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Semilinear PDEs for valuation under credit, collateral and funding: theory and numerical case studies.

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

Black, Scholes and Merton, who deduced a formula for pricing an European option on an underlying whose price follows a log-normal diffusion process, contributed to major developments in modern finance. Assumptions made in their theory were very restricted such as constant volatility, and many important features in finance were not included, for instance, credit, collateral and funding. The aim of this project is to derive and study a pricing formula including those features mentioned previously.

Firstly, we introduce and derive the mathematical expression of value adjustments for credit, collateral and funding. Later on, we derive the semilinear parabolic Partial Differential Equation linked to these value adjustments. We show the solution exists and is unique under some assumptions. Finally, we study the resulting PDE numerically for simple cases such as call option and straddle contract and sensitivity analysis are run for parameters such as funding rate, default intensity etc.

Contents

List of Figures

Chapter 1

Introduction

1.1 History of XVA

The acronym XVA refers to valuation adjustments due to some financial phenomena such as: default (CVA), hedging/funding (FVA) and collateralization (ColVA), etc. Such adjustments are considered by banks when pricing derivatives in Over-The-Counter(OTC) markets, the importance of which has grown in recent years especially after the financial crisis in 2008.

The first member of the XVA family is called Credit Value Adjustment (CVA). The reason why CVA is born is very intuitive: banks are afraid that the counterparties will not be able to fulfil portions or the whole amount of cash payments incurred in a derivative contract. Through CVA, the credit risks of the counterparty in a derivative contract are priced and such amount is charged to the counterparty, making the contract cheaper than the default-free price. The term CVA first appeared in the late 90s, after the Asian Crisis in 1997, the default of Russia and the fall of Long-Term Capital Management in 1998, investment banks realized that credit risks of the counterparties must be taken into account by the pricing procedure of derivative contracts. In the early 00s, the OTC market has grown rapidly, the default of some big companies such as Enron, WorldCom and Parmalat alerted banks and CVA desks were created in all the banks. Calculations were simple, which were typically based on historic probabilities of default, and banks considered themselves as risk-free parties in derivative markets. CVA mark to market losses have contributed importantly to the 2008 crisis, as stated by the bank of international settlements.

One of the major causes of the financial crisis was the systematic mispricing of American mortgages and Mortgage-Backed Securities (MBSs). Those complex and structured products were mainly issued by mortgage providers such as Fannie Mae and Freddie Mac, which held a large amount of low-rated subprime loans, however, it was ignored as they were backed by high-rated issuers. Mortgage securitization is a fast way to recover capital and mitigate the risks to other market participants, and this particular feature attracts many financial entities in the market, among which, Lehman Brothers, one of the protagonists of the crisis. During 2007-2008, investors were losing faith on these securities due to the subprime mortgage crisis starting in 2007, which triggered a liquidity crisis subsequently. Lehman Brothers faced a huge loss due to the price decay of MBSs, which finally declared bankruptcy in September 2008. Other entities such as Merrill Lynch also received financial aids from Bank of America which led to the merge of these two companies.

After the crisis, investment banks raised the importance of CVA and no longer considered themselves to be risk-free, leading to the second member of XVA family: Debit Value Adjustment (DVA), which can be thought as the CVA charged from the counterparties' point of view, that is, both parties need to agree on the amount of CVA and DVA before entering into a contract. Furthermore, the calculation of CVA/DVA has evolved from historical default probabilities to a general pricing framework, check for example Brigo's book Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes [1].

Before the crisis in 2008, the term DVA was omitted and banks only calculated so-called unilateral CVA (UCVA), see for example in the general framework of Brigo and Masetti [2], which is later on, applied to price different products such as interest rates, equities and commodities by Brigo and Pallavini [3], Brigo and Bakkar [4] and Brigo and Chourdakis [5]. After the crisis in 2008, DVA has also been considered in pricing, leading to bilateral CVA. Brigo and Capponi [6] priced Credit Default Swaps (CDS), taking CVA into account. In 2011, Pallavicini, Perini and Brigo developed a framework for CVA, DVA, FVA and collateral [7], which served as the base in the pricing of defaultable bonds and simple call options, according to Brigo, Buescu and Rutkowski [8].

Apart from counterparties' credit risks, there are also valuation adjustments caused by other economic/financial aspects, for example:

- Banks need to hedge the risks in derivative contracts they trade on, which comes with a cost and will be accounted in the pricing of each contract, and this is called Funding Value Adjustment (FVA). The hedge is done by traders in a bank that deals with other market participants. Whenever needed, traders will need to borrow money from the treasury and the treasury from external funders. The interest rate incurred from borrowing is usually higher than risk-free rate due to the credit risks of the banks and their contribution to FVA.

- Another way to mitigate the credit risk is via collateralization, which is a common practice in the financial industry especially in OTC markets. The majority of the contracts are collateralized, and both parties will post collaterals if needed, which also comes with a cost, since collateral can be seen as capital and opportunity cost is present. Such cost is also calculated and called as Collateral Value Adjustment (ColVA).

When the term FVA was first introduced, it was controversial in the financial industry, and there were authors who did not agree on the inclusion of FVA in derivative pricing, for example Hull and White [9], Burgard and Kjaer [10]. The reasons they provide include that pricing should be separated from hedging which involves buying and selling securities in the market at a fair price, thus there is no additional cost. However, over time, such adverse voice disappeared and FVA became as prominent as CVA/DVA. Examples of the importance FVA can be found in the industry: according to Wall Street Journal on January 2014, the funding valuation adjustment costed J.P. Morgan Chase $1.5 billion in the fourth-quarter results.

There are lots of analysis of FVA in academia: Brigo, Pallavicini and Perini [7] constructed a general framework including CVA, collateral and funding cost, Brigo, Liu, Pallavicini and Sloth [11] discovered the non-linearity of pricing equation by introducing the funding cost, which is non-separable from credit risks. Burgard and Kjaer [12] calculated the FVA of funding strategies that only involved trading bonds, showing that different funding strategies would generate different economic value and hence FVA was not unique. Later on, in 2017 Burgard and Kjaer [13] extended the replicating strategy to multiple counterparties.

Nowadays, there are emerging terms of valuation adjustments. Capital Value Adjustment (KVA) has been raised due to the cost of capital one has to set aside, so as to be able to trade. There are theoretical discussions on it, see for the example of Green, Kenyon and Dennis [14]. Margin Value Adjustment (MVA) has been created to offset the funding cost of the initial margin required in trading. We will not focus on these adjustments in this project because in the industry, they are not as well established as previous adjustments and there is not a standard agreement on the calculation of these terms.

Focusing on the pricing theory, Black and Scholes [15] deduced a PDE formula for pricing an European option on an underlying whose price follows a log-normal diffusion process, contributing

to major developments in modern finance. The famous Black-Scholes formula was widely used in the industry, which, however, does not reflect the effect of CVA or other price adjustments. More advanced mathematical tools were introduced to deal with new financial terms. For example, El Karoui [16] introduced Backward Stochastic Differential Equation in the case of asymmetric interest rates. In 2014, Lou [17] extended the classic pricing formula to include funding costs. Brigo, Francischello and Pallavicini [18] managed to derive the Forward Backward Stochastic Differential Equation and its corresponding semilinear partial differential equation including credit risks, collateral and funding effects. They also showed the uniqueness and existence of viscosity solutions under some mild conditions. Later on, in 2019, Brigo, Francischello and Pallavicini [19] conducted a numerical analysis on the FBSDE using Monte Carlo simulation.
This dissertation will be based on the work done by Brigo, Francischello and Pallavicini [18], we will prove the existence and uniqueness of a classical solution of the semilinear PDE instead of a viscosity solution. Instead of building numerical solutions to FBSDE, we will follow another path and search numerical solutions to the semilinar PDE through finite difference method. We will then study a number of numerical cases to illustrate our numerical methods.

1.2 Structure of the dissertation

The rest of the dissertation is organized as follows. In section 2.1 we provide a general theoretical framework for the price adjustments mentioned previously, including a brief financial background and the derivation of their mathematical expression. Section 2.2 provides the derivation of Black-Scholes PDE with credit, collateral and funding adjustments. Furthermore, we present the existence and uniqueness theorems for the solution of the PDE and we verify the conditions for these theorems are hold in our case. In section 3, we first introduce the finite difference method to solve the PDE numerically. Then, we present two numerical studies: simple call option and straddle contract, we also run some sensitivity analysis on different parameters. Conclusion and futher work may be found in section 4.

Chapter 2

Black Scholes PDE with credit, collateral and funding

2.1 financial Background

In this section, we will derive the Partial Differential Equation (PDE) linked to a general pricing framework taking account the effect of credit, collateral and funding. It is noteworthy that throughout the section, unless otherwise stated, we price the product under the point of view of the "Bank", and the other party is treated as the "Counterparty", we will use X_B and X_C to refer any process X related to the "Bank" and "Counterparty" respectively. This pricing framework was first introduced by A. Pallavicini [7] in 2011, later on, D. Brigo, M. Francischello and A. Pallavicini extended that framwork[19], our work in this section will be based on their work and the lecture notes from the course Interest Rates Models at Imperial College London [20].
We will start by introducing the general notation we adapt in this chapter. Next, we will introduce and explain the mathematical expression for each of the value adjustments mentioned in the previous chapter. Finally, we will put all the terms together and derive the corresponding Forward Backward Stochastic Differential Equation (FBSDE) and Partial Differential Equation.

2.1.1 General Settings

First of all, we specify the probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ we will be working on. The probability measure $\mathbb{Q}$ is called the Risk-Neutral measure, i.e. the equivalent martingale measure such that the discounted price process of any non-dividend paying contingent claim is a martingale with respect to $\mathbb{Q}$.
We also need the probability space to equip a filtration that store the information available up to time t. There are couple of filtrations that are interesting to us. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the default free filtration. Under this filtration, $\mathcal{F}_t$ encodes the market information up to time t, however, $\mathcal{F}_t$ does not include the default intensity of either "Bank" or "Counterparty". The other filtration that is crucial is defined by $\mathcal{G}_t = \mathcal{F}_t \wedge \mathcal{D}_t$ where $\mathcal{D}_t$ only contains the information about the fault intensities at time t, we will go into details in the next section. We can call $(\mathcal{G}_t)_{t\geq 0}$ the complete filtration.
Denote by r_t by the short interest rate process. One of the products that we can trade in the market is simply depositing the money in the Treasury and earn the interest rate. Assuming continuous compounding, the dynamic of treasury account is given by:

$$dB_t = r_t B_t dt,\ B_0 = 1.$$

We use this to derive the discount factor $B_t/B_T = e^{-\int_t^T r_s ds}$. We will now adopt the notation $D(t,T,x) = e^{-\int_t^T x_s ds}$ for the discount factor with respect to the process x_t and for simplicity we assume $D(t,T) = D(t,T,r)$. The classic pricing theory states that: if the market is free of default and arbitrage, then for any derivative with payoff V_T at time to maturity T one has:

$$V_t = \mathbb{E}[D(t,T)V_T|\mathcal{F}_t]$$

where V_t is the price of the derivative at time t where $\mathbb{E}$ denotes the expectation with respect to probability measure $\mathbb{Q}$, since the only probability measure we will use is the Risk-Neutral measure, we simply omit $\mathbb{Q}$ in the expectation expression. In this project, we will assume that the final payoff of the derivative we try to price depends only on the price process S_t of the underlying asset at time T, i.e. $V_T = \Phi(S_T)$. We denote by Π_t the cash flow process incurred during the contract, if $\Pi_t > 0$, then we expect to receive money from the counterparty, otherwise, we make a payment to them. We assume that $\Pi_t = \Pi(S_t, t)$ depends only on the underlying asset process the current time t. This assumption is valid for standard products such as european options, interest swaps etc.

Furthermore, to simplify the problem, we will assume in dimension of S_t to be 1, that is, we assume the derivatives we price has only one particular underlying asset, it can be a particular stock or even a portfolio with constant weights. This also implies the market we are considering only includes 2 different assets: risk-free bond B_t and the underlying asset S_t any trading strategy including hedging will be based on this two products.

2.1.2 Credit risk: CVA and DVA

As explained in the previous section, CVA comes from the default possibility of the counterparty, this value adjustment usually makes the contract cheaper. By switching the point of view to counterparty, we can calculate the CVA due to the default from the bank. The debt value adjustment can be seen as the opposite of the CVA seen from the counterparty:

$$\text{DVA}_B = -\text{CVA}_C$$

In order to calculate them, we need also to clarify and model a couple of things. We start with the default scenario, a default situation happens when a party passes the payment deadline on a debt they were due to pay, it can either be due to insolvency (the party has not enough asset the pay its liabilities) or the party defaults intentionally, this is also called strategic default. We denote by τ_B and τ_C the default time of the bank and counterparty respectively. A very common practice in the industry is to model τ_i as an exponential distribution. Mathematically speaking, let ξ_B and ξ_C be two independent exponential distributions with mean equals to 1, we also assume $\mathcal{F}_t$-adapted, positive processes λ_t^B and λ_t^C, then we define:

$$\tau_i = \inf\left\{t \geq 0 \mid \int_0^t \lambda_s^i ds > \xi_i\right\},\ i \in \{B, C\}. \tag{2.1.1}$$

Defined in this way, we can check that conditional on $\mathcal{F}_t$, the distribution of τ_i behaves like a discounting factor:

$$\mathbb{Q}(\tau_i > t|\mathcal{F}_s) = \mathbb{E}(\mathbf{1}_{\tau_i > t}|\mathcal{F}_s) = e^{-\int_t^T \lambda_s^i ds}, \forall s \geq t,\ i \in \{B, C\}.$$

Also, due to the independent assumption of ξ_i's, we can further check that:

$$\mathbb{E}(\mathbb{1}_{\tau_B>t_1}\mathbb{1}_{\tau_C>t_2}|\mathcal{F}_s) = \mathbb{E}(\mathbb{1}_{\tau_B>t_1}|\mathcal{F}_s)\mathbb{E}(\mathbb{1}_{\tau_C>t_2}|\mathcal{F}_s)\ \forall t_1, t_2 \in [0, s]$$

that is, given information up to time s, λ^i are completely deterministic and we assume that λ^i contains of the dependence structure of τ_i. Since we are interested in the first default time regardless the party who defaulted, we set $\tau = \tau_B \wedge \tau_C$. Combining the previous results we can show that conditional on $\mathcal{F}_s$ the distribution of tau has still the form of a discount process:

$$\mathbb{Q}(\tau > t|\mathcal{F}_s) = \mathbb{E}(\mathbb{1}_{\tau>t}|\mathcal{F}_s) = \mathbb{E}(\mathbb{1}_{\tau_B>t}\mathbb{1}_{\tau_C>t}|\mathcal{F}_s) = e^{-\int_t^T \lambda_s^B+\lambda_s^C ds},\ \forall s > t.$$

For a shortcut, we can write $\mathbb{Q}(\tau > t|\mathcal{F}_s) = D(t, T, \lambda_s^B + \lambda_s^C)$. Check Duffle-Huang [21] for more detailed explanation.

Now, we are able to extend the risk-free filtration $\mathcal{F}_t$ to contain credit information of both parties up to time t. Mathematically, denote by

$$\mathcal{H}_t^B = \sigma(\mathbb{1}_{\tau_B<s},\ s < t),\ \mathcal{H}_t^C = \sigma(\mathbb{1}_{\tau_C<s},\ s < t),$$

the credit filtration, $\mathcal{H}_t^i$ contains the information whether a party defaults at or before time t. We can now specify the filtration $\mathcal{D}_t$ and the complete filtration $\mathcal{G}_t$ to be:

$$\mathcal{D}_t = \mathcal{H}_t^B \vee \mathcal{H}_t^C, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$$

We assume that both parties cannot default at the same time, i.e. $\mathbb{Q}(\tau_B = \tau_C) = 0$. Denote by $\bar{\tau} = \tau \wedge T$. Finally, we can write our first version of the pricing formula including credit risk of both parties:

$$\begin{aligned} V_t =& \mathbb{E}\Bigg(\underbrace{\mathbb{1}_{\tau>T} D(t,T)\Phi(S_T)}_{\text{Part A}} + \underbrace{\int_t^{\bar{\tau}} D(t,u)\Pi_u du}_{\text{Part B}} \ \Bigg|\ \mathcal{G}_t \Bigg) \\ &+ \mathbb{E}\Bigg(\underbrace{D(t,\tau)\mathbb{1}_{t<\tau<T}\big(\mathbb{1}_{\tau_C<\tau_B}(\mathrm{REC}_C(\varepsilon_{\tau_C})^+ - (-\varepsilon_{\tau_C})^+)\big)}_{\text{Part C}} \ \Bigg|\ \mathcal{G}_t \Bigg) \\ &+ \mathbb{E}\Bigg(\underbrace{D(t,\tau)\mathbb{1}_{t<\tau<T}\big(\mathbb{1}_{\tau_B<\tau_C}((\varepsilon_{\tau_B})^+ - \mathrm{REC}_B(-\varepsilon_{\tau_B})^+)\big)}_{\text{Part D}} \ \Bigg|\ \mathcal{G}_t \Bigg) \end{aligned} \tag{2.1.2}$$

Financially speaking, part A corresponds to the discounted final payoff where there is no default scenario between $[t, T]$. In integral in part B is the sum of all discounted cash flows incurred in the contract during $[t, \tau \wedge T]$.

The term ε_t used in part C and D is called the net close-out process this process, which represents the amount that is owed by one party to another in a default scenario at time t. According to the Gregory's book [22], in a general default scenario,

- Close-out is the right of the surviving party to terminate transactions with the defaulted counterparty and cease any contractual payments.
- Netting is the right to offset the Mark-To-Market value across transactions and determine a net balance which is the sum of positive and negative values. This will the single amount of money one party pay to another depending on its sign.

Close-out netting is very common in the industry since it reduces settlement risk, for example, one party A pays in USD and receives EUR, without close-out netting there is possibility of a sudden drop in value of EUR after A transferred USD to its counterparty. Another benefit of close-out

netting is that the survival party can immediately realize the gain (if any) and avoid a potential bankruptcy process of the counterparty.
We first look at part C where we assumed that the counterparty defaults at τ_C: if ε_t is positive, then we can only receive a fraction of this amount (REC_C times ε_{τ_C}), REC_i is called the recovery rate, normally is between 0.4 and 0.6. By contrast, if we owe money to the counterparty (ε_t is negative) we have to pay the whole amount. Similarly, from the point of view of the counterparty, we obtain the expression in part D. Part C and part D are called the CVA and DVA of the Bank respectively, we will improve this expression when collateralization is involved.
There are two different ways to calculate the net close-out process ε_t. The most intuitive way is to take the average of discounted future cash flows up to time T:

$$\varepsilon_\tau = \mathbb{E}\left(D(\tau, T)\Phi(S_T) + \int_\tau^T D(t,u)\Pi_u du \,\middle|\, \mathcal{G}_\tau \right)$$

This amount is often called the default-free close-out amount. One advantage of this setting is its fast tractability since it is straight forward. However, in the practice, this is not always true: we implicitly assumed that both parties will not default after time τ. Since we are discounting future cash flows, it is reasonable to assume that both parties can default in the future as if the current default does not happen. Under this setting, we have $\varepsilon_\tau = V_\tau$, we call it the replacement close-out and we will stick with this setting in out pricing framework.

2.1.3 Collateralisation

According to Gregory's book [22], a collateral is an asset supporting a risk in a legally enforceable way. Collateralisation is another commonly practiced tool in the industry to mitigate the default risk. If two parties enter into a contract with collateral agreement, then one party must post collateral to another party to support such exposure on credit risk. The ownership of asset being posted as collateral does not change unless the collateral giver party defaults, in that case, the collateral receiver party becomes the permanent economic owner of the collateral. Collateralization provides a really dynamic and flexible way to offset the loses in the event of default.
In a collateral agreement, both parties will have to concur on certain terms such as:

- Type of collateral: normally, cash or liquid tradable securities are accepted in order to eliminate liquidity risk.
- Thresholds: the amount below which collateral is not needed. The reason of setting threshold is to reduce potential operating risk and liquidity cost when the contract's value is volatile. Zero threshold means that collateral is always needed and an infinite threshold means no collated is required.
- Haircuts: depending on the type of collateral, specially in the case of illiquid assets, the value may be reduced in the counting of collateral required. The haircut is designed to reduce the market risk and liquidity risk driven by the volatility of the asset and its liquidity.

At the early stage of collateralisation, the collateral was segregated, the receiver party was not allowed to trade it in the market. Nowadays it is common that the collateral received can be traded by the receiver party, this is called re-hypotecation. In this way, the receiver party gives up some protection against credit risk to seek for higher return using the collateral.
Now, we introduce the mathematical model for the collateral. Denote by C_t the collateral process from the point or view of the bank. If $C_t < 0$, the bank posts such amount at time t to the counterparty, similarly if $C_t > 0$, the bank has gained exposure to credit risk and receives collateral from the counterparty. Although the collateralisation in the real practice is a discontinuous process

due to the minimum transfer threshold, haircuts and so on, we will treat it as a continuous process. We assume the asset posted as collateral is cash and can be re-hypotecated. If two parties enter in a derivative contract with collateral agreement, we describe the collateral process as follows:

- At the beginning, no party posts collateral since the derivative is traded at fair price. Hence we have $C_0 = 0$.
- Every day during the contract, both parties will calculate the MTM value of the contract and agree to an unique value. Then, they compare the value agreed with the value calculated in day before. The party whose contract has depreciated will have to post additional collateral to the other party.
- In the case of default by the collateral giver party, the collateral receiver party will become the permanent owner of the collateral.

In this way, the party with positive exposure to default risk will be protected by the collateral received up to day. Normally, this collateral has high liquidity in the market (such as cash or liquid securities) so the collateral receiver can offset the losses rapidly. Note that, collateralisation does not eliminate the risk, it transfers the credit risk to other risks such that market, operational and liquidity risk.
We can now improve our pricing formula to include the effect of collateralisation. We define the CVA and DVA term with collateral as:

$$\text{CVA}_{\text{COL}} = \mathbb{E}\Big(D(t,\tau)\mathbb{1}_{t<\tau<T}\big(\mathbb{1}_{\tau_C<\tau_B}(\text{REC}_B(\varepsilon_{\tau_C} - C_{\tau_C})^+ - (-\varepsilon_{\tau_C} + C_{\tau_C})^+)\big) \,\Big|\, \mathcal{G}_t \Big)$$

and

$$\text{DVA}_{\text{COL}} = \mathbb{E}\Big(D(t,\tau)\mathbb{1}_{t<\tau<T}\big(\mathbb{1}_{\tau_B<\tau_C}((\varepsilon_{\tau_B} - C_{\tau_B})^+ - (-\text{REC}_C\varepsilon_{\tau_B} + C_{\tau_B})^+)\big) \,\Big|\, \mathcal{G}_t \Big).$$

That is, we treat now the new close-out amount process as $\varepsilon_t - C_t$. There is one more thing to add to our pricing formula. From the point of view of the bank, unless the counterparty defaults, the collateral received (if any) still belongs to the counterparty. Therefore, if we keep this asset, we must remunerate it. Assuming continuously compounding, we denote by $c_t > 0$ the rate at which we remunerate the collateral, this is called the collateral remuneration rate. Hence, we need to include another term in our pricing formula, we call it the Collateral Valuation Adjustment:

$$\text{ColVA} = -\mathbb{E}\Big(\int_t^{\bar{\tau}} D(t,u)(c_u - r_u)C_u du \,\Big|\, \mathcal{G}_t \Big)$$

The financial explanation is simple: assume the bank receives the collateral ($C_t > 0$), then the bank will have to pay back $C_t c_t$ as compensation, also, the bank can invest the collateral in a risk-free way to earn $C_t r_t$ since we allow re-hypotecation. The converse case is derived similarly. Putting all this together, the equation 2.1.2 becomes:

$$V_t = \mathbb{E}\Big(\mathbb{1}_{\tau>T} D(t,T)\Phi(S_T) + \int_t^{\bar{\tau}} D(t,u)\Pi_u du \,\Big|\, \mathcal{G}_t \Big) + \text{CVA}_{\text{COL}} + \text{DVA}_{\text{COL}} + \text{ColVA}. \quad (2.1.3)$$

2.1.4 Hedging cost

In the financial industry, the role of an investment bank is to design financial products that fulfills clients' demand, price the product correctly, once the product is sold the bank tries to hedge the risk using basic financial products such as stocks, swaps etc. In our case, we suppose that the hedging strategy involves trading repurchase agreements (as known as repo). Repo can be seen

as a short-term, collateral-backed loan, if two parties enter in a repo, the dealer party sells its asset to the investor party and agrees to purchase it back (typically in the next day) with a higher price. The bank can use repo to gain/hedge exposure on the underlying by entering a repo as investor/dealer party.
Denote by H_t the value of the risky asset traded in the repo and C_t^H by the cash exchanged in the contract. We assume that the repo is traded at fair meaning that $H_t + C_t^H = 0$, we also assume that the bank can enter in a fair repo at any time. The trading strategy involving repo is as follows:

1. Starting at time t, suppose we want to hedge $H_t > 0$ exposure to stock's price movements, thus, we need to short the stock. We can do this by entering in a repo agreement.

2. We first borrow from treasury H_t amount of cash at rate f_t. This rate does not need to be the same as r_t, in this section we will see that it does not affect to the hedging cost.

3. We enter in a repo agreement, to obtain H_t/S_t amount of stock.

4. Then, we sell the stock to obtain H_t amount of cash and pays back to treasury.

5. At time $t+dt$, we need to close the repo agreement so we borrow from the treasury H_tS_{t+dt}/S_t amount of cash to buy from the market H_t/S_t amount of stock.

6. We close the repo agreement, and receive $(1+h_tdt)H_t$ of cash, we pay back this cash to cover what we have borrowed earlier.

Trading under this way, we can gained a negative exposure to stock's price movements, and the hedge is completed. The PnL of the portfolio is:

$$(1 + h_tdt)H_t - H_tS_{t+dt}/S_t = H_t + h_tH_tdt - H_t - H_t/S_tdS_t$$

where we used $S_{t+dt} = S_t + dS_t$). Under risk-neutral measure we get:

$$h_tH_t - H_tr_tdt = (h_t - r_t)H_tdt - \sigma dW_t = -((r_t - h_t)H_tdt + \sigma S_tdW_t)$$

where W_t is a $\mathbb{Q}-$Brownian Motion. Finally, the cost of hedging is calculated by taking expectation of the discounted PnL of our strategy up to $\bar{\tau}$, we give it the name of Hedging Valuationg Adjustment. Thus, we get:

$$\text{HVA} = -\mathbb{E}\left(\int_t^{\bar{\tau}} D(t,u)(r_u - h_u)H_u du \;\middle|\; \mathcal{G}_t \right)$$

Note that, the calculation of HVA is independent of f_t since in the trading strategy we designed, we assumed that we pay back immediately the cash we borrowed, hence no interest rate is involved.

2.1.5 Funding Cost

In the previous section, we designed a strategy to explain how hedging cost is calculated. In this section we will derive a full replicating strategy to the derivative and we will see that, apart from costs coming from collateral and repo agreement, there is an additional term to be considered. The main reason is because the treasury will not lend the money at risk-free rate. Let F_t be the funding account with the treasury. As usual, if F_t is positive/negative, it means we are borrowing/investing money from/at the treasury and let:

$$f_t = f_t^+ \mathbb{1}_{F_t>0} + f_t^- \mathbb{1}_{F_t<0}$$

be the interest rate at which we borrow/lend money with treasury.
We start by assuming that we can perfectly replicate the derivative V_t in the following way:

$$V_t = F_t + C_t + \frac{H_t}{S_t}S_t - C_t^H$$

where we know $H_t + C_t^H = 0$ since the repo is traded at fair. Note that, we also assumed the derivative is default-free, otherwise we need to include trading a credit default swap (CDS), see Brigo's paper [8]. But it does not matter since the default risk is already covered by CVA and DVA term defined previously.
Take the point of view of the derivative buyer (traders in the bank), we construct the following strategy during a time interval $[t, t+dt]$, a trader would repeat the same process until the time of maturity T of the contract.
At time t:

1. The trader borrows from the treasury V_t at interest rate f_t and uses it to buy a contract.
2. By entering in the contract, the collateral account C_t is present, it can either be positive or negative depending on the nature of the contract. If $C_t > 0$, the trader deposit it at treasury, if $C_t < 0$, the trader fund this amount from treasury.
3. Repeat the steps 2-4 in the previous section, assuming $H_t > 0$ to be the exposure to stock's price movements.

Thus, the net funding account we have is $F_t = V_t - C_t$. At time $t + dt$:

4. The funding account now becomes $F_{t+dt} = (V_t - C_t)(1 + f_t dt)$.
5. Repeat steps 5 and 6 in the previous section. The PnL of the repo operation is:

$$(1 + h_t dt)H_t - H_t S_{t+dt}/S_t = h_t H_t dt - H_t/S_t dS_t$$

   So the exposure to price movements is eliminated.

6. Now the trader sells the contract and obtains V_{t+dt} and pays back to the treasury.
7. By closing the contract, the trader also clears the collateral account C_t together with interest c_t.

After the whole produce, the funding account is now given by:

$$\begin{aligned} F_{t+dt} &= -V_{t+dt} + (V_t - C_t)(1 + f_t dt) + C_t(1 + c_t dt) \\ &= V_t(1 + f_t dt) + C_t(c_t - f_t)dt - V_{t+dt} \\ &= -dV_t + V_t f_t dt + C_t(c_t - f_t)dt \end{aligned}$$

And the total PnL is given by:

$$dV_t - V_t f_t dt - C_t(c_t - f_t)dt - \frac{H_t}{S_t}dS_t + h_t H_t dt$$

Taking risk-neutral expectation conditional on t of above expression we get:

$$\begin{aligned} &\mathbb{E}_t\left(dV_t - V_t f_t dt - C_t(c_t - f_t)dt - \frac{H_t}{S_t}dS_t + h_t H_t dt\right) \\ &= r_t V_t dt - d\varphi(t) - V_t f_t dt - C_t(c_t - f_t)dt - \frac{H_t}{S_t} r_t S_t dt + h_t H_t dt \\ &= (V_t - C_t)(r_t - f_t)dt - C_t(c_t - r_t)dt - H_t(r_t - h_t)dt - d\varphi(t) \end{aligned}$$

where we used $\mathbb{E}_t[dS_t] = r_T S_t dt$ and $\mathbb{E}_t[dV_t] = r_t V_t dt - d\varphi(t)$, where $d\varphi$ represents the funding cost. Since the trader starts with zero cash, he must earn nothing under risk neutral measure, hence we equal the above expression to zero and we get:

$$d\varphi(t) = \underbrace{(V_t - C_t)(r_t - f_t)dt}_{\text{Funding cost}} \underbrace{-C_t(c_t - r_t)dt}_{\text{ColVA}} \underbrace{-H_t(r_t - h_t)dt}_{\text{HVA}}$$

Indeed, we can check that $d\varphi(t)$ contains HVA and ColVA defined previously. Note also that, if $f_t = r_t$, then the funding cost will vanish as expected. This is done in the interval $[t, t+dt]$, the Funding Value Adjustment (FVA) is obtained by summing the discounted cash flow over $[t, \bar{\tau}]$, i.e.:

$$\text{FVA} = -\mathbb{E}\left(\int_t^{\bar{\tau}} D(t,u)(f_u - r_u)(V_u - C_u)du \,\middle|\, \mathcal{G}_t \right)$$

To sum up, we conclude that:

$$\begin{aligned} V_t =& \mathbb{E}\left(\underbrace{\mathbb{1}_{\tau > T} D(t,T)\Phi(S_T)}_{\text{Part A}} + \underbrace{\int_t^{\bar{\tau}} D(t,u)\Pi_u du}_{\text{Part B}} \,\middle|\, \mathcal{G}_t \right) \\ &+ \text{CVA}_{\text{COL}} + \text{DVA}_{\text{COL}} + \text{ColVA} + \text{HVA} + \text{FVA} \end{aligned} \tag{2.1.4}$$

2.1.6 External Funding Adjustments

As explained in the previous section, we borrow/lend money externally with funding rate f_t. This amount of money is not risk-free either, hence we need to consider the corresponding CVA and DVA related to the funding amount F_t, we denote them by CVA_F and DVA_F respectively. The mathematical modelling is similar to what we discussed in section 2.1.2, to simplify the problem, we assume that if we have spare money, we always invest into safe assets such as Treasury Bonds, therefore we have $\text{CVA}_F = 0$. For DVA_F we have:

$$\begin{aligned} \text{DVA}_F &= \mathbb{E}\left(D(t,\tau)\mathbb{1}_{t<\tau<T} \left(\mathbb{1}_{\tau_B<\tau_C}((F_{\tau_B})^+ - \text{REC}_C(-F_{\tau_B})^+)\right) \,\middle|\, \mathcal{G}_t \right) \\ &= \mathbb{E}\left(D(t,\tau)\mathbb{1}_{t<\tau<T} \left(\mathbb{1}_{\tau_B<\tau_C}((V_{\tau_B} - C_{\tau_B})^+ - \text{REC}_C(C_{\tau_B} - V_{\tau_B})^+)\right) \,\middle|\, \mathcal{G}_t \right) \end{aligned} \tag{2.1.5}$$

Finally, we can write the final expression for the price of the contract:

$$\begin{aligned} V_t =& \mathbb{E}\left(\underbrace{\mathbb{1}_{\tau > T} D(t,T)\Phi(S_T)}_{\text{Final Payoff}} + \underbrace{\int_t^{\bar{\tau}} D(t,u)\Pi_u du}_{\text{Cashflow}} \,\middle|\, \mathcal{G}_t \right) \\ &+ \text{CVA}_{\text{COL}} + \text{DVA}_{\text{COL}} + \text{ColVA} + \text{HVA} + \text{FVA} + \text{DVA}_{\text{F}} \end{aligned} \tag{2.1.6}$$

This is the equation we base to derive the semilinear Partial Differential Equation.

2.2 Derivation

Starting with the expression V_t defined in equation 2.1.6, we will derive the Black-Scholes typed Stochastic Differential Equation and its corresponding Partial Differential Equation. Note that, V_t appears in both sides of the equation 2.1.6 because we assumed $\epsilon_t = V_t$ under replacement close-out setting, this will lead to what so called Forward Backward Stochastic Differential Equation.
A FBSDE is formed by an ordinary SDE and a terminal condition. It is not a time reversed SDE since measurability of the solution is affected by the flow of time. Mathematically, a FBSDE is an

equation in two variable processes (Y_t, Z_t) such that:

$$dY_t = f(Y_t, Z_t, t)dt - Z_t dW_t,\ Y_T = \xi$$

check El Karoui, Peng and Quenez [16] for more details. We are only interested in the price of the contract before default i.e. pre-default price, since the contract ends at default and the value will be zero afterwards. In other words, we want to calculate $\mathbb{1}_{\tau > t} V_t$ for any $t \in [0, T]$. Generally, V_t will be a $\mathcal{G}_t-$adapted process, however, we can always find a $\mathcal{F}_t$ process $\tilde{V}_t$ such that $\mathbb{1}_{\tau > t} V_t = \mathbb{1}_{\tau > t} \tilde{V}_t$ and this will give us a way to simplify the problem since the filtration $\mathcal{F}_t$ is much less complex than $\mathcal{G}_t$ and we are allowed to only analyse $\tilde{V}_t$. This result is based on the following lemma:

Lemma 2.2.1. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ be a filtered space and let $\mathcal{D}_t = \sigma(\mathbb{1}_{\tau < s},\ s < t)$ where τ is defined in equation 2.1.1. Let $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$, then for any $A \in \mathcal{G}_t$ there exists $C \in \mathcal{F}_t$ such that:*

$$A \cap \{t < \tau\} = C \cap \{t < \tau\}$$

Proof. We will prove $\mathcal{F}_t \vee \mathcal{D}_t = \mathcal{C}_t$ where

$$\mathcal{C}_t = \Big\{ A \in \mathcal{H}_t \mid \exists\, C \in \mathcal{F}_t \ s.t.\ A \cap \{t < \tau\} = C \cap \{t < \tau\} \Big\}$$

It is easy to see $\mathcal{C}_t$ is a $\sigma-$algebra. By construction we have $\mathcal{C}_t \subseteq \mathcal{F}_t \vee \mathcal{D}_t$. Also, note that $\mathcal{F}_t \subseteq \mathcal{C}_t$, and for any $s \leq t$:

$$\{\tau \leq s\} \cap \{t < \tau\} = \emptyset \cap \{t < \tau\}$$

which implies $\{\tau \leq s\} \in \mathcal{C}_t$ for all $s \leq t$. Therefore we have:

$$\mathcal{F}_t \cup \Big\{ \{\tau \leq s\} \mid s \leq t \Big\} \subseteq \mathcal{C}_t$$

Finally, we have:

$$\mathcal{F}_t \vee \mathcal{D}_t = \sigma\Big(\mathcal{F}_t \cup \Big\{ \{\tau \leq s\} \mid s \leq t \Big\} \Big) \subseteq \sigma(\mathcal{C}_t) = \mathcal{C}_t$$

Hence, $\mathcal{C}_t = \mathcal{F}_t \vee \mathcal{D}_t$. □

This lemma enables us to construct $\tilde{V}_t$:

Lemma 2.2.2. *Let V_t be an $\mathcal{G}_t$ adapted process, then there exists an $\mathcal{F}_t$ adapted process $\tilde{V}_t$ such that:*

$$\mathbb{1}_{\{t < \tau\}} V_t = \mathbf{1}_{\{t < \tau\}} \tilde{V}_t$$

Proof. Note first if $\mathbf{1}_{\{t < \tau\}} = 0$ then the equality holds, so we assume $\mathbb{1}_{\{t < \tau\}} = 1$. Let x be in the image of $\mathbf{1}_{\{t < \tau\}} V_t$, then the pre-image must have the form $A \cap \{t < \tau\}$ with $A \in \mathcal{G}_t$, by lemma 2.2.1, we obtain $C \in \mathcal{F}_t$ such that $A \cap \{t < \tau\} = C \cap \{t < \tau\}$. Thus, we define:

$$\tilde{V}_t(\omega) = \begin{cases} V_t & \text{if } \omega \in C, \\ 0 & \text{if } \omega \notin C \end{cases}$$

we repeat the same process for all x in the image of $\mathbb{1}_{\{t < \tau\}} V_t$ and we are done. □

Now, we present the key lemma in the credit risk pricing:

Lemma 2.2.3. *For any $\mathcal{A}-$measurable random variable X and for any $t \in \mathbb{R}^+$ we have:*

$$\mathbb{E}[\mathbf{1}_{\{\tau > t\}} X | \mathcal{G}_t] = \mathbf{1}_{\{\tau > t\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau > t\}} X | \mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t]} \tag{2.2.1}$$

Proof. Note that we can rewrite the equation 2.2.1 as follows:

$$\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[X|\mathcal{G}_t]\mathbb{E}[\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t] = \mathbb{E}[X\mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]|\mathcal{G}_t]$$

now we let:

$$Y = \mathbb{E}[X\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{\{\tau>t\}}, \quad Z = X\mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t]$$

so the problem reduces to show Y is the conditional expectation of Z given the filtration $\mathcal{G}_t$. First of all, note that $\mathbb{E}[Y]$ is bounded. Secondly, $\mathbb{1}_{\{\tau>t\}}$ is $\mathcal{G}_t$ measurable by definition of τ, $\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}]$ is also $\mathcal{G}_t$ measurable because it is $\mathcal{F}_t$ measurable, hence Y is $\mathcal{G}_t$ measurable. Finally, for any $A \in \mathcal{G}_t$, there is a such set $C \in \mathcal{F}_t$ by lemma 2.2.1, now, using tower property of conditional expectation we have:

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_A Y] &= \mathbb{E}\big[\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{A\cap\{\tau>t\}}\big] \\
&= \mathbb{E}\big[\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{1}_{C\cap\{\tau>t\}}\big] \\
&= \mathbb{E}\big[\mathbb{1}_C\mathbb{E}[X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\big] \\
&= \mathbb{E}\big[\mathbb{E}[X\mathbb{1}_{C\cap\{\tau>t\}}\mathbb{E}[\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t]|\mathcal{F}_t]\big] \\
&= \mathbb{E}\big[X\mathbb{1}_{C\cap\{\tau>t\}}\mathbb{E}[\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t]\big] \\
&= \mathbb{E}\big[X\mathbb{1}_{A\cap\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]\big] \\
&= \mathbb{E}[\mathbb{1}_A Z]
\end{aligned}$$

and the proof is complete. □

This lemma gives us a way to price products conditional on a simpler filtration $\mathcal{F}_t$ and the only thing we do is to divide by $\mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]$ whose closed form is known and equals to $D(t, T, \lambda_t^B + \lambda_t^C)$. Now we want to extend this result to calculate integrals involving stopping time, for example the HVA and FVA term in the equation 2.1.6. Also, we want to compute the expectation of stopped process such as net close-out amount ε_τ at default.

Lemma 2.2.4. *Suppose φ_u is a $\mathcal{G}_u$ adapted process. Let τ be a default time with intensity λ. Then:*

$$\mathbb{E}\bigg[\int_t^{\bar{\tau}} \varphi_u du \,\Big|\, \mathcal{G}_t\bigg] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\bigg[\int_t^T D(t,u,\lambda)\tilde{\varphi}_u du \,\Big|\, \mathcal{F}_t\bigg]$$

where $\tilde{\varphi}_u$ is an $\mathcal{F}_u-$adapted process such that $\mathbb{1}_{\{\tau>u\}}\varphi_u = \mathbb{1}_{\{\tau>u\}}\tilde{\varphi}_u$.

Proof. We first note that, the LHS is equivalent to:

$$\mathbb{E}\bigg[\int_t^{\bar{\tau}} \varphi_u du \,\Big|\, \mathcal{G}_t\bigg] = \mathbb{E}\bigg[\int_t^T \mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>u\}}\varphi_u du \,\Big|\, \mathcal{G}_t\bigg] = \int_t^T \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau>u\}}\varphi_u|\mathcal{G}_t]du$$

where the last equality holds because of linearity of conditional expectation and Dominated Convergence theorem. Now, we apply lemma 2.2.3 and the RHS becomes:

$$\int_t^T \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[\mathbb{1}_{\{\tau>u\}}\varphi_u|\mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]}du = \mathbb{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[\mathbb{1}_{\{\tau>u\}}\varphi_u|\mathcal{F}_t]D(0,t,\lambda)^{-1}du$$

now, let $\tilde{\varphi}_u$ be such an $\mathcal{F}_t$-adapted process, we get:

$$\begin{aligned}
\mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[\mathbf{1}_{\{\tau>u\}}\tilde{\varphi}_u|\mathcal{F}_t]D(0,t,\lambda)^{-1}du &= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\tau>u\}}\tilde{\varphi}_u|\mathcal{F}_u]|\mathcal{F}_t]D(0,t,\lambda)^{-1}du \\
&= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\tau>u\}}|\mathcal{F}_u]\tilde{\varphi}_u|\mathcal{F}_t]D(0,t,\lambda)^{-1}du \\
&= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[D(0,u,\lambda)\tilde{\varphi}_u|\mathcal{F}_t]D(0,t,\lambda)^{-1}du \\
&= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[\tilde{\varphi}_u|\mathcal{F}_t]D(t,u,\lambda)du \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big[\int_t^T \tilde{\varphi}_u D(t,u,\lambda)du \;\Big|\; \mathcal{F}_t\Big]
\end{aligned}$$

this completes the proof. □

Lemma 2.2.5. *Suppose that φ_u is an $\mathcal{F}-$predictable process. Let τ_B and τ_C be two default times with intensities λ_t^B, λ_t^C respectively. Then:*

$$\mathbb{E}[\mathbb{1}_{\{t<\tau<T\}}\mathbf{1}_{\{\tau_B<\tau_C\}}\varphi_\tau \mid \mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}} = \mathbb{E}\Big[\int_t^T D(t,u,\lambda^B+\lambda^C)\lambda_u^B\varphi_u du \;\Big|\; \mathcal{F}_t\Big]$$

Proof. See Bieleki, Jeanblanc-Picque, Rutkowski 2009. □

Now, we can apply lemma 2.2.4 to cashflow term, HVA, FVA and ColVA to get:

$$\begin{aligned}
\text{Cashflow} &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)\Pi_u du \;\Big|\; \mathcal{F}_t\Big) \\
\text{HVA} &= -\mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)(r_u-\tilde{h}_u)\tilde{H}_u du \;\Big|\; \mathcal{F}_t\Big) \\
\text{FVA} &= -\mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)(\tilde{f}_u-r_u)(\tilde{V}_u-C_u)du \;\Big|\; \mathcal{F}_t\Big) \\
\text{ColVA} &= -\mathbf{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)(c_u-r_u)C_u du \;\Big|\; \mathcal{F}_t\Big)
\end{aligned}$$

Where $\tilde{H}_t$, $\tilde{V}_t$, $\tilde{h}_t$, $\tilde{f}_t$ are $\mathcal{F}-$adapted processes stated in lemma 2.2.2. Note that, Π_t and C_t are $\mathcal{F}-$adapted since they are not related to the credit risk of either parties thus there is no need to change them.

Furthermore, we can apply lemma 2.2.5 to CVA, DVA and DVA_F:

$$\begin{aligned}
\text{CVA} &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)\lambda_u^C(\text{REC}_B(\varepsilon_{\tau_C}-C_{\tau_C})^+ - (-\varepsilon_{\tau_C}+C_{\tau_C})^+)du \;\Big|\; \mathcal{F}_t\Big) \\
\text{DVA} &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)\lambda_u^B((\varepsilon_{\tau_B}-C_{\tau_B})^+ - (-\text{REC}_C(\varepsilon_{\tau_B}+C_{\tau_B})^+)du \;\Big|\; \mathcal{F}_t\Big) \\
\text{DVA}_\text{F} &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\int_t^T D(t,u,r+\lambda)\lambda_u^B((\varepsilon_{\tau_B}-C_{\tau_B})^+ - \text{REC}_C(C_{\tau_B}-\varepsilon_{\tau_B})^+)) \;\Big|\; \mathcal{F}_t\Big)
\end{aligned}$$

Finally, using lemma 2.2.3 the final payoff becomes:

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{\tau>T\}}D(t,T)\Phi(S_T)\mid\mathcal{G}_t] &= \mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}D(t,T)\Phi(S_T)\mid\mathcal{G}_t] \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}D(t,T)\Phi(S_T)\mid\mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mid\mathcal{F}_t]} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau>T\}}D(t,T)\Phi(S_T)\mid\mathcal{F}_t]D(0,t,\lambda)^{-1} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{E}[\mathbb{1}_{\{\tau>T\}}D(t,T)\Phi(S_T)\mid\mathcal{F}_T]\mid\mathcal{F}_t]D(0,t,\lambda)^{-1} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\Phi(S_T)\mathbb{E}[\mathbb{1}_{\{\tau>T\}}\mid\mathcal{F}_T]\mid\mathcal{F}_t]D(0,t,\lambda)^{-1} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\Phi(S_T)D(0,T,\lambda)\mid\mathcal{F}_t]D(0,t,\lambda)^{-1} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\Phi(S_T)D(0,T,\lambda)D(0,t,\lambda)^{-1}\mid\mathcal{F}_t] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\Phi(S_T)D(t,T,\lambda)\mid\mathcal{F}_t] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T,r+\lambda)\Phi(S_T)\mid\mathcal{F}_t]
\end{aligned}$$

Summing all the terms together, we get a new expression for V_t:

$$\begin{aligned}
V_t =& \mathbb{1}_{\{\tau>t\}}\mathbb{E}[D(t,T,r+\lambda)\Phi(S_T)\mid\mathcal{F}_t] + \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Bigg[\int_t^T D(t,u,r+\lambda)\times \\
&(\Pi_u-(c_u-r_u)C_u-(\tilde{f}_u-r_u)(\tilde{V}_u-C_u)-(r_u-\tilde{h}_u)\tilde{H}_u)du\mid\mathcal{F}_t\Bigg] \\
&+\mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big[D(t,u,r+\lambda)(\tilde{\theta}_u+\mathrm{LGD_B}\lambda_u^B(\tilde{V}_u-C_u)^+)du\mid\mathcal{F}_t\Big]
\end{aligned} \tag{2.2.2}$$

where

$$\tilde{\theta}_u = \varepsilon_u\lambda_u - \mathrm{LGD}_C\lambda_u^C(\varepsilon_u-C_u)^+ + \mathrm{LGD}_B\lambda_u^B(\varepsilon_u-C_u)^-$$

By including the credit risk in the default-free filtration the indicator term $\mathbb{1}_{\{\tau>t\}}$ becomes an extra funding cost term $D(t,T,\lambda)$. Financially, this is reasonable since vulnerable contracts must have lower value than default-free contracts and we modelled the default intensity deliberately to have to form of discounted factor.
By lemma 2.2.2 we have $\mathbb{1}_{\{\tau>t\}}V_t = \mathbb{1}_{\{\tau>t\}}\tilde{V}_t$ for some $\mathcal{F}-$adapted stochastic process $\tilde{V}_t$, by comparing this with equation 2.2.2, we can deduce that:

$$\begin{aligned}
\tilde{V}_t =& \mathbb{E}[D(t,T,r+\lambda)\Phi(S_T)\mid\mathcal{F}_t] + \mathbb{E}\Bigg[\int_t^T D(t,u,r+\lambda)\times \\
&(\Pi_u-(c_u-r_u)C_u-(\tilde{f}_u-r_u)(\tilde{V}_u-C_u)-(r_u-\tilde{h}_u)\tilde{H}_u)du\mid\mathcal{F}_t\Bigg] \\
&+\mathbb{E}\Bigg[\int_t^T D(t,u,r+\lambda)(\tilde{\theta}_u+\mathrm{LGD_B}\lambda_u^B(\tilde{V}_u-C_u)^+)du\mid\mathcal{F}_t\Bigg]
\end{aligned} \tag{2.2.3}$$

Now, are can going to derive the BSDE of $\tilde{V}_t$ that is equivalent to equation 2.2.3.

Proposition 2.2.6. *The process $\tilde{V}_t$ satisfies the following BSDE:*

$$\begin{aligned}
\tilde{V}_t =& \Phi(S_T) + \int_t^T [\Pi_u - (f_u+\lambda_u)\tilde{V}_u + \tilde{\theta}_u + (f_u-c_u)C_u \\
&-(r_u-\tilde{h}_u)\tilde{H}_u + LGD_B\mathbb{1}_{\{\tilde{V}_u-C_u>0\}}\lambda_u^B(\tilde{V}_u-C_u)]du - \int_t^T Z_u dWu
\end{aligned}$$

Proof. We introduce the process:

$$X_t = \int_0^t D(0,u,r+\lambda)\Pi_u du + \int_0^t D(0,u,r+\lambda)(\tilde{\theta}_u + \mathrm{LGD}_B \lambda_u^B (\tilde{V}_u - C_u)^+) du$$
$$- \int_0^t D(0,u,r+\lambda)[(c_u - r_u)C_u + (f_u - r_u)(\tilde{V}_u - C_u) + (r_u - \tilde{h}_u)\tilde{H}_u] du.$$

Note that:

$$\begin{aligned} D(0,t,r+\lambda)\tilde{V}_t + X_t &= \mathbb{E}[D(0,T,r+\lambda)\Phi(S_T) \mid \mathcal{F}_t] + \mathbb{E}\Big[\int_0^T D(0,u,r+\lambda)\times \\ &(\Pi_u - (c_u - r_u)C_u - (\tilde{f}_u - r_u)(\tilde{V}_u - C_u) - (r_u - \tilde{h}_u)\tilde{H}_u) du \mid \mathcal{F}_t\Big] \\ &+ \mathbb{E}\Big[\int_0^T D(0,u,r+\lambda)(\tilde{\theta}_u + \mathrm{LGD_B}\lambda_u^B(\tilde{V}_u - C_u)^+) du \mid \mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[X_T + D(0,T,r+\lambda)\tilde{V}_T \mid \mathcal{F}_t\Big] \end{aligned} \tag{2.2.4}$$

Thus, $X_T + D(0,T,r+\lambda)\tilde{V}_T$ is a $\mathcal{F}-$martingale. Now, we divide by $D(0,t,r+\lambda)$ and apply Ito's lemma to both sides of equation 2.2.4 to get:

$$\begin{aligned} \tilde{V}_t + \frac{X_t}{D(0,t,r+\lambda)} =& \tilde{V}_0 + \frac{X_0}{D(0,0,r+\lambda)} + \int_0^t d\tilde{V}_u \\ &+ \int_0^t X_u d\frac{1}{D(0,u,r+\lambda)} + \int_0^t \frac{1}{D(0,u,r+\lambda)} dX_u \end{aligned}$$

and

$$\begin{aligned} \frac{\mathbb{E}[X_T + D(0,T,r+\lambda)\tilde{V}_T \mid \mathcal{F}_t]}{D(0,t,r+\lambda)} =& \frac{X_0 + D(0,0,r+\lambda)\tilde{V}_0}{D(0,0,r+\lambda)} \\ &+ \int_0^t X_u d\frac{1}{D(0,u,r+\lambda)} \\ &+ \int_0^t D(0,u,r+\lambda)\tilde{V}_u d\frac{1}{D(0,u,r+\lambda)} \\ &+ \int_0^t \frac{1}{D(0,u,r+\lambda)} d\mathbb{E}[X_T + D(0,T,r+\lambda)\tilde{V}_T \mid \mathcal{F}_u] \end{aligned}$$

combining these two expressions and note that $X_0 = 0$, $D(0,0,r+\lambda) = 1$ and $d\frac{1}{D(0,u,r+\lambda)} = \frac{(r_u+\lambda_u)du}{D(0,u,r+\lambda)}$ we get:

$$\begin{aligned} \tilde{V}_t + \int_0^t \frac{1}{D(0,u,r+\lambda)} dX_u =& \tilde{V}_0 + \int_0^t \tilde{V}_u (r_u + \lambda_u) du \\ &+ \underbrace{\int_0^t \frac{1}{D(0,u,r+\lambda)} d\mathbb{E}[X_T + D(0,T,r+\lambda)\tilde{V}_T \mid \mathcal{F}_u]}_{\mathcal{M}_t} \end{aligned}$$

Since $\mathbb{E}[X_T + D(0,T,r+\lambda)\tilde{V}_T \mid \mathcal{F}_t]$ is also a $\mathcal{F}-$martingale, we have that $\mathcal{M}_t$ is a local martingale. By martingale representation theorem we can write

$$\mathcal{M}_t = \int_0^t Z_u dW_u$$

for some $\mathcal{F}-$predictable process Z_u. Thus, we have for any $t \in [0,T]$:

$$\tilde{V}_t = \tilde{V}_0 - \int_0^t \frac{1}{D(0,u,r+\lambda)} dX_u + \int_0^t \tilde{V}_u(r_u + \lambda_u)du + \int_0^t Z_u dW_u$$

Also,

$$\tilde{V}_T - \tilde{V}_t = -\int_t^T \frac{1}{D(0,u,r+\lambda)} dX_u + \int_t^T \tilde{V}_u(r_u + \lambda_u)du + \int_t^T Z_u dW_u$$

By noting also that $\tilde{V}_T = \Phi(S_T)$, substitute for dX_u in the previous equation we get the result. □

The proposition 2.2.6 gives us a general framework of the BSDE as $\tilde{V}_t$ depends on the future dynamics of itself.
Before deriving the PDE subject to this BSDE, we make some assumptions:

1. Π_t is a deterministic function of time t and the underlying asset S_t and is Lipschitz continuous in S_t.

2. $r_t, f_t^{\pm}, c_t^{\pm}, \lambda^B, \lambda^C, h_t^{\pm}$ are deterministic continuous bounded functions of time.

3. The collateral posted/received is a fraction of the value of the contract: $C_t = \alpha_t \tilde{V}_t$ where $0 < \alpha_t < 1$ is a deterministic continuous function of time.

4. We assume the hedging strategy $\tilde{H}_t = H(t, S_t, \tilde{V}_t, Z_t)$ is a deterministic function, Lipschitz continuous in $\tilde{V}_t, Z_t$ and uniformly continuous in t. Furthermore, we assume $H(t, s, 0, 0)$ is continuous in s.

5. The dynamic of S_t is described by the SDE:

$$dS_t = r_t S_t dt + \sigma S_t dW_t$$

with constant volatility under the risk-neutral measure.

The Lipschitz continuous condition is crucial for the existence and uniqueness of the solution to BSDE. Under these assumptions, the dynamics of $\tilde{V}_t$ can be written as follows:

$$\begin{aligned}
&dS_t^{q,s} = r_t S_t^{q,s} + \sigma S_t dW_t, \ q < t < T \\
&S_q = s, \ 0 \le t \le q \\
&dV_t^{q,s} = -B(t, S_t^{q,s}, V_t^{q,s}, Z_t^{q,s})dt + Z_t^{q,s} dW_t \\
&V_T^{q,s} = \Phi(S_T^{q,s})
\end{aligned} \tag{2.2.5}$$

where

$$\begin{aligned}
B(t, S_t^{q,s}, V_t^{q,s}, Z_t^{q,s}) = &[\Pi_t + \theta_t + ((1-\alpha_t)(\text{LGD}_B \mathbb{1}_{\{V_t^{q,s} > 0\}} \lambda_t^B - f_t) \\
&- \lambda_t - c_t \alpha_t) V_t^{q,s} - (r_t - h_t) H(t, S_t^{q,s}, V_t^{q,s}, Z_t^{q,s})]
\end{aligned}$$

and the super-script q, s denotes the stochastic process with initial condition $S_q = s$. We omit all the tildes now and assume that all processes are $\mathcal{F}_t-$ adapted.
To derive the PDE subject to this BSDE, we assume that $V_t^{q,s} = u(t, S_t^{q,s})$ is a function of t and S_t and we apply Ito's lemma to it:

$$\begin{aligned}
du(t, S_t^{q,s}) = &\Big(\partial_t u(t, S_t^{q,s}) + r_t S_t^{q,s} \partial_s u(t, S_t^{q,s}) + \frac{1}{2}\sigma^2 S_t^{q,s2} \partial_{ss}^2 u(t, S_t^{q,s})\Big) dt \\
&+ \sigma S_t^{q,s} \partial_s u(t, S_t^{q,s}) dW_t
\end{aligned}$$

comparing with equation 2.2.5 and equation dt and dW_t terms we get:

$$\partial_t u(t, S_t^{q,s}) + r_t S_t^{q,s} \partial_s u(t, S_t^{q,s}) + \frac{1}{2}\sigma^2 S_t^{q,s2} \partial_{ss}^2 u(t, S_t^{q,s}) = -B(t, S_t^{q,s}, V_t^{q,s}, Z_t^{q,s})$$
$$\sigma S_t^{q,s} \partial_s u(t, S_t^{q,s}) = Z_t^{q,s}$$

Thus $\tilde{V}_t$ satisfies the PDE:

$$\begin{aligned} &\partial_t u(t,s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 u(t,s) + r_t s \partial_s u(t,s) + B(t, s, u(t,s), \sigma s \partial_s u(t,s)) = 0 \\ &u(T,s) = \Phi(s) \end{aligned} \tag{2.2.6}$$

We can simplify further this equation by assuming that the funding rate and repurchase rate are the same: $f_t = h_t$. Also, if we adopt the delta-hedging strategy, i.e.

$$H = S_t^{q,s} \partial_s u(t, S_t^{q,s}) = S_t^{q,s} \frac{Z_t^{q,s}}{\sigma(t, S_t^{q,s})}$$

then, we notice that terms involving r_t in equation 2.2.6 are eliminated, so the PDE does not depend on r_t. This is a desirable property since the short rate r_t is unobservable in the market. It turns out that the PDE can be writte as:

$$\begin{aligned} &\partial_t u(t,s) - f_t^+(-u(t,s) + s\partial_s u(t,s) + \alpha_t u(t,s))^+ + f_t^-(u(t,s) - s\partial_s u(t,s) - \alpha_t u(t,s))^+ - \\ &\lambda_t u(t,s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 u(t,s) + \Pi_t - c_t \alpha_t u(t,s) + \theta_t + \mathrm{LGD_B} \lambda_t^B (u(t,s) - \alpha_t u(t,s)) = 0 \\ &u(T,s) = \Phi(s) \end{aligned} \tag{2.2.7}$$

This equation is not linear, indeed, it is a semilinear parabolic PDE. We will discuss two different cases below:

1. The simplified case, we just assume $f_t = f_t^+ = f_t^-$. In this case, the system in equation 2.2.7 becomes a linear parabolic PDE:

$$\begin{aligned} &\partial_t u(t,s) - f_t u(t,s) - \lambda_t u(t,s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 u(t,s) + f_t s \partial_s u(t,s) + \\ &\Pi_t + (f_t - c_t)\alpha_t u(t,s) + \theta_t + \mathrm{LGD_B} \lambda_t^B (u(t,s) - \alpha_t u(t,s)) = 0 \\ &u(T,s) = \Phi(s) \end{aligned} \tag{2.2.8}$$

   and is guaranteed to have one and only one solution as we will discuss in section 2.3.

2. The realistic case, we assume $f_t = f_t^+$ if the process multiplied by f_t is positive, and $f_t = f_t^-$ if the process multiplied by f_t is negate. In this case, the PDE is no longer linear parabolic because $\partial_s u(t,s)$ is also multiplied by f_t¿ In fact, the system becomes semilinear and the best we can obtain is the existence and uniqueness of a viscosity solution. This is explained in sections 2.4.

Finally, the numerical procedure to compute the solution of equation 2.2.8 also depends on those cases mentioned earlier, this will be explained in section 3.1.

2.3 Existence and Uniqueness of the solution with symmetric funding rate

In this section, we show that the parabolic partial differential equation 2.2.8 stated in the previous section has one and only one solution under some mild conditions provided $f_t = f_t^+ = f_t^{+-}$.

The work was done by Avner Friedman in his book Partial Differential Equation of Parabolic Type [23]. We will give a brief summary of the result, check the book for more details. We adapt a different notation from previous sections and we introduce some definitions related the mathematical analysis.

Definition 2.3.1. A function f is said to be Hölder continuous of exponent $0 < \alpha \leq 1$ if there exists a positive constant K such that:

$$|f(x) - f(y)| \leq K|x - y|^\alpha \tag{2.3.1}$$

for all x, y in the domain of f. The smallest K such that 2.3.1 holds is called the Hölder coefficient. If $\alpha = 1$, we simply say that f is Lipschitz continuous.

Denote by $C^{m,\alpha}$ to be the space to $m-$th continuously differentiable, real valued functions such that their $m-$th derivative is also $\alpha-$Hölder continuous, $C^{m,\alpha}$ is also called Hölder space. We will discuss solution functions in this space.
Let $u(x, t)$ be a two-dimensional function with domain $D \times [0, T]$, we denote the differential operator by L as follows:

$$Lu = \frac{\partial u}{\partial t} + a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u \tag{2.3.2}$$

where $a(x, t)$, $b(x, t)$ and $c(x, t)$ are continuous and bounded functions with the same domain as $u(x, t)$. We will assume that the domain of D is a closed bounded set.
We consider the semilinear partial differential equation as follows:

$$\begin{aligned} &Lu(x,t) = f(x,t,u,\partial_x u) \quad \text{for } (x,t) \in D \times [0,T] \\ &u(x,T) = \Phi(x) \end{aligned} \tag{2.3.3}$$

where D is the domain for variable x. We assume that the function f is Lipschitz continuous in u. The work of Friedman provides a more general framework and includes the set of semilinear PDEs. We are stating stronger theorems for linear parabolic PDEs in this section, these theorems will also be used in the next section where the PDE becomes semilinear. We state now the uniqueness theorem for the PDE defined in 2.3.3:

Theorem 2.3.2. *Consider the equation 2.3.3, if f is further uniformly continuous with respect to $(x, t, u, \partial_x u)$ in the closure of domain $D \times [0, T]$, then there exists at most one solution of the equation 2.3.3.*

Proof. See Friedman's book, Chapter 7, Section 4, Theorem 6 [23]. □

In order to prove existence of the solution, we need some stronger assumptions. First of all, we shall introduce some new definitions. We define the notion of distance in the domain space $D \times [0, T]$ as follows:

Definition 2.3.3. For any two points $P = (x_1, t_1)$ and $Q(x_2, t_2)$ in the domain $D \times [0, T]$, the distance between them is defined as:

$$d(P, Q) = [|x_1 - x_2|^2 + |t_1 - t_2|]^{1/2}$$

We adapt this notion of distance when defining Hölder continuity for functions in two dimensions by simply replacing equation 2.3.1 to

$$|f(P) - f(Q)| \leq K d(P, Q)^\alpha$$

for any $P, Q \in D \times [0, T]$.
Now, let $u(x, t)$ be a function in the domain $D \times [0, T]$. We define:

$$|u|_0^{D\times[0,T]} = \sup_{P \in D\times[0,T]} |u(P)|$$
$$H_\alpha^{D\times[0,T]}(u) = \sup_{P,Q \in D\times[0,T]} \frac{|u(P) - u(Q)|}{d(P,Q)^\alpha}$$
$$|u|_\alpha^{D\times[0,T]} = |u|_0^{D\times[0,T]} + H_\alpha^{D\times[0,T]}(u)$$

Based on this, we defined so called $(1 + \alpha)$ norms that involves also the norm of the derivative of the function:

$$|u|_{1+\alpha}^{D\times[0,T]} = |u|_\alpha^{D\times[0,T]} + |u_x|_\alpha^{D\times[0,T]}$$
$$|u|_{1-0}^{D\times[0,T]} = |u|_0^{D\times[0,T]} + L^{D\times[0,T]}[u]$$
$$|u|_{2-0}^{D\times[0,T]} = |u|_{1-0}^{D\times[0,T]} + |u_x|_{1-0}^{D\times[0,T]}$$

where

$$L^{D\times[0,T]}[u] = \sup_{(x,t),(x',t') \in D\times[0,T]} \frac{|u(x,t) - u(x',t')|}{|x - x'| + |t - t'|}$$

We say that a function u belongs to $C_{1+\alpha}(S),\ C_{1-0}(S),\ C_{2-0}(S)$ if $|u|_{1+\alpha}^S,\ |u|_{1-0}^S,\ |u|_{2-0}^S$ are finite respectively.
The existence of solution of equation 2.3.3 exists under the following conditions:

1. $a(x, t),\ b(x, t)$ and $c(x, t)$ are $\alpha-$Hölder continuous in the closure of D, that is:

$$|a|_\alpha^{D\times[0,T]} + |b|_\alpha^{D\times[0,T]} + |c|_\alpha^{D\times[0,T]} \leq H$$

   for some constant H.

2. $a(x, t)$ is in $C_{1-0}(D \times [0, T])$.

3. There exists a constant M_0 such that for all $M \geq M_0$ we have:

$$2K|f(x, t, u, u_x)| \leq M, \quad (x, t) \in D \times [0, T] \qquad (2.3.4)$$

   for all functions $u = u(x, t)$ satisfying $|u|_{1+\alpha}^{D\times[0,T]} \leq M$ and K is a constant such that:

$$|u|_{1+\alpha}^{D\times[0,T]} \leq K|f|_0^{D\times[0,T]}$$

4. Finally, $\Phi \in C_{2+\delta}$ for some $\alpha < \delta < 1$ and $L\Phi = f(x, t, \Phi, \Phi_x)$ on the set $\partial D \times \{T = 0\}$.

We have:

Theorem 2.3.4. *If above conditions are hold, then there is a solution to the Equation 2.3.3.*

Proof. See Friedman's book, Chapter 7, Section 4, Theorem 8 [23]. □

Next, we shall verify that the Black-Scholes PDE we derived in equation 2.2.8 satisfies conditions stated above. The first thing we shall discuss is the domain of the stock price S_t, in theory, S_t can take any positive value, however, we are only interested in the behaviour of the contract price up to some level. We assume that, the stock price rises sharply to an extreme value (for example the 99%-percentile) then one of the parties will enter into default scenario, indeed, this statement is

true for standard products such as forwards or call options.
Now, we write the the PDE 2.2.8 in the form of 2.3.3, we have that:

$$\begin{aligned} a(x,t) &= \frac{1}{2}\sigma^2 x^2 \\ b(x,t) &= f_t x \\ c(x,t) &= (f_t - c_t)\alpha_t + (1-\alpha_t)(\mathrm{LGD_B}\lambda_t^B - \mathrm{LGD_C}\lambda_t^C) - f_t \\ f(x,t,u,\partial_x u) &= -\Pi_t \end{aligned} \tag{2.3.5}$$

We assumed Π_t to be bounded, Lipschitz continuous in x, t and it does not depend on $u, \partial_x u$. Thus, the uniform continuity in (x,t) is implied by Lipschtiz continuity (x,t). The functions $a(x,t)$, $b(x,t)$ and $c(x,t)$ are all continuous and bounded functions due to the boundness and continuity of processes λ_t, α_t, c_t and f_t. Thus, all the conditions for uniqueness are satisfied and there is at most one solution of the equation 2.2.8.
For existence, we divide the verification in the subsequent lemmas.

Lemma 2.3.5. *$a(x,t)$, $b(x,t)$ and $c(x,t)$ are $\alpha-$Hölder continuous in the closure of D.*

Proof. We start with $a(x,t)$, let $P = (x_1,t_1)$, $Q = (x_2,t_2)$ be two points in $D \times [0,T]$ and let $0 < \alpha < 1$ we have:

$$d(P,Q)^\alpha = [|x_1 - x_2|^2 + |t_1 - t_2|]^{\alpha/2} \geq |x_1 - x_2|^\alpha$$

and

$$|a(P) - a(Q)| = \frac{1}{2}\sigma^2|x_1^2 - x_2^2| = \frac{1}{2}\sigma^2|x_1 - x_2||x_1 + x_2|$$

therefore,

$$\frac{|a(P) - a(Q)|}{|x_1 - x_2|^\alpha} = \frac{1}{2}\sigma^2|x_1 - x_2|^{1-\alpha}|x_1 + x_2| \leq K_1$$

for some constant K_1 since x_1, x_2 are bounded. Finally we have

$$\frac{|a(P) - a(Q)|}{d(P,Q)^\alpha} \leq \frac{|a(P) - a(Q)|}{|x_1 - x_2|^\alpha} \leq K_1$$

and the Hölder continuity holds.
For $b(x,t)$, we have that, for any P, Q in $D \times [0,T]$ and $0 < \alpha < 1$:

$$|b(P) - b(Q)| = |f_{t_1}x_1 - f_{t_2}x_2|$$

we can assume $x_1 > x_2$ w.l.o.g. and we divide in two cases:

1. If $f_{t_2} \geq f_{t_1}$ then:

$$|f_{t_1}x_1 - f_{t_2}x_2| = |f_{t_2}|\left|\frac{f_{t_1}}{f_{t_2}}x_1 - x_2\right| \leq |f_{t_2}||x_1 - x_2|$$

   in this case, we can follow the steps for $a(x,t)$ to find such K_2 for Hölder continuity.

2. If $f_{t_1} \geq f_{t_2}$ then:

$$\begin{aligned} (f_{t_1}x_1 - f_{t_2}x_2)^2 &= f_{t_1}^2 x_1^2 + f_{t_2}^2 x_2^2 - 2f_{t_1}f_{t_2}x_1x_2 \\ &\leq f_{t_1}^2(x_1 - x_2)^2 + 2f_{t_1}^2 x_1 x_2 - 2f_{t_1}f_{t_2}x_1x_2 \\ &= f_{t_1}^2(x_1 - x_2)^2 + 2f_{t_1}x_1x_2(f_{t_1} - f_{t_2}) \\ &= f_{t_1}^2(x_1 - x_2)^2 + 2f_{t_1}x_1x_2 L|t_1 - t_2| \\ &\leq K_3[|x_1 - x_2|^2 + |t_1 - t_2|] \end{aligned}$$

for some constant K_3 and L where we used Lipschitz continuity of f_t. Thus, we have $b(x,t)$ is Lipschitz continuous with respect to (x,t) together with boundedness we get Hölder continuity in this case.

Finally, we just pick $K_4 = \max(K_2, K_3)$ and we are done.
The Hölder continuity for $c(x,t)$ holds because $f_t,\ \alpha_t,\ c_t,\ \lambda_t$ are all bounded, Lipschitz continuous functions. We have that any sum and product of bounded and Lipschitz continuous functions are also bounded Lipschitz continuous and hence Hölder continuous. $\square$

Lemma 2.3.6. *$a(x,t)$ is in $C_{1-0}(D\times[0,T])$.*

Proof. We need to show that $L^{D\times[0,T]}[a]$ is finite, i.e.:

$$\sup_{(x,t),(x',t')\in D\times[0,T]} \frac{|a(x,t)-a(x',t')|}{|x-x'|+|t-t'|} < \infty.$$

Let (x,t) and (x',T') be two points in $D\times[0,T]$, we have:

$$\frac{|a(x,t)-a(x',t')|}{|x-x'|+|t-t'|} \leq \frac{|a(x,t)-a(x',t')|}{|x-x'|} = \frac{1}{2}\sigma^2|x+x'| < K$$

for some constant K, since $x,\ x'$ are bounded. The result follows directly. $\square$

The condition 3 is automatically satisfied because $f(x,t,u,\partial_x u)$ in our case does not depend on u and $\partial_x u$ and it is a bounded function so we can always find such constant M_0.
Finally, the last condition to be satisfied depends on the choice of Φ, the boundary conditions we adapt for our system, in particular, the product we choose to price. Fortunately, since we assumed D is one dimensional, the boundary of D just consists of two points which has Lebesgue measure 0, hence, even if the condition are not satisfied we do not have to worry about it as it does not affect the solution. Thus, we can apply theorem 2.3.4 and the existence is guaranteed.

2.4 Existence and Uniqueness of the solution with asymmetric funding rate

In this section, we discuss the case where f_t is variable and depends on the sign of $x\partial_x u-(1-\alpha_t)u$. In this case, the system becomes semilinear since now $f(x,t,u,\partial_x u)$ depends on both u and $\partial_x u$. Similar to previous section, the object we study is a two dimensional function $u(x,t)$ in the domain $D\times[0,T]$ where D is an one-dimensional closed bounded set. We rewrite the equations in 2.3.5 as follows:

$$a(x,t) = \frac{1}{2}\sigma^2 x^2$$
$$b(x,t) = 0$$
$$c(x,t) = (1-\alpha_t)(\mathrm{LGD_B}\lambda_t^B - \mathrm{LGD_C}\lambda_t^C) - c_t\alpha_t$$
$$f(x,t,u,\partial_x u) = -(\Pi_t - f_t^+(x\partial_x u(x,t)-(1-\alpha_t)u(x,t))^+ + f_t^-((1-\alpha_t)u(x,t)-x\partial_x u(x,t))^+$$

We notice that, $a(x,t)$ remains unchanged, there is nothing to check for $b(x,t)$ since it is zero, finally, $c(x,t)$ has a reduced form from the previous section, hence we don't have to check the conditions for $a(x,t)$, $b(x,t)$ and $c(x,t)$ again.
The only thing we need to check is $f(x,t,u,\partial_x u)$, we need $f(x,t,u,\partial_x u)$ to be uniformly continuous for uniqueness and the condition in 2.3.4 for existence. These conditions may not be guaranteed to be satisfied unless we make the following simplifications: we assume $f_t^+ = f^+$ and $f_t^- = f^-$ to

be positive constant functions and $\partial_x u$ to be a bounded function in $D \times [0,T]$. Also, for simplicity of notation we write $f(x,t,u,\partial_x u)$ as $f(x,t,u,w)$ where u and w are just some numbers.
Starting with uniform continuity, we have:

Lemma 2.4.1. *If f_t^+ and f_t^- are positive constant processes, then $f(x,t,u,w)$ is uniformly continuous.*

Proof. Suppose $P = (x_1,t_1,u_1,w_1)$ and $P(x_2,t_2,u_2,w_2)$ are two different points in the space. There are four difference scenarios depending on the sign of $g(x,t,u,w) = xw - (1-\alpha_t)u$, by symmetry, we only analyse two different cases:

1. Suppose that $g(P) > 0$ and $g(Q)$ does not change the sign, we have:

$$\begin{aligned}
|f(P) - f(Q)| &= |\Pi_{t_1} - \Pi_{t_2} + f^+(x_1w_1 - x_2w_2 - (1-\alpha_t)(u_1-u_2))| \\
&\leq |\Pi_{t_1} - \Pi_{t_2}| + f^+(1-\alpha_t)|u_1-u_2| + f^+x_1|w_1-w_2| + f^+|w_2||x_1-x_2| \\
&\leq L|t_1-t_2| + f^+(1-\alpha_t)|u_1-u_2| + f^+x_1|w_1-w_2| + f^+|w_2||x_1-x_2| \\
&\leq K_1\left(|t_1-t_2| + |x_1-x_2| + |u_1-u_2| + |w_1-w_2|\right) \\
&\leq 2K_1\left(|t_1-t_2|^2 + |x_1-x_2|^2 + |u_1-u_2|^2 + |w_1-w_2|^2\right)^{\frac{1}{2}} \\
&= 2K_1 d(P,Q)
\end{aligned}$$

   where in each line we used triangle inequality, Lipschitz continuity of Π_t, and boundedness of the parameters/variables and Cauchy-Schwarz inequality respectively.

2. Suppose that $g(P) > 0$ and $g(Q)$ changes sign:

$$\begin{aligned}
|f(P) - f(Q)| &= |\Pi_{t_1} - \Pi_{t_2} - f^+(x_1w_1 - (1-\alpha_t)u_1) - f^-((1-\alpha_t)u_2 - x_2w_2)| \\
&\leq |\Pi_{t_1} - \Pi_{t_2}| + |f^+(x_1w_1 - (1-\alpha_t)u_1) + f^-((1-\alpha_t)u_2 - x_2w_2)| \\
&\leq |\Pi_{t_1} - \Pi_{t_2}| + f^+|(x_1w_1 - (1-\alpha_t)u_1) + ((1-\alpha_t)u_2 - x_2w_2)| \\
&\leq L|t_1-t_2| + f^+(1-\alpha_t)|u_1-u_2| + f^+x_1|w_1-w_2| + f^+|w_2||x_1-x_2| \\
&\leq K_2\big(|t_1-t_2| + |x_1-x_2| + |u_1-u_2| + |w_1-w_2|\big) \\
&\leq 2K_2\big(|t_1-t_2|^2 + |x_1-x_2|^2 + |u_1-u_2|^2 + |w_1-w_2|^2\big)^{\frac{1}{2}} \\
&= 2K_2 d(P,Q)
\end{aligned}$$

   where we used the same reasoning as before and $f^+ \geq f^-$.

Similarly, we can obtain other two constants K_3 and K_4 for the rest of situations. Finally, by choosing $K = \max(K_1,K_2,K_3,K_4)$ we obtained Lipschitz continuity of f in this case and hence uniform continuity. □

With lemma 2.4.1, we obtain the uniqueness of solution. Next, we need to prove the following result:

Lemma 2.4.2. *The function $f(x,t,u,w)$ satisfies condition 2.3.4 with correspondent constant K and M.*

Proof. Just note that, f is a linear function in u as long as f_t is a constant function in t. We know that t is bounded by T and u is bounded by some constant L_1 as long as T and $S_{\max}$ are bounded, since the price of a contract can not be infinite, one party would just enter into default before the price spikes. Similarly, the term Π_t is also bounded by L_2, there is no contract with infinite cashflows over the period. The only crossterm is xw, however, we know the range of x, depending on the scenario we can bound f by substituting x by 0 or $S_{\max}$. Summarising above, we have:

$$\begin{cases} f(x,t,u,w) \leq |L_2 + S_{\max} w - (1-\alpha_t)L_1| & \textit{if } xw - (1-\alpha_t)u \geq 0 \\ f(x,t,u,w) \leq |L_2 + (1-\alpha_t)L_1| & \textit{if } xw - (1-\alpha_t)u \leq 0 \end{cases}$$

The second case is just a constant and we are done. In the first case, we need to bound w which we assumed in the simplification step. Note that, if w is unbounded then the condition 2.3.4 is clearly unsatisfied, we would require

$$M \leq K|f|_0^{D\times[0,T]} < 2K|f|_0^{D\times[0,T]} \leq M$$

and this is a contradiction. With w bounded, then f is bounded in both cases, we can just choose large M_0 and this is satisfied. □

Remark 2.4.3. Note that, we used the boundedness of w *a-priori* to complete the proof, however, we do not know it until we derived such solution. Rigorously speaking, we will need to prove that there exists at least one solution with bounded w, otherwise we could be stating an empty theorem. The existence of such solution is still an open question in the academia and goes beyond the scope of this dissertation. We only state this theorem as an indicative theorem, the numerical examples we provide does not suffer from this assumption as we avoided infinity in the discrete setting.

To sum up, we have constructed the results we need despite under strict conditions. If we remove the simplifications, the best we can get is the existence and uniqueness of a viscosity solution, for more details check theorem 5.3 in Brigo, Francischello and Pallavicini [18]. In chapter 3, we will use numerical methods to solve this semilinear parabolic PDE and use it as an empirical verification to our theorem stated in this section.

Chapter 3

Numerical methods

In this section, we will use numerical methods to solve both the semilinear parabolic PDE and FBSDE we derived in the previous section. Concretely, we will use finite difference method to solve the equation 2.2.8 numerically and we will adopt a deep learning approach to solve the FBSDE. We divide this section in three parts. In the first part, we describe the methodology adopted, i.e. the finite difference method. Secondly, we present a case study with constant funding rate f. Finally, we present the same case study with f as a function of hedging strategy. We will analyse the price function as well as sensitivity behaviour to parameters.

3.1 PDE approach: finite difference method

The finite difference method is a classical approach used to solve PDEs numerically. The key idea is to replace the partial derivative term by a differential quotient and convert the PDE to a difference equation. Mathematically, let $u(t,s) : [0,T] \times [0,\infty) \to [0,\infty)$ be a differentiable function in t and twice differentiable function in s. The partial derivative of u with respect to t is defined as:

$$\partial_t u(t,s) = \lim_{h \to 0} \frac{u(t+h,s) - u(t,s)}{h}$$

and we obtain a similar expression for s. In the Black-Scholes theory, the dynamic of S_t is given by $dS_t = rS_t dt + \sigma S_t dW_t$, it is log-normally distributed and it can take any value between $[0,\infty)$. In practice, we set an upper bound S_{MAX} for S_t to be the α%th quantile so that S_t is unlikely to exceed S_{MAX}. From now, we assume that S_t can only take values between $[0, S_{\text{MAX}}]$. In the finite difference setting, we first divide the domain of t and s into equidistant points, define:

$$t_i = i\Delta t,\ i = 0,1,\ldots,N,\ \Delta t = T/N$$
$$s_j = j\Delta s,\ j = 0,1,\ldots,M,\ \Delta s = S_{\text{MAX}}/M.$$

If we can find the solution function $u(t,s)$ evaluated at those grid-points, we can roughly build up an estimate of the true solution via linear interpolation, we denote by

$$u_{i,j} = u(t_i, s_j)$$

discretized values of the function u.

In the next step, we approximate the partial derivative of u at t_i, s_j by forward difference quotient, namely:

$$\partial_t u_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta t},\ 0 \le i \le N-1,\ 0 \le j \le M$$

and

$$\partial_s u_{i,j} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s},\ 0 \le i \le N,\ 1 \le j \le M-1$$

also, for the second order derivative we have:

$$\partial_{ss} u_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta s)^2},\ 1 \le j \le M-1$$

this is not valid for the edge cases, we need to use forward and backward difference:

$$\partial_t u_{0,j} \approx \frac{u_{1,j} - u_{0,j}}{\Delta t},\ \partial_s u_{i,0} \approx \frac{u_{i,1} - u_{i,0}}{\Delta s}$$
$$\partial_t u_{N,j} \approx \frac{u_{N,j} - u_{N-1,j}}{\Delta t},\ \partial_s u_{i,M} \approx \frac{u_{i,M} - u_{i,M-1}}{\Delta s}$$

similarly, for second order derivatives we have:

$$\partial_{ss} u_{i,0} \approx \frac{u_{i,2} - 2u_{i,1} + u_{i,0}}{(\Delta s)^2},\ \partial_{ss} u_{i,M} \approx \frac{u_{i,M} - 2u_{i,M-1} + u_{i,M-2}}{(\Delta s)^2}.$$

Recall our system in equation 2.2.7, we have the boundary condition $u_{N,j} = \Phi(s_j)$ for all j. We also need to specify boundary conditions for s, i.e. $u_{i,0}$ and $u_{i,M}$ for all i, which depends on the contract. Once the boundary conditions are set, we can work backwards to get $u_{0,j}$. We will discuss two different scenarios of the equation 2.2.7: when f_t is a symmetric process and when f_t is asymmetric and depends on the sign of the process to which it multiplies.
For the first scenario, the algorithm is straightforward. We just replace $\partial_t u$, $\partial_s u$ and $\partial_{ss} u$ in the equation 2.2.8. Formally, we can construct the following recursive relation. For any $1 \le i \le N-1$ and $1 \le j \le M-1$:

$$\begin{aligned} &\frac{u_{i+1,j} - u_{i,j}}{\Delta t} + f_t s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s} + \frac{1}{2}\sigma^2 s_j^2 \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta s)^2} - (f_t + \lambda_t) u_{i,j} = \\ &-(\Pi_t + (f_t - c_t)\alpha_t u_{i,j} + \theta_t + \mathrm{LGD}_B \lambda_t^B (1-\alpha_t) u_{i,j})) \end{aligned} \tag{3.1.1}$$

where

$$\theta_t = \lambda_t u_{i,j} - \mathrm{LGD}_C \lambda_t^C (1-\alpha_t) u_{i,j}$$

This can be written as:

$$A_j u_{i,j-1} + B_j u_{i,j} + C_j u_{i,j+1} = u_{i+1,j} + D \tag{3.1.2}$$

with

$$\begin{aligned} A_j &= \Delta_t \left[\frac{f_t s_j}{2\Delta s} - \frac{\sigma^2 s_j^2}{2\Delta s^2} \right] \\ B_j &= 1 + \Delta t \left[\frac{\sigma^2 s_j^2}{\Delta s^2} - (f_t - c_t)\alpha_t - \mathrm{LGD}_B \lambda_t^B (1-\alpha_t) + \mathrm{LGD}_C \lambda_t^C (1-\alpha_t) + f_t \right] \\ C_j &= -\Delta_t \left[\frac{f_t s_j}{2\Delta s} + \frac{\sigma^2 s_j^2}{2\Delta s^2} \right] \\ D &= \Delta_t \Pi_t \end{aligned}$$

for any $1 \leq i \leq N-1$ and $1 \leq j \leq M-1$. With matrix notation, we can write the above expression as:

$$\begin{bmatrix} B_1 & C_1 & 0 & \dots & 0 \\ A_2 & B_2 & C_2 & \ddots & 0 \\ 0 & A_3 & B_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & C_{M-2} \\ 0 & \dots & 0 & A_{M-1} & B_{M-1} \end{bmatrix} \begin{bmatrix} u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,M-2} \\ u_{i,M-1} \end{bmatrix} = \begin{bmatrix} u_{i+1,1} \\ u_{i+1,2} \\ \vdots \\ u_{i+1,M-2} \\ u_{i+1,M-1} \end{bmatrix} + \begin{bmatrix} D - A_1 u_{i,0} \\ D \\ \vdots \\ D \\ D - C_{M-1} u_{i,M} \end{bmatrix} \tag{3.1.3}$$

hence, by knowing the terminal condition $[u_{N,1}, \dots, u_{N,M}]^T$ and boundary conditions $[u_{0,0}, \dots, u_{N,0}]^T$ and $[u_{0,M}, \dots, u_{N,M}]^T$ we can work backwards to get the vector $[u_{0,1}, \dots, u_{0,M}]^T$.
In the second scenario, we set $f_t = f_t^+$ if the process multiplied by f_t is positive, and $f_t = f_t^-$ if the process multiplied by f_t is negate. With this in mind, we replace the derivatives in the equation 2.2.7 to get:

$$\begin{aligned} &\frac{u_{i+1,j} - u_{i,j}}{\Delta t} + \frac{1}{2}\sigma^2 s_j^2 \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta s)^2} \\ &- f_t^+ (s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s} - (1-\alpha_t) u_{i,j})^+ \\ &+ f_t^- ((1-\alpha_t) u_{i,j} - s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s})^+ - \lambda_t u_{i,j} = \\ &- \Pi_t + c_t \alpha_t u_{i,j} + \theta_t + \mathrm{LGD}_B \lambda_t^B (u_{i,j} - \alpha_t u_{i,j})) \end{aligned}$$

we can see that f_t depends on the sign of $s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s} - (1-\alpha_t) u_{i,j}$ thus we decompose f_t into piecewise constant functions $f_{i,j}$ and we derive again the recursive relation in equation 3.1.2 as follows:

$$A_j u_{i,j-1} + B_j u_{i,j} + C_j u_{i,j+1} = u_{i+1,j} + D$$

with

$$\begin{aligned} A_j &= -\Delta_t \left[\frac{f_{i,j} s_j}{2\Delta s} + \frac{\sigma^2 s_j^2}{2\Delta s^2} \right] \\ B_j &= 1 + \Delta t \left[\frac{\sigma^2 s_j^2}{\Delta s^2} - (1-\alpha_t) f_{i,j} + c_t \alpha_t - \mathrm{LGD}_B \lambda_t^B (1-\alpha_t) + \mathrm{LGD}_C \lambda_t^C (1-\alpha_t) + f_{i,j} \right] \\ C_j &= \Delta_t \left[\frac{f_{i,j} s_j}{2\Delta s} - \frac{\sigma^2 s_j^2}{2\Delta s^2} \right] \\ D &= \Delta_t \Pi_t \end{aligned} \tag{3.1.4}$$

where:

$$f_{i,j} = \begin{cases} f_t^+ & s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s} - (1-\alpha_t) u_{i,j} \geq 0 \\ f_t^- & s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s} - (1-\alpha_t) u_{i,j} < 0 \end{cases}$$

This recursive relation is more complex because, although we know f_t^+ and f_t^- at the beginning, we can not determine f_t unless we solve the system of equations in 3.1.3. To solve this problem, we do the following thing:

1. To calculate $[u_{i,1}, \dots, u_{i,M}]^T$ for any index i, we start assuming that $f_{i,j} = f_t^+$ regardless the sign of the multiplied process.

2. We solve the backward equation 3.1.3.

3. Once we obtained the result for $[u_{i,1}, \dots, u_{i,M-1}]^T$ from $[u_{i+1,1}, \dots, u_{i+1,M-1}]^T$, we verify if

our assumption were actually hold, i.e. check the sign of

$$s_j \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta s} - (1-\alpha_t)u_{i,j}$$

and compare it with the value of $f_{i,j}$ for all $1 \leq j \leq M-1$.

4. We correct those $f_{i,j}$ which mismatch with its proper definition, then we repeat the Step 2-4 until every $f_{i,j}$ matches its own definition.

5. We proceed to the calculation of $[u_{i-1,1}, \dots, u_{i-1,M-1}]^T$ until we get $[u_{0,1}, \dots, u_{0,M-1}]^T$.

3.2 Case study 1: European Option with symmetric funding rate

In this section, we deliver a special case study: an European option with payoff $(S_T - K)^+$ at maturity. The following table summarizes the numerical values of our parameters:

T	f	σ	α	c	λ^B	LGD_B	LGD_C	M
1 year	0.5%	40%	0.5	0.2%	2%	0.6	0.6	1000

Table 3.1: Parameters for a call option

To simplify the problem we assume that all the parameter processes are constant in time. We can extend easily to deterministic processes by evaluating them at each grid-point (t_i, s_j).
We assume that the initial stock price is $S_0 = 100$ and the option is at the money ($K = 100$), also, we set $S_{\text{MAX}} = 300$. Finally, we set terminal and boundary conditions in order to proceed the finite difference method in the following way:

1. In the case where $S_t = 0$, then the option loses its value hence $u(t, 0) = 0$ for all $t \in [0, T]$.

2. In the case where $S_t = S_{\text{MAX}}$ we assume that the counterparty, i.e. the option seller will immediately enter into a default scenario to avoid this payment. Then, the contract ends at time t and the payment we receive is just

$$\alpha V_t + \text{REC}_C(S_{\text{MAX}} - K - \alpha V_t)$$

therefore, we set $u(t, S_{\text{MAX}}) = \alpha u(t, S_{\text{MAX}}) + \text{REC}_C(S_{\text{MAX}} - K - \alpha u(t, S_{\text{MAX}}))$, solving the equation we obtain:

$$u(t, S_{\text{MAX}}) = \frac{(1 - \text{LGD}_C)(S_{\text{MAX}} - K)}{1 - \alpha \text{LGD}_C}, \text{ for all } t \in [0, T]$$

3. At time $t = T$, the contract ends and $u(T, s) = (s - K)^+$ for all $s \in [0, S_{\text{MAX}})$.

The last parameter we need to set is the default intensity process λ_t^C. The intensity can also be thought as the default probability in the near future $[t, t + dt]$ given that the party has not defaulted before. To be realistic, we assume that the default intensity depends only on the stock price. Mathematically, we assume the counterparty's default intensity to be:

$$\lambda^C(s) = \begin{cases} 0.04 & s < 100 \\ \frac{0.96}{200}x - 0.44 & s > 100 \end{cases}$$

so λ^C start with 4% and approaches to 100% as the stock price increases.

Note that, there is a discrepance at (t_N, S_M) between terminal and boundary conditions, the discrepancy disappears if $\alpha = 1$ i.e. the contract is fully protected by collateral. We give preference to the boundary condition since the counterparty is very likely to default at extreme scenarios regardless the life of the contract.

We will use empirical results to show that this method is indeed convergent. To do this, we increment N progressively and calculate $u(0, S_0)$ for each N. For instance, we let N to take values between $[100, 250, 500, 1000, 2000, 3600]$.

3.3 Result and analysis

In this section we present the result of our numerical method. As shown in Figure 3.1, we can see a similar trend of the Credit, Collateral and Funding adjusted price as time step N increases. For Out-of-The-Money(OTM) options, i.e. $K > S_0$ we can see that the adjusted price is close to Black-Scholes price meanwhile deep In-The-Money(ITM) option prices show a divergent behaviour.

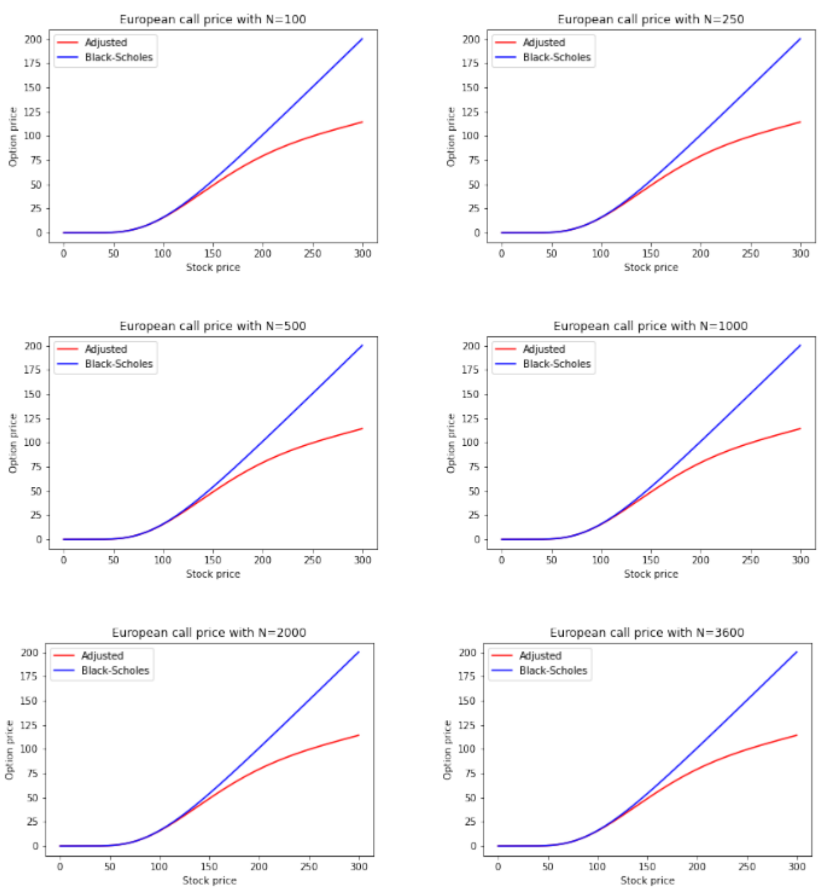

Figure 3.1: European Call price at different time step N

Now, we focus on some particular values of initial stock prices. Figures 3.2, 3.3, 3.4, 3.5 show the European option valued at different initial stock prices ($S_0 = 50, 100, 150, 250$ respectively).

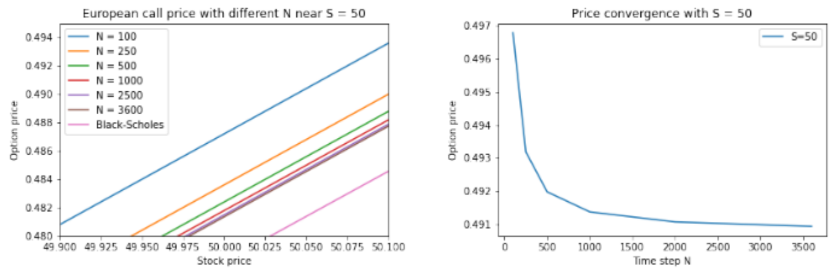

Figure 3.2: Convergence result at $S_0 = 50$

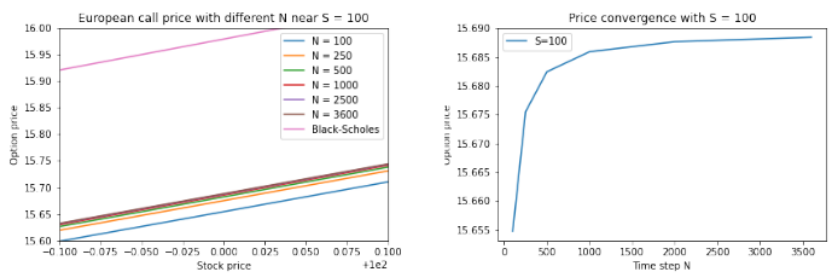

Figure 3.3: Convergence result at $S_0 = 100$

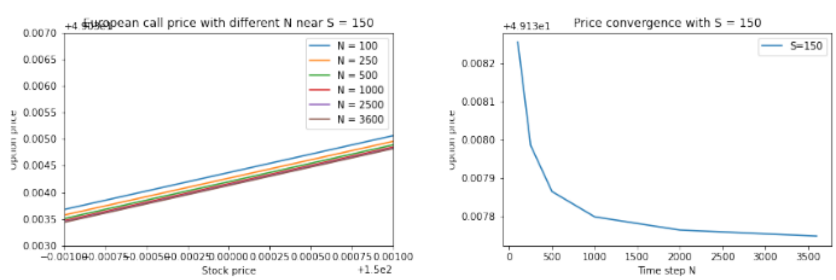

Figure 3.4: Convergence result at $S_0 = 150$

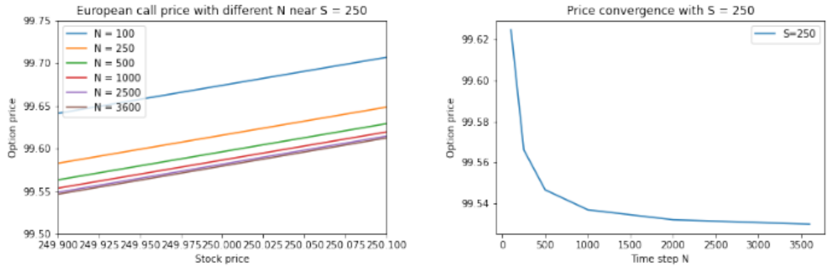

Figure 3.5: Convergence result at $S_0 = 250$

In an OTM setting (Figure 3.2) we can see the adjusted price dicreases as a function of time step N and converges near 0.491. By contrast, in an At-The-Money setting (Figure 3.3) the price starts increasing as N increases and moves towards to the Black-Scholes price. Finally, in an ITM setting, the price becomes again decreasing as a function of N, however, the price moves in an opposite direction to Black-Scholes price. In all these cases, we can clearly see a converging behaviour of the adjusted prices calculated using Finite-Difference method.
Next, we are going to explore the sensitivity of the option's adjusted price to its parameters.

3.3.1 Sentisivity to collateral amount

Figure 3.6 shows the option price when the collateral posted α is equal to 0.2, 0.4, 0.6 and 0.8 times the value of the contract respectively. It is natural to expect that, as α increases, the price of ITM options also increases since more value is protected by the collateral. There is not big notable difference at OTM and ATM scenarios since the counterparty do not need to post any collateral in those cases.

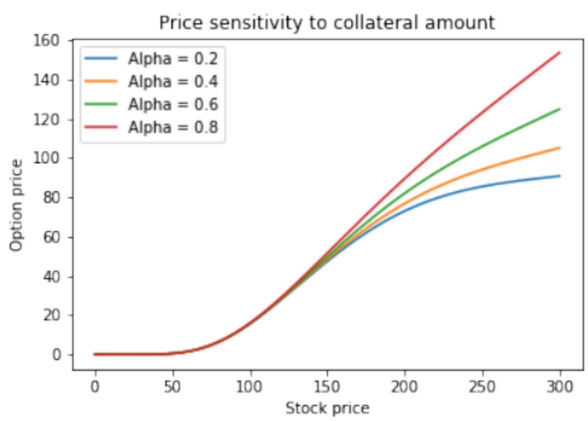

Figure 3.6: Option prices with different collateral amount

Similarly, we focus on the sensitivity behaviour at specific values of S_0. In Figure 3.7 we present the relationship of option value with respect to the collateral amount α at different initial stock prices $S_0 = 50,\ 100,\ 150$ and 200. In the OTM scenario $S_0 = 50$, the relationship is linear. As S_0 increases, the price function $u(\alpha; S_0)$ presents positive convexity. This is not surprising because as

S_0 increases the option becomes ITM, hence, the counterparty will very likely default, in this case, collateral plays an important role and the relationship is more than linear.

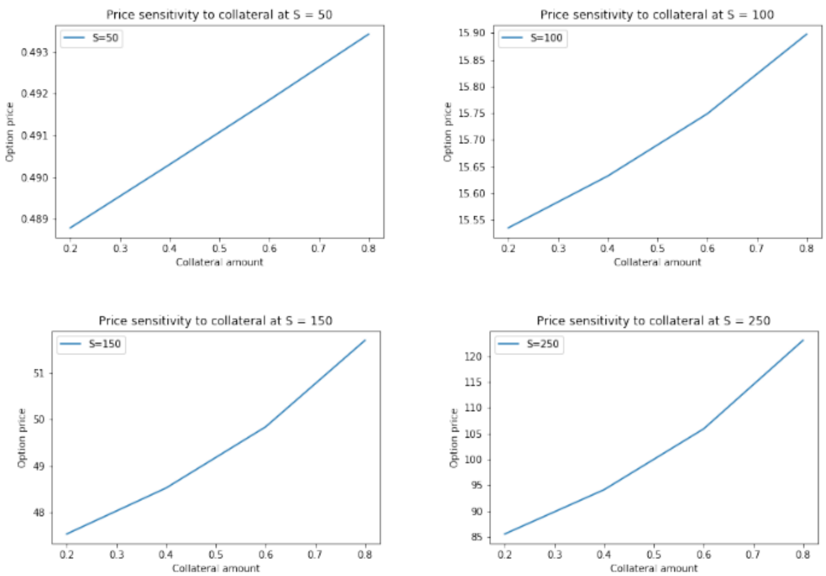

Figure 3.7: Sensitivity to α at $S_0 = 50,\ 100,\ 150,\ 200$

3.3.2 Sensitivity to funding rate

Funding rate will affect the hedging cost as the hedging strategy described in section 2.1.5. The option seller (in this case the counterparty) would hedge it's exposure to stock price changes, hence, the higher funding rate is, the more expensive the option is. Figure 3.8, shows the contract price as a function of initial stock price with different funding rates ranging from 0.5% to 9.5%. Indeed, the contract price is increasing as a function of funding rate. Note that, the effect of funding rate f is mild even though we used exaggerated values for f, this is because, f is not appearing in the boundary and terminal conditions. Therefore, we expect that the Funding Value Adjustment shall be small.

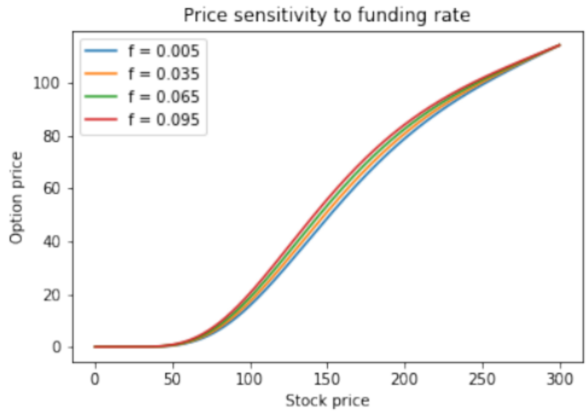

Figure 3.8: Option prices with different funding rates

Next, we focus on the sensitivity behaviour to funding rate f at specific values of S_0. Figure 3.9 shows the relationship of option value with respect to the funding rate f at different initial stock prices $S_0 = 50,\ 100,\ 150$ and 200. We can see that, the relationship is approximately linear at ATM and ITM scenarios ($S_0 = 100$ and $S_0 = 150$ respectively). In the OTM scenario $S_0 = 50$, the value function $u(f; S_0)$ exhibits a positive convexity. By contrast, at deep ITM scenario $S_0 = 250$ a negative convexity is present. It is hard to find a financial explanation of this change of convexity behaviour.

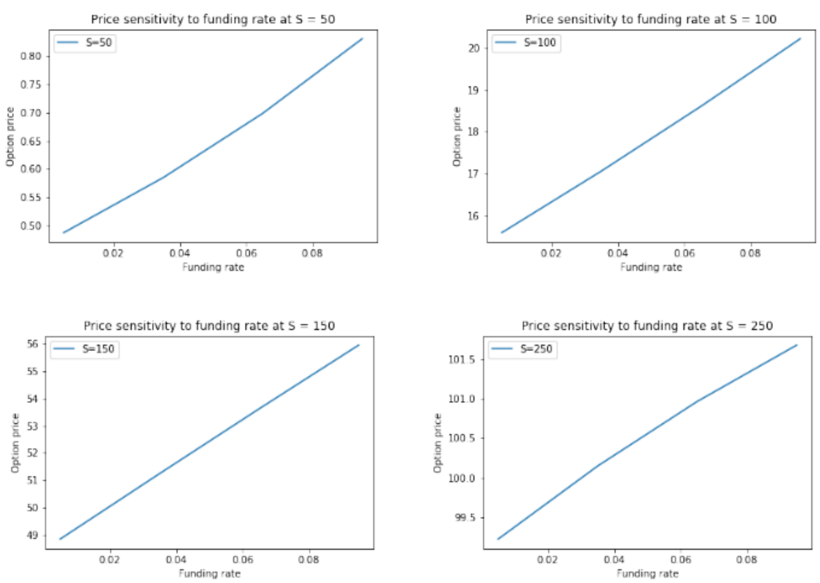

Figure 3.9: Sensitivity to funding rate f at $S_0 = 50,\ 100,\ 150,\ 200$

3.3.3 Sensitivity to default intensity

In this section, we explore the affect of changes in counterparty's default intensity. We let the counterparty's default intensity to range from $\lambda_c = [0.08,\ 0.16,\ 0.24,\ 0.32,\ 0.4]$, Figure 3.10 shows the option price dynamic with different values of default intensity. We can check that, as the default intensity increases, the price of the contract decays for all S_0. As an option buyer the more likely the counterparty will default, the less we are willing to pay for that contract. The local relationship of the contract price with respect to counterparty's default intensity is shown in the Figure 3.11. The relationship is linear and decreasing in all scenarios.

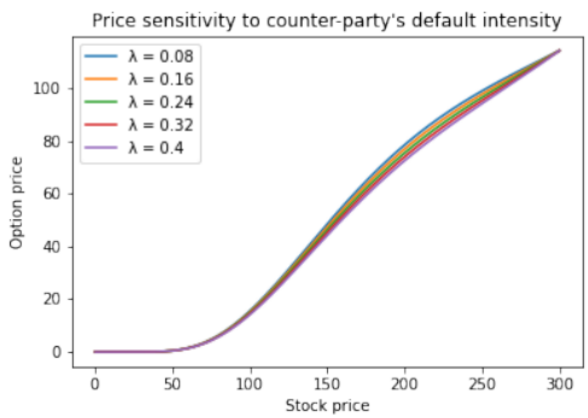

Figure 3.10: Option prices with different default intensity

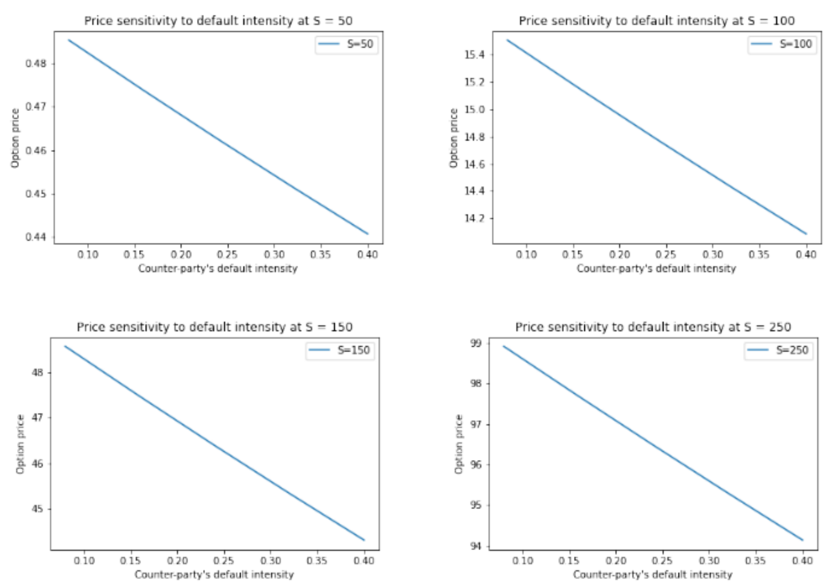

Figure 3.11: Sensitivity to counterparty's default intensity at $S_0 = 50,\ 100,\ 150,\ 200$

3.3.4 Sensitivity to loss given default

Loss Given Default (LGD) when the counterparty defaults is also a measure of the credit risk, it plays an important role in our model since we assumed the counterparty for always default at S_{MAX}. Figure 3.12 shows the option price when the Loss Given Default ranges from 0.2 to 0.8. The trend is clear at ITM scenarios where the contract price decays sharply as LGD incresases. In OTM and ATM scenarios, the trend is still decreasing as LGD increases however, the effect is mild.

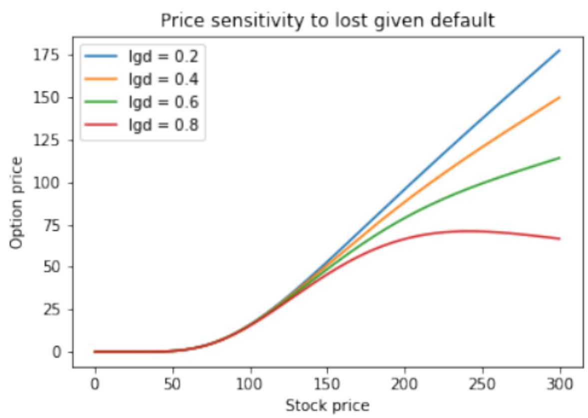

Figure 3.12: Option prices with different loss given default

The above observation is again verified by Figure 3.13 where we focus on the sensitivity behaviour at specific values of S_0. We notice that the relationship starts as linear decreasing trend at OTM scenario $S_0 = 50$. The trend becomes negatively convex at ATM and ITM scenarios ($S_0 \geq 100$).

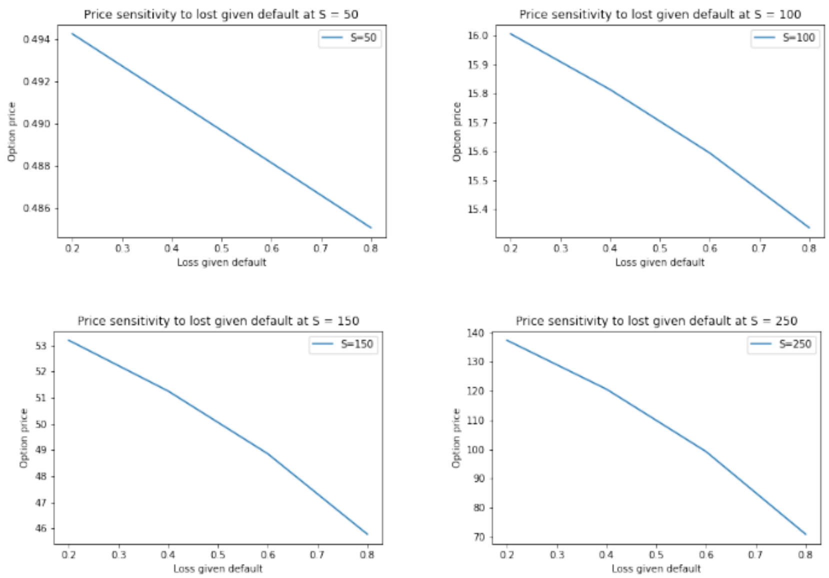

Figure 3.13: Sensitivity to loss given default at $S_0 = 50,\ 100,\ 150,\ 200$

3.3.5 Sensitivity to strike price

Strike prices is also an important parameter in option valuation theory. We explore the sensitivity of option price to the strike price by letting $K = 80,\ 90,\ 100,\ 110,\ 120$. As shown in Figure 3.14, it is natural to see that, as strike price increases the option value decreases.
We now focus on the sensitivity behaviour at specific values of S_0. As shown in 3.15, in the OTM scenario $S_0 = 50$, the function $u(K; S_0)$ presents positive convexity. As S_0 increases, the trend becomes linear and decreasing. This is not surprising because as S_0 increases the option becomes ITM, hence, the payoff function becomes $S_T - K$ and the option becomes a forward contract, the classic pricing theory yields $u = S_0 - Ke^{-rT}$ which is linear in K.

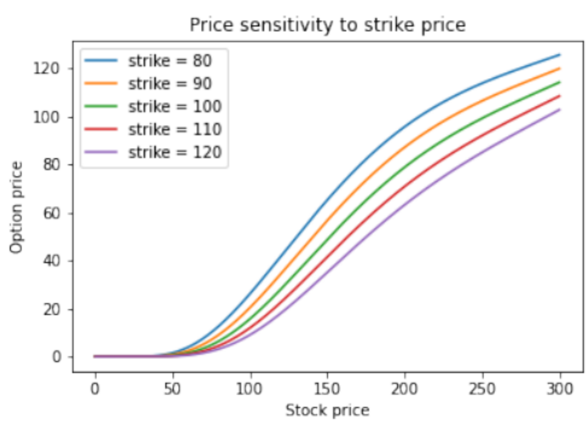

Figure 3.14: Option prices with different strike price

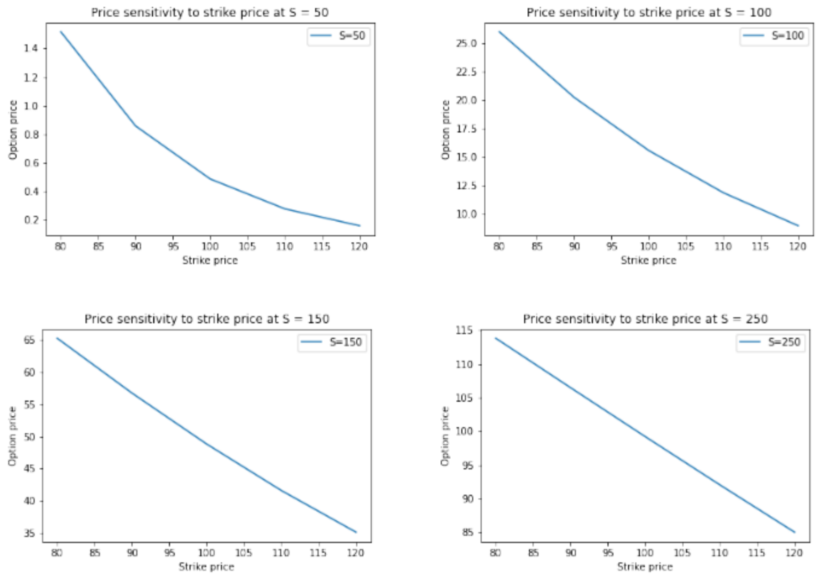

Figure 3.15: Sensitivity to strike price at $S_0 = 50,\ 100,\ 150,\ 200$

3.4 Asymmetry of Put option

We wonder now, does the same phenomenon happen in the case of put option? To answer this question, we calculate the numerical solution of a put option using the same parameter as before:

T	f	σ	α	c	λ^B	$\mathrm{LGD_B}$	$\mathrm{LGD_C}$	M
1 year	0.5%	40%	0.5	0.2%	2%	0.6	0.6	1000

Table 3.2: Parameters for a call option

To recover the full symmetry as the case 1, we set the strike price to $K = 200$. Note that, this is a really irrealistic case if we assumed the stock price to follow a lognormal distribution with boundaries $[0, 300]$, instead, we only focus on the qualitative behaviour of value adjustments.
The default intensity, terminal condition and boundary conditions are different from the simple call scenario since the counterparty will default if the stock price is approaching 0. We present the conditions in the following way:

1. We assume the counterparty's default intensity is linear decreasing in S_t, namely

$$\lambda^C(s) = 1 - \frac{0.96}{300}s$$

2. In the case where $S_t = 0$ the counterparty will immediately enter into a default scenario to avoid buying those stock at strike. Then, the contract ends at time t and the payment we receive is:

$$\alpha V_t + \mathrm{REC}_C(K - \alpha V_t)$$

again, setting $u(t,0) = \alpha u(t,0) + \text{REC}_C(K - \alpha u(t,0))$, solving the equation we obtain:

$$u(t,0) = \frac{(1 - \text{LGD}_C)K}{1 - \alpha \text{LGD}_C}, \text{ for all } t \in [0,T]$$

3. At time $t = T$, the contract ends and $u(T,s) = (K - s)+$ for all $s \in (0, S_{\text{MAX}})$.

Figure 3.16 shows the price process obtained by classical Black-Scholes formula and PDE in equation 2.2.8:

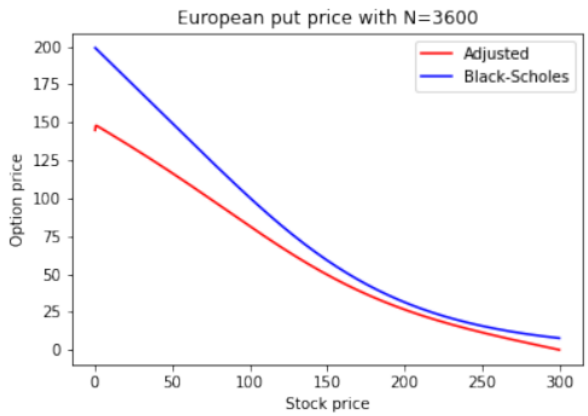

Figure 3.16: Put option price

The result is surprising, we observe that, unlike the case of call option, the adjusted price is not far from the Black-Scholes price when S_0 is approaching to zero there is neither negative convexity observed in call case. In other words, the default scenario we set at the boundary $0 \times [0,T]$ does not contribute much in the pricing equation, the difference in price is mainly explained by adjustment parameters such as λ_t^c, f_t and LGD.

Inspecting into the equation 3.1.3 we notice that, the boundary condition is multiplied by A_1, namely:

$$A_1 = \Delta_t \left[\frac{f_t s_1}{2\Delta s} - \frac{\sigma^2 s_1^2}{2\Delta s^2} \right] = \frac{\Delta_t}{2}(f_t - \sigma^2)$$

this amount is extremely small and reduces the effect of the boundary condition. Clearly, this behaviour does not concur with reality, nobody is willing to pay this option by \$200 when the stock price is small, we can regard this as a major drawback of our method as it reduces versatility. However, the behaviour of the solution in central range is well explained and the method is still valuable for the analysis in common situations.

We will not run the sensitivity tests in this case as they are included in the second case study.

3.5 Case study 2: Straddle with dynamic funding rate

In this case study, we price a synthetic product: a straddle contract. Entering in a long position of straddle means we buy a call option and put option of the same underlying asset at the same time and at the same strike price. The payoff of this product and maturity T is given by $|S_t - K|$, that is, we bet on the market being more volatile, we make profit out of price movements regardless in which direction.

Similar to the first case study, we assume that the stock price is between $[0, S_{\text{MAX}}]$ with $S_{\text{MAX}} = 300$. Also, we set the initial stock price to be $S_0 = 100$ and the options are at the money, i.e. $K = 100$. Aditionally, we assume that if we borrow money ($F_t > 0$) from external funding source, we are charged at a rate f^+ which is bigger than the risk-free rate r_t. Conversely, if we lend money ($F_t < 0$), we assume that we will always receive $f^- = r_t$ as interest rate as we always buy risk-free securities such as Treasury bond to avoid the additional calculation of credit adjustments. The following table summarizes the numerical values of our parameters:

T	f^-	f^+	σ	α	c	λ^B	LGD_B	LGD_C	M
1 year	0.4%	0.7%	40%	0.5	0.2%	2%	0.6	0.6	5000

Table 3.3: Parameters

Again, we assume all the parameters but counterparty's default intensity are constant. The only difference here is that we increased the number of grid-points by increasing M, we divide the interval $[0, S_{\text{MAX}=300}]$ into 5000 subintervals instead of 1000 in the previous case. This is because, if we assume f_t to be dependent on the hedging strategy, we must be more sensitive to the change in stock price.

The terminal condition and boundary conditions are different from the simple call scenario since the counterparty will default also if the stock price is approching 0. We present these conditions and the process λ^C in the following way:

1. In the case where $S_t = S_{\text{MAX}}$ the put option loses its value and we assume that the counterparty, i.e. the option seller will immediately enter into a default scenario to avoid the payment of the call. Therefore, we retreive the case of the simple call and we have:

$$u(t, S_{\text{MAX}}) = \frac{(1 - \text{LGD}_C)(S_{\text{MAX}} - K)}{1 - \alpha \text{LGD}_C}, \text{ for all } t \in [0, T]$$

2. In the case where $S_t = 0$ the counterpaty will also default due to the put contract. Then, the contract ends at time t and the payment we receive is:

$$\alpha V_t + \text{REC}_C(K - \alpha V_t)$$

again, setting $u(t, 0) = \alpha u(t, 0) + \text{REC}_C(K - \alpha u(t, 0))$, solving the equation we obtain:

$$u(t, 0) = \frac{(1 - \text{LGD}_C)K}{1 - \alpha \text{LGD}_C}, \text{ for all } t \in [0, T]$$

3. Since the counterparty will default in both directions of S_t, we assume the default intensity λ^C is linear decreasing on $s \in [0, 100)$ and linear increasing on $s \in [100, 300]$, mathematically we have:

$$\lambda^C(s) = \begin{cases} 1 - 0.0096s & s < 100 \\ \frac{0.96}{200}s - 0.44 & s > 100 \end{cases}$$

4. At time $t = T$, the contract ends and $u(T, s) = |s - K|$ for all $s \in (0, S_{\text{MAX}})$.

We will use empirical results to show that this method is also convergent. To do this, we increment N progressively and calculate $u(0, S_0)$ for each N. For instance, we let N to take values between $[100, 250, 500, 1000, 2000, 3600]$.

3.6 Result and analysis

In this section we present the result of our numerical method applied to straddle product.
As shown in figure 3.17, the pattern of the contract price as a function of underlying stock price is similar with increasing number of time-grids set, this provides an empirical evidence of convergence to the theoretical solution of the equation despite being a viscosity solution. The adjusted price function behaves symmetrically around the strike price $K = 100$ in the range $S = [0, 200]$. However, this is not because the payoff function is symmetric around $K = 100$. In fact, the contract price at $S_0 \in [0, 100]$ is reduced because of default intensity and the default scenario at boundary is not affecting its price as discussed previously. By contrast, the price at $S_0 \in [100, 200]$ is both affected by the default intensity and boundary condition.

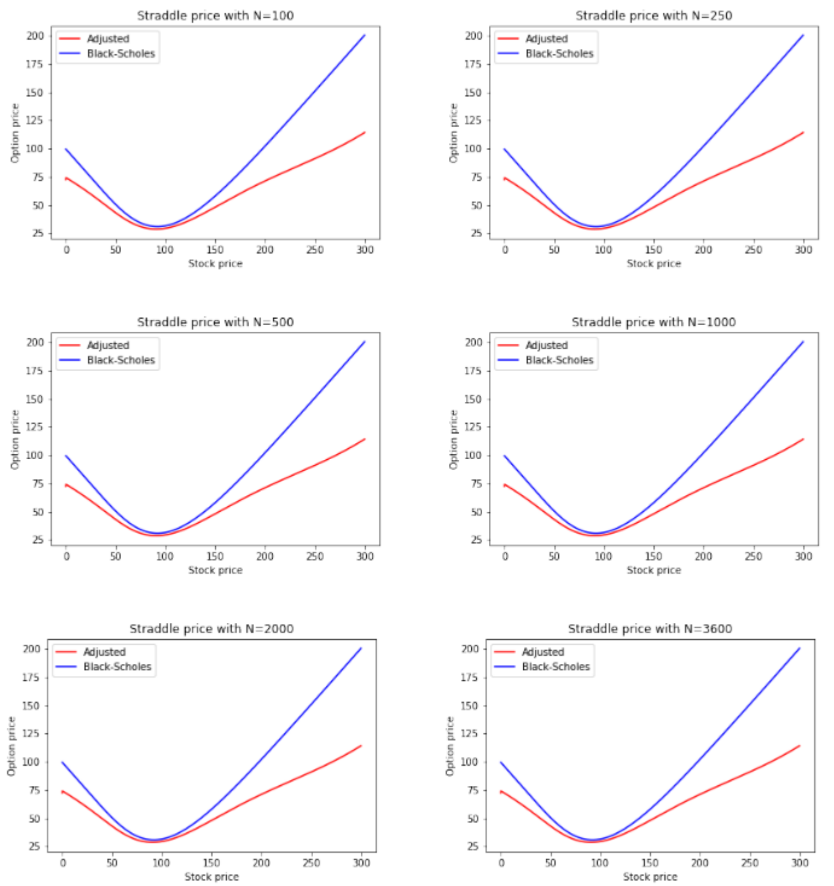

Figure 3.17: European Call price at different time step N

Furthermore, we can notice that the trend is also different between the ITM range of call side and put side. When the option is deep ITM due to the put option, we see that the trend is linear and decreasing. Also, there is a drastic price change when the underlying stock price approaches to zero by observing the peak the all of these graphs in figure 3.17. By contrast, under the ITM scenario due the call option, the contract price behaves smoothly when the stock price approaches its left boundary and presents a change in convexity. All these observations indicate an asymmetry

between the price of call and put options with the same strike and time to maturity.
Next, we show in more detail the convergence of the solution by focusing on some particular values of underlying stock prices ($S_0 = 50, 100, 150, 250$ respectively). Figures 3.18, 3.19, 3.20, 3.21 show the convergence behaviour of the contract price at stock prices stated previously.

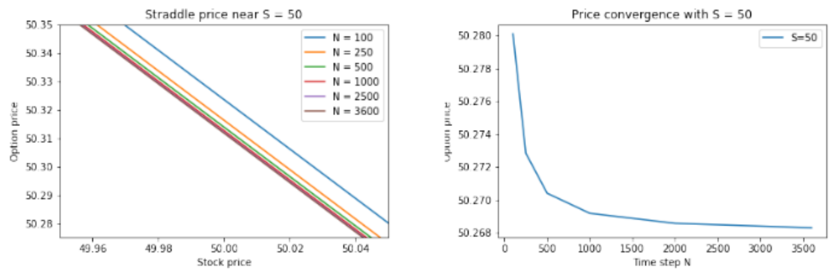

Figure 3.18: Convergence result at $S_0 = 50$

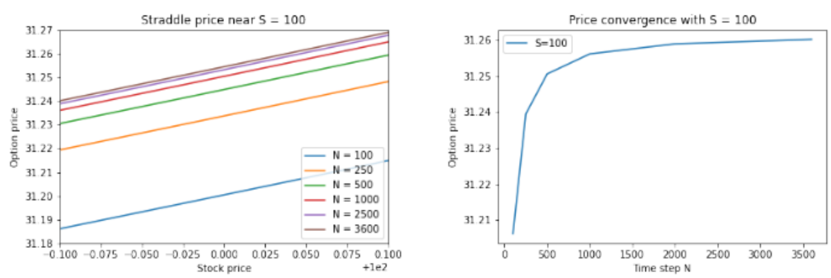

Figure 3.19: Convergence result at $S_0 = 100$

In fact, we can observe that, as N increases, the contract decreases when $S = 50$ and $S = 250$, those cases represent deep ITM scenarios of the call option and put option. Since it is natural to expect that the contract price is less than the theoretical Black-Scholes price, we can deduce that: as N increases, our method exaggerates the effect of credit, funding and collateral adjustments thus makes the contract price to move in the opposite direction of theoretical price. By contrast, when the contract is at the money the convergence is increasing, approaching to its theoretical price. In all these cases, we can clearly see a converging behaviour of the adjusted prices calculated using Finite-Difference method.

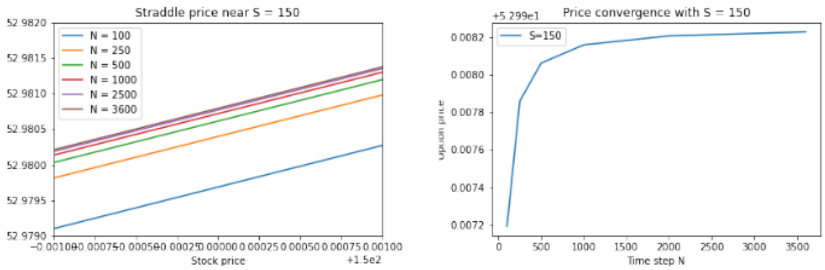

Figure 3.20: Convergence result at $S_0 = 150$

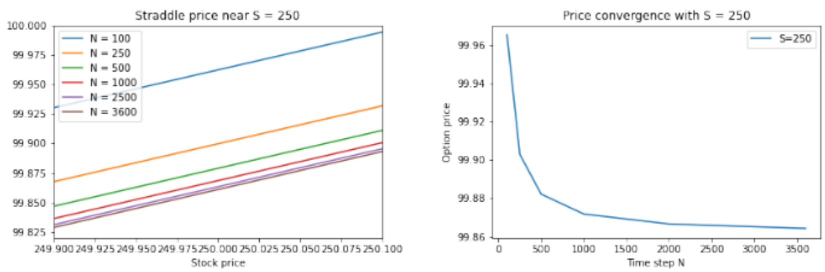

Figure 3.21: Convergence result at $S_0 = 250$

Next, we are going to explore the sensitivity of the contract's adjusted price to its parameters.

3.6.1 Sentisivity to collateral amount

Figure 3.22 shows the option price when the collateral posted α is equal to 0.2, 0.4, 0.6 and 0.8 times the value of the contract respectively.

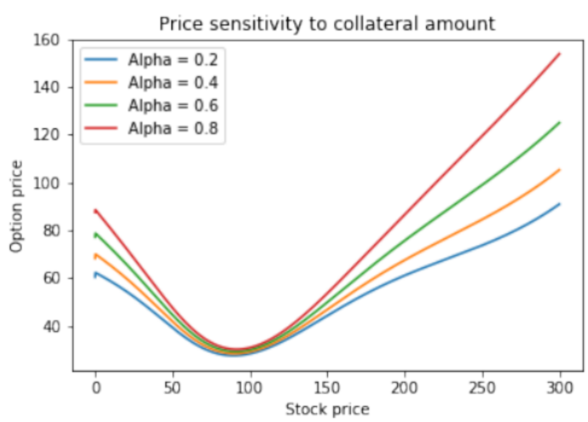

Figure 3.22: Option prices with different collateral amount

It is natural to see when the contract is deep in the money as S_0 is large, the higher the collateral amount is, the more valuable the contract is, since the buyer of the contract is well protected. When collateral posted is high ($\alpha = 0.8$) we observe a straight line as S_0 increases. However, when α is low, the contract price exhibits negative convexity around $S_0 = 200$ and switches to positive convexity near $S_0 = 250$. A financial explanation of this phenomenon could be as follows: we know that both CVA and ColVA reduced the value of the contract, although CVA is increased (in absolute value) by low amount of collateral posted, it is compensated by the reduction of ColVA (in absolute value). As a result, this trade-off between CVA and ColVA causes the convexity behaviour of straddle's price.
On the other hand, when the contract is ITM from the put side, the effect is also significant. The contract ends in different prices as S_0 approaches to zero although the boundary condition does not affect much on the pricing equation as discussed in 3.4.
To be more precise, we provide another point of view by concentrating in particular values of S. The figure 3.23 shows the relationship between the contract value and the collateral amount α at different initial stock prices $S_0 = 50$, 100, 150 and 200.

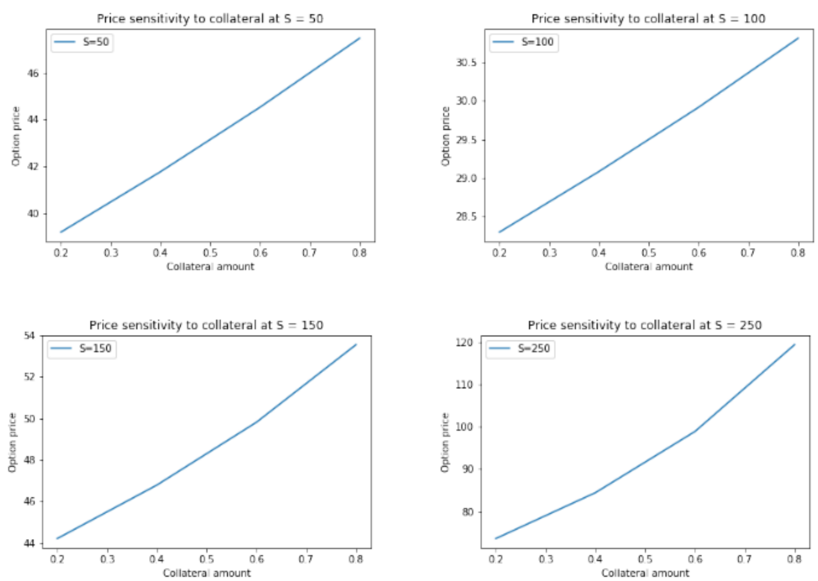

Figure 3.23: Sensitivity to α at $S_0 = 50,\ 100,\ 150,\ 200$

We can observe that, when the underlying stock price is small ($S = 50,\ 100$), the change due to collateral amount is mild (from 50 to 50.5 and from 31 to 31.5 respectively), the trend is increasing and linear. However, as S is large, the behaviour becomes positively convex and the price change is significant.

3.6.2 Sensitivity to funding rate

Funding rate will affect the hedging cost as the hedging strategy described in section 2.1.5. The option seller (in this case the counterparty) would hedge it's exposure to stock price changes, hence, the higher funding rate is, the more expensive the option is. In this case, we got two different rates: f^+ and f^-, we only alter the value of f^+ since we assume f^- is always the risk-free rate. Figure

3.24, shows the contract price as a function of initial stock price with different funding rates f^+ ranging from 0.5% to 15.5%. Indeed, the contract price is increasing as a function of funding rate. When the contract is ITM from the call option ($S_0 > 100$), we note that, the effect of funding rate f^+ is mild even though we used exaggerated values for f^+, this is because, f^+ is not appearing in the boundary and terminal conditions. Therefore, we expect that the Funding Value Adjustment shall be small. The behaviour is different in ITM scenarios when S_0 is small. Although is price is decreasing as α is decreasing (check figure 3.25), the effect is not significant. Another reason of this phenomenon is due to the size of S, if we inspect the equation 3.1.4, as s_j approaches to zero, the effect on funding rate will also approach to an amount less than 1. Then, by backwards induction in s_j all the terms related to credit, collateral and funding will converge to zero, as a consequence, the contract price done in this way will eventually be independent of these parameters when S is small. We will check this for other parameters.

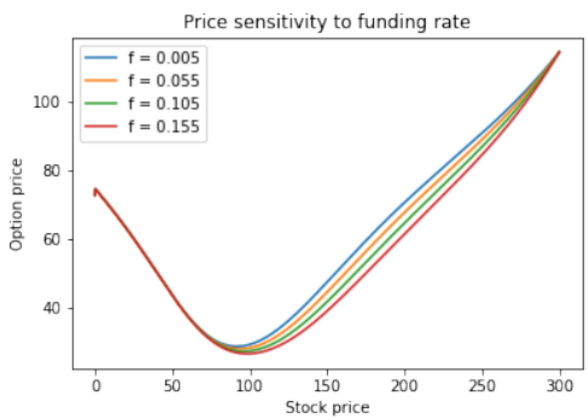

Figure 3.24: Option prices with different funding rates

Next, we focus on the sensitivity behaviour to funding rate f at specific values of S_0. Figure 3.9 shows the relationship of option value with respect to the funding rate f at different initial stock prices $S_0 = 50,\ 100,\ 150$ and 200. The relationship is approximately linear in all the cases.

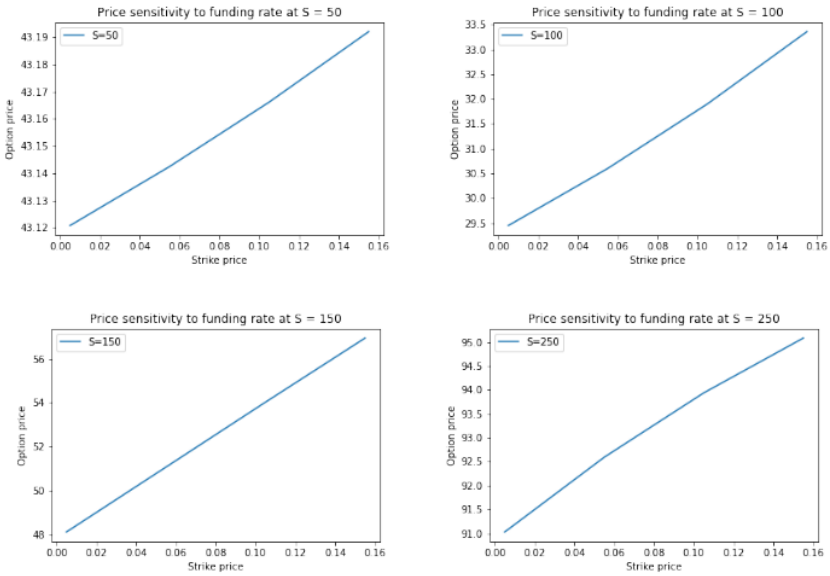

Figure 3.25: Sensitivity to funding rate f at $S_0 = 50,\ 100,\ 150,\ 200$

3.6.3 Sensitivity to default intensity

In this section, we explore the affect of changes in counterparty's default intensity. We let the minimum of counterparty's default intensity to range from $\lambda_c = [0.08, 0.16, 0.24, 0.32, 0.4]$, Figure 3.26 shows the option price dynamic with different values of default intensity. We can check that, as the default intensity increases, the price of the contract decays for all S_0. As an option buyer the more likely the counterparty will default, the less we are willing to pay for that contract. The effect of changing default intensity is relatively significant around $S_0 = 100$. This is not surprising, as S_0 diverges from the strike level $K = 100$, the default intensity of counterparty will increase and eventually converges to 1 at both ends, therefore no remarkable difference is observed in $S_0 = 0$ and $S_0 = 300$.

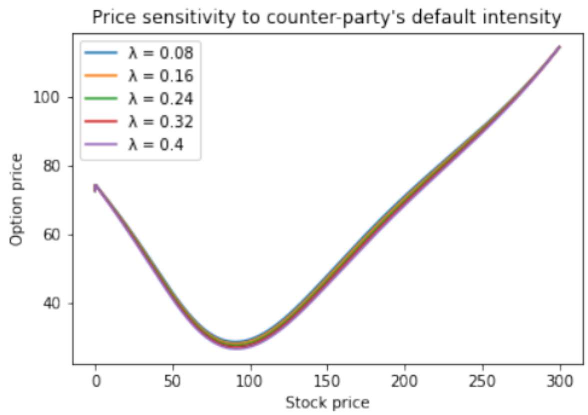

Figure 3.26: Option prices with different default intensity

The local relationship of the contract price with respect to counterparty's default intensity is shown in the figure 3.27. Again, the assertion of divergence is verified by noting that the spread in straddle value at $S_0 = 50$ $(50-45)$ is bigger than the case with $S_0 = 100$ $(31-28)$. The relationship is linear and decreasing in all scenarios with the presence of negative convexity at $S_0 = 250$.

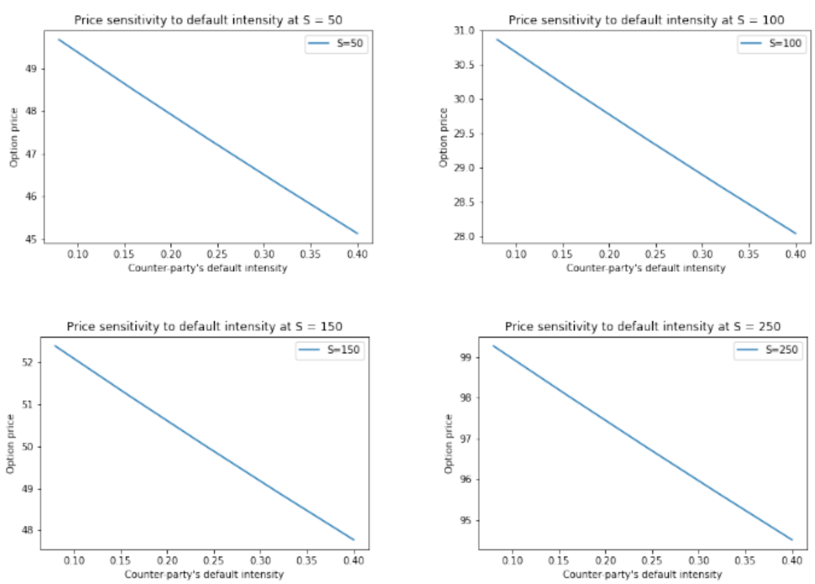

Figure 3.27: Sensitivity to counterparty's default intensity at $S_0 = 50,\ 100,\ 150,\ 200$

3.6.4 Sensitivity to loss given default

The amount Loss Given Default (LGD) when the counterparty defaults is also a measure of the credit risk, it plays an important role in our model since we assumed the counterparty for always default at $S_0 = 0$ and $S_0 = S_{\text{MAX}}$. Figure 3.28 shows the option price when the Loss Given Default

ranges from 0.2 to 0.8.
The trend is clear at ITM scenarios from the call option side where the contract price decays sharply as LGD incresases, however, this effect is less significant than simple call case. Straddles tends to be more valuable than simple call if LGD is high. Also, unlike the simple call option case, the effect of LGD is relevant also around strike level meanwhile in the call case, the prices start to diverge around $S_0 = 150$. This phenomemnon indicades straddle are more volatile to LGD, since LGD is an unobservable parameter (one cannot know it until the counterparty defaults), one can expect bigger bid-ask spreads in the market.
On the other side, as S_0 approaches to zero, the effect is mild as we discussed in section 3.4 and 3.6.2. We can clearly see the divergence of the prices around $S_0 = 200$ is much bigger than $S_0 = 0$ this is another evidence of the asymmetry mentioned in 3.4. Overall, we expect higher values of the straddle as LGD decreases.

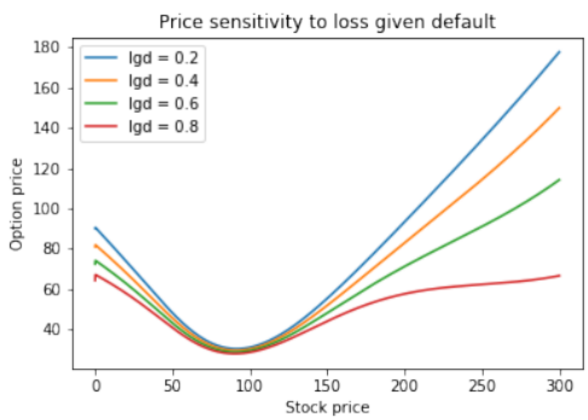

Figure 3.28: Option prices with different loss given default

The above observation is again verified by figure 3.29 where we focus on the sensitivity behaviour at specific values of $S_0 = 50,\ 100,\ 150$ and 250. We notice that the relationship starts being linear decreasing as S_0 is small and it becomes negatively convex at higher values of S_0. We can see the price reduces more than half at $S_0 = 250$, to explain why this situation happens, we can not separate the effect of LGD and increasing default intensity, the combination of these two effects is more than additive.

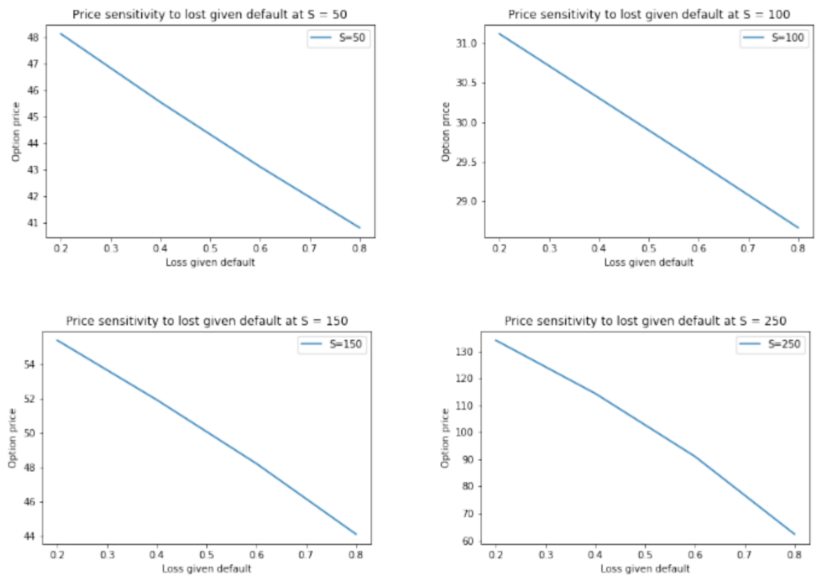

Figure 3.29: Sensitivity to loss given default at $S_0 = 50,\ 100,\ 150,\ 200$

3.6.5 Sensitivity to strike price

Strike prices is also an important parameter in option valuation theory. We explore the sensitivity of option price to the strike price by letting $K = 80$, 90, 100, 110, 120. As shown in figure 3.30, straddles with less strike price is more valuable in deep ITM scenarios for the call option as S gets large and less valuable in deep ITM scenatios for the put option as S approaches to zero. This phenomenon agrees with the nature of a straddle.

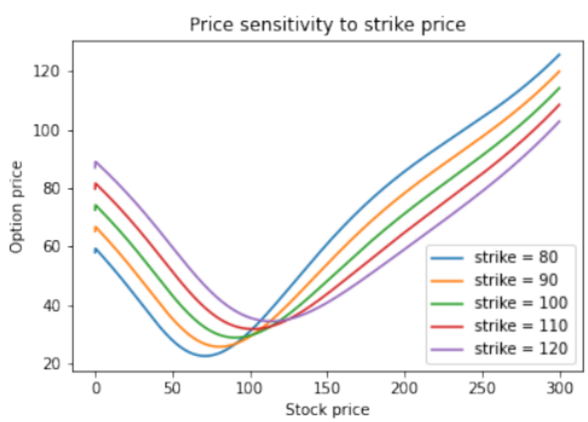

Figure 3.30: Option prices with different strike price

Also, we notice there is an upward shift, we can see the minimum of the contract value is ascending as the strike price increases. One possible reason for it is, as strike prices moves up, its effect on the put option part of straddle over-weights its effect on the call option part. Another reason could be the asymmetry of the domain $S \in [0, 300]$. Unfortunately, we can not choose $K = 150$ for the symmetry because if the contract is at the money with $K = 150$, we need to change the upper-bound S_{MAX} subsequently because the possibility of a log-normal random variable to double its value can not be neglected in this case.

Next, we now focus on the sensitivity behaviour at specific values of S_0 in figure 3.31. The asymptotic behaviour is not different from the simple call when S_0 is large as we expected. The behaviour of contract value when S_0 is around 100 is of a parabolic shape. We deduce there is a global minimum, we can potentially observe arbitrage opportunities if there is a inversion of the shape in the market when S_0 is around 100. Finally, when $S_0 = 50$ the relation is linear and increasing just like a simple put option.

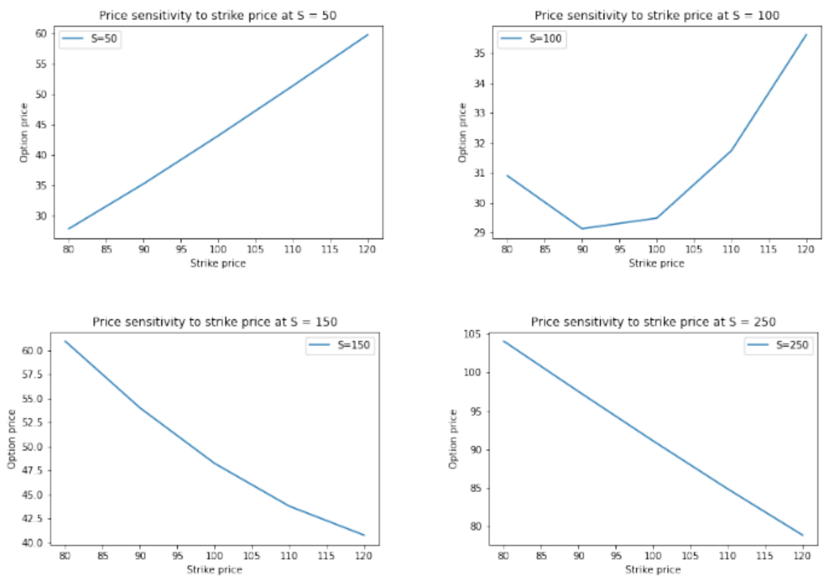

Figure 3.31: Sensitivity to strike price at $S_0 = 50,\ 100,\ 150,\ 200$

Chapter 4

Conclusion

In this dissertation, we managed to derive the pricing formulas for CVA, DVA, ColVA, FVA and HVA following the general framework done by Perini, Pallavicini, Brigo [7]. We deduced the corresponding FBSDE and PDE related to the contract's price dynamic. We discussed the existence and uniqueness of the PDE in equation 2.2.7 in two different cases: when f_t is symmetric and asymmetric with respect to borrowing/lending. We showed under some strict conditions the PDE in equation 2.2.8 has unique solution in a closed bounded domain, in a more general settings we state the existence and uniqueness theorems of viscosity solutions discussed in [19].
Next, we used finite difference method to solve numerically the PDE and delivered two case studies: simple call option and a straddle. We showed evidences of asymmetry between call and put options in the straddle case. We found out that under finite difference method, the effect of default scenario on the boundary $S_0 = 0$ fades away but remains significant on the boundary $S_0 = 300$ causing the asymmetry. Furthermore, we ran sensitivity analysis with different parameters related to credit, funding and collateral, we observed that straddles are in general more sensitive to those parameters.
There are lots of further research directions inspired by this dissertation. In the theoretical part, as stated in remark 2.4.3, finding a solution with bounded derivative to the equation 2.2.7 is crucial for the rigorous proof of lemma 2.4.2, unfortunately, this problem remains unsolved in the academia. Furthermore, we imposed very strict conditions on the funding rate f_t to be constant in time to get the existence and uniqueness of the classical solution for the pricing PDE, one might try loosing this condition to continuously differentiable functions. In the numerical part, we may choose other numerical methods such as Galerkin method [24], finite element [25] or upper-lower solutions method [26] to eliminate the drawback on boundary conditions of finite difference method.
Finally, we have not discussed much about the FBSDE derived in this dissertation, solving FBSDE numerically would be an sensible alternative to obtain the price function. In academia, solving SDEs numerically is a topic that has gained attention recently, there are lots of methods available: from classical ones such as Monte Carlo simulation to the use of Neural Networks see for example Han, Jentzen and Weinan [27].

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GRADEMARK REPORT

FINAL GRADE

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GENERAL COMMENTS

Instructor

PAGE 1

PAGE 2

PAGE 3

PAGE 4

PAGE 5

PAGE 6

PAGE 7

PAGE 8

PAGE 9

PAGE 10

PAGE 11

PAGE 12

PAGE 13

PAGE 14

PAGE 15

PAGE 16

PAGE 17

PAGE 18

PAGE 19

PAGE 20

PAGE 21

PAGE 22

PAGE 23

PAGE 24

PAGE 25

PAGE 26

PAGE 27

PAGE 28

PAGE 29

PAGE 30

PAGE 31

PAGE 32

PAGE 33

PAGE 34

PAGE 35

PAGE 36

PAGE 37

PAGE 38

PAGE 39

PAGE 40

PAGE 41

PAGE 42

PAGE 43

PAGE 44

PAGE 45

PAGE 46

PAGE 47

PAGE 48

PAGE 49

PAGE 50

PAGE 51

PAGE 52

PAGE 53

PAGE 54

PAGE 55

PAGE 56

PAGE 57

PAGE 58