha,\beta}(t) - \int_0^t \left(X^\pi(s)\alpha(s) + \pi'(s)\beta(s)\right)ds$ is a super-martingale if we assume that it is bounded below, i.e.

$$E\left[X^\pi(T)Y^{(y,\alpha,\beta)}(T) - \int_0^T \left(X^\pi(s)\alpha(s) + \pi'(s)\beta(s)\right)ds\right] \leq x_0 y \tag{1.1.11}$$

The primal optimization problem can be written as

$$\max_\pi E\left[\int_0^T \left(-f\left(t, X^\pi(t), \pi(t)\right) - \Psi_K(\pi(t))\right)dt - g\left(X^\pi(T)\right)\right]$$

and the dual functions are given by

$$\begin{cases} \phi(t,\alpha,\beta) = \sup_{x,\pi}\left\{-f(t,x,\pi) - \Psi_K(\pi) + x\alpha + \pi'\beta\right\} \\ m_T(y) = \sup_x(-g(x) - xy) \end{cases}$$

By the dual functions above and (1.1.11) we can see that

$$\begin{aligned} &\max_\pi E\left[\int_0^T \left(-f\left(t, X^\pi(t), \pi(t)\right) - \Psi_K(\pi(t))\right)dt - g\left(X^\pi(T)\right)\right] \\ &\leq \min_{y,\alpha,\beta}\left\{x_0 y + E\left[\int_0^T \phi(t,\alpha(t),\beta(t))dt + m_T(Y^{(y,\alpha,\beta)}(T))\right]\right\} \end{aligned}$$

i.e.$-J(\hat{\pi}) \leq \tilde{\Psi}(\hat{y}, \hat{\alpha}, \hat{\beta})$

1.1.3 Special Case

If we further assume that the admissible control is the whole Euclidean space and $R(t) = 0$, we can see that the dual control set is 0 which means $\beta = 0$. Thus we can find a more simpler HJB for the dual control problem i.e.

$$W_t + \inf_{(\alpha,\beta)\in\mathcal{A}}\left[\left(\alpha(t) - r(t)y\right)W_y + \frac{1}{2}\left|\sigma(t)^{-1}\beta(t) - \theta(t)y\right|^2 W_{yy} + \frac{\alpha^2(t)}{2Q(t)}\right] = 0$$

Using the same ansatz as the Section 1.1.2 we can see that

$$\begin{cases} \dot{h}_1 - 2Q(t)h_1^2 + \left(|\theta(t)|^2 - 2r(t)\right)h_1 = 0 \\ \dot{h}_2 - \left(2Q(t)h_1 + r(t)\right)h_2 = 0 \\ \dot{h}_3 - \dfrac{1}{2}Q(t)h_2^2 = 0 \end{cases}$$

with the same terminal conditions as before, but this time we can see that the first equation is a Riccati equation which is decoupled, thus we can solve the system of ODEs analytically:

$$\begin{cases} h_1(t) = -\dfrac{\exp\left(\displaystyle\int 2r(t) - |\theta(t)|^2 dt\right)}{2\displaystyle\int Q(t)\exp\left(\displaystyle\int 2r(t) - |\theta(t)|^2 dt\right)dt + A} \\ h_2(t) = B\exp\left(\displaystyle\int 2Q(t)h_1(t) + r(t)dt\right) \\ h_3(t) = \dfrac{1}{2}\displaystyle\int Q(t)h_2^2(t)dt + C \end{cases}$$

where A, B and C are constants determined by the terminal conditions.

1.2 FBSDE Method

The theory of backward stochastic differential equation (BSDE) is popular nowadays because of its connection to stochastic control and partial differential equations. In this section we will derive the BSDE to get the forward backward stochastic differential equation (FBSDE) system of the unconstrained stochastic control problem on the finite time horizon. BSDE can also provide a probabilistic representation of nonlinear PDEs, which extends the Feynman-Kac formula to non-linear PDEs. We will discuss this further in Chapter 3.

The following definition and theorem are taken from [Pham, 2009b]. We consider the framework of a stochastic control problem on a finite horizon as defined in Theorem 1.1.2, let the associated process $X^\pi(t)$ on $\mathbb{R}^n$ be

$$dX = b(t, X^\pi(t), \pi(t))dt + \sigma(t, X^\pi(t), \pi(t)))dW(t)$$

where $W(t)$ is a d-dimensional Brownian process and $\pi(t) \in \mathcal{A}$, we are going to minimize the functional

$$J(t, x, \pi) = E\left[\int_t^T f\left(s, X^\pi(s), \pi(s)\right)ds + g\left(X^\pi(s)\right)\Big|\mathcal{F}_t\right]$$

Definition 1.2.1. (Hamiltonian and associated BSDE)

For the state process and objective functional we stated above, we define Hamiltonian $\mathcal{H} : [0, T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ as

$$\mathcal{H}(t, x, \pi, p, q) := b(t, x, \pi) \cdot p + \mathrm{tr}\left(\sigma'(t, x, \pi)q\right) - f(t, x, \pi)$$

also we assume that $\mathcal{H}$ is differentiable in x, for each $\pi \in \mathcal{A}$, we can write the adjoint BSDE

$$\begin{cases} -dp(t) = D_x\mathcal{H}\left(t, X^\pi(t), \pi_t, p(t), q(t)\right)dt - q(t)dW(t) \\ p(T) = -D_x g\left(X^\pi(T)\right) \end{cases}$$

Theorem 1.2.2. ***(Stochastic maximum principle)*** *Let $\hat{\pi} \in \mathcal{A}$ and X^π the associated controlled diffusion. Suppose that there exists a solution $(\hat{p}, \hat{q})$ to the associated BSDE such that*

$$\mathcal{H}\left(t, X^{\hat{\pi}}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t)\right) = \min_{\pi\in A}\mathcal{H}\left(t, X^\pi(t), \pi(t), \hat{p}(t), \hat{q}(t)\right), \quad 0 \le t \le T, \ a.s.$$

and $(x, \pi) \to \mathcal{H}\left(t, x, \pi, \hat{p}(t), \hat{q}(t)\right)$ is a concave function for all $t \in [0, T]$.

Then $\hat{\pi}$ is an optimal control, i.e.

$$J(\hat{\pi}) = \inf_{\pi\in\mathcal{A}} J(\pi)$$

1.2.1 FBSDE Primal Problem

According to the dynamic of X and the correlated value function V we can get the Hamiltonian

$$H_1(t,x,\pi,p_1,q_1) = (r(t)x + \pi'\sigma(t)\theta(t))\, p_1 + q_1'\sigma'(t)\pi - \frac{1}{2}Q(t)x^2 - \frac{1}{2}\pi' R(t)\pi$$

where $p_1 \in \mathcal{S}^2(0,T;\mathbb{R})$ and $q_1 \in \mathcal{H}^2\left(0,t;\mathbb{R}^2\right)$

Thus we have the BSDE

$$\begin{cases} dp_1(t) = \left[-r(t)p_1(t) + Q(t)X^\pi(t)\right]dt + q_1'(t)dW(t) \\ p_1(T) = -aX^\pi(T) - c \end{cases}$$

Theorem 1.2.3. *Let $\hat{\pi} \in \mathbb{A}$, then $\hat{\pi}$ is the optimal control of the primal problem if and only if the solution $\left(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1\right)$ of the FBSDE*

$$\begin{cases} dX^{\hat{\pi}}(t) = \left[r(t)X^{\hat{\pi}}(t) + \hat{\pi}'(t)\sigma(t)\theta(t)\right]dt + \hat{\pi}'(t)\sigma(t)dW(t) \\ X^{\hat{\pi}}(0) = x_0 \\ d\hat{p}_1(t) = \left[-r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t)\right]dt + \hat{q}_1'(t)dW(t) \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c \end{cases} \tag{1.2.1}$$

satisfies

$$\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) - R(t)\hat{\pi}(t) = 0 \tag{1.2.2}$$

for $(\mathbb{P}\otimes Leb)-a.e.(\omega,t) \in \Omega\times[0,T]$ *and* $\pi \in R^2$

The condition (1.2.2) can be derived from the stochastic maximum principle. By the condition we can get

$$\hat{\pi} = R^{-1}(t)\sigma(t)\left(p_1(t)\theta(t) + q_1(t)\right) \tag{1.2.3}$$

From the terminal condition we can make the ansatz that

$$\hat{p}_1(t) = \phi_1(t)X^{\hat{\pi}}(t) + \psi_1(t) \tag{1.2.4}$$

substitute the dynamic of $X^\pi(t)$ into $dp_1(t)$ then we can get the dynamic of $p_1(t)$

$$d\hat{p}_1(t) = \left[\dot{\phi}_1(t)X_t + \phi_1(t)\left(r(t)X_t + \pi'(t)\sigma(t)\theta(t)\right) + \dot{\psi}_1(t)\right]dt + \phi_1(t)\pi'(t)\sigma(t)dW(t)$$

compare the drift and local martingale part with the dynamic of $\hat{p}_1(t)$ we stated in Theorem 1.2.3 we can get

$$\begin{cases} -r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t) = \dot{\phi}_1(t)X_t + \phi_1(t)\left(r(t)X_t + \pi'(t)\sigma(t)\theta(t)\right) + \dot{\psi}_1(t) \\ \hat{q}_1(t) = \phi_1(t)\sigma'(t)\hat{\pi}(t) \end{cases} \tag{1.2.5}$$

By (1.2.3) and the second equation of (1.2.5) we can see that

$$\begin{aligned} \hat{\pi}(t) &= \hat{p}_1(t)\left(\sigma^{-1}(t)R(t) - \phi_1(t)\sigma'(t)\right)^{-1}\theta(t) \\ &= \phi_1(t)\left(\sigma^{-1}(t)R(t) - \phi_1(t)\sigma'(t)\right)^{-1}\theta(t)X_t + \psi_1(t)\left(\sigma^{-1}(t)R(t) - \phi_1(t)\sigma'(t)\right)^{-1}\theta(t) \end{aligned}$$

Compare with $\hat{\pi}$ with (1.1.2) we can see that

$$\begin{cases} \phi_1(t) = -V_{xx} \\ \phi_1(t)\hat{X}_t + \psi_1(t) = -V_x \end{cases}$$

Thus we can conclude that

$$\begin{cases} g_1(t) = -\dfrac{\phi_1(t)}{2} \\ g_2(t) = -\psi_1(t) \\ g_3(t) = \dfrac{1}{2}\displaystyle\int \theta'(t)\sigma'(t)\left(- \sigma(t)\sigma'(t)\phi_1(t) + R(t)\right)^{-1} \sigma(t)\theta(t)\psi_1(t)dt + C_1 \end{cases}$$

where constant C_1 can be determined by $g_3(T) = 0$.

And finally we can get

$$\begin{aligned} V(t,x) = -\frac{\phi_1(t)}{2}x^2 - \psi_1(t)x - \frac{1}{2}\int \theta'(t)\sigma'(t) \\ \left(- \sigma(t)\sigma'(t)\phi_1(t) + R(t)\right)^{-1} \sigma(t)\theta(t)\psi_1(t)dt + C_1 \end{aligned}$$

So to solve the value function of the primal problem we only need to solve $\phi_1(t)$ and $\psi_1(t)$

From the first equation of (1.2.5) we can see that

$$\left(\dot{\phi}_1(t) + 2r(t)\phi_1(t) - Q(t)\right) X^{\hat{\pi}}(t) + \dot{\psi}_1(t) + r(t)\psi_1(t) + \phi_1(t)\hat{\pi}'(t)\sigma(t)\theta(t) = 0 \tag{1.2.6}$$

Thus we have

$$\begin{cases} \dot{\phi}_1(t) + \theta'(t)\left(K^{-1}(t) - \phi_1(t)\right)^{-1}\theta(t)\phi_1^2(t) + 2r(t)\phi_1(t) - Q(t) = 0 \\ \dot{\psi}_1(t) + \left(r(t) + \theta'(t)\left(K^{-1}(t) - \phi_1(t)\right)^{-1}\theta(t)\phi_1(t)\right)\psi_1(t) = 0 \end{cases}$$

with terminal condition

$$\begin{cases} \phi_1(T) = -a \\ \psi_1(T) = -c \end{cases}$$

1.2.2 FBSDE Dual Problem

According to the dynamic of the dual process Y and the dual value function W we can get the Hamiltonian

$$H_2(t,y,\alpha,\beta,p_2,q_2) = (\alpha - r(t)y)\,p_2 + q_2'\left(\sigma^{-1}(t)\beta - \theta(t)y\right) - \frac{1}{2}\left(\frac{\alpha^2}{Q(t)} + \beta' R^{-1}(t)\beta\right)$$

where $p_2 \in \mathcal{S}^2(0,T;\mathbb{R})$ and $q_2 \in \mathcal{H}^2\left(0,t;\mathbb{R}^2\right)$

Thus we have the BSDE

$$\begin{cases} dp_2(t) = \left[r(t)p_2(t) + q_2'(t)\theta(t)\right]dt + q_2'(t)dW(t) \\ p_2(T) = -\dfrac{Y^{(y,\alpha,\beta)}(T) + c}{a} \end{cases}$$

Theorem 1.2.4. *Suppose $Q(t)$ and $R(t)$ are positive definite and their inverse are uniformly bounded, then $(\hat{y},\hat{\alpha},\hat{\beta})$ is the optimal of the dual problem if and only if the solution $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2\right)$ of the FBSDE*

$$\begin{cases} dY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \left[\hat{\alpha}(t) - r(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]dt + \left[\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]' dW(t) \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y} \\ d\hat{p}_2(t) = \left[r(t)\hat{p}_2(t) + \hat{q}_2'(t)\theta(t)\right]dt + \hat{q}_2'(t)dW_t \\ \hat{p}_2(T) = -\dfrac{Y^{(\hat{y},\hat{\alpha}\hat{\beta})}(T) + c}{a} \end{cases} \tag{1.2.7}$$

satisfies

$$\begin{cases} \hat{p}_2(0) = x_0 \\ \left(\hat{p}_2(t), [\sigma']^{-1}(t)\hat{q}_2(t)\right) \in \partial\phi(\hat{\alpha}(t), \hat{\beta}(t)) \end{cases} \tag{1.2.8}$$

for $(\mathbb{P} \otimes Leb)-a.e.(\omega, t) \in \Omega \times [0, T]$

Similar to the primal problem, we can use the second condition of (1.2.8) to find the optimal control α and β, since

$$\partial\phi(\hat{\alpha}(t), \hat{\beta}(t)) = \left(\frac{\hat{\alpha(t)}}{Q(t)}, R^{-1}(t)\hat{\beta}(t)\right)$$

we can get

$$\begin{cases} \hat{\alpha} = \hat{p_2}(t)Q(t) \\ \hat{\beta} = R(t)[\sigma']^{-1}(t)\hat{q_2}(t) \end{cases} \tag{1.2.9}$$

Comparing with the terminal condition of p_2 we can make the ansatz

$$p_2(t) = \phi_2(t)Y^{(y,\alpha,\beta)}(t) + \psi_2(t) \tag{1.2.10}$$

Substitute the dynamic of $Y^{(y,,\beta)}(t)$ into $dp_2(t)$ as we assumed as (1.2.10), we can get the dynamic of $p_2(t)$

$$dp_2(t) = \left(\dot{\phi}_2(t)\hat{Y}(t) + \phi_2(t)\left(\hat{\alpha}(t) - r(t)\hat{Y}(t)\right) + \dot{\psi}_2(t)\right)dt + \phi_2(t)\left(\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)\hat{Y}(t)\right)' dW(t)$$

Compare the dt and $dW(t)$ terms with the dynamic of $p_2(t)$ we stated in Theorem 1.2.4 we can get

$$\begin{cases} r(t)\hat{p}_2(t) + \hat{q}_2'(t)\theta(t) = \dot{\phi}_2(t)\hat{Y}(t) + \phi_2(t)\left(\hat{\alpha}(t) - r(t)\hat{Y}(t)\right) + \dot{\psi}_2(t) \\ \hat{q_2}(t) = \phi_2(t)\left(\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)\hat{Y}(t)\right) \end{cases} \tag{1.2.11}$$

From the second equation of (1.2.11) and the optimized β in (1.2.9) we have

$$\hat{q_2}(t) = \left(K^{-1}(t) - \frac{I}{\phi_2(t)}\right)^{-1}\theta(t)\hat{Y}_t \tag{1.2.12}$$

and

$$\hat{\beta}(t) = \left(\sigma^{-1}(t) - \frac{\sigma'(t)R^{-1}(t)}{\phi_2(t)}\right)^{-1}\theta(t)\hat{Y}_t \tag{1.2.13}$$

Also by our ansatz we can write

$$\hat{\alpha}(t) = Q(t)\left(\phi_2(t)\hat{Y}_t + \psi_2(t)\right) \tag{1.2.14}$$

Compare the $\hat{\beta}$ and $\hat{\alpha}$ with (1.1.8), we can see that

$$\begin{cases} \phi_2(t) = -W_{yy} \\ \phi_2(t)\hat{Y}_t + \psi_2(t) = -W_y \end{cases}$$

Thus we can conclude that

$$\begin{cases} h_1(t) = -\dfrac{\phi_2(t)}{2} \\ h_2(t) = -\psi_2(t) \\ h_3(t) = \dfrac{1}{2}\displaystyle\int Q(t)\psi_2^2(t)dt + C_2 \end{cases}$$

where constant C_2 is determined by $h_3(T) = \dfrac{c^2}{2a}$ And finally we can have

$$W(y,t) = -\frac{\phi_2(t)}{2}y^2 - \psi_2(t)y + \frac{1}{2}\int Q(t)\psi_2^2(t)dt + C$$

so to get the value function of the dual problem we only need to solve $\phi_2(t)$ and $\psi_2(t)$

Substitute $\hat{\alpha}(t)$ and (1.2.12) into the first equation of (1.2.11) we have

$$\begin{aligned}&\left(\dot{\phi}_2(t) + Q(t)\phi_2^2(t) - 2r(t)\phi_2(t) - \theta'(t)\left(K^{-1}(t) - \frac{I}{\phi_2(t)}\right)^{-1}\theta(t)\right)\hat{Y}_t\\&+\dot{\psi}_2(t) + \Big(\phi_2(t)Q(t) - r(t)\Big)\psi_2(t) = 0\end{aligned} \tag{1.2.15}$$

Summing the terms in the power of $\hat{Y}_t$ we have

$$\begin{cases}\dot{\phi}_2(t) + Q(t)\phi_2^2(t) - 2r(t)\phi_2(t) - \theta'(t)\left(K^{-1}(t) - \dfrac{I}{\phi_2(t)}\right)^{-1}\theta(t) = 0\\ \dot{\psi}_2(t) + \Big(\phi_2(t)Q(t) - r(t)\Big)\psi_2(t) = 0\end{cases}$$

with terminal condition

$$\begin{cases}\phi_2(T) = -\dfrac{1}{a}\\ \psi_2(T) = -\dfrac{c}{a}\end{cases}$$

1.2.3 Relations Between FBSDE Primal and Dual Problem

Actually we can derive the primal optimal control process and its associated process by solving the dual FBSDE, the following theorems are taking from [Li and Zheng, 2018]

Theorem 1.2.5. *Suppose $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is the optimal control process for the dual problem and $\big(Y^{(\hat{y},\hat{\alpha},\beta)}, \hat{p}_2, \hat{q}_2\big)$ be the associated process which is the solution to FBSDE (1.2.7), then*

$$\hat{\pi}(t) := [\sigma']^{-1}(t)\hat{q}_2(t) \tag{1.2.16}$$

is the optimal control of the primal problem and the associated process satisfies

$$\begin{cases}X^{\hat{\pi}}(t) = \hat{p}_2(t)\\ \hat{p}_1(t) = Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\\ \hat{q}_1(t) = \sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\end{cases} \tag{1.2.17}$$

This is easy to proof. Suppose that $(\hat{y}, \hat{\alpha}, \hat{\beta}) \in \mathbb{B}$ is the optimal control to the dual control problem, we can see that the process $\Big(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t), \hat{p}_2(t), \hat{q}_2(t)\Big)$ solves FBSDE (1.2.7) satisfies the conditions (1.2.8) by Theorem 1.2.3.

Define $\hat{\pi}(t)$ and $\big(X^{\hat{\pi}}(t), \hat{p}_1(t), \hat{q}_1(t)\big)$ in the form of (1.2.16) and (1.2.17). Again by Theorem 1.2.3 we can see that

$$\big(X^{\hat{\pi}}(t), \hat{\pi}(t)\big) \in \partial\phi(\hat{\alpha}(t), \hat{\beta}(t))$$

Since $\phi(\omega, t, \alpha, \beta)$ is the convex conjugate of $f(\omega, t, x, \pi)$ we can see that

$$(\hat{\alpha}(t), \hat{\beta}(t)) \in \partial f\left(X^{\hat{\pi}}(t), \hat{\pi}(t)\right)$$

which leads to

$$\hat{\alpha}(t) = Q(t)X^{\hat{\pi}}(t) \tag{1.2.18}$$

$$\hat{\beta}(t) = R(t)\hat{\pi}(t) \tag{1.2.19}$$

for $(\mathbb{P} \otimes Leb)$ -a.e. $(\omega, t) \in \Omega \times [0, T]$.

Using (1.2.16) , (1.2.17) and (1.2.18) we can solve the FBSDE of the primal problem and also by applying (1.2.17) and (1.2.19) we can derive the condition of the primal FBSDE (1.2.2).

Also we can derive the dual control process and its associated process inversely from the primal stuffs using the following theorem

Theorem 1.2.6. *Suppose $\hat{\pi}$ is the optimal control process for the dual problem and $\left(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1\right)$ be the associated process which is the solution to FBSDE (1.2.1), then*

$$\begin{cases} \hat{y} = \hat{p}_1(0) \\ \hat{\alpha}(t) = Q(t)X^{\hat{\pi}}(t) \\ \hat{\beta}(t) = \sigma(t)\left[\hat{q}_1(t) + \theta(t)\hat{p}_1(t)\right] \end{cases} \tag{1.2.20}$$

are the optimal control process of the dual control problem and the associated process satisfies

$$\begin{cases} Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \hat{p}_1(t) \\ \hat{p}_2(t) = X^{\pi}(t) \\ \hat{q}_2(t) = \sigma'(t)\hat{\pi}(t) \end{cases}$$

The proof is similar to Theorem 2.3

From (1.2.3) and the third equation in (1.2.20) we can see that actually

$$\hat{\pi}(t) = R^{-1}(t)\hat{\beta}(t)$$

Since we have

$$\begin{cases} X^{\hat{\pi}}(t) = \hat{p}_2(t) \\ Y^{\hat{y},\hat{\alpha},\hat{\beta}}(t) = \hat{p}_1(t) \\ \hat{p}_1(t) = \phi_1(t)X^{\hat{\pi}}(t) + \psi_1(t) \\ \hat{p}_2(t) = \phi_2(t)Y^{\hat{y},\hat{\alpha},\hat{\beta}}(t) + \psi_2(t) \end{cases}$$

We can easily find the relation

$$\begin{cases} \phi_1(t) = \dfrac{1}{\phi_2(t)} \\ \psi_1(t) = -\dfrac{\psi_2(t)}{\phi_2(t)} \end{cases}$$

which means that is we can solve $\phi_2(t)$ and $\psi_2(t)$ for the dual problem, we can directly solve $\phi_1(t)$ and $\psi_1(t)$ to derive the value function of the primal problem.

1.3 Comparison

To compare the result of the 4 methods, we need to plot the graphs of the value functions not only for the initial time 0 but the whole time horizon $[0, T]$, since there is no analytical solution to our ODEs systems derived from all the 4 methods, we need to use some numerical analysing methods.

To ensure the accuracy of the numerical solution, we would introduce the Runge-Kutta methods instead of the classical Euler method, see [Ascher and Petzold, 1998].

Runge-Kutta method: *Suppose that we are solving the ordinary differential equation* $\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0$ *where* $y(t)$ *is a differentiable function of* t*, then we can take step-size* h *to approximate* $y(t)$ *in the following way. For* $n = 0, 1, 2, ...$ *define:*

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right) \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right) \\ k_4 &= f(t_n + h, y_n + hk_3) \end{aligned}$$

and iterated by

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\ t_{n+1} &= t_n + h \end{aligned}$$

Here y_{n+1} *is the Runge-Kutta approximation of* $y(t_{n+1})$*, we can see that* y_{n+1} *is given by* y_n *plus the weighted average of* $k_1, k_2, k3$ *and* k_4 *where they are actually the slope at different points on the interval* $[t_n, t_{n+1}]$ *and it gives larger weight to the slope at the mid-point of the interval, if we only average on* k_1 *the Runge-Kutta method is indeed the Euler method*

We can see that the local truncation error of Runge-Kutta method is of order $O(h^5)$ *where Euler method is of order* $O(h^3)$*, which means Runge-Kutta method is much more accurate than the Euler method*

Assuming we are using parameters $a = 1,\ c = 3$ with Runge-Kutta step-size $h = \frac{T}{n}$, where T is the upper-bound of time t, n is the number of intervals we divided in the interval $[0,\ T]$, to ensure the accuracy of calculation we choose $n = 1000$ and we also simply assume that

$$Q(t) = t + 1$$

$$R(t) = \begin{pmatrix} t+1 & 1 \\ 1 & t + \frac{1}{2} \end{pmatrix}$$

$$\sigma(t) = \begin{pmatrix} t+1 & 1 \\ 0 & t+2 \end{pmatrix}$$

$$b'(t) = \left(\frac{t+1}{100}, \frac{t+2}{100}\right)$$

$$r(t) = \frac{t}{100}$$

The following figure 1.1 shows the differences between the results simulated by python, with $t \in [0, 1]$, starting from different initial values $x_0 = 0.1,\ 0.3,\ 0.5,\ 1$ and using HJB and FBSDE methods on both primal and dual problems

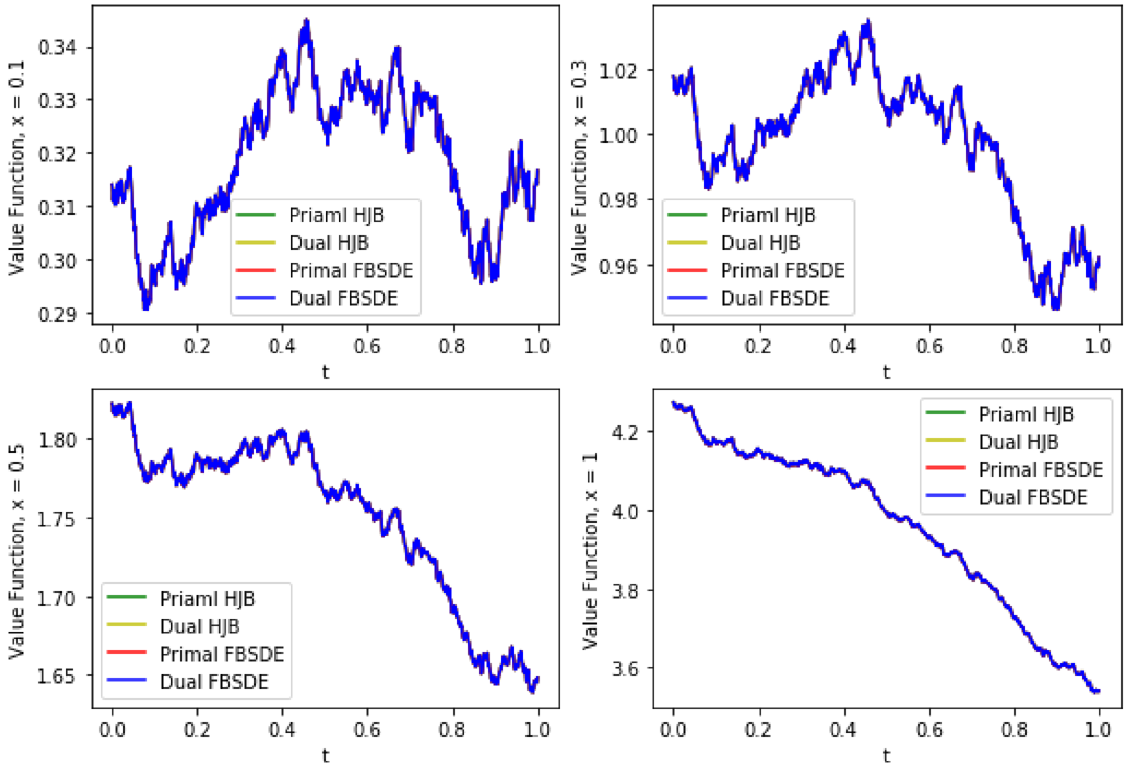

Figure 1.1: Comparison of $V(t, X^{\hat{\pi}}(t))$ with fixed x_0

It seems that there is only little difference between different methods which may be caused by the truncated error from the Runge-Kutta method, which suggests that the 4 methods can indeed get the same result. We can see that as the initial value x_0 gets larger, the value function tends to be "smoother", this might be the case that x_0 outweighs the stochastic variations in our case.

Also we can set different initial values, x_0 and plot the value function at time $t = 0$. The following Figure 1.2 illustrate the differences among the curves of $V(0, X^{\hat{\pi}}(0))$ plotted using 4 methods with $x_0 \in [1, 10]$ at different time t, this time we only separated the interval of x_0 into 100 subintervals, since it needs much more calculation to generated 1000 paths at the same time.

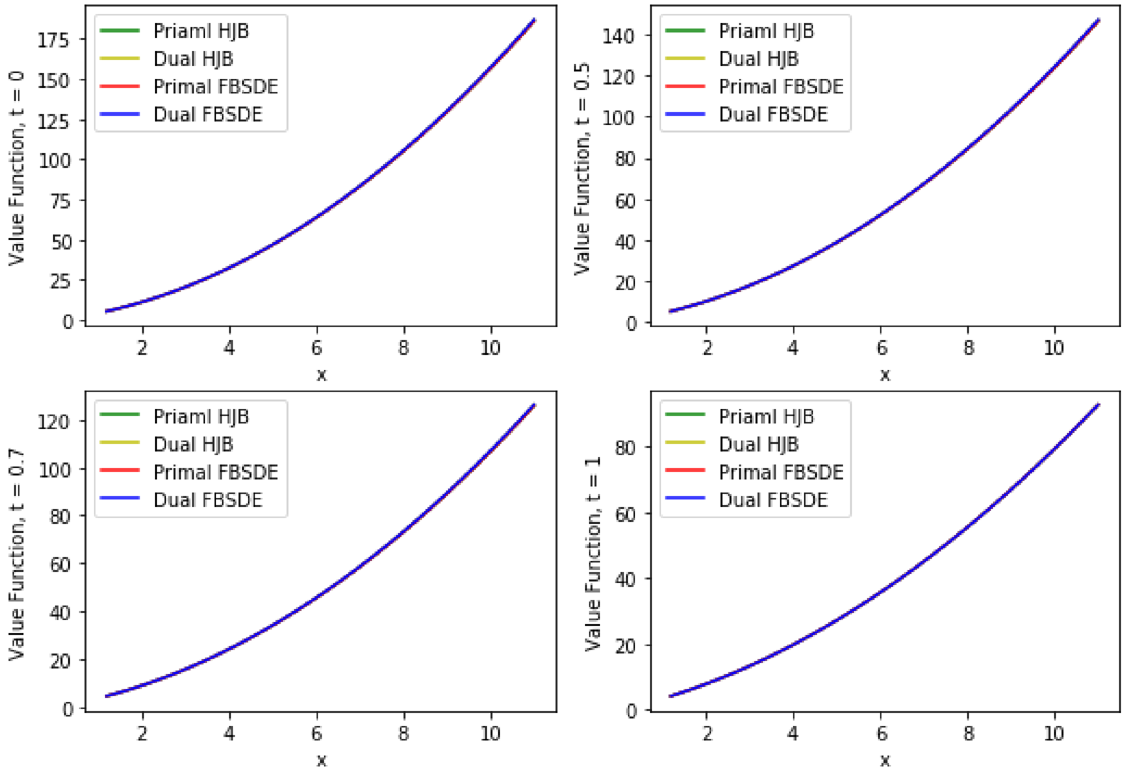

Figure 1.2: Comparison of $V(0, X^{\hat{\pi}}(0))$ at different t

Although there is still only little difference between the lines we can still find the equivalences of the 4 methods to some extent. Also we can figure out that for all $t \in [0,\ T]$, the value function $V(0, X^{\hat{\pi}}(0))$ is of a quadratic form which is consistent with our ansatz $V(t,x) = x^2 g_1(t) + x g_2(t) + g_3(t)$ which is also a quadratic function.

To observe the differences between the control processes and associated state processes calculated by HJB and FBSDE method we have the following figures.

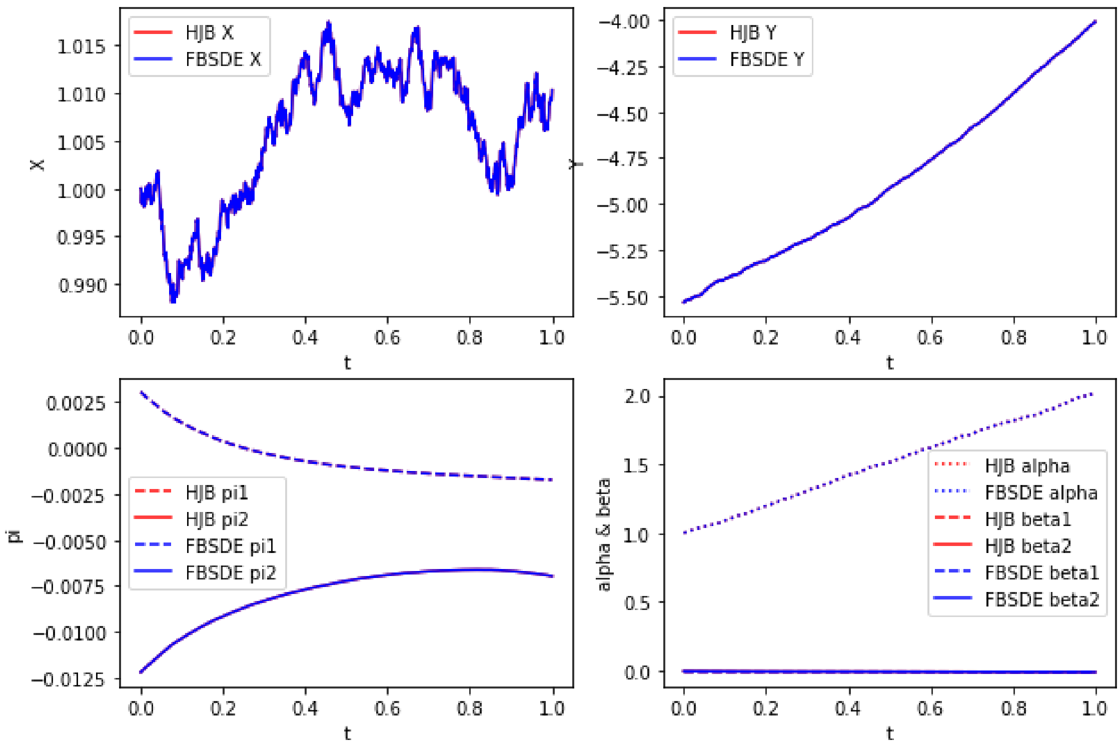

Figure 1.3: Comparison of associated processes and controls

There is still little difference between different curves. In the bottom-right subgraph, the dual control $\beta(t)$ seems to be 0, but it is not strictly 0 for both entry. This is because $\alpha(t) = Q(t)X^\pi(t)$ is largely influenced by the associated process $X^\pi(t)$ thus influenced by the initial value x_0, here the chosen $x_0 = 1$ might be relatively large and enlarged the scale of the plot, to show this is we can zoom in the plot of $\beta(t)$.

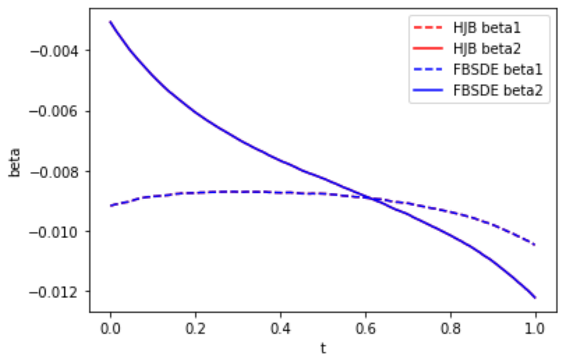

Figure 1.4: β in the unconstrained problem

We can also plot the functions that only depends on t in the 4 methods, i.e. $h_n(t)$, $g_n(t)$ for $n = 1,\ 2,\ 3$, $\phi_1(t)$, $\psi_1(t)$ and $\phi_2(t)$, $\psi_2(t)$.

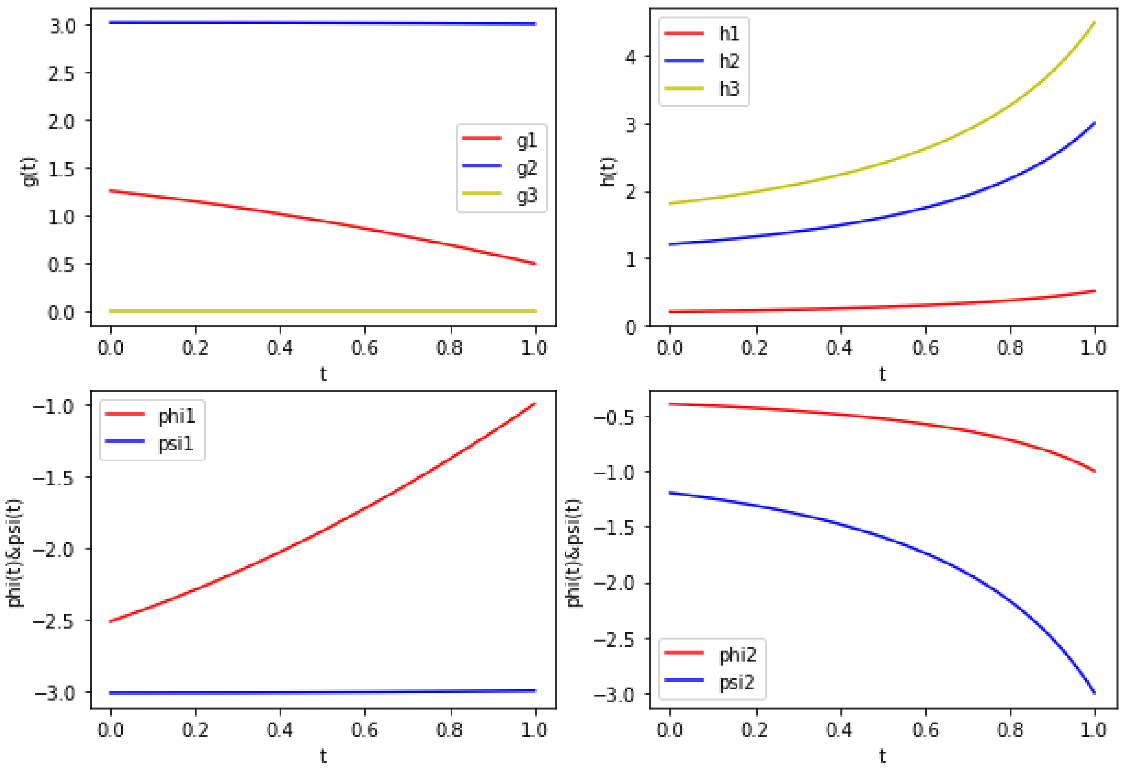

Figure 1.5: Functions depend on t in Value functions

Similar to the dual control $\beta(t)$ in figure 1.3, although the functions $g_2(t)$, $g_3(t)$ and $\psi_1(t)$ seem to be constant, there does exist changes. This might be due to the changes from the stochastic variation are relatively small compare with the influence of the large terminal values. We plot the functions of the same system to show that the change of $g_1(t)$ has larger influence to the value function compared with $g_2(t)$ and $g_3(t)$ on the value function, and similar for the $\phi_1(t)$ compared with $\psi_1(t)$, this can be expected since we have deduced that $g_1(t) = -\dfrac{\phi_1(t)}{2}$. We also plot $g_2(t)$, $g_3(t)$ and $\psi_1(t)$ to show its own trend.

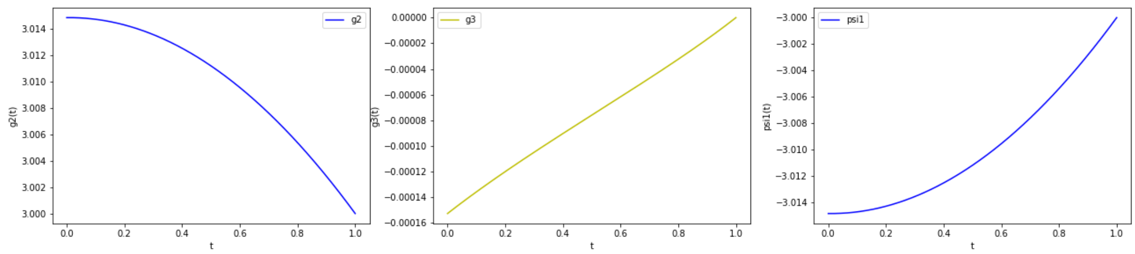

Figure 1.6: $g_1(t), g_2(t)$ and $\psi_1(t)$

Chapter 2

Constrained Problem

In this section we would still consider the previous optimization problem in Chapter 1, i.e. minimize $J : \mathcal{A} \to \mathbb{R}$ such that

$$J(\pi) := E\left[\int_0^T f\left(t, X^\pi(t), \pi(t)\right) dt + g\left(X^\pi(T)\right) \Big| X^\pi(0) = x_0\right] \tag{2.0.1}$$

But this time we would add constraint conditions to the admissible control set, the second entry of the optimal control $\pi(t)$ is set to be 0, i.e.

$$\mathcal{A} := \left\{\pi \in \mathcal{H}^2\left(0, T; \mathbb{R}^2\right) : \pi(t) \in K \text{ for } t \in [0, T] \text{ a.e. } \right\},$$

where $K = \mathbb{R} \times \{0\}$ which is not the whole space anymore.

2.1 HJB method

For the HJB method, the primal problem is easier to solve compared with the unconstrained problem, it can be regarded as degenerating to a 1-dimensional optimization problem, but the dual problem is more complicated since we need to recalculate the dual system under the constrained condition.

2.1.1 Primal problem

We have the same HJB equation as before i.e.

$$V_t + \frac{1}{2}Q(t)X^\pi(t)^2 + r(t)X^\pi(t)V_x + \inf_{\pi \in K} L_v(t) = 0$$

with terminal condition

$$V(T, X^\pi(T)) = \frac{a}{2}x^2 + cx$$

but this time we need to minimize

$$L_v(t, \pi_1) = \pi_1(t)\left[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)\right]V_x + \frac{1}{2}\pi_1^2(t)\left[\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right)V_{xx} + R_{11}(t)\right]$$

which is a much simpler compared with the unconstrained primal problem.

By the first order condition we have

$$\hat{\pi}_1 = -\frac{\left[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)\right]V_x}{\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right)V_{xx} + R_{11}(t)} \tag{2.1.1}$$

According to the constrained condition we have the optimal control $\hat{\pi} = (\hat{\pi}_1, 0)$

Consider the terminal condition is unchanged, We still make the ansatz

$$V(t,x) = x^2 g_1(t) + x g_2(t) + g_3(t).$$

Substitute $\hat{\pi}(t)$ and our ansatz into the HJB equation and sum up each term in terms of the power of X we can get the system of ODEs

$$\begin{cases} \dot{g}_1 - \dfrac{2\left[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)\right]^2}{2\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right) g_1 + R_{11}(t)} g_1^2 + 2r(t) g_1 + \dfrac{1}{2} Q(t) = 0 \\ \dot{g}_2 + \left(r(t) - \dfrac{2\left[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)\right]^2}{2\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right) g_1 + R_{11}(t)} g_1 \right) g_2 = 0 \\ \dot{g}_3 - \dfrac{\left[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)\right]^2}{4\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right) g_1 + 2R_{11}(t)} g_2^2 = 0 \end{cases}$$

with terminal condition

$$\begin{cases} g_1(T) = \dfrac{a}{2} \\ g_2(T) = c \\ g_3(T) = 0 \end{cases}$$

2.1.2 Dual Problem

As the previous Section 1.1.2, we can do the same dual method, we still set

$$m_0(y) := \sup_{x \in \mathbb{R}} \left\{ xy - \Psi_{\{x_0\}}(x) \right\} = x_0 y$$

$$m_T(\omega, y) := \sup_{x \in \mathbb{R}} \{ -xy - g(\omega, x) \} = \frac{(y+c)^2}{2a}$$

$$\phi(\omega, t, \alpha, \beta) := \sup_{x \in \mathbb{R}, \pi \in K} \{ x\alpha + \pi'\beta - f(\omega, t, x, \pi) \}$$

where

$$f(\omega, t, x, \pi) := \frac{1}{2}\left[Q(t)x^2 + \pi' R(t) \pi \right]$$

The only difference is that $K = \mathbb{R} \times \{0\}$ but not $\mathbb{R}^2$ anymore. Thus we have

$$\begin{aligned} \phi(\omega, t, \alpha, \beta) :&= \sup_{x \in \mathbb{R}, \pi_1 \in \mathbb{R}} \left\{ x\alpha + \pi_1 \beta_1 - \frac{1}{2}\left[Q(t)x^2 + \pi_1^2 R_{11}(t) \right] \right\} \\ &= \frac{1}{2}\left[\frac{\alpha^2(t)}{Q(t)} + \beta'(t) A(t) \beta(t) \right] \end{aligned}$$

Where $A(t)$ is the matrix

$$\begin{pmatrix} \dfrac{1}{R_{11}(t)} & 0 \\ 0 & 0 \end{pmatrix}$$

Thus we have the HJB equation

$$W_t - r(t) Y^{y,\alpha,\beta}(t) W_y + \inf_{(\alpha,\beta) \in \mathcal{B}} L_w(t) = 0$$

with the terminal condition

$$W(T, Y^{(y,\alpha,\beta)}(T)) = \frac{1}{2a} y^2 + \frac{c}{a} y + \frac{c^2}{2a}$$

where

$$L_w(t,\alpha,\beta) = \alpha(t)W_y + \frac{1}{2}\left|\sigma(t)^{-1}\beta(t) - \theta(t)Y^{y,\alpha,\beta}(t)\right|^2 W_{yy} + \frac{1}{2}\left[\frac{\alpha^2(t)}{Q(t)} + \beta'(t)A(t)\beta(t)\right]$$

Assuming that the matrix $\sigma(t)^{-1} + \dfrac{\sigma'(t)A(t)}{W_{yy}}$ is invertible. Take the partial derivative of $L_w(t,\alpha,\beta)$ respect to α and β, by the first order condition we have

$$\begin{cases} \hat{\alpha}(t) = -Q(t)W_y \\ \hat{\beta}(t) = \left(\sigma(t)^{-1} + \dfrac{\sigma'(t)A(t)}{W_{yy}}\right)^{-1} \theta(t)y \end{cases}$$

According to the terminal condition we can make the same ansatz as the unconstrained dual problem we discussed in Section 1.1.2, i.e.

$$W(t,y) = y^2 h_1(t) + y h_2(t) + h_3(t)$$

Substitute $\hat{\alpha}(t)$, $\hat{\beta}(t)$ and our ansatz into the HJB equation we would get the system of ODEs

$$\begin{cases} \dot{h}_1 - 2Q(t)h_1^2 - 2r(t)h_1 + \left|\left(I + \dfrac{K_A(t)}{2h_1}\right)^{-1}\theta(t) - \theta(t)\right|^2 h_1 + \\ \dfrac{1}{2}\left|\left(I + \dfrac{K_A(t)}{2h_1}\right)^{-1}\theta(t)\right|^2_{K_A(t)} = 0 \\ \dot{h}_2 - \left(2Q(t)h_1 + r(t)\right)h_2 = 0 \\ \dot{h}_3 - \dfrac{1}{2}Q(t)h_2^2 = 0 \end{cases}$$

with terminal condition

$$\begin{cases} h_1(T) = \dfrac{1}{2a} \\ h_2(T) = \dfrac{c}{a} \\ h_3(T) = \dfrac{c^2}{2a} \end{cases}$$

where $K_A(t) = \sigma'(t)A(t)\sigma(t)$

Minimize the function $\tilde{\Psi}(y, \hat{\alpha}(0), \hat{\beta}(0)) = x_0 y + W(y, \hat{\alpha}(0), \hat{\beta}(0))$ according to y, we can get

$$\hat{y} = -\frac{h_2(0) + x_0}{2h_1(0)}$$

which is still unchanged compared with the unconstrained dual problem. Thus by comparing the ansatz of the primal problem and dual problem we still have

$$\begin{cases} g_1(t) = \dfrac{1}{4h_1(t)} \\ g_2(t) = \dfrac{h_2(t)}{2h_1(t)} \\ g_3(t) = \dfrac{h_2^2(t)}{4h_1(t)} - h_3(t) \end{cases}$$

2.2 FBSDE Method

Similar to the HJB method, since now we are considering the constrained control space with $\pi_2 = 0$, we can just consider the first entry of the optimal control in the primal problem, but this does mean that the second entry of the dual control β also equal to 0, we need to do more derivation to figure out β in dual problem.

2.2.1 Primal Problem

Using the same method as the previous Section 1.2.1, we can get the Hamiltonian of the primal problem, i.e.

$$\begin{aligned} H_1(t, x, \pi_1, p_1, q_1) = &[r(t)x + \pi_1(\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t))]\, p_1 \\ &+ \pi_1[q_{11}\sigma_{11}(t) + q_{12}\sigma_{12}(t)] - \frac{1}{2}Q(t)x^2 - \frac{1}{2}\pi_1^2 R_{11}(t) \end{aligned}$$

thus we have the BSDE

$$\begin{cases} dp_1(t) = [-r(t)p_1(t) + Q(t)X^\pi(t)]\, dt + q_1'(t)dW(t) \\ p_1(T) = -aX^\pi(T) - c \end{cases}$$

As the previous equation, we can have the following theorem

Theorem 2.2.1. *Let $\hat{\pi} \in \mathbb{A}$, then $\hat{\pi}$ is the optimal control of the primal problem if and only if the solution $\left(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1\right)$ of the FBSDE*

$$\begin{cases} dX^{\hat{\pi}}(t) = \left[r(t)X^{\hat{\pi}}(t) + \hat{\pi}'(t)\sigma(t)\theta(t)\right] dt + \hat{\pi}'(t)\sigma(t)dW(t) \\ X^{\hat{\pi}}(0) = x_0 \\ d\hat{p}_1(t) = \left[-r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t)\right] dt + \hat{q}_1'(t)dW(t) \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c \end{cases} \tag{2.2.1}$$

satisfies

$$[\hat{\pi}' - \pi']\,[\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) - R(t)\hat{\pi}(t)] \geq 0 \tag{2.2.2}$$

for $(\mathbb{P} \otimes Leb)-a.e.(\omega, t) \in \Omega \times [0, T]$ and $\pi \in K$

The condition is different from Theorem 1.2.3 since now $K = R \times \{0\}$, actually K can be any closed convex set containing 0 in this theorem.

Since the second entry of π equal to 0, from condition (2.2.2) we can get

$$(\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t))\, p_1(t) + q_{11}(t)\sigma_{11}(t) + q_{12}(t)\sigma_{12}(t) - R_{11}(t)\pi_1(t) - \hat{\pi}_1(t)R_{11}(t) = 0$$

i.e.

$$\hat{\pi}_1 = \frac{[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)]\, p_1(t) + q_{11}(t)\sigma_{11}(t) + q_{12}(t)\sigma_{12}(t)}{R_{11}(t)} \tag{2.2.3}$$

Using the same ansatz as (1.2.4), i.e. $p_1(t) = \phi_1(t)X_t + \psi_1(t)$, we can get

$$\begin{cases} -r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t) = \dot{\phi}_1(t)X_t + \phi_1(t)\left[r(t)X_t + \hat{\pi}_1(t)(\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t))\right] + \dot{\psi}_1(t) \\ \hat{q}_1(t) = \phi_1(t)\hat{\pi}_1(t)\sigma_1(t) \end{cases} \tag{2.2.4}$$

where $\sigma_1 = (\sigma_{11},\ \sigma_{12})'$

By (2.2.3) and the second equation of (2.2.4) we can see that

$$\hat{\pi}_1 = \frac{\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)}{R_{11}(t) - \phi_1(t)\,[\sigma_{11}^2(t) + \sigma_{12}^2(t)]}\left(\phi_1(t)X_t + \psi_1(t)\right)$$

Compared with (2.1.1) we can conclude that

$$\begin{cases} \phi_1(t) = -V_{xx} \\ \phi_1(t)\hat{X}_t + \psi_1(t) = -V_x \end{cases}$$

Thus we have

$$\begin{cases} g_1(t) = -\dfrac{\phi_1(t)}{2} \\ g_2(t) = -\psi_1(t) \\ g_3(t) = \dfrac{1}{2}\displaystyle\int \frac{[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)]^2}{-\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right)\phi_1 + R_{11}(t)}\psi_1^2 dt + C_1 \end{cases}$$

where constant C_1 can be determined by $g_3(T) = 0$.

Substitute $\hat{\pi}$ into the first equation of (2.2.4) and summing the terms in the power of $X^{\hat{\pi}}(t)$ we can get

$$\begin{cases} \dot{\phi}_1(t) + \dfrac{[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)]^2}{R_{11}(t) - \phi_1(t)\left[\sigma_{11}^2(t) + \sigma_{12}^2(t)\right]}\phi_1^2(t) + 2r(t)\phi_1(t) - Q(t) = 0 \\ \dot{\psi}_1(t) + \left(r(t) + \dfrac{[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)]^2}{R_{11}(t) - \phi_1(t)\left[\sigma_{11}^2(t) + \sigma_{12}^2(t)\right]}\phi_1(t)\right)\psi_1(t) = 0 \end{cases}$$

where

$$\begin{cases} \phi_1(T) = -a \\ \psi_1(T) = -c \end{cases}$$

2.2.2 Dual Problem

The Hamiltonian of the dual problem is

$$H_2\left(t, y, \alpha, \beta, p_2, q_2\right) = (\alpha - r(t)y)p_2 + q_2'\left(\sigma^{-1}(t)\beta - \theta(t)y\right) - \frac{1}{2}\left(\frac{\alpha^2}{Q(t)} + \beta' A(t)\beta\right)$$

where $p_2 \in \mathcal{S}^2(0, T; R)$ and $q_2 \in \mathcal{H}^2\left(0, t; R^2\right)$

Thus we have the following theorem

Theorem 2.2.2. *Suppose $Q(t)$ and $R(t)$ are positive definite and their inverse are uniformly bounded, then $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is the optimal of the dual problem if and only if the solution $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2\right)$ of the FBSDE*

$$\begin{cases} dY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \left[\hat{\alpha}(t) - r(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]dt + \left[\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]' dW(t) \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y} \\ d\hat{p}_2(t) = \left[r(t)\hat{p}_2(t) + \hat{q}_2'(t)\theta(t)\right]dt + \hat{q}_2'(t)dW_t \\ \hat{p}_2(T) = -\dfrac{Y^{(\hat{y},\hat{\alpha}\hat{\beta})}(T) + c}{a} \end{cases} \tag{2.2.5}$$

satisfies

$$\begin{cases} \hat{p}_2(0) = x_0 \\ [\sigma']^{-1}(t)\hat{q}_2(t) \in K \\ \left(\hat{p}_2(t), [\sigma']^{-1}(t)\hat{q}_2(t)\right) \in \partial\phi(\hat{\alpha}(t), \hat{\beta}(t)) \end{cases} \tag{2.2.6}$$

for $(\mathbb{P} \otimes Leb)-a.e.(\omega, t) \in \Omega \times [0, T]$

Compared with Theorem 1.2.4, we add a new condition i.e. $[\sigma']^{-1}(t)\hat{q}_2(t) \in K$, where $K = \mathbb{R} \times \{0\}$ in this case, but actually K can be any subspace of $\mathbb{R}^2$ contains 0. By this condition we simplify H_2 to

$$H_2(t, y, \alpha, \beta, p_2, q_2) = [\alpha - r(t)y]\, p_2 + \frac{1}{d(t)}[\sigma_{22}(t)q_{21} - \sigma_{21}(t)q_{22}]\, \beta_1 - q_2'\theta(t)y - \frac{1}{2}\left[\frac{\alpha^2}{Q(t)} + \frac{\beta_1^2}{R_{11}(t)}\right]$$

where $d(t) = det(\sigma(t))$ is the determinant of matrix $\sigma(t)$

Since

$$\partial\phi(\hat{\alpha}(t), \hat{\beta}(t)) = \left(\frac{\hat{\alpha}(t)}{Q(t)}, \frac{\hat{\beta}_1(t)}{R_{11}(t)}\right)$$

Using the third condition of (2.2.6) we can get

$$\begin{cases} \hat{\alpha}(t) = \hat{p}_2(t)Q(t) \\ \hat{\beta}_1(t) = \dfrac{R_{11}(t)}{d}[\sigma_{22}(t)q_{21}(t) - \sigma_{21}(t)q_{22}(t)] \end{cases} \tag{2.2.7}$$

Since $\sigma_{11}q_{22} - \sigma_{12}q_{21} = 0$, we have

$$q_{22} = \frac{\sigma_{12}}{\sigma_{11}}q_{21}$$

By the second equation of (2.2.7) we can get

$$\hat{\beta}_1 = \frac{R_{11}(t)}{\sigma_{11}(t)}q_{21}(t) \tag{2.2.8}$$

and

$$q_2 = q_{21}\left(1, \frac{\sigma_{12}}{\sigma_{11}}\right)'$$

Make the same ansatz as the previous Section 1.2.2, i.e. $p_2(t) = \phi_2(t)Y(t) + \psi_2(t)$ we can get

$$\begin{cases} r(t)\hat{p}_2(t) + \hat{q}_2'(t)\theta(t) = \dot{\phi}_2(t)\hat{Y}(t) + \phi_2(t)\left(\hat{\alpha}(t) - r(t)\hat{Y}(t)\right) + \dot{\psi}_2(t) \\ \hat{q}_2(t) = \phi_2(t)\left(\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)\hat{Y}(t)\right) \end{cases} \tag{2.2.9}$$

By the second equation of (2.2.9) we can see that

$$\hat{\beta}_1 = \frac{1}{\phi_2(t)}\left[\sigma_{11}(t) + \frac{\sigma_{12}^2(t)}{\sigma_{11}(t)}\right] q_{21}(t) + [\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)]\, Y(t)$$

Combined with (2.2.8) we can get

$$q_2 = B(t)Y(t) \tag{2.2.10}$$

where

$$B(t) = \frac{\sigma_{11}(t)\,[\sigma_{11}(t)\theta_1(t) + \sigma_{12}(t)\theta_2(t)]}{R_{11}(t) - \dfrac{1}{\phi_2(t)}[\sigma_{11}^2(t) + \sigma_{12}^2(t)]}\left(1, \frac{\sigma_{12}}{\sigma_{11}}\right)'$$

where C_2 is determined by the terminal condition $h_3(T) = \dfrac{c^2}{2a}$

Similarly as previous Section 1.2.2 we can also conclude that

$$\begin{cases} h_1(t) = -\dfrac{\phi_2(t)}{2} \\ h_2(t) = -\psi_2(t) \\ h_3(t) = \dfrac{1}{2}\displaystyle\int Q(t)\psi_2^2(t)dt + C_2 \end{cases}$$

Substitute our ansatz p_2 and q_2 into the first equation of (2.2.9) and summing in the power of Y we can get the system of ODEs

$$\begin{cases} \dot{\phi}_2(t) + Q(t)\phi_2^2(t) - 2r(t)\phi_2(t) - B'\theta(t) = 0 \\ \dot{\psi}_2(t) + \Big(\phi_2(t)Q(t) - r(t)\Big)\psi_2(t) = 0 \end{cases}$$

where

$$\begin{cases} \phi_2(T) = -\dfrac{1}{a} \\ \psi_2(T) = -\dfrac{c}{a} \end{cases}$$

2.2.3 Relations Between FBSDE Primal and Dual Problem

Although we have added constrained condition $\pi_2 = 0$ to the optimization problem, we still have

$$\begin{cases} \hat{\pi}(t) := [\sigma']^{-1}(t)\hat{q}_2(t) \\ X^{\hat{\pi}}(t) = \hat{p}_2(t) \\ \hat{p}_1(t) = Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) \\ \hat{q}_1(t) = \sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) \end{cases}$$

and

$$\begin{cases} \hat{y} = \hat{p}_1(0) \\ \hat{\alpha}(t) = Q(t)X^{\hat{\pi}}(t) \\ \hat{\beta}(t) = \sigma(t)\left[\hat{q}_1(t) + \theta(t)\hat{p}_1(t)\right] \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \hat{p}_1(t) \\ \hat{p}_2(t) = X^{\pi}(t) \\ \hat{q}_2(t) = \sigma'(t)\hat{\pi}(t) \end{cases}$$

The proof is the same to the previous Section 1.2.3.

2.3 Comparison

Similar to the previous Section 1.3, we plot Figure 2.1 to show the differences between the results simulated by Python using the same functions and random seeds as the ones in the unconstrained problem, with $t \in [0, 1]$ and different initial values $x_0 = 0.1,\ 0.3,\ 0.5,\ 1$ with Runge-Kutta step-size $h = \frac{1}{1000}$ and using HJB and FBSDE methods on both primal and dual problem.

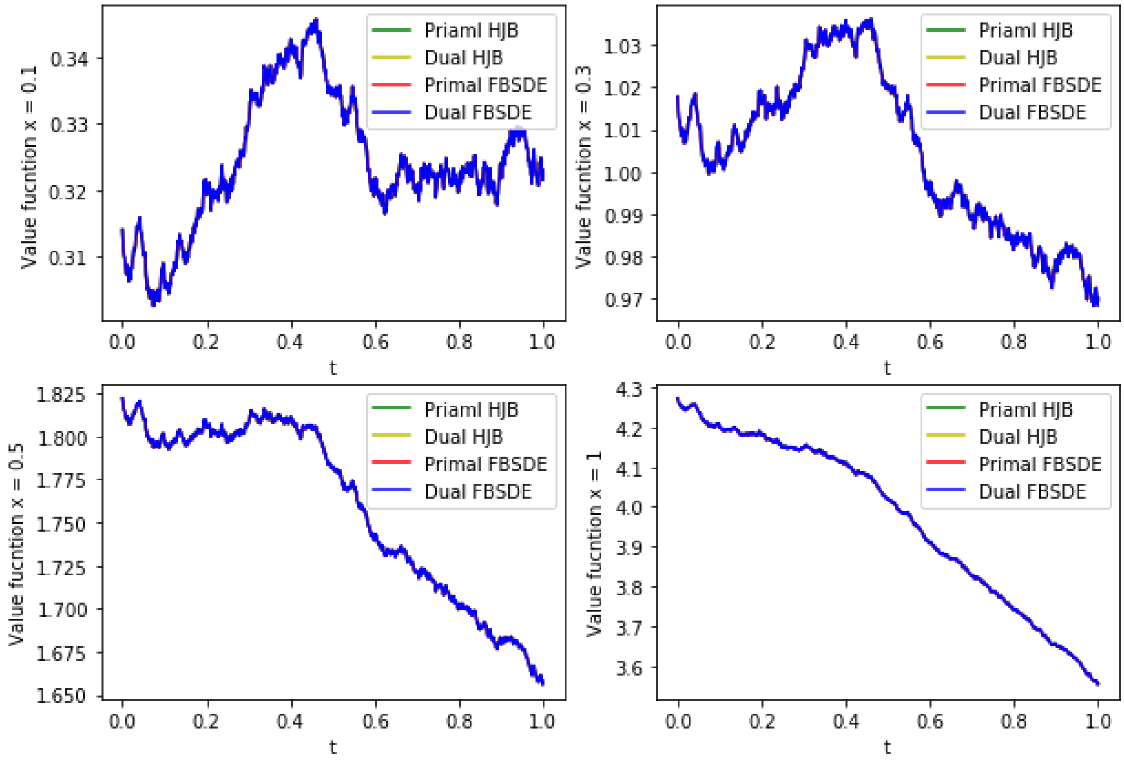

Figure 2.1: Comparison of $V(t, X^{\hat{\pi}}(t))$ with fixed x_0

There is only little difference between different methods which may caused by truncation error, which suggests that our calculations are indeed correct. We can see the value functions of the constrained problem and unconstrained problem for large x_0 show a similar trend, this might be that the correlation between the stocks we assumed are relatively high. We can still figure out that the value functions get "smoother" as x_0 gets larger, which is also present in the unconstrained problem.

We also set different x_0 again and present the value function at $t = 0$. The following Figure 2.2 shows the difference between the lines of $V(0, X^{\hat{\pi}}(0))$ plotted using 4 methods with $x_0 \in [1, 10]$ at different time t on the constrained problem.

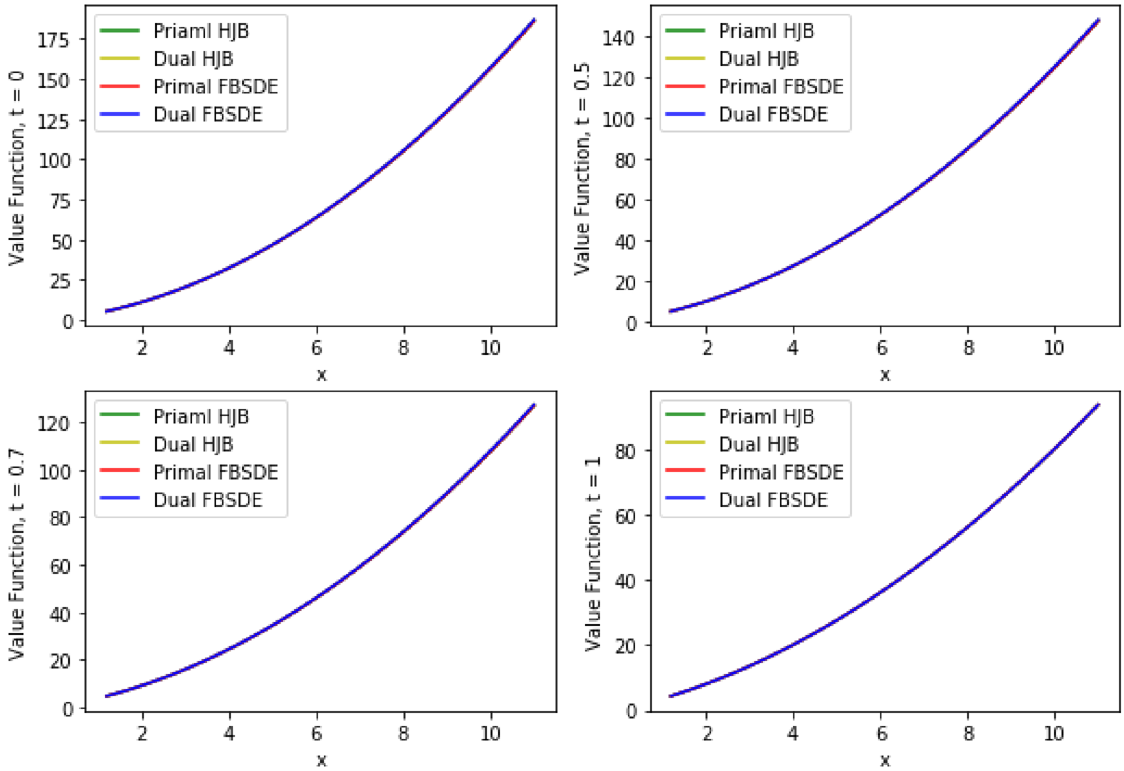

Figure 2.2: Comparison of $V(0, X^{\hat{\pi}}(0))$ at fixed t

Although there is still only little difference between the lines we can still find the equivalence of the 4 method in some extend. Also we can figure out that for all $t \in [0,\ T]$, the value function $V(0, X^{\hat{\pi}}(0))$ shows a quadratic since we still make the same ansatz $V(t,x) = x^2 g_1(t) + x g_2(t) + g_3(t)$ as we did in the unconstrained problem which is a quadratic function.

To observe the differences between the control processes and associated processes calculated by HJB and FBSDE methods we have the following figures.

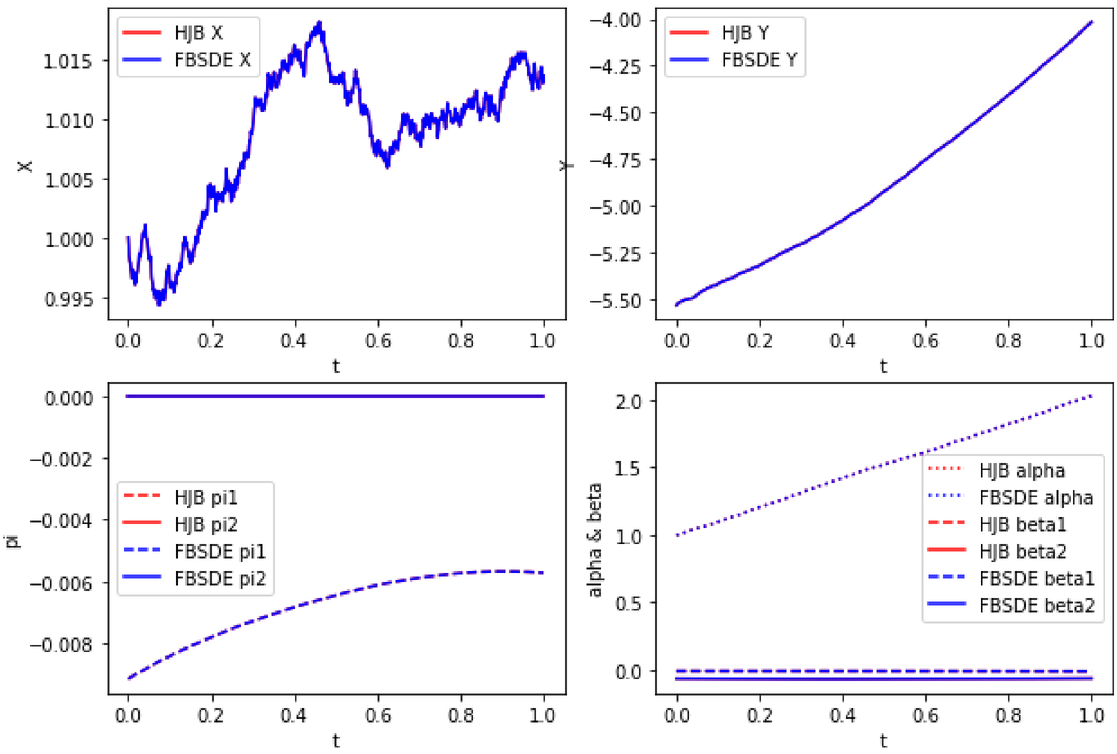

Figure 2.3: Comparison of associated processes and controls

There is still only little difference between different lines, the graphs suggest that different methods do give the same result at least numerically under the constrained condition. Similar to the unconstrained problem, in the subgraph on the bottom right, the dual control $\beta(t)$ seems to be 0 but is actually not. To show this is we can zoom in the plot of $\beta(t)$.

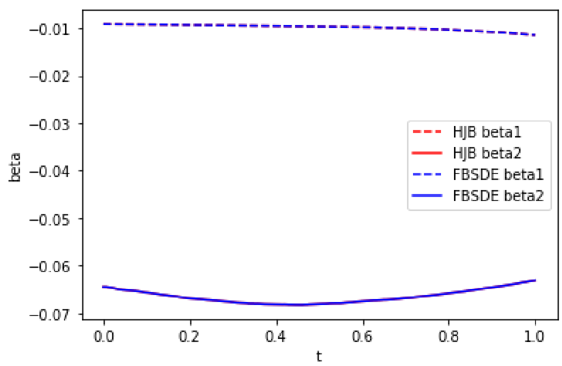

Figure 2.4: β in the constrained problem

We can see that although we have the constraint on $\pi(t)$ such that $\pi_2(t) = 0$, both entries of the dual control $\beta(t)$ are not equal to 0.

We also plot the functions in which only t varies in the 4 methods, i.e. $h_n(t)$, $g_n(t)$ for $n = 1, 2, 3$, $\phi_1(t)$, $\psi_1(t)$ and $\phi_2(t)$, $\psi_2(t)$.

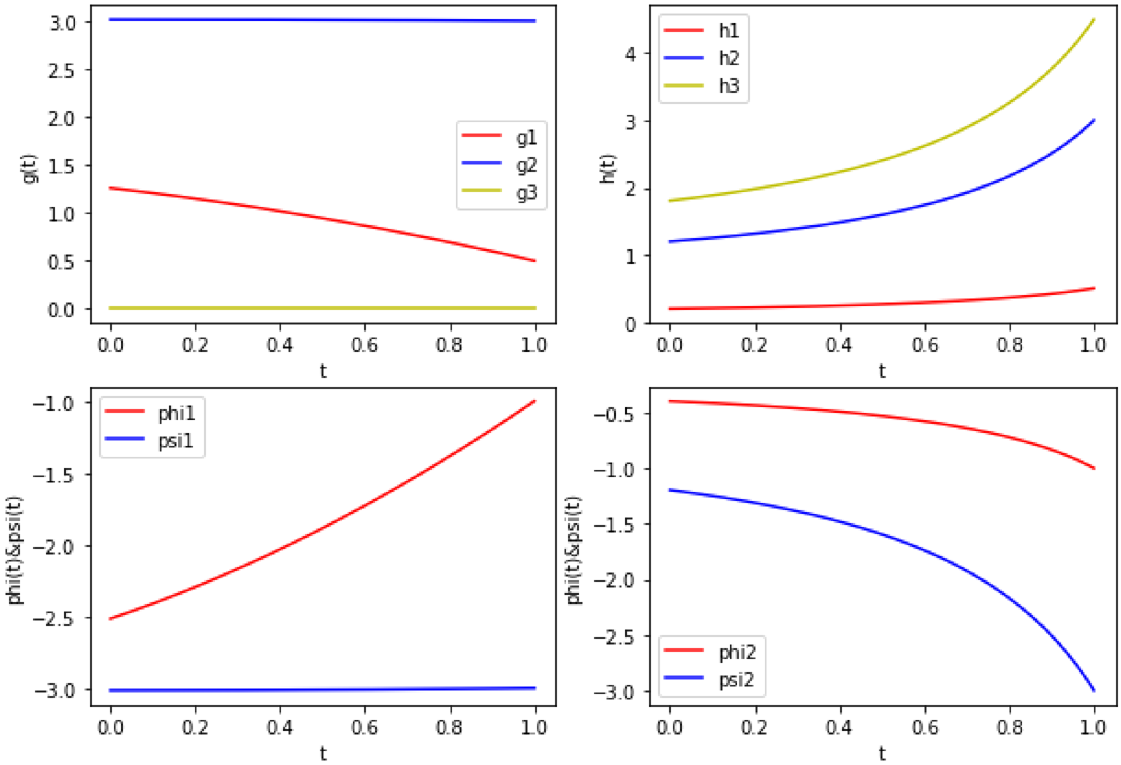

Figure 2.5: Parameter functions of Value functions

Similar to the unconstrained problem. We plot the functions of the same system to show that the change of $g_1(t)$ has larger influence to the value function compared with $g_2(t)$ and $g_3(t)$ on the value function, and similar for the $\phi_1(t)$ compared with $\psi_1(t)$. We also plot $g_2(t)$, $g_3(t)$ and $\psi_1(t)$ to show its own trend.

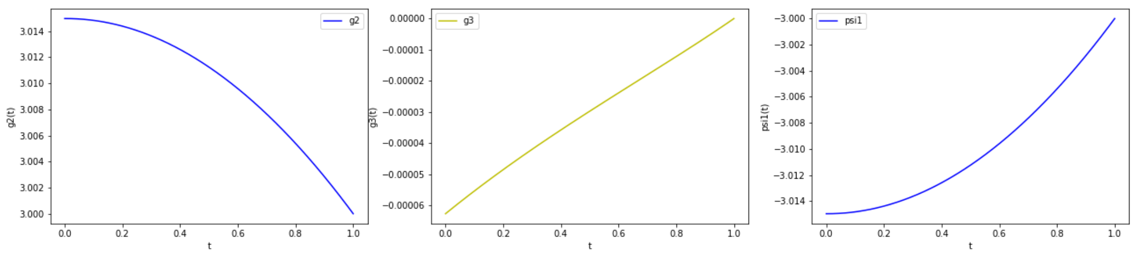

Figure 2.6: $g_1(t), g_2(t)$ and $\psi_1(t)$

We can see that the functions in which only t varies in Chapter 1 and Chapter 2 seems to have the same behaviour, this might be caused by the choice of functions, but we can still see that the value function and control processes show different trends.

Chapter 3

Random Drift Case

In this section we will consider the one-dimensional case i.e. $K = \mathbb{R}$ on the same optimization problem as the unconstrained problem, but this time we introduce the random drift which we set to be an Ornstein-Uhlenbeck process into the stock process.

Following [Doob, 1942] we have the definition

Definition 3.0.1. (Ornstein–Uhlenbeck process)

The Ornstein-Uhlenbeck process (OU process) $X(t)$ is defined by the following stochastic differential equation:

$$dX(t) = k\left(\mu - X(t)\right)dt + \sigma(t)dW(t)$$

where $W(t)$ denotes the Brownian motion, k and μ are deterministic constants

We still have the bank account process $\{S_0\}$, where

$$dS_0(t) = r(t)S_0(t)dt, 0 \leq t \leq T, S_0(0) = 1$$

but the stock price process become

$$dS(t) = S(t)H(t)dt + S(t)\sigma(t)dW(t), 0 \leq t \leq T, S(0) > 0$$

where $H(t)$ is an OU-process

$$\begin{cases} dH(t) = k\left[\mu - H(t)\right]dt + \sigma_H(t)dW_H(t) \\ H(0) = H_0 \end{cases}$$

and $dW(t)dW_H(t) = \rho dt$

Now we can rewrite the wealth process $X^\pi(t)$ as

$$\begin{cases} dX^\pi(t) = \Big[r(t)X^\pi(t) + \pi(t)\,\Big(H(t) - r(t)\Big)\Big]\,dt + \pi(t)\sigma(t)dW(t), \quad 0 \leq t \leq T \\ X^\pi(0) = x_0 \end{cases}$$

3.1 HJB method

Under the case with random drift in the stock process we need to include the infinitesimal generator of the random drift $H(t)$ in the HJB equation, which cause we need to include one more variable H in the PDE that we can not separate it from t, thus we will introduce the Feynman-Kac theorem and its nonlinear version

3.1.1 Primal problem

Applying DPP we can get the HJB equation

$$V_t + \frac{1}{2}Q(t)X^\pi(t)^2 + r(t)X^\pi(t)V_x + \mathcal{L}_H V + \inf_{\pi\in K} L_v(t) = 0$$

with terminal condition

$$V(T, H(T), X^\pi(T)) = \frac{a}{2}x^2 + cx \tag{3.1.1}$$

where

$$L_v(t) = \pi(t)\left[\Big(H - r(t)\Big)V_X + \rho\sigma(t)\sigma_H(t)V_{XH}\right] + \frac{1}{2}\pi^2(t)\left[\sigma^2(t)V_{XX} + R(t)\right]$$

and $\mathcal{L}_H$ is the infinitesimal generator of the OU process $H(t)$. Thus by optimizing L_v we have

$$\hat{\pi}(t) = -\frac{[H - r(t)]\,V_X + \rho\sigma(t)\sigma_H(t)V_{XH}}{\sigma^2(t)V_{XX} + R(t)}$$

Substitute (3.1.1) into the HJB we can get the PDE

$$V_t + \frac{1}{2}Q(t)X^\pi(t)^2 + r(t)X^\pi(t)V_x + \mathcal{L}_H V - \frac{\left[\Big(H - r(t)\Big)V_x + \rho\sigma(t)\sigma_H(t)V_{XH}\right]^2}{2\left[\sigma^2(t)V_{xx} + R(t)\right]} = 0 \tag{3.1.2}$$

Make the ansatz

$$V(t, x, H) = g_1(t, H)x^2 + g_2(t, H)x + g_3(t, H)$$

and substitute into PDE (3.1.2) we have

$$\begin{cases} \partial_t g_1 + 2g_1 r(t) + \mathcal{L}_H g_1 + \frac{1}{2}Q(t) - \dfrac{2\left[g_1\left(H - r(t)\right) + \rho\sigma(t)\sigma_H(t)\partial_H g_1\right]^2}{2\sigma^2(t)g_1 + R(t)} = 0 \\ \partial_t g_2 + r(t)g_2 + \mathcal{L}_H g_2 - \dfrac{2\left[g_1\left(H - r(t)\right) + \rho\sigma(t)\sigma_H(t)\partial_H g_1\right]\left[g_2\left(H - r(t)\right) + \rho\sigma(t)\sigma_H(t)\partial_H g_2\right]}{2\sigma^2(t)g_1 + R(t)} = 0 \\ \partial_t g_3 + \mathcal{L}_H g_3 - \dfrac{\left[g_2\left(H - r(t)\right) + \rho\sigma(t)\sigma_H(t)\partial_H g_2\right]^2}{4\sigma^2(t)g_1 + 2R(t)} = 0 \end{cases} \tag{3.1.3}$$

with terminal conditions

$$\begin{cases} g_1(T) = \dfrac{a}{2} \\ g_2(T) = c \\ g_3(T) = 0 \end{cases}$$

This system of PDEs is difficult to solve thus we introduce the Feynman-Kac theorem and its non-linear version. The following theorems are taken from [Pham, 2009b].

Theorem 3.1.1. ***(Feynman-Kac theorem)***

Assume that $v \in C^{1,2}\left([0,T)\times\mathbb{R}^n\right)$ and the stochastic process X_t are well defined, we have the linear parabolic PDE

$$\begin{cases} -\partial_t v - \mathcal{L}v - u(t,x)v(t,x) - f(t,x) = 0, \quad (t,x)\in[0,T)\times\mathbb{R}^n \\ v(T,x) = g(x), \quad x\in\mathbb{R}^n \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator

$$\mathcal{L}v = b(t,x)\cdot D_x v + \frac{1}{2}\operatorname{tr}\left(\sigma(t,x)\sigma'(x)D^2_{xx}v\right)$$

Then the solution to this PDE is

$$v(t,x)=E\left[\int_t^T e^{\int_t^s u(\tau,X_\tau)d\tau} f\left(s,X_s^{t,x}\right)ds+e^{\int_t^T u(\tau,X_\tau)d\tau} g\left(X_T^{t,x}\right)\Big|X_t=x\right] \tag{3.1.4}$$

where $\{X_s^{t,x}, t\le s\le T\}$ *is the solution to the SDE*

$$dX_s=b\left(s,X_s\right)ds+\sigma\left(s,X_s\right)dW_s,\quad t\le s\le T, X_t=x$$

Theorem 3.1.2. ***(Non-linear Feynman-Kac theorem)***

Suppose we have the semi-linear PDE

$$\begin{cases} -\partial_t v-\mathcal{L}v-f\left(t,x,v,\sigma' D_x v\right)=0, & (t,x)\in[0,T)\times\mathbb{R}^n \\ v(T,x)=g(x), \quad x\in\mathbb{R}^n \end{cases}$$

we can write the BSDE representation and its associated forward SDE of the PDE as

$$\begin{cases} -dY_s=f\left(s,X_s,Y_s,Z_s\right)ds-Z_s.dW_s, \quad t\le s\le T, \quad Y_T=g\left(X_T\right) \\ dX_s=b\left(s,X_s\right)ds+\sigma\left(s,X_s\right)dW_s \end{cases}$$

Then the solution to the PDE is

$$v(t,x):=Y_t^{t,x}$$

we can see that the first PDE only depends on g_1, we can write the BSDE representation of g_1

$$\begin{cases} dG_1(t)=\left[\dfrac{2\left[G_1(t)\left(H(t)-r(t)\right)+\rho\sigma(t)Z_1(T)\right]^2}{2\sigma^2(t)G_1(t)+R(t)}-2G_1(t)r(t)-\dfrac{1}{2}Q(t)\right]dt+Z_1(t)dW_H(t) \\ G_1(T)=\dfrac{a}{2} \\ dH(t)=k\left[\mu-H(t)\right]dt+\sigma_H(t)dW_H(t) \\ H(0)=H_0 \end{cases}$$

By non-linear Feynman-Kac theorem we can get

$$g_1(t,H)=G_1(t)$$

Substitute g_1 and $\partial_H g_1$ into the second equation of (3.1.3), we directly solve $g_2(t,H)$ by Feynman-Kac theorem. Suppose

$$b_1(t,H)=\frac{2\rho\sigma(t)\sigma_H(t)\left[g_1\left(H-r(t)\right)+\rho\sigma(t)\sigma_H(t)\partial_H g_1\right]}{2\sigma^2(t)g_1+R(t)}$$

$$u_1(t,H)=\frac{2\left(H-r(t)\right)\left[g_1\left(H-r(t)\right)+\rho\sigma(t)\sigma_H(t)\partial_H g_1\right]}{2\sigma^2(t)g_1+R(t)}-r(t)$$

We have

$$g_2(t,H)=E\left[ce^{-\int_0^T u_1(t,H(t))dt}\Big|H(0)=H_0\right]$$

where

$$dH(t)=\left[k\left(\mu-H(t)\right)-b_1(t,H)\right]dt+\sigma_H(t)dW_H(t)$$

Substitute g_1, g_2, $\partial_H g_1$ and $\partial_H g_2$ into the third equation of (3.1.3), it becomes a decoupled PDE, we can simply apply Feynman-Kac theorem to solve $g_3(t,H)$, where

$$g_3(t,H)=-E\left[\int_0^T\frac{\left[g_2\left(H(t)-r(t)\right)+\rho\sigma(t)\sigma_H(t)\partial_H g_2\right]^2}{4\sigma^2(t)g_1+2R(t)}dt\Big|H(0)=H_0\right]$$

with

$$dH(t)=k\left[\mu-H(t)\right]dt+\sigma_H(t)dW_H(t)$$

3.1.2 Dual Problem

Suppose

$$dY(t) = \alpha_1(t)dt + \beta_1(t)dW(t) + \gamma(t)dW_H(t)$$

Since

$$dX^\pi(t) = \Big[r(t)X^\pi(t) + \pi(t)\Big(H(t) - r(t)\Big)\Big]\,dt + \pi(t)\sigma(t)dW(t)$$

By Ito's formula we can get

$$dX(t)Y(t) = [X(t)\alpha(t) + \pi(t)\beta(t)]\,dt + local\ martingale$$

where

$$\begin{cases} \alpha(t) = \alpha_1(t) + r(t)Y(t) \\ \beta(t) = Y(t)\,[H(t) - r(t)] + \sigma(t)\,[\beta_1(t) + \rho\gamma(t)] \end{cases}$$

Thus we have

$$\begin{cases} \alpha_1(t) = \alpha(t) - r(t)Y(t) \\ \beta_1(t) = \dfrac{\beta(t) - Y(t)\,[H(t) - r(t)] - \rho\sigma(t)\gamma(t)}{\sigma(t)} \end{cases} \tag{3.1.5}$$

And we are facing the similar optimization problem as the unconstrained case in Section 1.1.2 but

$$\phi(\omega, t, \alpha, \beta) = \frac{1}{2}\left[\frac{\alpha^2(t)}{Q(t)} + \frac{\beta^2(t)}{R(t)}\right]$$

Thus we have the HJB equation

$$W_t - r(t)yW_y + \mathcal{L}_H W + \inf_{(\alpha,\beta,\gamma)\in\mathcal{B}} L_w(t) = 0 \tag{3.1.6}$$

with the terminal condition

$$W(T, H(T), Y^{(y,\alpha,\beta,\gamma)}(T)) = \frac{1}{2a}y^2 + \frac{c}{a}y + \frac{c^2}{2a}$$

Where

$$\begin{aligned} L_W(t) =& \alpha(t)W_y + \frac{[\beta(t) - Y(t)\,(H(t) - r(t)) - \rho\sigma(t)\gamma(t)]^2}{2\sigma^2(t)}W_{yy} \\ &+ \frac{\gamma^2(t)}{2}W_{yy} + \frac{\gamma(t)\,[\beta(t) - Y(t)\,(H(t) - r(t)) - \rho\sigma(t)\gamma(t)]}{\sigma(t)}W_{yy} \\ &+ \frac{\rho\sigma_H(t)\,[\beta(t) - Y(t)\,(H(t) - r(t)) - \rho\sigma(t)\gamma(t)]}{\sigma(t)}W_{Hy} \\ &+ \sigma_H(t)\gamma(t)W_{Hy} + \frac{\alpha^2(t)}{2Q(t)} + \frac{\beta^2(t)}{2R(t)} \end{aligned}$$

By the first order condition we can get $\hat{\alpha}(t)$ $\hat{\beta}(t)$ and $\hat{\gamma}(t)$, substitute back into (3.1.6), we can get the PDE form of the dual HJB.

Since we have random drift H, we make the ansatz

$$W(t, H, y) = y^2h_1(t, H) + yh_2(t, H) + h_3(t, H)$$

Summing in the power of y we can get the system of PDE

$$\begin{cases} \partial_t h_1 + \mathcal{L}_H h_1 + f_1(t, H, h_1, \sigma_H\partial_H h_1) = 0 \\ \partial_t h_2 + \mathcal{L}_H h_2 + f_2(t, H, h_1, h_2, \partial_H h_1, \partial_H h_2) = 0 \\ \partial_t h_3 + \mathcal{L}_H h_3 + f_3(t, H, h_1, h_2, \partial_H h_1, \partial_H h_2) = 0 \end{cases} \tag{3.1.7}$$

where

$$f_2(t, H, h_1, h_2, \partial_H h_1, \partial_H h_2) = b_2(t, H, h_1, \partial_H h_1)\partial_H h_2(t, H) + u_2(t, H, h_1, \partial_H h_1)h_2(t, H) + \tilde{f}_2(t, H, h_1, \partial_H h_1)$$

Since the functions f_1, f_2 and f_3 are quite long functions, we are not going to give the explicit form of them.

As Section 3.1.1 we can write the BSDE representation of the first PDE of (3.1.7)

$$\begin{cases} dH_1(t) = -f_1(t, H(t), H_1(t), Z_1(t))dt + Z_1(t)dW_H(t) \\ H_1(T) = \dfrac{1}{2a} \\ dH(t) = k\left[\mu - H(t)\right]dt + \sigma_H(t)dW_H(t) \\ H(0) = H_0 \end{cases}$$

By the non-linear Feynman-Kac theorem, we can get

$$h_1(t, H) = H_1(t)$$

Then we substitute h_1 and $\partial_H h_1$ into the second equation of (3.1.7), we can write

$$f_2(t, H, h_2, \partial_H h_2) = b_2(t, H)\partial_H h_2(t, H) + u_2(t, H)h_2(t, H) + \tilde{f}_2(t, H)$$

Thus we have

$$h_2(t, H) = E\left[\int_0^T e^{\int_0^t u_2(\tau, H_\tau)d\tau}\tilde{f}_2\left(t, H(t)\right)dt + \frac{c}{a}e^{\int_0^T u_2(t, H(t))dt}\Big|H(0) = H_0\right]$$

where

$$dH(t) = \left[k\left(\mu - H(t)\right) + b_2(t, H(t))\right]dt + \sigma_H(t)dW_H(t)$$

After substituting h_1, h_2, $\partial_H h_1$ and $\partial_H h_2$ into the third ODE in (3.1.7), since f_3 is independent of h_3 and $\partial_H h_3$, we can directly solve $h_3(t, H)$ by Feynman-Kac theorem

$$h_3(t, H) = E\left[\int_0^T f_3(t, H(t))dt + \frac{c^2}{2a}\Big|H(0) = H_0\right]$$

with

$$dH(t) = k\left[\mu - H(t)\right]dt + \sigma_H(t)dW_H(t)$$

Substitute W into $\tilde{\Psi}(Y(t), \alpha(t), \beta(t))$, we have $\tilde{\Psi}(y, \hat{\alpha}(0), \hat{\beta}(0)) = x_0 y + W(y, \hat{\alpha}(0), \hat{\beta}(0))$, take the minimum at time $t = 0$, we can get

$$\hat{y} = -\frac{h_2(0, H_0) + x_0}{2h_1(0, H_0)}$$

Through comparing the ansatz of the primal and dual problem we can conclude that

$$\begin{cases} g_1(t, H) = \dfrac{1}{4h_1(t, H)} \\ g_2(t, H) = \dfrac{h_2(t, H)}{2h_1(t, H)} \\ g_3(t, H) = \dfrac{h_2^2(t, H)}{4h_1(t, H)} - h_3(t, H) \end{cases}$$

3.2 FBSDE method

The derivation of the primal and dual problem using FBSDE method do not have many changes compare with Section 1.2 and Section 2.2. To apply the FBSDE method under the random drift situation, we need to assume the consistency of the measures, i,e. $W_H(t) = W(t)$ a.s., which leads to $\rho = 1$

As we introduced the random drift $H(t)$ into the stock process, it is difficult to find the explicit form in this case, and we will just derive the FBSDE system of the primal and dual problem without going further on solving them by the reinforcement learning.

3.2.1 Primal problem

Since we have a stochastic drift in the asset process, we have the Hamiltonian

$$H_1(t,x,\pi,p_1,q_1) = \left[r(t)x + \pi\left(H(t)-r(t)\right)\right]p_1 + \sigma(t)\pi q_1 - \frac{1}{2}Q(t)x^2 - \frac{1}{2}R(t)\pi^2 \tag{3.2.1}$$

Thus we have the BSDE

$$\begin{cases} dp_1(t) = \left[-r(t)p_1(t) + Q(t)X^{\hat{\pi}}(t)\right]dt + q_1(t)dW(t) \\ p(T) = -aX^{\pi}(T) - c \end{cases}$$

Theorem 3.2.1. *Let $\hat{\pi} \in \mathbb{A}$, then $\hat{\pi}$ is the optimal control of the primal problem if and only if the solution $\left(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1\right)$ of the FBSDE*

$$\begin{cases} d\hat{p}_1(t) = \left[-r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t)\right]dt + \hat{q}_1'(t)dW(t) \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c \\ dX^{\hat{\pi}}(t) = \Big[r(t)X^{\hat{\pi}}(t) + \hat{\pi}(t)\,\Big(H(t)-r(t)\Big)\Big]dt + \hat{\pi}(t)\sigma(t)dW(t) \\ X^{\hat{\pi}}(0) = x_0 \\ dH(t) = k\left[\mu - H(t)\right]dt + \sigma_H(t)dW(t) \\ H(0) = H_0 \end{cases} \tag{3.2.2}$$

satisfies

$$\hat{p}_1(t)\left[H(t)-r(t)\right] + \sigma(t)\hat{q}_1(t) - R(t)\hat{\pi}(t) = 0 \tag{3.2.3}$$

By condition (3.2.3)

$$\hat{\pi}(t) = \frac{1}{R(t)}\left[p_1(t)\left(H(t)-r(t)\right) + \sigma(t)q_1(t)\right]$$

3.2.2 Dual problem

Since now we are facing the case that $\rho = 1$, we have the dual process

$$\begin{cases} dY^{(y,\alpha,\beta)}(t) = [\alpha(t) - r(t)Y^{(y,\alpha,\beta)}(t)]dt + \dfrac{\beta(t) - Y^{(y,\alpha,\beta)}(t)\left[H(t)-r(t)\right]}{\sigma(t)}dW(t) \\ Y(0) = y \end{cases}$$

which is similar to Section 1 and Section 2 where the control $\gamma(t) = 0$

Considering the stochastic drift $\{H(t)\}$ and dual cash process $\{Y(t)\}$, we can get the Hamiltonian for the dual FBSDE problem

$$H_2(t,y,\alpha,\beta,p_2,q_2) = [\alpha - r(t)y]p_2 + \frac{1}{\sigma(t)}\Big[\beta - y\left(H(t)-r(t)\right)\Big] - \frac{\alpha^2}{2Q(t)} - \frac{\beta^2}{2R(t)}$$

Thus we have the BSDE

$$\begin{cases} dp_2(t) = \left[r(t)p_2(t) + \dfrac{H(t)-r(t)}{\sigma(t)}\hat{q}_{21}\right]dt + q_2(t)dW(t) \\ p_2(T) = -\dfrac{Y^{(y,\alpha,\beta)}(T) + c}{a} \end{cases}$$

Theorem 3.2.2. *Suppose $Q(t)$ and $R(t)$ are positive definite and their inverse are uniformly bounded, then $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is the optimal of the dual problem if and only if the solution $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2\right)$*

of the FBSDE

$$\begin{cases} d\hat{p}_2(t) = \left[r(t)\hat{p}_2(t) + \dfrac{H(t)-r(t)}{\sigma(t)}\hat{q}_{21}\right]dt + \hat{q}_2(t)dW(t) \\ \hat{p}_2(T) = -\dfrac{Y^{(\hat{y},\hat{\alpha}\hat{\beta})}(T)+c}{a} \\ dY^{(\hat{y},\hat{\alpha}\hat{\beta})}(t) = [\alpha(t) - r(t)Y^{(\hat{y},\hat{\alpha}\hat{\beta})}(t)]dt + \dfrac{\beta(t) - Y^{(\hat{y},\hat{\alpha}\hat{\beta})}(t)\,[H(t)-r(t)]}{\sigma(t)}dW(t) \\ Y(0) = y \\ dH(t) = k\,[\mu - H(t)]\,dt + \sigma_H(t)dW(t) \\ H(0) = H_0 \end{cases}$$

satisfies

$$\begin{cases} \hat{p}_2(0) = x_0 \\ \left(\hat{p}_2(t), \dfrac{\hat{q}_2(t)}{\sigma(t)}\right) \in \partial\phi(\hat{\alpha}(t), \hat{\beta}(t)) \end{cases} \tag{3.2.4}$$

for $(\mathbb{P}\otimes Leb)-a.e.(\omega,t) \in \Omega\times[0,T]$

By the condition (3.2.4)

$$\begin{cases} \hat{\alpha}(t) = \hat{p}_2(t)Q(t) \\ \hat{\beta}(t) = \dfrac{R(t)\hat{q}_2(t)}{\sigma(t)} \end{cases}$$

3.2.3 Relations Between FBSDE Primal and Dual Problem

Although we cannot solve the FBSDE problem, we can still find the relations between the primal FBSDE and dual FBSDE

Even we have introduced the random drift into the stock process we still have

$$\begin{cases} \hat{\pi}(t) := \dfrac{\hat{q}_2(t)}{\sigma(t)} \\ X^{\hat{\pi}}(t) = \hat{p}_2(t) \\ \hat{p}_1(t) = Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) \\ \hat{q}_1(t) = \dfrac{\hat{\beta}(t)}{\sigma(t)} - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) \end{cases}$$

and

$$\begin{cases} \hat{y} = \hat{p}_1(0) \\ \hat{\alpha}(t) = Q(t)X^{\hat{\pi}}(t) \\ \hat{\beta}(t) = \sigma(t)\,[\hat{q}_1(t) + \theta(t)\hat{p}_1(t)] \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \hat{p}_1(t) \\ \hat{p}_2(t) = X^{\pi}(t) \\ \hat{q}_2(t) = \sigma(t)\hat{\pi}(t) \end{cases}$$

The proof is the same to the unconstrained problem, the only difference is that the 2-dimensional problem has been reduced to 1-dimensional

Conclusion

In this paper, we discussed the stochastic linear quadratic control problem and used HJB equation to solve both its primal problem and its dual problem. We also derived the FBSDE for both of the primal and dual problem and stated the necessary and sufficient conditions for the optimal solution. We prove the relations of the optimal portfolio strategy and the associated cash processes of the primal problem and the adjoint processes of the dual problem, and vice versa. We use the HJB and FBSDE methods to solve the primal and dual SLQ problems under the constrained condition where the second entry of the portfolio strategy is equal to 0. We introduced the random drift into the stock process and still used the 4 methods to derive the optimal control and the associated process even though we didn't solve the FBSDE to the end even under the limited case.

There are still many unsolved problems. For example, can we solve the dual problem if the constrained condition is not convex, more specifically when it is not a convex cone? Can we find the explicit form of the FBSDE under the random drift case if we assume that the correlation of the Brownian motion in the OU process and the cash process is not equal to 1?

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