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Department of Mathematics

A Jump-Diffusion Model for Credit Risk with Endogenous Contagion

Author: Niklas Walter (CID: 01791707)

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Abstract

We propose a structural credit model for large portfolios of defaultable assets. In particular, we consider that each company's asset value process follows a jump-diffusion with a non-linear drift arising from a contagion mechanism. Motivated by Bush et al. (2011), the processes are connected via global market factors, represented by a common noise and jump process. The joint density of the portfolio is described by a stochastic partial differential equation (SPDE). We derive the regarding SPDE and find an explicit expression for its solution by studying the law of a conditional McKean-Vlasov problem. Via a calibration exercise, we evaluate the contribution of the contagion for a post-crisis period.

Key words: portfolio credit modelling, contagious jump-diffusion, SPDE, structural model

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Introduction

Credit derivatives are financial instruments which help investors to secure themselves from credit risk arising from a portfolio of defaultable assets. Before the credit crisis, they have formed the second largest product class in the global over-the-counter (OTC) market. Until today the outstanding notional on these products is immense. This fact gives rise to establish mathematical models for pricing purposes and risk management. Traditionally, there are two main types of models to capture the default risk of a single obligor: reduced-form and structural models. While the first ones are widely used in the industry, the second type has the advantage to be directly linked to a economic grounding. However, for credit basket derivatives, like a CDO index, one needs a model capturing the default risk for multiple names. Following this fact, a structural model for this purpose is proposed in [2]. Here a firm's asset value is mainly driven by two diffusion processes and the dependence between the companies arises from an exposure to a common market process. This setup gets extended in [1], by adding a common compound Poisson process. Therefore, the firm values are described by jump-diffusions. Independent of this, [3] proposed to include a contagion mechanism on top of the diffusion model. The idea is that defaults within the system have a negative impact on the asset value of the remaining companies. In this thesis, we study a firm value process that consists of a jump-diffusion and a contagion. Therefore, it can be seen as part of a family of works extending the model presented in [2]. One feature that all extensions have in common is that the joint density, determining the losses of a defaultable portfolio, can be described by a stochastic partial differential equation (SPDE). In particular, the solution of the latter turns out to be the limit of the empirical measure in case that the population of the underlying system is large enough. This thesis can be distinguished into one part studying the upper SPDE for our model setup and one trying to numerically implement it. The latter aims to evaluate the benefits arising from the contagion term while matching market spreads of credit derivatives.

The thesis is structured as follows. The first chapter is of an introductory character. It clarifies the terminology of systemic and credit risk and presents how both are modelled in the state-of-the-art literature. Moreover, we give a brief overview of some credit derivatives and their regarding pricing functions. In Chapter 2, we present structural models for a single or multiple names following the presentation given in [1, Sections 3 and 4] and extend them by adding a contagion mechanism as proposed in [4]. Finally, we demonstrate how the multi-name model can be used to approximate the distances-to-default of companies considered to be particles within a finite particle system. In Chapter 3, we mainly study the case that the population size of the system tends towards infinity and derive a SPDE for the joint density of this large basket case. Moreover, we illustrate the impact of each parameter in the limiting model by plotting the density evolution for different market scenarios. In Chapter 4, we provide a short outline on how to calibrate the model so that it fits spreads of a CDO index observed in the market. In addition, we compare our results to the ones presented in [1] to analyse if the contagion mechanism provides more flexibility to match spreads especially for dates after the crisis of 2007.

Chapter 1

Systemic and Credit Risk

In this chapter, we give an introduction to systemic and credit risk. Besides stating how these notions are defined in the current literature, we also present methods to model both types of risks mathematically. Moreover, we derive pricing formulas for common credit derivatives, which we partly use later in a calibration exercise.

1.1 Systemic Risk

Following [5, Section 1.1], there are five main categories of risk faced by financial institutions, in the following called banks for simplicity:

1. **Credit Risk:** The risk of not being payed a stipulated repayment on an outstanding investment caused by the default of a counterparty on their contractual obligations.
2. **Market Risk:** The risk of an unexpected change in the value of a financial position triggered by changes in the market prices or value of the respective underlying such as a stock or index.
3. **Liquidity Risk:** The risk that adjusting financial positions in the market might impact its price since it cannot be bought or sold fast enough due to a lock of marketability.
4. **Operational Risk:** The risk of losses resulting from fraud, external events or most importantly inadequate internal processes, people and operations.
5. **Systemic Risk**: The risks stemming from the interlinkages in the financial market, where the failure of a single bank might cause a cascading failure of other market participants.

It is noteable that the first four categories above only concern banks individually, while systemic risk relates to the entire financial market. Therefore, the latter can be seen from a more macroeconomic perspective although it can have dramatic consequences for individual market members. In the following paragraph we introduce systemic risk in more detail.
In general, there are not one but many definitions of systemic risk emphasizing different aspects of the same underlying idea of this risk category. However, as stated in [6, Section 1.4] one can agree that at least the following three ingredients are essential to speak of systemic risk:

1. **Triggering Event:** This can be any event, either of internal or external nature, impairing the banks in the system. In case the trigger is caused by something outside of the system we speak of exogenous shock, like a natural disaster. On the contrary, the event is called endogenously if it arises from within the system. A failure in being able to pay back obligations due to mismanagement is one example for the latter.
2. **Propagation of stress within the system:** The shock initialised by an event as described above begins to spread through the the system of banks. Via direct or indirect interdependence between the latter, other members are affected by the initial event. This phenomenal is comparable with the contagion of a disease within in a group of humans. Due to this similarity one also speaks of financial contagion. In [6] it is written that there are four main channels for contagion within a financial system. The first, but more indirect, reason for

contagion is the asset allocation found in modern financial markets. Different banks tend to invest in common and similar assets. This leads to correlated portfolios and hence to susceptibility to similar types of shocks. However, this is not a channel for contagion in a narrow sense, it is a tinderbox for a financial market especially in combinations with the following channels. A main type of financial contagion is called default contagion. The wide range of modern financial products like swaps, derivatives and securities cross-holdings lead to many linkages between different banks. In case of the default of one bank, other banks need to write off possible interbank assets. This can cause major losses due to the disrupted promised payments from the defaulted bank. Such shocks are the channel for default contagion. As a consequence of the many links between the institutions, this can even create a default cascade, i.e. the bankruptcy of one bank can trigger the default of many others. The next channel of contagion concerns with liquidity. One speaks of funding illiquidity if a bank has not enough liquid assets to meet short term obligations towards others. As a reaction it recalls or not rolls over their issued loans. However, this will bring other banks in the same situation and so this behaviour will spread through the market. This is the channel for liquidity contagion. Lastly, the last channel for financial contagion are asset fire sales. In case some shock in the market diminishes the capital of at least one bank.It begins to sell some of its held assets to meet for example regulatory leverage constraints. However, in case the amount of sold assets is big enough, the price of the asset will drop by a simple supply and demand argument. This would not only reduce the asset's value of the affected bank but also brings other institutions in the same position. The latter is in particular boosted by the asset correlation explained above.

3. **Significant macroeconomic impact:** Lastly, one should be aware of the fact that the financial system serves several purposes for the entire economy, for example the supply of credit or the provision of liquidity in the market. Therefore, a failure of wider parts of the financial industry has also a huge impact on the non-financial economy.

The similarity between financial contagion and the contagion of a disease between humans was well illustrated in the speech [7] by the economist Haldane in 2009. He compared the financial crisis of 2007/08 with the SARS epidemic in 2002/03. He stated that both events were triggered by external event strikes. The resulting fear about the uncertain future caused panic and overreaction in the market and society respectively. In both cases, the global collateral damage was immense and demonstrated how a "flap of a butterfly's wing" can trigger a wide spread of the initially local shock. He continues by explaining that both phenomenal were the result of the behaviour of a stressed complex network web with a big amount of interconnections. A similar analogy could be observed concerning the current COVID-19 epidemic that caused even more dramatic reactions than the similar SARS virus. However, at today's state of knowledge, one can not say if the regarding reactions from politics and economy can be seen as an overreaction. Nevertheless, it is observable how the growth of uncertainty in the market goes along with the spread of the disease. Events, like the crises described above, demonstrate how important the understanding of contagion dynamics and external shocks is. However, before we present three main streams of research in the field of systemic risk, it is important to highlight the difference between the latter and the often confused systematic risk. While systemic risk is the risk that the failure of a single or a group of market members can cause negative effects on the entire system, systematic risk can be seen similar to market risk. It describes how market instability can cause losses to for example certain investments. One can replace the terminology with the more common one of the volatility in the market. Besides the underlying fluctuations due to trading activity, it also incorporates macroeconomic effects like recession or inflation.

1.2 Approaches to model Systemic Risk

As already mentioned, the mortgage crisis of 2007/08 demonstrated the importance of the mathematical understanding of complex financial networks. In this section, we briefly highlight the existing research on this field. One can distinguish between three main lines of approaches to model systemic risk within a system of banks. The first and probably oldest line of research in this field is based on the concept of quantitative risk measures like the value-at-risk (VaR) and expected-shortfall (ES). As an review on monetary risk measures we refer to [8, Chapter 6] or

[9, Chapter 4]. The VaR, as the most famous measure of risk used by banks, only focuses on individual institutions ignoring its interdependence with others in the financial system. In [10], the conditional value-at-risk (CoVaR) is introduced. As the name indicates, the CoVaR is defined as the VaR of some bank i given that a certain shock occurred to another bank j. Therefore, this measure does not look at banks as isolated members of the financial market but incorporates their interlinkages. While the CoVaR applies the concept of the classic VaR, one can also introduce systemic risk measure relying on the ES. To state examples, the marginal expected-shortfall (MES) and systemic expected-shortfall (SES) are defined in [11]. To understand the first one, we consider the classical ES of the entire banking system. The MES is given by the partial derivative of the ES with respect to the exposure of a single bank. Therefore, it can be seen as the expected short-term losses of the bank given that the market takes a loss greater then its VaR. Related to this, the SES is the penchant of a single bank to hold too less capital in case the system itself is undercapitalised. Hence, it can be seen as a measure for a bank's contribution to the overall systemic risk. Besides the techniques inspired by classic monetary risk measures, there is also a more axiomatic approach, which is for instance described in [12]. The method involves considering a multi-dimensional random variable representing certain risk factors within the financial system. Via aggregation functions those different factors are summarised in a single univariate variable. Measuring the risk in the system then reduces to apply a so-called base risk measure which can be seen as measure for a single institution. Therefore, the idea is to aggregate the certain factors so that the entire system can be treated as a single representative bank. A similar approach is also presented in [13].

The second main approach to understand systemic risk and contagion incorporates models based on a network structure. The financial system containing several banks can be considered as a graph whose nodes represent the institutions and the edges their connections established by different types of contractual obligations. The probably most famous model was introduced by Eisenberg and Noe in [14]. It represents all liabilities between the nodes in the system by a so called nominal liability matrix L. Moreover, it considers exogenous operating cash flows for each institution represented by a vector e. Those two quantities are sufficient to characterise the financial system, denoted by (L, e). By adding a clearing vector the model can be used to understand the consequences of defaults of banks and the dynamics of the clearing process. For further reading, we refer to [6] or the study in [15]. The main drawbacks of network-based models are that they are in general static and one needs data of the entire financial system considered to calibrate them.

Those two disadvantages can be avoided by using the third approach of modelling systemic risk, dynamic continuous-time mean fields. One of the illustrative works, using a mean-field limit to establish a model of lending and borrowing banks, is given by [16]. They represent the log-capital of several banks by a diffusion process respectively. Correlations between the institutions are modeled using correlated Brownian motions driving the diffusion processes. Fouque and Sun introduce dynamics, so that the log-capital processes converge to independent Ornstein-Uhlenbeck processes, in case the number of banks in the system tends to infinity. This model is studied in more detail in [17]. In contrary to the network-based models, mean field models offer greater dynamics and stochastic structure. However, they often do not incorporate the interlinkages between the banks in the detail that network models do. Nevertheless, this can be an advantage when it comes to calibrating the model to data. Due to considering the mean field limit, aggregated data is often sufficient. The model introduced in this thesis can be seen as mean field model, build on a structural mechanism for the banks' default. The structural approach is often used in credit risk models and can be seen as the main alternative to intensity models. In section 1.4, we give a short introduction to the idea behind structural models.

1.3 Credit Risk

Following the explanations in [18, Chapter 1], credit risk is any risk associated with all kind of events linked to credit. These includes changes in the credit rating or quality of an institution, variations of credit spreads and the risk that a default event happens. Moreover, it can be distinguished into the two subtypes *reference credit risk* and *counterparty credit risk*. The first one considers a contract between two default-free entities which depends on the default event of a third reference party not being an active part of the contract. One example for such a contract is a credit default swap (CDS) between two not defaultable parties. This type of financial agreement will be introduced in more detail in one of the following sections. On the other hand, counterparty credit risk plays an important role within the OTC market. Contracts traded this way are not backed by any clearinghouse or public exchange. Therefore, in case that both contract parties are defaultable, they are exposed to the risk that the respective counterparty defaults and is not able to meet its obligations. This uncertainty is often overcome by some form of collateral agreement. Nevertheless, to assess the fair value of any contract it is essential to quantify the default risk, which is exactly the risk that a counterparty can not meet its contractural obligations. Following [19, Page 1], credit risk is defined as "the risk that an obligor does not honour his payment obligations". Note that here the idea of default and credit risk coincide. Moreover, one can distinguish between three main components of credit risk:

1. **Arrival Risk:** This risk describes the uncertainty whether a default will occur to a counterparty or a reference entity or not. This risk is quantified by the probability of a default. By denoting the default time of interest by τ and considering a contract of time horizon T, the probability of default within the contract life time is given by $\mathbb{P}(\tau < T)$.

2. **Timing Risk:** In contrast to the arrival risk, it captures not only whether or not a default happens but the uncertainty about when precisely the latter occurs. Therefore, one is now interested in $\mathbb{P}(\tau \in [t_{k-1}, t_k])$ for any partition $0 = t_0,\ t_1, ...,\ t_{n-1}, t_n = T$. Note that those first two components of credit risk are described by the probability density function of the time of default.

3. **Recovery Risk:** It describes the unknown severity of the loss in case of a default. The latter may depend on the type of default or possible agreements within a contract. Conventionally, in many contracts, which are exposed to default risk, a so called recovery rate (REC) is defined. This quantity is expressed by a fraction of the contract's notional value which is paid to the creditor in case of a default. For example, if one assumes REC=40%, the creditor receives 40% of the outstanding payment by the defaulting counterparty. Since the recovery can vary depending on the market situation, the recovery risk is measured by the conditional distribution of the recovery rate, i.e. by $\mathbb{P}(\text{REC} = x \mid \tau < T)$.

In this thesis, we will focus less on this third component but much more on the occurrence of default events and its consequences for a system of different individual banks. In the following section, we describe one of the main ideas of modelling default events.

1.4 Structural Models

The models used to describe default events can be mainly distinguished into intensity and structural models. The first type does not explain the reason for an occurring default, whereas the second focuses more on studying why a bank or firm in general defaults. The model proposed in this thesis belongs to the class of (multi-dimensional) structural models. Therefore, we only give an introduction to this family of models. For an overview of intensity models or in general a more detailed survey we refer to [18], [19] or [20]. The following statements are mainly based on the summary given in [21, Section 1.2]. The idea behind this kind of models is that a company defaults in case its asset value falls below a certain default barrier. Merton assumes that a firm's life is linked to its ability to repay obligations to counterparties. Therefore, he considers that default can only happen at a terminal time point $T \geq 0$, e.g. the maturity of an issued bond. Until today many structural models are inspired by this original idea. In the basic model [22], the asset value of a company is described by a stochastic process $(A_t)_{t\geq 0}$ described by

$$dA_t = \mu A_t dt + \sigma A_t dW_t,$$

where μ is the mean rate of returns, σ the asset's volatility and W is a standard Brownian motion. Under the risk neutral measure it is assumed that the mean of the returns equals the risk-free rate r. Therefore, the risk-neutral dynamics are given by

$$dA_t = rA_t dt + \sigma A_t dW_t.$$

As already mentioned, defaults can only occur at time T in case the terminal asset value hits a certain barrier level D. It follows that the time of default, denoted by τ, is defined by

$$\tau = \begin{cases} T \text{ , if } A_T \leq D \\ \infty \text{ , otherwise.} \end{cases}$$

By a simple application of Itô's formula, we get the well-know explicit expression

$$A_T = A_0 \cdot \exp\left\{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right\}.$$

Therefore, it follows easily that the probability of default under the risk-neutral setting is given by

$$\mathbb{P}(A_T \leq D) = \Phi\left(\frac{\log\left(\frac{D}{A_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

where $\Phi(\cdot)$ is the standard normal distribution function. To lose the dependence on σ and D, it is common to introduce the distance-to-default notion

$$X_t := \frac{1}{\sigma}(\log(A_t) - \log(D)).$$

In particular, we get

$$X_t = \frac{1}{\sigma}\left(\log(A_0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t - \log(D)\right) = X_0 + \beta t + W_t,$$

where $\beta := \frac{1}{\sigma}(r - \frac{\sigma}{2})$ and we used the fact that $X_0 = \frac{1}{\sigma}(\log(A_0) - \log(D))$. Using the notion of the distance-of-default, we can rewrite the time of default as

$$\tau = \begin{cases} T \text{ , if } X_T \leq 0 \\ \infty \text{ , otherwise} \end{cases}$$

with new probability of default

$$\mathbb{P}(X_T \leq 0) = \Phi\left(\frac{-X_0 - \beta T}{\sqrt{T}}\right).$$

In 1976, Black and Cox proposed in [23] a variant of Merton's model to incorporate defaults not only at maturity time. Their approach is based on the theory of passage times. The idea is that we are not only interested in the terminal distance-to-default X_T, but on the quantity over a whole period. In other words, the time of default considered now is given by

$$\tau = \inf\{t > 0 : X_t = 0\}.$$

The infimum indicates that the default happens at the first passage time of the distance-to-default at the origin. Clearly, one big advantage of structural models and especially Merton's model is that the assumption and the underlying dynamics allow to asses the value of the assets and credit risk in general. In particular, as shown above, there exists a range of quantities which can be computed by a closed-form formula. However, there are some disadvantages to the Merton model. One main is that it only considers defaults to happen at a terminal time point T. This problem was solved by Black and Cox as presented above. Nevertheless, both models assume a continuous process to model the asset value of a firm. Our model tries to overcome this shortcoming by adding jump processes to the underlying process. Inspired by [1], we consider a Poisson process in addition to the Brownian motion. Lastly, one disadvantage of structural model in general is that they do not capture the capital structure of a bank in detail. For example, the debts are often considered to be constant or follow a specific evolution in time.

1.5 Credit Derivatives

Being interested in default events and probabilities, we now give a brief introduction on credit derivatives. They are financial instruments which transfer credit risk from a protection buyer to a protection seller. The contracts can be either based on a single third reference entity or a pool of defaultable assets. Credit default swaps (CDS) are possibly the most popular representative of the first group. As stated in [21], they are designed to offer protection against the default of a reference entity in exchange for a periodically paid premium. While a CDS depends only on a single name, collateralized debt obligations (CDOs) are based on a basket of names. In the following subsections, we briefly state the most important characteristics of those two products and introduce pricing formulas. The subsequent explanations are inspired by [2], [21], [24] and [25].

1.5.1 Credit Default Swaps (CDS)

By purchasing a CDS, the buyer ensures that the regarding seller of the contract compensates the loss given the default of a third entity. Therefore, this agreement involves three different parties, while one of them is not an active part but only acts as a reference on whose possible default the contract depends on. In exchange for the insurance against possible losses, the buyer of the CDS has to periodically pay a fee, called CDS spread and denoted by S_{CDS}, to the initiator of the agreement. The payments are made until maturity of the contract or the actual default of the reference firm. If the latter occurs, the seller has to pay the loss-given-default LGD to the buyer. Assuming a unit notional, we have that $LGD = 1 - REC$. Following the notation in [21, Section 3.4.1], we denote by T_a the starting time of the contract and T_b the maturity. Moreover, from now on T_i with $i = a+1, ..., b$ are the payment dates for the CDS spread. The payment intervals are given by $\Delta_j := T_j - T_{j-1}$ and the value of the bank account at time t is denoted by $b(t)$. Furthermore, we assume that the risk-free interest rate r and the recovery rate REC are constant and that trading is frictionless and continuous in time. In this case, we can write the discounted running cashflow at initial time $T_a = 0$ seen from the payer side of the CDS as

$$\Pi_{a,b} = LGD \sum_{i=a+1}^{b} \frac{1}{b(T_i)} \mathbb{1}_{\{T_{i-1} < \tau \leq T_i\}} - S_{CDS} \sum_{i=a+1}^{b} \frac{\Delta_i}{b(T_i)} \mathbb{1}_{\{\tau > t\}}.$$

The first term is called the *protection leg* of the CDS and the second one *fee leg*. To get the fair value of both legs respectively, we need to take expectation under an appropriate pricing measure $\mathbb{P}$. Hence, for the fee leg of the CDS it follows that its value is given by

$$V_{\text{fee-leg}} = S_{CDS} \sum_{i=a+1}^{b} \frac{\Delta_i}{b(T_i)} \mathbb{P}(\tau > t),$$

where the quantity $\mathbb{P}(\tau > t)$ is called the survival probability of the reference entity under the pricing measure $\mathbb{P}$. On the other side, it follows similarly for the protection leg that

$$V_{\text{protection-leg}} = LGD \sum_{i=a+1}^{b} \frac{1}{b(T_i)} \mathbb{P}(T_{i-1} < \tau \leq T_i).$$

Since we know that at time $T_a = 0$ the fair price of the contract must ensure that the value of both legs must be equal, we can follow that the fair CDS spread is given by

$$S_{CDS} = \frac{LGD \sum_{i=a+1}^{b} \frac{1}{b(T_i)} \mathbb{P}(T_{i-1} < \tau \leq T_i)}{\sum_{i=a+1}^{b} \frac{\Delta_i}{b(T_i)} \mathbb{P}(\tau > t)}. \tag{1.1}$$

1.5.2 Colletarlized Debt Obligation (CDO)

A CDO is a financial instrument consisting of fixed-income securities whose payments depend on credit events within a portfolio of defaultable assets. A portfolio can consist, for example, of loans, mortgages or CDSs. In this thesis, we focus on a so-called synthetic CDO, whose underlying pool of assets only consists of CDSs. The seller of such a CDO chooses the portfolio of companies and a maturity for the product beforehand. Easily spoken, the originator sells CDSs on each company within the portfolio, where each CDS has the same maturity as the CDO itself. The principal of the latter equals the sum of the individual CDS principals. The seller of this product has cash inflows in form of regular payments, called CDO spreads. On the other hand, there are cash outflows in case a company in the underlying portfolio defaults. A CDO is structured in a way so that different risk profiles are offered to possible investors. The choice of the riskiness of an investment then depends on the individual risk appetite. The risk profiles are called tranches. In general, one can distinguish between three main categories of tranches, which then may be divided into several subcategories. The tranche connected to the most risk is often called *equity tranche*. The underlying assets are often unrated or at least highly speculative. The middle risk profile is called *mezzanine tranche*, which is of investment-grade. The lowest risk is summarised in the *senior tranche*, whose underlying asstes can have a AAA rating. The different tranches are characterised by an attachment point a and detachment point $d > a$. Normally, those quantities refer to percentages of the underlying portfolio notional. The liabilities for the cash outflows depend on the seniority of the different tranches. In case of a default in the company portfolio, the CDO originator needs to make insurance payments to the counterparties of the respective CDSs. If the amount of these payments with respect to the CDO principal lie beneath the detachment point of the equity tranche, only investors of this risk profile are responsible. In case it is so high that it is between the attachment and detachment point of the mezzanine tranche, the payments are carried out on the equity tranche entirely and partly one the mezzanine tranche. This liability rule is called waterfall and is characteristic for many asset-backed securities. An investor investing into a specific tranche of the CDO periodically pays a spread, denoted by S_{CDO}, on the outstanding notional of the tranche.

Following [1] and [2], we introduce pricing formulas for a single CDO tranche and a CDO index. For the latter we consider an underlying portfolio consisting of N CDSs. Each CDS has a notional of $N_0 = 1/N$. At any time t the outstanding notional of the portfolio is given by

$$\widetilde{N}_t = N_0 \sum_{i=1}^{N} \mathbb{1}_{\{\tau^i > t\}},$$

where τ_i denotes the default time of the i-th reference entity. We denote the spread of the CDO index by S_{IND}. Using the notations introduced in the last section about CDSs, we can write the discounted payoff at initial time $T_a = 0$ from the view of the CDO originator as

$$\Pi_{a,b} = S_{IND} \sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} \widetilde{N}_{T_j} - (1 - REC) \sum_{j=a+1}^{b} \frac{1}{b(T_j)} (\widetilde{N}_{T_{j-1}} - \widetilde{N}_{T_j}),$$

where we assumed that the recovery rates of the portfolio companies are the same and constant. We can call the first term again fee leg and the second protection leg. By taking risk-neutral expectation we obtain the fair value of both legs respectively. For the fee leg we get

$$V_{\text{fee-leg}} = S_{IND} \sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} \mathbb{E}\left[\widetilde{N}_{T_j}\right] = S_{IND} \cdot N_0 \sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} \sum_{i=1}^{N} \mathbb{P}(\tau_i < T_j),$$

where $\mathbb{P}(\tau_i < T_j)$ is the probability of company i to survive the payment date T_j. Similarly, we get for the protection leg

$$V_{\text{protection-leg}} = (1 - REC) \sum_{j=a+1}^{b} \frac{1}{b(T_j)} \mathbb{E}\left[\widetilde{N}_{T_{j-1}} - \widetilde{N}_{T_j}\right].$$

Again using the fact that at time 0 both legs need to be of the same fair value, we finally get the CDO index spread

$$S_{IND} = \frac{(1 - REC) \sum_{j=a+1}^{b} \frac{1}{b(T_j)} \mathbb{E}\left[\widetilde{N}_{T_{j-1}} - \widetilde{N}_{T_j}\right]}{\sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} \mathbb{E}\left[\widetilde{N}_{T_j}\right]}. \tag{1.2}$$

In the next step, we want to present a formula for a single CDO tranche. Here we assume that the recovery rate may differ between the firms in the portfolio. First, we define the total loss process at time t of the entire portfolio by

$$L_t := \sum_{i=1}^{N} L_i \mathbb{1}_{\{\tau_i < t\}}, \tag{1.3}$$

where $L_i = N_0(1 - REC_i)$ for REC_i being the recovery rate of entity i. The outstanding notional of the tranche is then given by

$$Z_t := [d - L_t]^+ - [a - L_t]^+ \tag{1.4}$$

and the tranche loss as

$$Y_t := [L_t - a]^+ - [L_t - d]^+.$$

Using those quantities, we get that the discounted cashflows seen from the CDO originator's view is given by

$$\Pi_{a,b} = S_{CDS} \sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} Z_{T_j} - \sum_{j=a+1}^{b} \frac{1}{b(T_j)} (Z_{t_{j-1}} - Z_{T_j}).$$

By taking risk-neutral expectation, we can identify the fair value of the fee and protection leg respectively. For the first we obtain

$$V_{\text{fee-leg}} = S_{CDS} \sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} \mathbb{E}[Z_{T_j}]$$

and for the protection

$$V_{\text{protection-leg}} = \sum_{j=a+1}^{b} \frac{1}{b(T_j)} \mathbb{E}\left[Z_{t_{j-1}} - Z_{T_j}\right].$$

Again using the fact that the par spread makes the CDO fair at the initial time 0 finally gives

$$S_{CDO} = \frac{\sum_{j=a+1}^{b} \frac{1}{b(T_j)} \mathbb{E}\left[Z_{t_{j-1}} - Z_{T_j}\right]}{\sum_{j=a+1}^{b} \frac{\Delta_j}{b(T_j)} \mathbb{E}[Z_{T_j}]}. \tag{1.5}$$

This shows that to find the par spread of the tranche of a CDO, it is essential to know the distribution of the outstanding notional of the tranche, defined in (1.4). The formula demonstrates that this boils down to work out the distribution of the loss process L as given in (1.3). This motivates that the model introduced in this thesis should be used to model the distribution of this process.

Chapter 2

Extending Merton's Model

In this chapter, we introduce the main model approaches used in this thesis. In the first section, we state the dynamics of the asset value of a single name. For this purpose, we use the jump-diffusion model introduced by Merton in 1976. Although we model the assets by a geometric Brownian motion like in the classical Black-Scholes framework, we introduce a jump process, represented by a Poisson process with normally distributed jump heights. Expanding the study of one single bank, we consider a system of banks in the following section. To incorporate the interdependence between the institution, established for example by borrowing and lending, we include a dependence structure by introducing a system-wide noise on top of the idiosyncratic driver, which only influences the individual banks respectively. Moreover, we introduce a contagion mechanism as proposed in [4]. Lastly, we derive a notion of the distance-to-default for each company in the system. Those quantities are finally expressed in form of a finite particle system.

2.1 The Single Name Structural Model

In this section, we introduce Merton's approach to model the asset value of a single bank. For this we follow the explanations given in [1, Section3]. In the following, let the process $(A_t)_{t\geq 0}$ describe the evolution of the latter over time. We assume that under the real world measure it follows the dynamics

$$dA_t = (\mu - \lambda\nu)A_t dt + \sigma A_t dW_t + A_{t\_} dJ_t, \tag{2.1}$$

where $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and $(J_t)_{t\geq 0}$ a compound Poisson process of the form

$$J_t := \sum_{k=1}^{N_t} (Y_k - 1). \tag{2.2}$$

with $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $\nu := \mathbb{E}[Y_1 - 1]$. In the upper definition, the process $(N_t)_{t\geq 0}$ denotes a Poisson jump process with intensity $\lambda > 0$ and $Y_1, Y_2, ...$ are i.i.d random variables denoting the jump heights of the compound Poisson process. Following Merton, we assume the jump height to be log-normally distributed, i.e.

$$\log(Y_k) \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \text{ for all } k = 1, 2, 3, ...$$

Since we are especially interested in negative jumps, one can assume that $\mu_Y < 0$. One objective of this thesis is to give a model, which can be used to price some credit derivatives. Therefore, we need to establish a risk neutral setting. While the existence of an equivalent martingale measure $\mathbb{P}$ follows directly from Girsanov's theorem, the market is not complete. In other words there exists a range of possible choices for the measure $\mathbb{P}$. In fact, we get by a direct computation that

$$\begin{aligned} d(e^{-rt}A_t) &= e^{-rt}(\mu - r)A_t dt + e^{-rt}\sigma A_t dW_t + e^{-rt}A_{t\_}(dJ_t - \lambda\nu dt) \\ &= e^{-rt}(\mu - r - \lambda\nu)A_t dt + e^{-rt}\sigma A_t(dW_t^{\mathbb{P}} - \theta dt) + e^{-rt}A_{t\_}dJ_t \\ &= e^{-rt}(\mu - r - \lambda\nu - \sigma\theta + \lambda^{\mathbb{P}}\nu^{\mathbb{P}})A_t dt + e^{-rt}\sigma A_t dW_t^{\mathbb{P}} + e^{-rt}A_{t\_}d(J_t - \lambda^{\mathbb{P}}\nu^{\mathbb{P}}), \end{aligned}$$

where θ is such that

$$W_t^{\mathbb{P}} := W_t + \theta t \ , \ t \geq 0$$

is a $\mathbb{P}$-Brownian motion by Girsanov's theorem, $\lambda^{\mathbb{P}}$ is the new arrival rate of the jump process and $\nu^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[Y_1 - 1]$. Hence, we need

$$\mu - r - \lambda\nu - \sigma\theta + \lambda^{\mathbb{P}}\nu^{\mathbb{P}} = 0$$

so that the discounted asset value is a martingale under $\mathbb{P}$. Following Merton, we choose $\lambda^{\mathbb{P}} = \lambda$ and $\nu^{\mathbb{P}} = \nu$ and therefore obtain the market price of risk by

$$\theta = \frac{\mu - r}{\sigma}.$$

In particular, we get that the dynamics of the process $(A_t)_{t\geq 0}$ under $\mathbb{P}$ are given by

$$dA_t = (r - \lambda\nu)A_t dt + \sigma A_t dW_t + A_{t\_} dJ_t. \tag{2.3}$$

In the next step, we would like to find an explicit expression for the asset value at a given time t. Before we do that, we introduce the jump measure corresponding to a jump process $(J_t)_{t\geq 0}$ by

$$\mathcal{P}_J(\omega, \cdot, \cdot) := \sum_{t\geq 0}^{\Delta J_t \neq 0} \delta_{(t, \Delta J_t)}, \tag{2.4}$$

where δ_x denotes the Dirac measure centered around x. Intuitively, for any measureable set $B \subset \mathbb{R}$ we have that the quantity $\mathcal{P}_J([0,t], B)$ gives the number of jumps of J occurring between time 0 and t with a jump height lying in B. Before applying Itô's formula for jump-diffusion processes to the logarithmic asset value process, we first note that

$$dA_t^c := (r - \lambda\nu)A_t dt + \sigma A_t dW_t$$

denotes the continuous part of the asset dynamics and on the contrary

$$dA_t^d := A_{t\_} dJ_t$$

the discontinuous one. So we finally obtain by Itô's formula, incorporating the discontinuity arising from the jumping part, that

$$\log(A_t) = \log(A_0) + \left(r - \lambda\nu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \int_0^t \int_{\mathbb{R}} \log(A_{s\_} + x) - \log(A_{s\_})\mathcal{P}_J(ds, dx),$$

which can be rewritten as

$$\log(A_t) = \log(A_0) + \left(r - \lambda\nu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{k=1}^{N_t} \log(A_{t\_} + (Y_k - 1)A_{t\_}) - \log(A_{t\_}).$$

By rearranging and using the basic logarithmic laws, this finally leads to the explicit expression

$$A_t = A_0 \exp\left\{\left(r - \lambda\nu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\} \prod_{k=1}^{N_t} Y_k. \tag{2.5}$$

The last expression indicates that the change of the asset value in case of a jump is meant in a multiplicative sense and hence is a relative change in the value. Let from now on $T_k \geq 0$ denote the time of the k-th jump of the process. Then we indeed observe that for $k = 1, 2, 3, ...$

$$\frac{A_{T_k}}{A_{T_{k\_}}} = Y_k.$$

2.2 The Multi-Name Structural Model

Extending the setup of only considering the asset of a single bank, we now consider a system of N banks, as in [1, Section 4]. The asset value of bank $1 \leq i \leq N$ is described by the process $(A_t^i)_{t\geq 0}$. While we keep consistent with the dynamics (2.3) of one individual bank, we need to incorporate some dependence structure within the system. We now assume that the process A^i can be described by

$$dA_t^i = (r - \lambda\nu)A_t dt + \sigma A_t d\widetilde{W}_t^i + A_{t\_} dJ_t^0,$$

where we consider the Brownian motion to be defined as a sum with the aim to establish correlation using the idea of a one-factor model. In particular, we define

$$\widetilde{W}_t^i := \rho W_t^0 + \sqrt{1-\rho^2} W_t^i \ , \ t \geq 0,$$

where the processes W^0 and W^i are independent standard Brownian motion respectively and $\rho \in [0, 1)$ a parameter capturing the correlation. Moreover, we have that J^0 is of the form

$$J_t^0 = \sum_{k=1}^{N_t} (Y_k - 1) \ , \ t \geq 0$$

with again N denoting a Poisson process with respective intensity rate λ and $Y_1, Y_2, ...$ i.i.d. random variables with $\log(Y_1) \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Note that the correlation between the processes is established by the dependence on the common jump process J^0. To highlight the dependence structure between the different asset value processes, we from now on consider the following dynamics under the martingale measure $\mathbb{P}$

$$dA_t^i = (r - \lambda\nu) A_t dt + \sigma A_t \left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t^i\right) + A_{t\_} dJ_t^0. \tag{2.6}$$

Note that the martingale property is achieved by the fact that the compound Poisson process is compensated within the drift term. Indeed, it holds by Wald's equation that

$$\mathbb{E}\left[J_t^0\right] = \mathbb{E}\left[\sum_{k=1}^{N_t} (Y_k - 1)\right] = \mathtt{E}\left[N_t\right] \mathbb{E}[Y_1 - 1] = \lambda t \mathbb{E}[Y_1 - 1] =: \lambda t \nu.$$

The processes W^i is a idiosyncratic factor, which has effects on each bank's asset values individually. It is used to model the more gradual effects on the asset, like the management of the regarding bank. Similarly, the factors W^0 and J^0 represent system-wide developments. Examples for continuous underlying factors are unemployment rates or the general economic health, for instance indicated by the S&P500. On the other hand, political decisions or pandemics may cause a sudden drop in the asset value of all banks in the system. The intensity of N, λ, can be seen as measuring the frequency of such economy-wide shocks. In a later section, we introduce a possible extension to the model which considers idiosyncratic jumps.

2.3 Introducing a Contagion Mechanism

Following [4, Section 1.3], one main objective of this thesis is to incorporate contagion within the system of banks. To do this, we first need to define a method to capture how much influence a single bank has on the entire system, i.e. to measure its systemic importance. For that we assign a weight to each institution. The weights are defined by

$$a_i^N := \frac{a_i(X_0^i)}{\sum_{j=1}^N a_j(X_0^j)}, \tag{2.7}$$

for $i = 1, ..., N$. The parameters $a_i(\cdot)$ are assumed to be bounded by some positive constants $c, C > 0$ so that $c < a_i(\cdot) < C$. Moreover, as already implied by their definition they depend on a quantity X_0^i, which indicated the healthiness of bank i at the initial time of the studied time period. Trivially, we observe that $\sum_{i=1}^N a_i^N = 1$, which shows that the importance should be understood in a relative sense. Lastly, we get by the constant bounds that

$$a_i^N = \frac{a_i(X_0^i)}{\sum_{j=1}^N a_j(X_0^j)} \leq \frac{C}{cN} = \mathcal{O}\left(\frac{1}{N}\right), \text{ as } N \to +\infty.$$

This behavior demonstrates that by construction not a single bank alone can have too much impact in the entire system, in case the amount of institutions is big enough.
In the following, we introduce a contagion mechanism to model the (negative) effect of a bank's default on others in the financial system. Instead of assuming this possible loss in form of jumps, we consider a gradual loss occurring within a short time period after the default of lets say bank j. In order to model the magnitude of loss faced by a bank $i \neq j$, we use the relative weight a_j^N, displaying the relative importance, and a new constant contagion parameter $\alpha \geq 0$. The gradual realisation of the loss is modelled by the process

$$\mathcal{L}_t^{j,N} := \int_0^t k(t-s)\mathbb{1}_{\{s \geq \tau_j\}} ds, \tag{2.8}$$

where the functional $k \in \mathcal{C}^\infty(\mathbb{R}_0^+)$ is assumed to satisfy $\|k\|_1 = 1$. This last property demonstrates the density-like behavior of k. Due to this fact, we refer to the latter as the *impact kernel* from now on. The monetary loss in the asset value is modelled by means of discounting. In particular, the asset value before the default gets updated by the rule

$$A_\cdot^i \mapsto \hat{A}_\cdot^i := \exp\left\{-a_j^N \int_0^\cdot \alpha d\mathcal{L}_s^{j,N}\right\} A_\cdot^i \ , \ i \neq j. \tag{2.9}$$

To better understand this gradual discounting, we present two special cases in the following remarks.

Remark 2.1. Assume that $t < \tau_j$, i.e. the default of bank j has not occurred yet. Trivially, we see that the indicator inside the loss process at time t is always zero for all $s \in [0,t]$ and so the impact on the other banks is also zero.

Remark 2.2. Now assume that $t \geq \tau_j + \epsilon$ for an arbitrary small $\epsilon > 0$. Moreover, we assume that $\text{supp}(k) = [0, \epsilon]$, i.e. the impact kernel only "lives" for a short period of time. Then we observe that

$$\mathcal{L}_t^{j,N} = \int_0^t k(t-s)\mathbb{1}_{\{s \geq \tau_j\}} ds = \int_{\tau_j}^t k(t-s) ds.$$

Moreover, we observe that for all $s < t - \epsilon$ we have $t - s > \epsilon$. Therefore, for all such s the functional k is zero due to its support. It follows that

$$\int_{\tau_j}^{t-\epsilon} k(t-s) ds = 0$$

and so by change of variable to $r := t - s$, we obtain

$$\mathcal{L}_t^{j,N} = \int_{t-\epsilon}^t k(t-s) ds = \int_0^\epsilon k(r) dr = 1.$$

The last equation follows from the assumption that $\|k\|_1 = 1$. Hence, regarding the discounted asset value at time $t = \tau_j + \epsilon$ we get

$$\hat{A}_t^i = \exp\left\{-a_j^N \int_0^t \alpha d\mathcal{L}_s^{j,N}\right\} A_t^i = \exp\left\{-\alpha a_j^N\right\} A_t^i.$$

In the case that N is large enough, we have that a_j^N is small due to the fact that $a_j^N = \mathcal{O}\left(\frac{1}{N}\right)$. Therefore, using a Taylor expansion of the exponential functional, we get that $\exp\left\{-\alpha a_j^N\right\} \approx \left(1 - \alpha a_j^N\right)$, which finally leads to

$$\exp\left\{-\alpha a_j^N\right\} A_t^i \approx \left(1 - \alpha a_j^N\right) A_t^i. \tag{2.10}$$

This illustrates that in case that the impact kernel restricts the negative impact of another bank's default to a short period after the regarding default, we have that by time $\tau_j + \epsilon$ the asset value of bank $i \neq j$ is lowered by the proportion αa_j^N from its initial value.

$\triangle$

As time passes by, more banks may default. Therefore, by continuing to apply the discounting procedure, we get in general for $i = 1, ..., N$ that

$$\hat{A}_t^i = \prod_{j \neq i} \exp\left\{-a_j^N \int_0^t \alpha d\hat{\mathcal{L}}_s^{j,N}\right\} = \exp\left\{-\sum_{j \neq i} a_j^N \int_0^t \alpha d\hat{\mathcal{L}}_s^{j,N}\right\}, \tag{2.11}$$

where $\hat{\mathcal{L}}_t^{j,N} = 0$ in case that bank j has not defaulted by time t and the updated process

$$\hat{\mathcal{L}}_t^{j,N} := \int_0^t k(t-s) \mathbb{1}_{\{s \geq \hat{\tau}_j\}} ds$$

with the new defined time of default

$$\hat{\tau}_j := \inf\left\{t > 0 \, : \, X_t^j \leq 0\right\},$$

where we have used the distance-to-default

$$X_t^j = \log(\hat{A}_t^j) - \log(D_t^j).$$

In the next section, we put emphasize on the resulting model for the distance-to-default process X^j. Before we do this, we introduce the *loss process* as

$$L_\cdot^N := \sum_{j=1}^N a_j^N \mathbb{1}_{\{\cdot \geq \tau_j\}}. \tag{2.12}$$

Evaluating this process at time t gives the proportion of banks in the system, which have already defaulted by time t and hence are considered to be dropped out of the system. We can use L_t^N to rewrite the discounted asset value of bank i for $t \leq \hat{\tau}_i$ as

$$\begin{aligned}
\hat{A}_t^i &= \exp\left\{-\sum_{j \neq i} a_j^N \int_0^t \alpha d\hat{\mathcal{L}}_s^{j,N}\right\} A_t^i = \exp\left\{-a_j^N \int_0^t \sum_{j \neq i} \alpha d\hat{\mathcal{L}}_s^{j,N}\right\} A_t^i \\
&= \exp\left\{-\int_0^t \alpha d\left(\int_0^t k(t-s) \sum_{j=1}^N a_j^N \mathbb{1}_{\{t \geq \tau_j\}} ds\right)\right\} A_t^i \\
&= \exp\left\{-\int_0^t \alpha d\left(\int_0^t k(t-s) L_t^N ds\right)\right\} A_t^i.
\end{aligned}$$

Therefore, by defining

$$\mathcal{L}_t^N := \int_0^t k(t-s) L_s^N ds \tag{2.13}$$

we end up with the expression

$$\hat{A}_t^i = \exp\left\{-\int_0^t \alpha d\mathcal{L}_s^N\right\} A_t^i. \tag{2.14}$$

2.4 Modelling the Distances-To-Default

In this section, we want to derive the dynamics for the individual distances-to-default, which finally lead to a finite particle system. For this purpose, recall that the healthiness of bank i at time t is given by X_t^i. In the following, we consider the system of banks as a number of interacting particles. At time t, each such particle is described by

$$X_t^i = \log(\hat{A}_t^i) - \log(D^i). \tag{2.15}$$

Using the expression for the updated asset value (2.14), we derive

$$X_t^i = \log\left(\exp\left\{-\int_0^t \alpha d\mathcal{L}_s^N\right\} A_t^i\right) - \log(D_t^i) = \log(A_t^i) - \log(D_t^i) - \int_0^t \alpha d\mathcal{L}_s^N.$$

Note that the first two terms describe the distance-to-default without having discounted the asset value at time t, which is given by Y_t^i. Hence, for $t < \hat{\tau}_i$ we get the dynamics

$$dX_t^i = dY_t^i - \alpha d\mathcal{L}_t^N, \quad \text{for } i = 1, ..., N.$$

Our goal is to rewrite the above equation in terms of the independent Brownian motions W^0, W^i and the Poisson process N. To do this, we use the dynamics of a single asset value for any i as above

$$\log(A_t^i) = \log(A_0) + \left(r - \lambda\nu - \frac{\sigma^2}{2}\right)t + \sigma\left(\rho W_t^0 + \sqrt{1-\rho^2}W_t^i\right) + \sum_{k=1}^{N_t}\log(Y_k),$$

where the last term is again a compound Poisson process. Since we assume the default barrier to be constant, we note that $X_0^i = \log(A_0^i) - \log(D^i)$. This follows from the assumption that no bank has defaulted by the initial time. In particular, this yields

$$\log(A_t^i) - \log(D^i) = X_0^i + \left(r - \lambda\nu - \frac{\sigma^2}{2}\right)t + \sigma\left(\rho W_t^0 + \sqrt{1-\rho^2}W_t^i\right) + \sum_{k=1}^{N_t}\log(Y_k).$$

For simplicity, we use the general drift parameter $\mu := (r - \lambda\nu - \frac{\sigma^2}{2})$ and denote the jump height by $\Pi := \log(Y)$. Note that the default time of bank i can now be written as $\tau_i := \inf\{t > 0 : X_t^i \leq 0\}$. Using those notations, we can state a finite system of interacting particles by

$$\begin{cases} dX_t^i = \mu dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i\right) + \Pi dN_t - \alpha d\mathcal{L}_t^N \\ \mathcal{L}_t^N = \int_0^t k(t-s)L_s^N ds \text{ , where } L_t^N := \sum_{j=1}^N a_j^N \mathbb{1}_{\{t \geq \tau_j\}} \\ \tau_i := \inf\{t > 0 : X_t^i \leq 0\}. \end{cases}$$

The upper dynamics depend on some parameter, which can be interpreted as followed:

- μ models the return of the assets in the absence of any contagion in the system.
- σ describes the volatility of a bank's assets also in absence of any interaction.
- ρ demonstrates how correlated the assets are in a contagion-free market. In particular, it reflects the influence of the common noise process W^0.
- Π is the random jump height of the underlying common Poisson process, which leads to a perfect jump correlation.
- α is the already mentioned contagion parameter, which captures that in case of bank i's default the others lose a proportion αa_i^N of their asset value.

Chapter 3

From the Finite System to the Large Basket Limit

This chapter builds the theoretical cornerstone of the thesis. As initially studied in [2], we are interested in the case that the population size of the particle system tends to infinity. We hope to find an appropriate approximation for this large basket case. In [1], the model from [2] gets extended by a jump processes and [4] added a contagion mechanism. As already indicated in the last chapter, we will combine those two extensions in one model. Firstly, we show how the finite system, shortly mentioned above, can be expressed using an empirical measure denoted by ν^N. Moreover, we give arguments for the well-posedness and uniqueness of the resulting system. After proving the evolution equation for the measure ν^N in form of a weakly formulated SPDE, we study how the latter behaves in the limit, i.e. in case $N \to +\infty$. In addition, we demonstrate how the same results can be received by considering the law of a conditional McKean-Vlasov problem. Lastly, we illustrate how the different parameters of the model affect the movements of the particles by heat plotting the evolution of their density.

3.1 Empirical Measure

Similarly to what is stated in [2, Section 3], we introduce the empirical measure of the particles X_t^i, $i = 1, .., N$, at time t as

$$\nu_t^N := \sum_{i=1}^{N} a_i^N \mathbb{1}_{\{t<\tau_i\}} \delta_{X_t^i}, \tag{3.1}$$

where δ_x denotes the Dirac measure centered at x. Note at this point that the mapping

$$\nu_\cdot^N = \sum_{i=1}^{N} a_i^N \mathbb{1}_{\{\cdot<\tau_i\}} \delta_{X_\cdot^i} : t \mapsto \nu_t^N$$

takes values in the sub-probability measures on $\mathbb{R}$. For any subset $S \subseteq \mathbb{R}$, $\nu_t^N(S)$ then simply gives the proportion of the particles that have survived until time t and lie in S while each particle is weighted by the individual weight a_i^N. In particular, we can observe the following relationship with the loss process

$$\nu_t^N(0,\infty) = \sum_{i=1}^{N} a_i^N \mathbb{1}_{\{t<\tau_i\}} \delta_{X_t^i}(0,\infty) = \sum_{i=1}^{N} a_i^N \mathbb{1}_{\{t<\tau_i\}} = 1 - L_t^N.$$

We can use this identity to rewrite the finite particle system explicitly depending on the measure ν_t^N. To make things more general, we from now assume that the jump process is of the form

$$J_t^0 = \sum_{k=1}^{N_t^0} Y_k, \tag{3.2}$$

where N^0 denotes a Poisson process with intensity rate λ and $(Y_k)_{k\geq 1}$ is a sequence of i.i.d. random variables being independent of N^0 and following a distribution ϖ. As before, the regarding Poisson random measure is defined by

$$\mathcal{P}_{J^0}(ds, dx) = \sum_{0<s\leq t} \delta_{(s,\Delta J_s^0)}(ds, dx). \tag{3.3}$$

Having this in mind, the finite particle system can be expressed as

$$\begin{cases} dX_t^i = \mu dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i + dJ_t^0 - \alpha d\mathcal{L}_t^N \\ \mathcal{L}_t^N = (k \star L^N)_t \ , \ L_t^N := 1 - \nu_t^N(0,\infty) \\ \nu_t^N := \sum_{i=1}^N a_i^N \mathbb{1}_{\{t<\tau_i\}}\delta_{X_t^i} \ , \ \tau_i := \inf\{t > 0 : X_t^i \leq 0\}. \end{cases} \tag{3.4}$$

3.2 Well-Posedness and Uniqueness of the Finite Particle System

Considering the system as stated in (3.4), we now provide arguments that the latter is well-posed and admits an unique solution for any $N \geq 1$. To see this, we write $\mathbf{X}^N = (X^1, ..., X^N)$ and state the system as a vector-valued SDE

$$d\mathbf{X}_t^N = \mu dt + \sigma\left(\rho \mathbf{W}_t^0 + \sqrt{1-\rho^2}d\mathbf{W}_t\right) + dJ_t^0 - \alpha d\mathcal{L}_t^N, \tag{3.5}$$

where $\mathbf{W}_t^0 = (W_t^0, ..., W_t^0) \in \mathbb{R}^N$ and $\mathbf{W}_t = (W_t^1, ..., W_t^N) \in \mathbb{R}^N$. The following argumentation follows similar ideas as presented in [4, Section 3]. It is mainly based on the fact that on every interval between the defaults and jumps in the system, the random variable L_t^N is $\mathcal{F}_0$-measurable. So we can find an unique solution of the system for the intervals lying between the default and jump times of the particles. Due to this fact, we can solve the system inductively by constructing its solution on every such interval by local continuous systems denoted by $\mathbf{X}^{(k),N}$, $k = 1, 2, ...$. Lastly, the uniqueness follows since those systems have an unique solution by [4] and the fact that there are only finitely many defaults and jumps in the system. Before we go into more detail, we introduce some crucial random times. From now on, $\tau^{i,N}$ denotes the default time of particle i, $i = 1, ..., N$, within the system $\mathbf{X}^N$. Similar to this, we introduce the first time a particle in the m-th local systems defaults by

$$\varrho_1^{(m),N} := \inf\{t > 0 \mid \exists i \in \{1, ..., N\} : X_t^{i,(m),N} \leq 0\}. \tag{3.6}$$

Lastly, we introduce T_n, which denotes the n-th jump time of the process J^0. At time 0, we are starting the first local system $\mathbf{X}^{(1),N}$. The evolution of its particles is described by the processes $X^{i,(1),N}$, $i = 1, ..., N$. Since there are no defaults at time 0, we have that $L_t^N = 0$ until the first default or jumps occurs. Hence, the first system moves continuously without any contagion term to incorporate. In particular, the dynamics of the system can be described by

$$d\mathbf{X}_t^{(1),N} = \mu dt + \sigma\left(\rho \mathbf{W}_t^0 + \sqrt{1-\rho^2}d\mathbf{W}_t\right), \tag{3.7}$$

where $\mathbf{X}_t^{(1),N} = (X_t^{1,(1),N}, ..., X_t^{N,(1),N}) \in \mathbb{R}^N$, $\mathbf{W}_t^0 = (W_t^0, ..., W_t^0) \in \mathbb{R}^N$ and $\mathbf{W}_t = (W_t^1, ..., W_t^N)$ $\in \mathbb{R}^N$. As a next step, we need to distinguish two cases. First, assume that $\varrho_1^{(1),N} < T_1$, which indicates that at time $\varrho_1^{(1),N}$ the first particle in $\mathbf{X}^{(1),N}$ defaults before the first jump occurs. Without loss of generality we give this particle the index i_1. Then we can easily construct the first default time of the system $\mathbf{X}^N$ by $\tau^{i_1,N} = \varrho_1^{(1),N}$. This event causes the particle to drop out of the system $\mathbf{X}^N$. Therefore, we need to update the loss process L^N accordingly, so that $L_{\tau_{i_1}}^N = a_{i_1}$. Now that this process does not equal zero anymore, the contagion mechanism enters the system and has to be taken into account when starting the second local system $\mathbf{X}^{(2),N}$. Moreover, the latter only consists of $N-1$ particles, which we denote by $X^{i_j,(2),N}$, $j = 2, ..., N$, with initial value $X_0^{i_j,(2),N} = X_{\varrho_1^{(1),N}}^{i_j,(1),N}$. The contagion at time t is captured by $\mathcal{L}_t^{(2),N} = (k \star L^{(2),N})_t$, where $L^{(2),N}$

is constantly equal to a_{i_1} for the lifetime of this second system. In the following, we assume that $\text{supp}(k) = [0, \varepsilon]$ for any $\varepsilon > 0$. The continuous dynamics of the new system can be expressed by

$$d\mathbf{X}_t^{(2),N} = \mu dt + \sigma\left(\rho \mathbf{W}_t^0 + \sqrt{1-\rho^2} d\mathbf{W}_t\right) - \alpha d\mathcal{L}_t^{(2),N}, \tag{3.8}$$

where $\mathbf{X}_t^{(2),N} = (X_t^{i_2,(1),N}, ..., X_t^{i_N,(1),N}) \in \mathbb{R}^{N-1}$, $\mathbf{W}_t^0 = (W_t^0, ..., W_t^0) \in \mathbb{R}^{N-1}$ and $\mathbf{W}_t = (W_t^{i_2}, ..., W_t^{i_N}) \in \mathbb{R}^{N-1}$. Next, we consider the case that $T_1 \leq \varrho_1^{(1),N}$. In this case, the system $\mathbf{X}^N$ jumps before the first particle in the first local system hits zero. However, the jump itself might push some of the particles below or equal to zero. Therefore, we need to check if some defaults get triggered or not. If the latter is the case, we can just start the second local system moving continuously and without any contagion. Equivalently to the last case, the particles of this system are denoted by $X^{i_j,(2),N}$, $j = 1, ..., N$, with initial condition $X_0^{i_j,(2),N} = X_{T_{1\_}}^{i_j,(1),N} + \Delta J_{T_1}^0$. The actual dynamics are then described by

$$d\mathbf{X}_t^{(2),N} = \mu dt + \sigma\left(\rho \mathbf{W}_t^0 + \sqrt{1-\rho^2} d\mathbf{W}_t\right), \tag{3.9}$$

where $\mathbf{X}_t^{(2),N} = (X_t^{i_2,(1),N}, ..., X_t^{i_N,(1),N}) \in \mathbb{R}^{N-1}$, $\mathbf{W}_t^0 = (W_t^0, ..., W_t^0) \in \mathbb{R}^{N-1}$ and $\mathbf{W}_t = (W_t^{i_2}, ..., W_t^{i_N}) \in \mathbb{R}^{N-1}$. However, if the jump at T_1 has caused the default of k particles indexed by $i_1, ..., i_k$ it follows that

$$\tau^{i_1,N} = ... = \tau^{i_k,N} = T_1.$$

Moreover, the loss process is updated so that $L_{T_1}^N = \sum_{j=1}^k a_{i_j}$ and the regarding particles are removed from $\mathbf{X}^N$ so that the second system $\mathbf{X}^{(2),N}$ has only a population size of $N-k$. Therefore, following the prior notation it consists of the particles $X^{i_j,(2),N}$, $j = k+1, ..., N$ with $X_0^{i_j,(2),N} = X_{T_{1\_}}^{i_j,(1),N} + \Delta J_{T_1}^0$. The value of the contagion at time t is given by $\mathcal{L}_t^{(2),N} = (k \star L^{(2),N})_t$, where $L^{(2),N}$ is constantly equal to $\sum_{j=1}^k a_{i_j}$ during the lifetime of the system. Incorporating this contagion term, we then have

$$d\mathbf{X}_t^{(2),N} = \mu dt + \sigma\left(\rho \mathbf{W}_t^0 + \sqrt{1-\rho^2} d\mathbf{W}_t\right) - \alpha d\mathcal{L}_t^{(2),N}, \tag{3.10}$$

where $\mathbf{X}_t^{(2),N} = (X_t^{i_{k+1},(1),N}, ..., X_t^{i_N,(1),N}) \in \mathbb{R}^{N-k}$, $\mathbf{W}_t^0 = (W_t^0, ..., W_t^0) \in \mathbb{R}^{N-k}$ and $\mathbf{W}_t = (W_t^{i_{k+1}}, ..., W_t^{i_N}) \in \mathbb{R}^{N-k}$.

Continuing in the same way, we get that in both cases above the system $\mathbf{X}^{(2),N}$ runs until the first of its particles hits zero or the time a jump in $\mathbf{X}^N$ occurs. As a next step, we would start the third local system $\mathbf{X}^{(3),N}$ similar to above. Note that from now on, one has to check if the new local system starts before the contagion of the prior has been completely materialised in case it got stopped due to a default. Taking the start of the third system as an example, this means that we need to check if $\varrho_1^{(2),N} < \varepsilon$. If this is the case, the system $\mathbf{X}^{(3),N}$ inherits the remaining contagion term in form of the additional drift term $\mathcal{L}_t^{(2),N} - \mathcal{L}_{\varrho_1^{(2),N}}^{(2),N}$. As said in the beginning, we use the local systems $\mathbf{X}^{(1),N}$, $\mathbf{X}^{(2),N}$, $\mathbf{X}^{(3),N}$,... to inductively construct the unique solution of the particle system $\mathbf{X}^N$ on each interval between a default or jump.

3.3 Evolution Equation of the Empirical Measure

Following the ideas from [4, Section 3.1], we want to find an equation for the evolution of the empirical measure. Firstly, for a continuous function ψ and measure ζ_t we write

$$\langle \zeta_t, \psi \rangle = \int \psi(x) \zeta_t(dx). \tag{3.11}$$

In the following, we want to focus on continuous functions that are rapidly decreasing. For this purpose, we introduce the so-called Schwartz space $\mathcal{S}$ as

$$\mathcal{S} := \{\phi \in C^\infty(\mathbb{R}) : \|\phi\|_{\alpha,\beta} < \infty \; \forall \alpha, \beta \in \mathbb{N}\}, \tag{3.12}$$

where we have the semi-norm

$$\|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha \phi^{(\beta)}(x)|$$

with $\phi^{(\beta)}$ denoting the β-th derivative of ϕ. Note that this derivatives actually exists for all $\beta \in \mathbb{N}$ by the assumption that $\phi \in C^\infty(\mathbb{R})$. We equip the space $\mathcal{S}$ with the topology induced by the semi-norms $\|\cdot\|_{\alpha,\beta}$. Moreover, we denote by $\mathcal{S}'$ its topological dual space. The latter one is called the space of tempered distributions or space of slowly increasing distributions. Then we can consider the empirical measure ν^N living in $D_{\mathcal{S}'}[0,T]$, which consists of all $\mathcal{S}'$-valued right continuous processes with left limits (cádlág) on the interval $[0,T]$. Lastly, we from now on assume the test functions to live in the space

$$\mathcal{S}_0 := \{\phi \in \mathcal{S} : \phi(x) = 0 \;\forall x \in (-\infty, 0]\}. \tag{3.13}$$

This property of the test functions ϕ yields that for any $i = 1, ..., N$ and time $t \geq 0$ it holds

$$\phi(X_t^i)\mathbb{1}_{\{t<\tau^i\}} = \phi(X_{t\wedge\tau^i}^i).$$

Using this fact, we can write (3.11) in terms of a test function and the empirical measure as

$$\langle \nu_t^N, \phi\rangle = \sum_{i=1}^N a_i^N \mathbb{1}_{\{t<\tau_i\}} \int_{\mathbb{R}} \phi(x)\delta_{X_t^i}(dx) = \sum_{i=1}^N a_i^N \phi(X_{t\wedge\tau^i}^i). \tag{3.14}$$

In the following, we show how to obtain a single evolution equation for the dynamics of the upper empirical measure. Moreover, we can show that in the large population, the idiosyncratic noise W^i vanishes due to the averaging via the weights a_i^N. Those results are stated in the following proposition and proved below. The argumentation follows closely [4, Proposition 3.2] with the regarding changes arising from the jump process.

Proposition 3.1 (Finite evolution equation)**.** *For all $N \geq 1$ we have that*

$$\begin{aligned} d\langle \nu_t^N, \phi\rangle =& \langle \nu_t^N, \mu\partial_x\phi + \frac{1}{2}\sigma^2\partial_{xx}\phi\rangle dt + \langle \nu_t^N, \sigma\rho\partial_x\phi\rangle\, dW_t^0 \\ &- \langle \nu_t^N, \alpha\partial_x\phi\rangle d\mathcal{L}_t^N + \int_{\mathbb{R}} \langle \nu_{t_-}^N, \phi(\cdot + y) - \phi\rangle \mathcal{P}_{J^0}(dt, dy) + dI_t^N(\phi), \end{aligned}$$

for every test function $\phi \in \mathcal{S}_0$, where the idiosyncratic part, $I^N(\phi)$, satisfies

$$\mathbb{E}\left[\sup_{t\leq T} |I_t^N(\phi)|^2\right] = \mathcal{O}\left(\frac{1}{N}\right) \quad \textit{as } N \to \infty.$$

Proof. Recall from above that for any test function $\phi \in \mathcal{S}_0$ we have the expression

$$\langle \nu_t^N, \phi\rangle = \sum_{i=1}^N a_i^N \phi(X_{t\wedge\tau_i}^i).$$

Applying Itô's formula for jump-diffusions to the stopped process $(X_{t\wedge\tau_i}^i)_{t\geq 0}$ with the stopping time $\tau_i := \inf\{t > 0 \;:\; X_t^i \leq 0\}$ yields

$$\begin{aligned} \phi(X_{t\wedge\tau_i}^i) =& \phi(X_0^i) + \int_0^t \mu\partial_x\phi(X_{s\wedge\tau_i}^i)ds + \int_0^t \sigma\rho\partial_x\phi(X_{s\wedge\tau_i}^i)dW_s^0 \\ &+ \int_0^t \sigma\sqrt{1-\rho^2}\partial_x\phi(X_{s\wedge\tau_i}^i)dW_s^i - \int_0^t \alpha\partial_x\phi(X_{s\wedge\tau_i}^i)d\mathcal{L}_s^N \\ &+ \frac{1}{2}\int_0^t \sigma^2\partial_{xx}\phi(X_{s\wedge\tau_i}^i)ds + \int_0^t\int_{\mathbb{R}} [\phi(X_{s_-\wedge\tau_i}^i + y) - \phi(X_{s_-\wedge\tau_i}^i)]\mathcal{P}_{J^0}(ds, dy), \end{aligned}$$

where $\mathcal{P}_J^0(\omega, \cdot, \cdot)$ denotes the Poisson random measure of the jump process J^0. Applying the empirical measure to the upper process yields in differential notation

$$d\langle \nu_t^N, \phi\rangle = \langle \nu_t^N, \mu\partial_x\phi + \frac{1}{2}\sigma^2\partial_{xx}\phi\rangle dt + \langle \nu_t^N, \sigma\rho\partial_x\phi\rangle\, dW_t^0$$
$$- \langle \nu_t^N, \alpha\partial_x\phi\rangle d\mathcal{L}_t^N + \int_{\mathbb{R}} \langle \nu_{t_-}^N, \phi(\cdot + y) - \phi\rangle \mathcal{P}_{J^0}(dt, dy) + dI_t^N(\phi),$$

with

$$I_t^N(\phi) := \sum_{i=1}^N \int_0^t a_i^N \sigma\sqrt{1-\rho^2}\partial_x\phi(X_{s\wedge\tau^i}^i)dW_s^i.$$

This proves the first statement of the upper proposition. For the second part, we focus on the idiosyncratic part $I^N(\phi)$. By the Burkholder-Davis inequality, it holds

$$\mathbb{E}\left[\sup_{t\le T}|I_t^N(\phi)|^2\right] \le C\mathbb{E}\left[\langle I^N(\phi)\rangle_T\right]$$

for some constant C. The independence of the W^i's yields

$$\begin{aligned}
\mathbb{E}\left[\langle I^N(\phi)\rangle_T\right] &= \mathbb{E}\left[\left(\sum_{i=1}^N \int_0^t a_i^N \sigma\sqrt{1-\rho^2}\partial_x\phi(X_{s\wedge\tau^i}^i)dW_s^i\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^N \int_0^t (a_i^N)^2\sigma^2(1-\rho^2)(\partial_x\phi(X_{s\wedge\tau_i}^i))^2 ds\right] \\
&\le \|\partial_x\phi\|_\infty^2 \sum_{i=1}^N \mathbb{E}\left[\int_0^t (a_i^N)^2\sigma^2(1-\rho^2)ds\right] \\
&\le \sigma^2 t\|\partial_x\phi\|_\infty^2 \sum_{i=1}^N (a_i^N)^2 = \mathcal{O}\left(\frac{1}{N}\right),
\end{aligned}$$

as $N \to +\infty$. This gives the second result of the proposition and ends the proof. □

3.4 The Limit SPDE

In this section, we finally want to study the case that the size of the particle system tends to infinity in more detail. Similar to [4, Section 4.2], we want to state the limit SPDE of $\nu := \lim_{N\to+\infty} \nu^N$ in its weak formulation, i.e. with respect to a test function $\phi \in \mathcal{S}_0$. It is important to mention that we consider the weak convergence of the empirical measure with respect to the Skorokhod $M1$ topology on the space $D_{\mathcal{S}'}$. For a detailed introduction of this topology applied to the setting of distribution-valued processes we refer to [26].

Theorem 3.2 (The limiting SPDE). *The sequence $(\nu^N)_{N\ge1}$ is tight on the space $(D_{\mathcal{S}'[0,T]}, M1)$ for any $T > 0$. Moreover, for each limit (ν, W^0, J^0) the following SPDE is satisfied almost surely*

$$\begin{aligned}
\langle \nu_t, \phi\rangle = \langle \nu_0, \phi\rangle + \int_0^t \langle \nu_s, \mu\partial_x\phi\rangle ds + \int_0^t \langle \nu_s, \sigma\rho\partial_x\phi\rangle dW_s^0 + \frac{1}{2}\int_0^t \langle \nu_s, \sigma^2\partial_{xx}\phi\rangle ds \\
- \int_0^t \langle \nu_s, \alpha\partial_x\phi\rangle d\mathcal{L}_s + \int_0^t \int_{\mathbb{R}} \langle \nu_{s_-}, \phi(\cdot + y) - \phi(\cdot)\rangle \mathcal{P}_{J^0}(ds, dy),
\end{aligned} \tag{3.15}$$

where $\mathcal{L}_t := \int_0^t k(t-s)L_s ds$ and $L_t := 1 - \nu_t(0,\infty)$ for all $t \in [0,T]$ and arbitrary test function $\phi \in \mathcal{S}_0$.

3.4.1 Outline of the proof of Theorem 3.2

We give a detailed outline of the argumentation for the proof of the theorem above. Note that the following is not a rigorous proof but only presenting the main ideas while leaving out some of the topological details. We divide the explanations into four steps to give them a certain structure.

Step 1: In the first step, we want to emphasise why the sequence $(\nu^N)_{N\geq 1}$ is tight on the space $(D_{\mathcal{S}'}[0,T], M1)$ for any $T > 0$. By the first part of Theorem 3.2. in [26], it is sufficient to show the $M1$ tightness of the process $t \mapsto \langle \nu_t^N, \phi\rangle$ on $D_{\mathbb{R}}$ for any function $\phi \in \mathcal{S}_0$. This can be done by using Theorem 12.12.3 from [27]. As stated there, we need to check two sufficient conditions. The first one follows directly from the fact that for any $t \in [0,T]$ we have

$$|\langle \nu_t^N, \phi\rangle| \leq \sum_{i=1}^N |a_i^N \phi\left(X^i_{t\wedge\tau^i}\right)| \leq \|\phi\|_\infty \sum_{i=1}^N a_i^N = \|\phi\|_\infty$$

for all $N \geq 1$. For the second condition, we define for any $t \in [0,T]$

$$\hat{\nu}_t^N := \sum_{i=1}^N a_i^N \delta_{X^i_{t\wedge\tau^i}}. \tag{3.16}$$

Then we end up with the decomposition

$$\langle \nu_t^N, \phi\rangle = \langle \hat{\nu}_t^N, \phi\rangle + \phi(0) L_t^N. \tag{3.17}$$

Note that the process L^N is monotonically increasing and $\phi(0)$ constant. Therefore, we were able to write $\langle \nu_t^N, \phi\rangle$ as a sum with one monotone part, which is negligible to the $M1$ modulus of continuity. So it is sufficient to work with the normal modulus of continuity of $\hat{\nu}^N$. So to control the increments it is mainly important to show that for all $s,t \in [0,T]$ it holds

$$\mathbb{E}[|J_t^0 - J_s^0|^2] = \mathcal{O}(t-s),$$

which follows by the property of the Poisson process J^0. Moreover, like in the same proposition we obtain that one can conclude the weak convergence of the loss process, i.e. $L^N \Rightarrow L^*$, which can be written as $L^* = 1 - \nu^*(0,\infty)$ in case one has $\nu^N \Rightarrow \nu^*$ and that the limit ν^* is a subprobability measure. Since we have tightness of $(\nu^N)_{N\geq 1}$ on the space $(D_{\mathcal{S}'}[0,T], M1)$, it follows naturally that $(\nu^N, L^N, W^0, (N^0, Y^0))$ is tight on the product space $(D_{\mathcal{S}'}[0,T], M1) \times (D_{\mathbb{R}}, M1) \times (D_{\mathbb{R}}, \text{unif-top}) \times ((D_{\mathbb{R}}, \text{unif-top}) \times (\mathbb{R}^N, \text{prod-top}))$, where we mean the uniform and product topology. Note that the process J^0 is fully described by the processes N^0 and Y^0.

Step 2: In this rather short part of the argumentation, we note that the sequence $(\nu^N)_{N\geq 1}$ is relatively compact on $(D_{\mathbb{S}'}[0,T], M1)$. This follows directly from the second part of Theorem 3.2. in [26].

Step 3: The relative compactness from above enables us to extract a weakly convergent subsequence, which we denote by $(\nu^N, L^N, W^0, (N^0, Y^0))$ and its weak limit by $(\nu^*, L^*, W^0, (N^0, Y^0))$. In the following, we have a closer look on the convergence behaviour. Firstly, we note that by $\nu^N \Rightarrow \nu^*$ on $(D_{\mathcal{S}'}[0,T], M1)$ we also have that $\langle \nu_t^N, \phi\rangle \Rightarrow \langle \nu_t^*, \phi\rangle$ in $(D_{\mathbb{R}}, M1)$ for any $\phi \in \mathcal{S}$. From now on, we restrict us to test functions $\phi \in \mathcal{S}_0$. As argued in Remark 4.6. in [4], we get that the weakly convergent subsequence has a further subsequence with the almost sure Skorokhod representation property. Therefore, we can consider a sequence $(\widetilde{\nu}^N, \widetilde{L}^N, W^N, (N^N, Y^N))$, which has the almost sure pointwise limit $(\widetilde{\nu}^*, \widetilde{L}^*, W^*, (N^*, Y^*))$ in $(D_{\mathcal{S}'}[0,T], M1) \times (D_{\mathbb{R}}, M1) \times (D_{\mathbb{R}}, \text{unif-top}) \times ((D_{\mathbb{R}}, \text{unif-top}) \times (\mathbb{R}^N, \text{prod-top}))$. For every $N \geq 1$, we have that $(\widetilde{\nu}^N, \widetilde{L}^N, W^N, (N^N, Y^N))$ has the same law as $(\nu^N, L^N, W^0, (N^0, Y^0))$. The same holds for the regarding limiting processes. With this in mind, note that by the decomposition (3.17) we see that $\langle \nu^N, \phi\rangle$ can only jump when the compound Poisson process J^0 jumps. Therefore, also the process $\langle \widetilde{\nu}^N, \phi\rangle$ is almost surely continuous except from the jumps

$$\sum_{0<s\leq t} \Delta\langle \widetilde{\nu}_s^N, \phi\rangle = \sum_{l=1}^{N_t^N} \langle \widetilde{\nu}^N_{T^N_{l-}}, \phi(\cdot + Y_l^N) - \phi\rangle,$$

where T_l^N denotes the l-th jump time of the Poisson process N^N. In addition, we see that for the time between the jumps, the paths of $\langle \widetilde{\nu}^N, \phi\rangle$ behave like a Brownian motion with bounded drift coming from the constant μ and the bounded contagion term. This again holds since this is especially almost surely true for $\langle \nu^N, \phi \rangle$. Those two properties of the converging subsequence give us, in combination with the definition of the $M1$-convergence, that jumps in the limiting process $\langle \widetilde{\nu}^*, \phi \rangle$ have to arise from matching jumps in the approximating $\langle \widetilde{\nu}^N, \phi \rangle$. In other words, assume that there is a jump in the limit at time t. Then the upper result states that this is only possible if there is a sequence $t_N \to t$ with $\langle \widetilde{\nu}^N_{t_N}(\omega), \phi \rangle \to \langle \widetilde{\nu}^*_t(\omega), \phi \rangle$ and $\langle \widetilde{\nu}^N_{t_N-}(\omega), \phi \rangle \to \langle \widetilde{\nu}^*_{t-}(\omega), \phi \rangle$ for all $\omega \in \Omega$. In particular, it holds that $\Delta \langle \widetilde{\nu}^N_{t_N}(\omega), \phi \rangle \to \Delta \langle \widetilde{\nu}^*_t(\omega), \phi \rangle$. One should note at this point that under the $M1$-convergence it is not natural that there are matched jumps. Rather it is possible that jumps can appear in the limit of purely continuous processes unlike if one works with the uniform or $J1$-convergence. So it is crucial to understand that the matched jump can only arise from the fact that a jump in the limit can not be approximated by the Brownian motion part with bounded drift of the sequence $\langle \widetilde{\nu}^N, \phi \rangle$. Thanks to this elementary observation, we know precisely the time and magnitude of the jumps of $\langle \widetilde{\nu}^N, \phi \rangle$. In particular, assume that t^* is a jump time of the limit process $\langle \widetilde{\nu}^*, \phi \rangle$. Then it follows that for all $\omega \in \Omega$ it holds

$$\langle \widetilde{\nu}^N_{T_l^N-}(\omega), \phi(\cdot + Y_l^N(\omega)) \rangle = \langle \widetilde{\nu}^N_{T_l^N}(\omega), \phi \rangle \longrightarrow \langle \widetilde{\nu}^*_{t^*}(\omega), \phi \rangle$$

and

$$\langle \widetilde{\nu}^N_{T_l^N-}(\omega), \phi \rangle \longrightarrow \langle \widetilde{\nu}^*_{t^*}(\omega), \phi \rangle$$

with $T_l^N(\omega) \to t^*$ for some integer $l \geq 1$. Clearly this implies $T_l^* = t^*$ in the light of the uniform convergence $N^N(\omega) \to N^*(\omega)$. Furthermore, by shifting the test function $\phi \in \mathcal{S}_0$ by the limiting height of the l-th jump, the test function still lives in the space of test functions, i.e. $\psi := \phi(\cdot + Y_l^*) \in \mathcal{S}_0$. Then it follows that

$$\langle \widetilde{\nu}^N_{T_l^N-}(\omega), \psi \rangle \longrightarrow \langle \widetilde{\nu}^*_t(\omega), \psi \rangle = \langle \widetilde{\nu}^*_t(\omega), \phi(\cdot + Y_l^*(\omega)) \rangle$$

and therefore

$$\begin{aligned}
&\langle \widetilde{\nu}^N_{T_l^N-}(\omega), \phi(\cdot + Y_l^N(\omega)) \rangle = \langle \widetilde{\nu}^N_{T_l^N-}(\omega), \psi + \phi(\cdot + Y_l^N(\omega)) - \psi \rangle \\
=&\langle \widetilde{\nu}^N_{T_l^N-}(\omega), \psi \rangle + \langle \widetilde{\nu}^N_{T_l^N-}(\omega), \phi(\cdot + Y_l^N(\omega)) - \psi \rangle \longrightarrow \langle \widetilde{\nu}^*_{t^*}(\omega), \phi(\cdot + Y_l^*(\omega)) \rangle.
\end{aligned}$$

The last convergence result arises from the fact that

$$\begin{aligned}
&\left| \langle \widetilde{\nu}^N_{T_l^N-}(\omega), \phi(\cdot + Y_l^N(\omega)) - \psi \rangle \right| \leq \langle \widetilde{\nu}^N_{T_l^N-}(\omega), |\phi(\cdot + Y_l^N(\omega)) - \psi| \rangle \\
\leq& \left\| \phi(\cdot + Y_l^N(\omega)) - \phi(\cdot + Y_t^*(\omega)) \right\|_\infty \leq C |Y_l^N(\omega) - Y_l^*(\omega)| \longrightarrow 0
\end{aligned}$$

for some constant $C > 0$. In conclusion, the jumps of the limiting process $\langle \widetilde{\nu}^*, \phi \rangle$ happen with probability 1 precisely at the limiting jump times of $\langle \widetilde{\nu}^N, \phi \rangle$, i.e. at $T_1^*, T_2^*, \ldots$. Moreover, they are of the form

$$\Delta \langle \widetilde{\nu}^*_{T_l^*}, \phi \rangle = \langle \widetilde{\nu}^*_{T_l^*-}, \phi(\cdot + Y_l^*) \rangle - \langle \widetilde{\nu}^*_{T_l^*}, \phi \rangle.$$

Step 4: Now that we understand the behavior of the convergence better, we focus on the weakly convergent sequence $(\nu^N)_{N \geq 1}$. In the following, we present a martingale argument to see that this finally leads to the desired limit SPDE. The structure and idea of the argumentation are inspired by [4, Section 4.2]. First of all, we define for a fixed time $t \leq T$ and any $\phi \in \mathcal{S}_0$

$$\Psi(\nu^N) := \int_0^t \langle \nu_s^N, G\phi \rangle ds \tag{3.18}$$

and

$$\Phi(\nu^N, L^N) := \int_0^t \langle \nu_s^N, H\phi \rangle d(k * L^N)_s, \tag{3.19}$$

where G and H are constants representing α, μ or $\frac{\sigma^2}{2}$. Then we get for any bounded Lipschitz function f and appropriate constant $C > 0$ that

$$\begin{aligned}&\left|\mathbb{E}[f(\Psi(\nu^N))] - \mathbb{E}[f(\Psi(\nu^*))]\right| = \left|\mathbb{E}[f(\Psi(\nu^N)) - f(\Psi(\nu^*))]\right| \\ \leq &C\mathbb{E}\left[|\Psi(\nu^N) - \Psi(\nu^*)|\right] = CG\mathbb{E}\left[\left|\int_0^t \langle \nu_s^N - \nu_s^*, \phi\rangle ds\right|\right].\end{aligned}$$

One can ensure that the last term tends towards zero using the weak convergence. The same can be shown for Φ. This implies that we have the weak convergence results $\Psi(\nu^N) \Rightarrow \Psi(\nu^*)$ and $\Phi(\nu^N, L^N) \Rightarrow \Phi(\nu^*, L^*)$ on $\mathbb{R}$, since $(\nu^N, L^N) \Rightarrow (\nu^*, L^*)$. Note at this point that for some $\phi \in \mathcal{S}_0$ also the first two derivatives live in this space. So that the convergence of the integrals holds also for them. Next, we define the process

$$\mathcal{M}(\nu^N, L^N)_t = \langle \nu_t^N, \phi\rangle - \langle \nu_0^N, \phi\rangle - \int_0^t \langle \nu_s^N, \frac{\sigma^2}{2}\partial_{xx}^2\phi + \mu\partial_x\phi\rangle ds - \int_0^t \langle \nu_s^N, \alpha\partial_x\phi\rangle d\mathcal{L}_s. \tag{3.20}$$

Using the weak convergence of the integrals shown before, we can use the same arguments as in Prob. 4.8. in [4] to deduce that $\mathcal{M}(\nu^N, L^N) \Rightarrow \mathcal{M}(\nu, L)$ in $\mathbb{R}$. Next, we introduce the process

$$\mathcal{J}(\widetilde{\nu}^N, Y^N, N^N)_t = \sum_{l=1}^{N_t^N} \langle \widetilde{\nu}^N_{T_{l\_}^N}, \phi(\cdot + Y_l^N)\rangle - \langle \widetilde{\nu}^N_{T_{l\_}^N}, \phi\rangle. \tag{3.21}$$

Using the results from the previous step, we directly get that

$$\mathcal{J}(\widetilde{\nu}^N, Y^N, N^N) \Rightarrow \mathcal{J}(\widetilde{\nu}^*, Y^*, N^*),$$

which naturally yields

$$\mathcal{J}(\nu^N, Y^0, N^0) \Rightarrow \mathcal{J}(\nu^*, Y^0, N^0).$$

Moreover, we get for any $t \leq T$

$$\mathcal{J}(\nu^N, Y^0, N^0)_t = \sum_{0<s\leq t} \Delta\mathcal{M}(\nu^N, L^N)_s$$

and the same for the limit (ν^*, N^0, Y^0). Then again similar to Prob. 4.8. [4] and using the weak convergence results one can show that

$$\mathcal{M}(\nu^*, L^*) - \mathcal{J}(\nu^*, Y^0, N^0) \tag{3.22}$$

is a continuous martingale. In the same manner, we can deduce the same for

$$\left(\mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, J^0, N^0)\right)^2 - \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds,$$

as well as

$$\left(\mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, J^0, N^0)\right) W^0 - \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds.$$

Finally, using (3.22) we see that

$$\mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, J^0, N^0) - \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0 \tag{3.23}$$

is a martingale. By applying the Doob-Meyer decomposition, we deduce by the martingale properties above that

$$\left\langle \mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, J^0, N^0)\right\rangle = \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds$$

and

$$\left\langle \mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, J^0, N^0), W^0 \right\rangle = \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds.$$

Finally, we observe that

$$\begin{aligned}
&\left\langle \mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, Y^0, N^0) - \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0 \right\rangle \\
=&\langle \mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, Y^0, N^0)\rangle - 2\left\langle \mathcal{M}(\nu^*, L) - \mathcal{J}(\nu^*, Y^0, N^0), \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0 \right\rangle \\
+&\left\langle \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0, \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0 \right\rangle \\
=&\int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds - 2\int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds + \int_0^\cdot \langle \nu^*, \sigma\rho\partial_x\phi\rangle^2 ds = 0.
\end{aligned}$$

This shows that the process (3.23) is a continuous martingale with zero quadratic variance and hence constant. This in particular implies

$$\begin{aligned}
0 &= \mathcal{M}(\nu^*, L)_t - \mathcal{J}(\nu^*, Y^0, N^0) - \int_0^t \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0 \\
&= \langle \nu_t^*, \phi\rangle - \langle \nu_0^*, \phi\rangle - \int_0^t \langle \nu^*, \frac{\sigma^2}{2}\partial_{xx}^2\phi + \mu\partial_x\phi\rangle dW_s^0 - \int_0^t \langle \nu_s^*, \alpha\partial_x\phi\rangle d\mathcal{L}_s \\
&- \mathcal{J}(\nu^*, Y^0, N^0) - \int_0^t \langle \nu^*, \sigma\rho\partial_x\phi\rangle dW_s^0,
\end{aligned}$$

which finally gives the limit SPDE and proves the theorem, since

$$\mathcal{J}(\nu^*, Y^0, N^0) = \sum_{l=1}^{N_t} \langle \nu_{T_{l\_}}^*, \phi(\cdot + Y_l)\rangle - \langle \nu_{T_{l\_}}^*, \phi\rangle = \int_0^t \int_{\mathbb{R}} \langle \nu_{s\_}^*, \phi(\cdot + y)\rangle - \langle \nu_{s\_}^*, \phi\rangle \mathcal{P}_{J^0}(ds, dy).$$

△

3.5 Stochastic McKean–Vlasov Problem

Inspired by [4], we show that the unique solution of the limit SPDE, denoted by ν, can also be established as the conditional law of a 'conditional' McKean-Vlasov type jump-diffusion with absorption at the origin, given the processes W^0 and J^0. This demonstrates the interesting fact that you can approach the problem from both a SPDE and McKean-Vlasov point of view and still end up with the same solution. The following proposition formulates this mathematically. Moreover, the advantage of this second perspective is that it is easier to find an explicit expression of the density process of the solving measure. The latter is stated and proved in Lemma 3.5.

Proposition 3.3 (McKean-Vlasov problem)**.** *Consider the probability measure*

$$\nu_t^\star = \mathbb{P}(X_t^\star \in \cdot\,,\ t < \tau^\star \mid W^0,\ J^0) \text{ for } \tau^\star := \inf\{t > 0\ :\ X_t^\star \le 0\}, \tag{3.24}$$

where the process $X^\star$ is a solution to the following conditional McKean-Vlasov jump-diffusion problem

$$\begin{cases} X_t^\star = X_0 + \mu t + \sigma\rho W_t^0 + \sigma\sqrt{1-\rho^2}W_t - \alpha\mathcal{L}_t^\star + J_t^0 \\ \mathcal{L}_t = (k \star L^\star)_t,\ L_t^\star = \mathbb{P}(\tau^\star \le t \mid W^0,\ J^0) \\ X_0 \sim \nu_0 \end{cases}$$

Then the measure $\nu^\star$ satisfies the same SPDE as the limiting measure ν in Theorem 3.2.

Before we can give the proof of the proposition, we need to prove the following lemma, which states a result on the interchanging of stochastic integrals and conditional expectation as presented in [3, Section 8].

Lemma 3.4. *Let W be a Brownian motion with natural filtration $\{\mathcal{F}_t^W\}_{t\geq 0}$. Let H be a real-valued adapted process with*

$$\mathbb{E}\left[\int_0^T H_s^2 ds\right] < \infty.$$

Then we have

$$\mathbb{E}\left[\int_0^t H_s dW_s \;\middle|\; \mathcal{F}_t^W\right] = \int_0^t \mathbb{E}\left[H_s \mid \mathcal{F}_t^W\right] dW_s.$$

Moreover, for a Brownian motion $W^\perp$ independent of W we get

$$\mathbb{E}\left[\int_0^t H_s dW_s^\perp \;\middle|\; \mathcal{F}_t^W\right] = 0.$$

Proof. Assume that H is a simple process and of the form

$$H_t := Z\mathbb{1}_{\{t_1 < t \leq t_2\}} \text{ for } 0 \leq t \leq T,$$

where Z is a $\mathcal{F}_{t_1}$-measurable random variable. Then by independence of the increments of the Brownian motion W it follows

$$\begin{aligned}
\mathbb{E}\left[\int_0^T H_s dW_s \;\middle|\; \mathcal{F}_T^W\right] &= \mathbb{E}\left[\int_{t_1}^{t_2} Z dW_s \;\middle|\; \mathcal{F}_T^W\right] \\
&= \mathbb{E}\left[Z(W_{t_2} - W_{t_1}) \mid \mathcal{F}_T^W\right] \\
&= \int_0^T \mathbb{E}[Z \mid \mathcal{F}_s^W]\mathbb{1}_{\{t_1 < s \leq t_2\}} dW_s \\
&= \int_0^T \mathbb{E}[H_s \mid \mathcal{F}_s^W] dW_s.
\end{aligned}$$

In addition, by the fact that Z is $\mathcal{F}_{t_1}$-measurable, the tower property yields

$$\begin{aligned}
\mathbb{E}\left[\int_0^T H_s dW_s^\perp \;\middle|\; \mathcal{F}_T^W\right] &= \mathbb{E}\left[\int_{t_1}^{t_2} Z dW_s^\perp \;\middle|\; \mathcal{F}_T^W\right] \\
&= \mathbb{E}\left[Z(W_{t_2}^\perp - W_{t_1}^\perp) \mid \mathcal{F}_T^W\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Z(W_{t_2}^\perp - W_{t_1}^\perp) \mid \sigma(\mathcal{F}_{t_1}, \mathcal{F}_T^W)\right] \mid \mathcal{F}_T^W\right] \\
&= \mathbb{E}\left[Z\mathbb{E}\left[(W_{t_2}^\perp - W_{t_1}^\perp)\right] \mid \mathcal{F}_T^W\right] = 0.
\end{aligned}$$

Note that we have used that the process $W^\perp$ and especially its increments are independent of the sigma-algebra $\sigma(\mathcal{F}_{t_1}, \mathcal{F}_T^W)$. The statement of the lemma follows now immediately by measure theoretical induction.

□

Proof of Proposition 4.2.1. First, we observe that for any test function $\phi \in \mathcal{S}_0$ it holds

$$\langle \nu_t, \phi \rangle = \mathbb{E}\left[\phi(X_t^\star)\mathbb{1}_{\{\tau^\star \leq t\}} \mid W^0,\ J^0\right] = \mathbb{E}\left[\phi(X_{t\wedge\tau^\star}^\star) \mid W^0,\ J^0\right].$$

Therefore, before we evaluate the upper conditional expectation, we apply Itô's formula to the stopped process $(X_{t\wedge\tau^\star}^\star)_{t\geq 0}$. In particular, we get

$$
\begin{aligned}
\phi(X^\star_{t\wedge\tau^\star}) &= \phi(X^\star_0) + \int_0^t \mu\partial_x\phi(X^\star_{s\wedge\tau^\star})ds + \int_0^t \sigma\rho\partial_x\phi(X^\star_{s\wedge\tau^\star})dW^0_s \\
&\quad + \int_0^t \rho\sqrt{1-\rho^2}\partial_x\phi(X^\star_{s\wedge\tau^\star})dW_s - \int_0^t \alpha\partial_x\phi(X^\star_{s\wedge\tau^\star})d\mathcal{L}^\star_s \\
&\quad + \frac{1}{2}\int_0^t \sigma^2\partial_{xx}\phi(X^\star_{s\wedge\tau^\star})ds + \int_0^t\int_{\mathbb{R}}[\phi(X^\star_{s_-\wedge\tau^\star}+y) - \phi(X^\star_{s_-\wedge\tau^\star})]\mathcal{P}_{J^0}(ds,dy),
\end{aligned}
$$

where $\mathcal{P}_J(\omega,\cdot,\cdot)$ denotes again the jump measure of an arbitrary jump process J. By taking expectation conditional on W^0 and J^0 we obtain that with the identity

$$
\nu^\star_{t_-} = \mathbb{P}(X^\star_t \in \cdot\,,\ t \le \tau^\star \mid W^0,\ J^0)
$$

and using the above result

$$
\begin{aligned}
\langle \nu^\star_t, \phi\rangle &= \langle \nu^\star_0, \phi\rangle + \int_0^t \langle \nu^\star_s, \mu\partial_x\phi\rangle ds + \int_0^t \langle \nu^\star_s, \sigma\rho\partial_x\phi\rangle dW^0_s + \frac{1}{2}\int_0^t \langle \nu^\star_s, \sigma^2\partial_{xx}\phi\rangle ds \\
&\quad - \int_0^t \langle \nu^\star_s, \alpha\partial_x\phi\rangle d\mathcal{L}_s + \int_0^t\int_{\mathbb{R}} \langle \nu^\star_{s_-}, \phi(\cdot+y) - \phi\rangle \mathcal{P}_{J^0}(ds,dy).
\end{aligned}
$$

Note that we have used the result of Lemma 3.4 for the integrals with respect to the Brownian motions respectively. For the deterministic integrals, the upper results directly from Fubini's theorem. Lastly, the integral with respect to the jump measure $\mathcal{P}_{J^0}$ arises from the fact that the expectation is conditional on the natural sigma-algebra $\sigma(J^0)$ and therefore the values of the process are given. By differentiation the results of the proposition follows. □

It is intuitive to think of the measure-valued SPDE in Theorem 3.2 in terms of a density process. The existence and formulation of the density is given in the following lemma. Its proof is inspired by the steps presented [3, Section 9] followed by an integration-by-parts argument.

Lemma 3.5. *Let ν be a limit point as described Theorem 3.2. Then there exists a density process V in $L^2(0,\infty)$ and with respect to the Lebesgue measure Leb, which satisfies*

$$
\begin{aligned}
V_t(x) &= \int_0^t \frac{1}{2}\sigma^2\partial_{xx}V_s(x)ds - \int_0^t \mu\partial_x V_s(x)ds - \int_0^t \rho\sigma\partial_x V_s(x)dW^0_s \\
&\quad + \int_0^t \alpha\partial_x V_s(x)d\mathcal{L}_s + \int_0^t\int_{\mathbb{R}} \left[V_{s_-}(x-y) - V_{s_-}(x)\right]\mathcal{P}_{J^0}(ds,dy).
\end{aligned}
$$

Proof. We have by Proposition 3.3 that the SPDE satisfied by the limit measure is also followed by the sub-probability of the conditional McKean-Vlasov problem. By the uniqueness of the limit measure, it follows that they are indeed the same. So for any Borel set $A \in \mathcal{B}(0,\infty)$ and by denoting the density of W_t by p_t, we have

$$
\begin{aligned}
\nu^\star_t(A) &= \mathbb{P}\left(X_0 + \mu t + \sigma\rho W^0_t + \sigma\sqrt{1-\rho^2}W_t - \alpha\mathcal{L}_t + J^0_t \in A\ ,\ t < \tau \mid W^0,\ J^0\right) \\
&\le \mathbb{P}\left(X_0 + \mu t + \sigma\rho W^0_t + \sigma\sqrt{1-\rho^2}W_t - \alpha\mathcal{L}_t + J^0_t \in A \mid W^0,\ J^0\right) \\
&= \int_A\int_0^\infty p_{\sigma^2(1-\rho^2)t}\left(x - x_0 - \mu t - \sigma\rho W^0_t + \alpha\mathcal{L}_t - J^0_t\right)\nu^\star_0(dx_0)dx,
\end{aligned}
$$

where we have used the fact that $\sigma(1-\rho^2)^{1/2}W_t \overset{d}{=} W_{\sigma(1-\rho^2)^{1/2}t}$. Moreover, we made use of the independence of X_0, W, W^0 and J^0 as well as the W^0-measurability of $\mathcal{L}$. It follows that

$$
\nu^\star_t(A) \le \frac{1}{\sqrt{2\pi\sigma^2(1-\rho^2)t}} Leb(A)
$$

for all $A \in \mathcal{B}(0,\infty)$. Therefore, we see that $Leb(A) = 0 \Rightarrow \nu_t^\star(A)$, which shows that $\nu^\star \ll Leb$. Finally, by the Radon-Nikodym theorem we get the existence of the density process of $\nu^\star$ with respect to Leb, which we will denote by V.
Now that we have proven the existence of a density, we get for any test function $\phi \in \mathcal{S}_0$ that

$$\langle \nu_t^\star, \phi\rangle = \int_{\mathbb{R}} \phi(x) V_t(x) dx. \tag{3.25}$$

Using this identity, we can conclude that

$$\begin{aligned}
\int_0^t \langle \nu_s^\star, \mu\partial_x\phi\rangle ds &= \int_0^t \int_{\mathbb{R}} \mu\partial_x\phi V_s(x) dx ds \\
&= [\mu\phi(x)V_t(x)]_{\mathbb{R}} - \int_0^t \int_{\mathbb{R}} \mu\partial_x V_s(x)\phi(x) dx ds \\
&= -\int_0^t \int_{\mathbb{R}} \mu\partial_x V_s(x)\phi(x) dx ds,
\end{aligned}$$

where we have used integration by parts and the fact that the support of the test function ϕ is compact in $\mathbb{R}$ by the fact that the set

$$C_c^\infty := \{\phi \in C^\infty : \text{supp}(\phi) \text{ is compact}\}$$

is dense in $\mathcal{S}_0$. In the same manner, we can derive the identities

$$\begin{aligned}
\frac{1}{2}\int_0^t \langle \nu_s^\star, \sigma^2\partial_{xx}\phi\rangle dt &= -\frac{1}{2}\int_0^t \int_{\mathbb{R}} \sigma^2\partial_{xx} V_s(x)\phi(x) dx ds, \\
\int_0^t \langle \nu_s^\star, \sigma\rho\partial_x\phi\rangle dW_s^0 &= -\int_0^t \int_{\mathbb{R}} \sigma\rho\partial_x V_s(x)\phi(x) dx dW_s^0
\end{aligned}$$

and

$$\int_0^t \langle \nu_s^\star, \alpha\partial_x\phi\rangle d\mathcal{L}_s = -\int_0^t \int_{\mathbb{R}} \alpha\partial_x V_s(x)\phi(x) dx d\mathcal{L}_s.$$

This covers all continuous parts of the limit SPDE of the measure ν. Lastly, we need to incorporate the discontinuous part arising from the jump process J^0. We get that

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} \langle \nu_{s_-}^\star, \phi(\cdot + y) - \phi(\cdot)\rangle \mathcal{P}_{J^0}(ds, dy) &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} [\phi(x+y) - \phi(x)]\, V_{s_-}(x) dx \mathcal{P}_{J^0}(ds, dy) \\
&= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} V_{s_-}(x-y)\phi(x) ds \mathcal{P}_{J^0}(ds, dy) \\
&\quad - \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} V_{s_-}(x)\phi(x) dx \mathcal{P}_{J^0}(ds, dy),
\end{aligned}$$

where the shift in the density is possible due to the fact that $V \in L^2(0,\infty)$. Therefore, it finally follows by the limit SPDE (3.15) that

$$\begin{aligned}
\int_{\mathbb{R}} V_t(x)\phi(x) dx = \int_{\mathbb{R}} \Bigg[&\frac{1}{2}\int_0^t \sigma^2\partial_{xx} V_s(x) ds - \int_0^t \mu\partial_x V_s(x) ds \\
&- \int_0^t \sigma\rho\partial_x V_t(x) dW_s^0 - \int_0^t \alpha\partial_x V_t(x) d\mathcal{L}_s \\
&+ \int_0^t \int_{\mathbb{R}} \left[V_{s_-}(x-y) - V_{s_-}(x)\right] \mathcal{P}_{J^0}(ds, dy) \Bigg] \phi(x) dx.
\end{aligned}$$

Since the upper result holds for all test functions, the statement of the proposition follows by the fundamental lemma of calculus of variations. ☐

3.6 Financial Contagion Illustrated

By construction of the model, the health of the banking system decisively depends on the impact of the common noise W^0, the market wide shocks J^0 and the non-linear negative effect of the contagion process $\mathcal{L}$. This setup captures mathematically what was observed during the credit crunch in 2007-2009. Namely that the portfolios of many financial institutions are facing a big amount of systemic risk beside the idiosyncratic risks. For a detailed study we refer to [11]. To give a more heuristic understanding of how the three main processes of the model above drive the system, we illustrate the spatial and terminal evolution of the density function $(t,x) \mapsto V_t(x)$ stated in Lemma 3.5. We fix one market scenario by considering one specific simulation of the processes W^0 and J^0. Moreover, we assume an arbitrary initial health state of the system described by V_0. Plots of those three processes are displayed in the Appendix. One can observe that at the start of the time period, the system consists of two parts, where the better doing one is of a bigger size. The following figures consist of two plots. The left one illustrates how the healthiness of the system behaves over time in form of a heat plot. The right graphic displays the regarding loss process L. Lastly, note that for both the time window and the healthiness we consider the values to lie in the interval $[0, 1]$. The vertical axis of the heat plot presents how far each particle is away from a possible default. The closer the value is to 1.0, the better is the company doing in terms of how unlikely its the default is in the near future. By definition of the loss process at time t, it states the proportion of firms in the system which have defaulted by time t, while we assume equal weights for each company. It is natural to expect this process to be monotonically increasing since no new firms enter the system and once they default, they drop out of it. For the first Figure 3.1 we consider that the distances-to-default only depend on the common diffusion process W^0 and that there are no system-wide shocks. In addition, we do not assume any contagious effects within the system, i.e. the default of one company has no negative impacts on the well-being of the remaining ones. In this case, the limiting McKean-Vlasov problem can be written as

$$X_t = X_0 + \mu t + \sigma W_t^0. \tag{3.26}$$

As the system starts running, a few firms of the unhealthy part default due to the slightly negative trend of the underlying W^0 in the beginning. After a short recovery, the number of defaults steadily increases until the end of the period. The most important observation is that the healthier part is not affected by the increasing number of defaults. As a consequence, all of the those banks "survived" the period, which keeps the total number of defaults low.

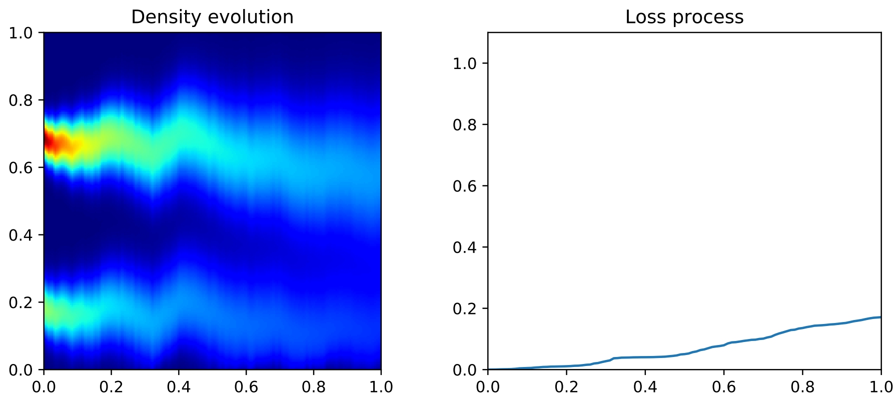

Figure 3.1: The left plot shows a heat plot of the density process $(t,x) \mapsto V_t(x)$ for a fixed realisation of the common noise W^0. We used the parameters $\alpha = 0$, $\rho = 0.4$, $\sigma = 0.2$ and $\mu = 0$. The impact kernel k is a triangle on $[0, 0.015]$ with height $2/0.015$. The right plot displays the evolution of the loss process $t \mapsto L_t$.

In Figure 3.2 we include the contagion term in the system. Therefore, now the default of a company adds a temporary non-linear negative drift to the rest of the system. The regarding limiting McKean-Vlasov problem can be written as

$$X_t = X_0 + \mu dt + \sigma W_t^0 - \alpha L_t\ ,\ L_t = \mathbb{P}(\tau \le t \mid W^0), \tag{3.27}$$

where $\tau = \inf\{t > 0 : X_t \leq 0\}$. One can easily observe that the first defaults, occurring around $t = 0.3$, have a negative impact on the remaining companies. Finally, with the unhealthy part "dying out" around $t = 0.5$, the system experiences a steep negative pull down, which is materialised over a short period after the triggering defaults. This lets the initial healthier part end up close to default. The loss process indicates that more than half of the system has defaulted at the end of the time period. The phenomenon taking place between $t = 0.4$ and $t = 0.6$ is often referred to as *default clustering* in existing literature.

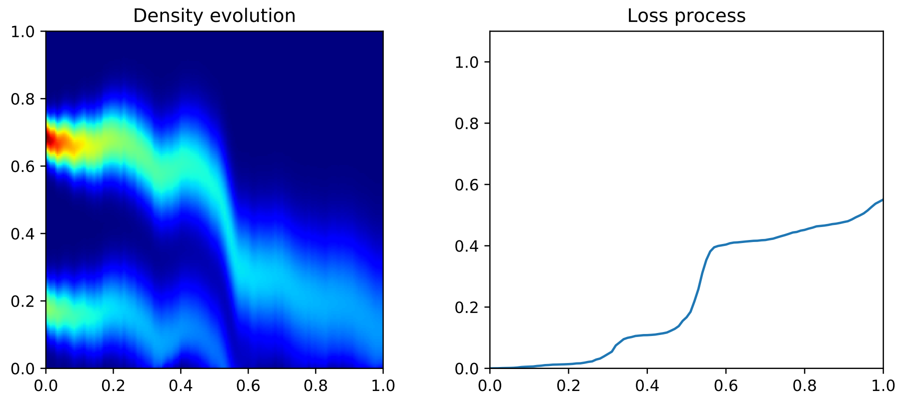

Figure 3.2: The left plot shows a heat plot of the density process $(t, x) \mapsto V_t(x)$ for a fixed realisation of the common noise W^0. We used the parameters $\alpha = 1.5$, $\rho = 0.4$, $\sigma = 0.2$ and $\mu = 0$. The impact kernel k is a triangle on $[0, 0.015]$ with height $2/0.015$. The right plot displays the evolution of the loss process $t \mapsto L_t$.

In the next scenario, displayed in Figure 3.3, we again consider a world without financial contagion, but where the system experiences some negative shocks now and then. As already stated they can be interpreted as system-wide crises. In this case, the limiting McKean-Vlasov problem is given by

$$X_t = X_0 + \mu dt + \sigma W_t^0 + J_t^0 \tag{3.28}$$

In contrast to the prior scenario, where the negative effects of the contagion materialises gradually, the jumps occur in form of discontinuous shifts of the particles in the system. The plot displays that there are two major shocks within the time period, while the third jump of J^0 is of negligible magnitude. The loss process indicates that in comparison to the pure diffusion case, the jumps lead to higher losses within in the system.

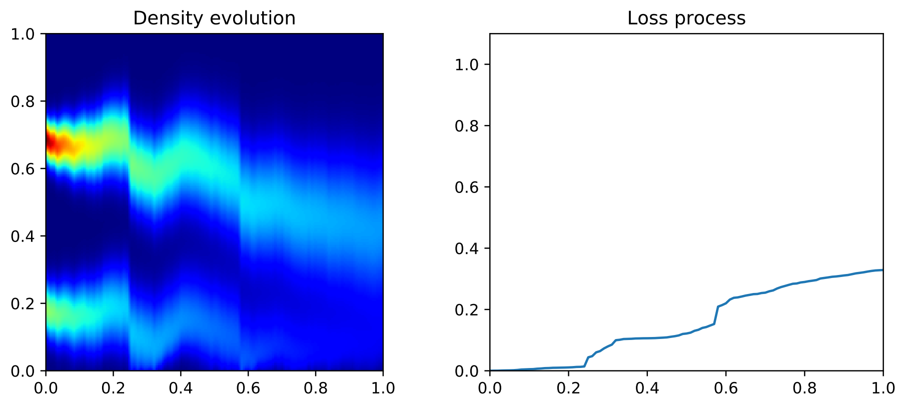

Figure 3.3: The left plot shows a heat plot of the density process $(t, x) \mapsto V_t(x)$, for a fixed realisation of the common noise W^0 and jump process J^0. We used the parameters $\alpha = 0$, $\rho = 0.4$, $\sigma = 0.2$, $\mu = 0$, $\lambda = 0.75$ and $\nu = -0.05$ with Gaussian jump heights. The impact kernel k is a triangle on $[0, 0.015]$ with height $2/0.015$. The right plot displays the evolution of the loss process $t \mapsto L_t$.

Lastly, for Figure 3.4 we also add contagion to the jump-diffusion case. So that the regarding McKean-Vlasov problem can be stated as

$$X_t = X_0 + \mu dt + \sigma W_t^0 + J_t^0 - \alpha L_t \ , \ L_t = \mathbb{P}(\tau \leq t \mid W^0, \ J^0). \tag{3.29}$$

One can observe that the defaults triggered by the first jump lead to a huge default cascade that entirely "kills" the unhealthy part of the system and puts the remaining one in a fairly bad position. Then, with the second jump occurring also the latter one dies out. This illustrates how a large contagion parameter paired with big negative jumps can cause the end of the whole system. As displayed by the loss process, shortly after half of the time period all firms have dropped out of the system.

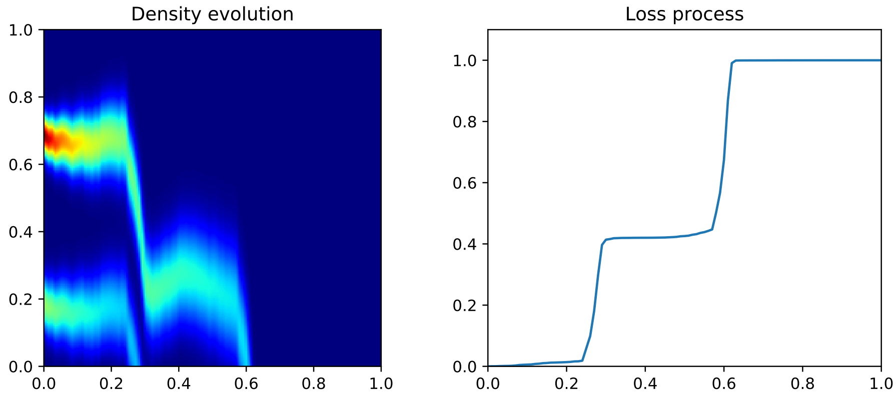

Figure 3.4: The left plot shows a heat plot of the density process $(t, x) \mapsto V_t(x)$, for a fixed realisation of the common noise W^0 and jump process J^0. We used the parameters $\alpha = 1.5$, $\rho = 0.4$, $\sigma = 0.2$, $\mu = 0$, $\lambda = 0.75$ and $\nu = -0.05$ with Gaussian jump heights. The impact kernel k is a triangle on $[0, 0.015]$ with height $2/0.015$. The right plot displays the evolution of the loss process $t \mapsto L_t$.

3.7 A Possible Model Extension: Idiosyncratic Jumps

The model described in the last sections only considers a common jump process capturing bigger system-wide shocks for the firms. An extensional approach could be to also include idiosyncratic jumps, which only have effects on single institutions or groups of institution. Inspired by [28], we introduce a way of incorporating such events. Let $I := \{1, ..., N\}$ be the index set of the initial banks in the system. Moreover, let $\mathcal{P}(I)$ denote the respective power set containing all subsets of I with $|\mathcal{P}(I)| = 2^N$. In the following, an element of the power set is denoted by π. For convenience, we define $\Pi(I) := \mathcal{P}(I) \setminus \{\emptyset, I\}$. Using this notation we introduce a new jump process by

$$J_t^i := \frac{1}{2^{N-1} - 1} \sum_{\pi \in \Pi(I)} \mathbb{1}_{\{i \in \pi\}} J_t^\pi, \tag{3.30}$$

where the processes J^π are independent compound Poisson processes of the form

$$J_t^\pi = \sum_{k=1}^{N_t^\pi} \psi_k^\pi. \tag{3.31}$$

In the latter expression, N^π is a Poisson process with intensity parameter λ^i. Moreover, we assume that the jump heights ψ_k^π, $\pi \in \Pi(I)$, $k = 1, 2, ...,$, are i.i.d. following some distribution function ϖ. Then it follows immediately that

$$\mathbb{E}\left[J_t^i\right] = \frac{1}{2^{N-1} - 1} \sum_{\pi \in \Pi(I)} \mathbb{1}_{\{i \in \pi\}} \mathbb{E}\left[J_t^\pi\right] = \frac{1}{2^{N-1} - 1} \sum_{\pi \in \Pi(I)} \mathbb{1}_{\{i \in \pi\}} \lambda^i \mathbb{E}[\psi_1^{\{N\}}] t = \lambda^i \mathbb{E}[\psi_1^{\{N\}}] t. \tag{3.32}$$

From now on, we use the notation $\nu^i := \mathbb{E}[\psi_1^{\{N\}}]$. By adding those idiosyncratic jumps to a particle i we get the dynamics

$$dX_t^i = \left(r - \lambda\nu - \lambda^i\nu^i\right) dt + \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}dW_t^i - \alpha d\mathcal{L}_t + dJ_t^0 + dJ_t^i. \tag{3.33}$$

Although we do not provide detailed results about the limiting behaviour as in the previous sections, one can give a certain intuition about how things behave after passing to the limit. The first thing to note is that, by a law of large number argument, we can expect that

$$J_t^i = \frac{1}{2^{N-1}-1} \sum_{\pi\in\Pi(I)} \mathbb{1}_{\{i\in\pi\}} J_t^\pi \xrightarrow{N\to+\infty} \mathbb{E}\left[J_t^{\{N\}}\right] = \lambda^i\nu^i t.$$

This shows that in case of an immensely increasing population of the banking system, the effect of the idiosyncratic jumps become similar to a drift. The same observation can be made by noting that J^i is a sum of independent compound Poisson processes and hence its intensity is given by the sum of the latter. Therefore, the intensity of the idiosyncratic jump process equals

$$\sum_{\pi\in\Pi(I)} \mathbb{1}_{\{i\in\pi\}}\lambda^i.$$

At the same time, the jumps of the process are scaled by the factor $\left(2^{N-1}-1\right)^{-1}$. This fact makes clear that a big population let the intensity of the processes also tend to infinity, while the size of the jumps are getting smaller and smaller. Hence, the are many jumps of small size, which can create the character of a drift. This result is in line with the intuition arising from the law of large number argument. Lastly, we see that the drift term the process J^i completely vanishes out by its compensator term. Therefore, the effect of the idiosyncratic jumps averages out in case the amount of banks gets big enough. One should note that this behaviour is similar to the idiosyncratic diffusion parts which also vanishes in the limit is proved in Proposition 3.1.

Chapter 4

Model Calibration

In this last chapter, we present how the model described in the last section can be used to reproduce quantities observed in the market and so in particular, study if the contagion feature actually adds any contribution. To do so we want to calibrate the model to market spreads of the iTraxx Europe Series index on the 31^{st} December 2016. We use [1, Section 6.3] as the main source for comparing our results. Since they only considered a jump-diffusion without a contagion mechanism, it is interesting to see what effect this new feature brings in. Before we state the main calibration problem, we give an explanation on how to find the density of the initial distances-to-default needed to simulate the system of particles. Moreover, we introduce a simple method to estimate the correlation within the market using real market equity data.

4.1 Finding the Initial Distribution of the Distances-To-Default

As already stated in the introduction of this chapter, it is sufficient to find the density V_0 of the initial distances-to-default, denoted by X_0^i, $i = 1, ...N$. Recall from earlier, that by definition it holds

$$X_0^i = \log(A_0^i) - \log(D_0^i), \tag{4.1}$$

where A_0^i denotes the asset value and D_0^i the amount of debt held by company i at the regarding reference date, which in our case is 31^{st} December 2016. To find those quantities we rely on the regarding balance sheet data provided by Yahoo Finance. Note that for this calibration exercise, we are interested in the companies listed in the iTraxx Europe Series and in theory N equals the population size of this index. In our case, we have $N = 58$, which is lower than the actual number. This difference arises from missing data points in the named data source. Using the financial statements, we can compute the vector $X_0 = (X_0^1, ..., X_0^{52})$. To estimate the unknown density V_0 of those samples we use a kernel density estimation (KDE). The actual estimator function is hence defined by

$$\hat{V}_{0,h}(x) := \frac{1}{hN} \sum_{i=1}^{N} K\left(\frac{x - X_0^i}{h}\right), \tag{4.2}$$

where K denotes the respective kernel function and $h > 0$ is a smoothing parameter called *bandwidth*. For the estimation we choose a Gaussian kernel, i.e.

$$K(t) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\}.$$

It remains to find a robust value for the bandwidth h. Following Silverman's rule [29], we take

$$h = \left(\frac{4\hat{\sigma}^5}{3N}\right)^{\frac{1}{5}} \approx 1.06\hat{\sigma}N^{-\frac{1}{5}}, \tag{4.3}$$

where $\hat{\sigma}$ is the empirical standard deviation of the samples $X_0^1, ..., X_0^{52}$. In our case, we end up with the choice $h = 0.7976$. The resulting density estimate and the underlying histogram is displayed in the following figure.

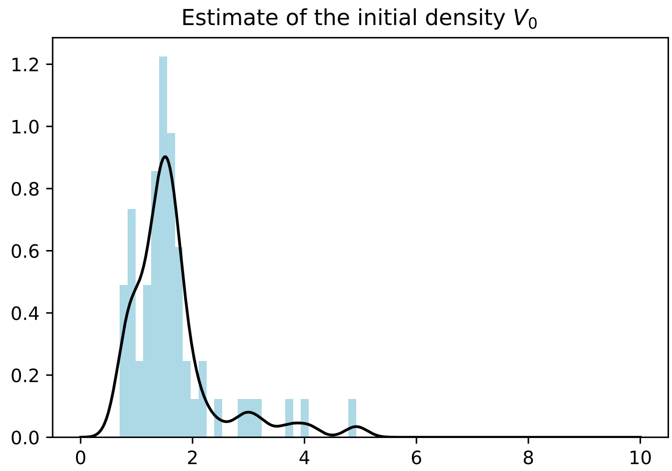

Figure 4.1: Plot of the kernel density estimate of the initial density function $x \mapsto V_0(x)$ of the distances-to-default samples X_i^0, $i = 1, ..., 52$ using a Gaussian kernel with bandwidth parameter $h = 0.3753$.

4.2 Estimating the Correlation in the Market

To find an accurate value for the correlation parameter ρ we do not want to rely on the calibration to the market spreads. Rather, we want it to be estimated by the empirical correlation observed in the market within one year before the effective date 31[st] December 2016. Recall that in our model ρ is an indicator for how much the distance-to-default is driven by moves in the overall market, which is represented by the Brownian motion W^0. For our analysis we take the S&P500 index as a proxy for the market. Then we are able to compute the empirical correlation between the moves in S&P500 and each company i in the iTraxx index over the year 2016. The latter is denoted by $\hat{\rho}^i$. Lastly, we simply take the arithmetic average to get an estimate for the correlation parameter by

$$\rho = \frac{1}{N} \sum_{i=1}^{N} \hat{\rho}^i. \tag{4.4}$$

In our case, this short analysis results in the observation that

$$\rho \approx \hat{\rho} = 0.1937. \tag{4.5}$$

Note that this method has also the advantage of reducing the number of parameters in the following calibration problem by one.

4.3 SPDE and Loss Simulation

To simulate the evolution of the density $V_t(x)$ over time we introduce both a space grid

$$0 = x_0, ..., x_j = x_0 + j\Delta x, ..., x_J = x_0 + J\Delta x = 9,$$

where we choose $\Delta x = (x_J - x_0)/J = 9/200$ and a time grid

$$0 = t_0, ..., x_i = x_0 + i\Delta t, ..., t_I = t_0 + I\Delta t = 10.$$

where $\Delta t = (t_I - t_0)/T = 100$. Using those notations, we define the approximation $V_j^m = V(t_m, x_j)$. Then for a given realisation of market factors we can approximate the loss process at time t_k, $k = 1, ..., I$, by

$$L_{t_k} = 1 - \int_0^{x_{max}} V(t_k, x)dx = 1 - \Delta x \sum_{j=1}^{J} V_j^k. \tag{4.6}$$

Recall from Section 1.5. that the spread of a CDO index is mainly driven by the risk-neutral expectation of the outstanding notional process $\widetilde{N}_t$. By choosing the notional as described, this process indicates how many companies have not defaulted by time t. Hence, using the approximation of the loss process above, we have at any time t_k, $k = 1, ..., I$, that $\widetilde{N}_{t_k} = 1 - L_{t_k}$. The corresponding expectation is estimated using a simple Monte Carlo estimation, i.e.

$$\mathbb{E}\left[\widetilde{N}_{t_k}\right] \approx \frac{1}{N_{Sim}} \sum_{i=1}^{N_{Sim}} \widetilde{N}_{t_k}(\phi^i) = \frac{1}{N_{Sim}} \sum_{i=1}^{N_{Sim}} (1 - L_{t_k}(\phi^i)), \tag{4.7}$$

where ϕ^i denotes the i-th sample of the triple $(W_{t_k}^0, W_{t_k}^1, ..., W_{t_k}^{52}, J_{t_k}^0)$ and N_{Sim} the number of simulations for the Monte Carlo routine.

4.4 Calibration Problem

In the next two sections, we use our model to estimate the remaining parameters by calibrating it to real market data of a CDO index. As shown in the subsequent result part we fit both the diffusion and the jump-diffusion model with and without contagion. However, we do not include the contagion parameter in the calibration problem below.

Problem 4.1. *Given market spreads at time $t = 0$ of a CDO index, denoted by $CI_0(T_i)$, for maturities T_i, $i = 1, ..., M$, we solve the minimisation problem*

$$\begin{aligned} \min_{\theta} \quad & \textstyle\sum_{i=1}^{M} \left(CI_0(T_i) - CI_0^{\theta, X_0}(T_i)\right)^2 \\ s.t. \quad & \sigma > 0,\ \lambda \geq 0,\ \mu_Y \leq 0,\ \sigma_Y \geq 0, \end{aligned} \tag{4.8}$$

where $CI_0^{\theta, X_0}(T_i)$ denotes the spreads computed under the model based on the parameters $\theta = (\sigma, \lambda, \mu_Y, \sigma_Y)$ and the vector of initial distances-to-default X_0.

In case we only want to fit the diffusion model, the jump parameters are chosen to zero. Hence, the problem reduces to calibrate the volatility parameter σ.

4.5 Calibration Results

As already written, we fit the model to the market spreads of the iTraxx Europe Series at the 31^{st} December 2016. For the data set we can observe that the index spreads increase with longer maturity. This is to be expected since in a longer time period more companies in the index may default. To compare the calibration results to the market data we use two ways to measure the errors of the model. In particular, the fit is evaluated by ARPE (Average Relative Percentage Error)

$$\text{ARPE} = \frac{1}{M} \sum_{i=1}^{M} \frac{|y_i - y_i^\theta|}{y_i} \tag{4.9}$$

and the RMSE (Root Mean Square Error)

$$\text{RMSE} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \left(y_i - y_i^\theta\right)^2}, \tag{4.10}$$

where y is the vector of the observed spreads, y^θ the vector of the spreads implied by the model and M the number of spreads computed. The following table displays the parameters calibrated

to the market spreads for both the diffusion and jump-diffusion case. For the simulation of the SPDE we assumed the magnitude of the jumps arising form the process J^0 to be normally distributed. Therefore, the jump component adds three additional parameters in comparison to the pure diffusion case.
As a first analysis, we compare the parameters presented in Table 4.1 to the results stated in [1] for the 22$^{\text{nd}}$ February 2007, which are given in the Appendix. For the diffusion we observe a slightly higher value for σ, which indicates that the market was considered to be more volatile. On the other side, the correlation parameter is lower. However, at this point one should bare in mind that the ρ presented in this thesis is estimated using historical data instead of including it in the calibration problem stated in the previous section. This also has the consequence that the same value is considered for the jump-diffusion. Similarly to what is shown in [1], the volatility parameter is smaller for the latter model specification. This is natural to expect, since the jumps add an additional downside potential for the system, so that this must not be captured by the Brownian motion alone. Therefore, the latter can be considered less volatile. Moreover, we observe a jump intensity of $\lambda = 0.1$, which indicates that a market crash is expected more or less every 10 years. In comparison, in Table A.1 the parameter lies only at $\lambda = 0.04$. So at out reference date the spreads imply that crises are expected more often than 9 years before. Additionally, in case a jump occurs then our results indicate that on average a company's asset value falls 15%, which is 8 percentage points more than found in 2007. Lastly, we get that the magnitude of the jumps is more volatile than in [1]. In conclusion, the analysis of the model parameters demonstrates that in general, the market is expected to be more unstable in 2016 than in 2007 before the credit crisis. This is also indicated by the higher spreads after the crisis and the fact that the differences of the prices between the maturities is bigger. For example in 2007, the gap between the 5Y index spread and 7Y spread was 9 basis points while in 2016 it was 22 basis points. Therefore, defaults are assumed to happen more often.

Parameters	Diffusion	Jump-Diffusion
σ	0.24	0.21
ρ	0.19	0.19
λ	-	0.1
μ_Y	-	-0.15
σ_Y	-	0.05

Table 4.1: Table of the estimated parameters for the 31st December 2016 for the diffusion and jump-diffusion model.

In Table 4.2 the spreads from the market are displayed along with the ones implied by the two model specifications. Note that the results presented do not include any contagion yet. Being in line with [1], we note that compared to the pure diffusion model both error measures are lower for the jump-diffusion. In particular, we see that the latter is more able to fit the short 3Y and long 10Y maturity.

Maturities	Market (bps)	Diffusion (bps)	Jump-Diffusion (bps)
3Y	43	57	53
5Y	72	76	75
7Y	94	87	88
10Y	110	93	100

Table 4.2: Table of the spreads resulting from the diffusion and jump-diffusion model without any contagious behaviour in the system ($\alpha = 0$). We assume that $r = 0.01$ and $R = 0.4$.

As one objective of this thesis is to study which contribution is made by the contagion mechanism, we also present model spreads for different values for the parameter α in Table 4.3 and 4.4. The regarding error measures demonstrate how the additional feature leads to better fits of the market spreads. In particular, for the diffusion both errors are the lowest for $\alpha = 0.5$ compared to the other values displayed. For the jump-diffusion this is the case for $\alpha = 0.2$. One should also note that the interplay between the jumps and contagious behavior yields that for the latter model type a slight change in the values for α can cause a big difference. On the other side, we have to choose more varying values for the diffusion to observe a discernible effect. It is important to mention that for now the choices of the contagion parameter are arbitrary and only have the purpose to substantiate the idea of [4] that adding a contagious character to the structural model brings additional benefits. Indeed, the results can be interpreted in a way that especially for the post-crisis world it might be of importance to include this feature. It remains a goal for further research to study in more detail how a suitable value for α can be extracted from market quantities.

Maturities	Market (bps)	$\alpha = 0.3$ (bps)	$\alpha = 0.5$ (bps)	$\alpha = 0.7$ (bps)
3Y	43	57	57	60
5Y	72	78	78	82
7Y	94	89	91	96
10Y	110	98	101	106

Table 4.3: Table of the spreads resulting from the diffusion model for different contagion parameter values α. We assume that $r = 0.01$ and $R = 0.4$.

Maturities	Market (bps)	$\alpha = 0.15$ (bps)	$\alpha = 0.2$ (bps)	$\alpha = 0.25$ (bps)
3Y	43	53	50	55
5Y	72	75	75	79
7Y	94	89	89	94
10Y	110	101	103	109

Table 4.4: Table of the spreads resulting from the jump-diffusion model for different contagion parameter values α. We assume that $r = 0.01$ and $R = 0.4$.

	Diffusion			
	$\alpha = 0$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$
RMSE	11.7260	10.0125	8.9722	10.1119
ARPE	0.1525	0.1428	0.1307	0.1480
	Jump - Diffusion			
	$\alpha = 0$	$\alpha = 0.15$	$\alpha = 0.2$	$\alpha = 0.25$
RMSE	7.8262	7.3314	5.8737	6.9642
ARPE	0.1072	0.1023	0.0835	0.0963

Table 4.5: Table of the error measures for the fit of the 31st December 2016.

4.6 Further Research Ideas for Numerical Analyses

We use this section to present two ideas for further research besides doing more analysis on the contagion parameter α mentioned above. While we have already introduced a possible model extension by idiosyncratic jumps, we now focus on numerical implementations.

4.6.1 Calibration to CDO Tranches

Taking up on the analysis presented before, a natural next step would be to not only calibrate to index spreads. In particular, one could study the impact of the contagion mechanism by considering also the tranches of a CDO index. As stated in [1], the resulting calibration problem for this analysis is given by

Problem 4.2. *Given market spreads at time $t = 0$ of a CDO index, $CI_0(T_i)$ and CDO tranches, $C_0^j(T_i)$, for maturities T_i, $i = 1, ..., M$ and tranches $j = 1, .., G$, we solve the minimisation problem*

$$\begin{aligned} \min_{\theta} \quad & \sum_{i=1}^{M} \sum_{j=1}^{G} \left(C_0^j(T_i) - C_0^{j,\theta,X_0}(T_i) \right)^2 + \sum_{i=1}^{M} \left(CI_0(T_i) - CI_0^{\theta,X_0}(T_i) \right)^2 \\ s.t. \quad & \sigma > 0,\ \lambda \geq 0,\ \mu_Y \leq 0,\ \sigma_Y \geq 0, \end{aligned} \tag{4.11}$$

where $CI_0^{\theta,X_0}(T_i)$ and $C_0^{j,\theta,X_0}(T_i)$ denote the spreads computed under the model based on the parameters $\theta = (\sigma, \lambda, \mu_Y, \sigma_Y)$ and the vector of initial distances-to-default X_0.

Moreover, performing the analysis for more reference dates would increase the robustness of the results and could give a more meaningful conclusion on the contribution of the contagion. Due to a lack of market data during the Corona-crisis it was not possible to carry out the upper ideas.

4.6.2 An Approach to Close-To-Market Estimations

The second idea for a further numerical analysis is mainly based on the form of a financial balance sheet and could be used to reduce the number of parameters to calibrate by estimating σ using market quantities. Following our initial introduction, consider that the firm i's, $i = 1, .., N$, asset value is described by the process $(A_t^i)_{t\geq 0}$, the value of all emitted shares by $(S_t^i)_{t\geq 0}$ and the amount of debt by $(D_t^i)_{t\geq 0}$. Then by assuming that the market capitalisation moves similar to the book value, the balance sheet at time t is of the form

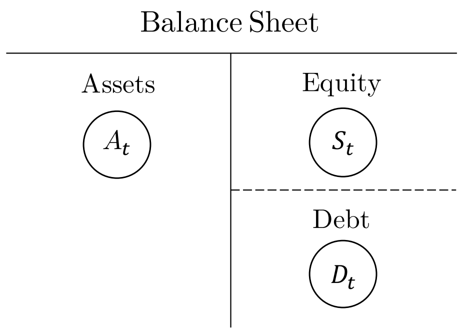

Figure 4.2: Picture of an arbitrary finacial balance sheet.

Considering an ideal Black-Scholes world, we assume that the asset process of company i satisfies the dynamics

$$\frac{dA_t^i}{A_t^i} = rdt + \sigma_A^i(t)d(\rho W_t^0 + \sqrt{1-\rho^2}W_t^i) \tag{4.12}$$

and similar for the debt process

$$\frac{dD_t^i}{D_t^i} = rdt, \tag{4.13}$$

where r denotes the risk-free rate, σ_A^i the volatility of the assets and W^0,W^i are two independent standard Brownian motions. Note that in theory it must hold

$$A_t^i = S_t^i + D_t^i. \tag{4.14}$$

Then in principle it follows that

$$\begin{aligned} dS_t^i = d(A_t^i - D_t^i) &= r(A_t^i - D_t^i)dt + \sigma_A^i(t)A_t^i d(\rho W_t^0 + \sqrt{1-\rho^2}W_t^i) \\ &= rS_t^i dt + \sigma_A^i(t)(S_t^i + D_t^i)dW_t \end{aligned}$$

with $W_t := \rho W_t^0 + \sqrt{1-\rho^2}W_t^i$. On the other side, since we assume a Black-Scholes setting, we have

$$dS_t^i = rS_t^i dt + \sigma_S^{imp}(t)S_t d\widetilde{W}_t,$$

were $\sigma_S^{imp}(t)$ is the implied volatility at time t arising from some at-the-money Call and Put options in the market and $\widetilde{W}$ is another standard Brownian motion. Then heuristically, we have in law that $\sigma_S^{imp}(t)S_t^i \sim \sigma_A^i(t)(S_t^i + D_t^i)$, which finally gives a proxy for the volatility of the assets at time t by

$$\sigma_A^i(t) \sim \sigma_S^{imp}(t)\frac{S_t^i}{S_t^i + D_t^i}.$$

Estimating the correlation as described in Section 4.2, all parameters of the diffusion model are already estimated by observed market quantities. Based on those parameters, we can initialise the system at any time 0 and can price credit derivatives in the near future. To evaluate the goodness of those predictions one can again use the measures introduced above.

On top of this rather simple setting, we can assume that from time to time some deteriorations in the asset value triggered by market imperfections and interdependence occur. In particular, we extend the model by exogenous Levy jumps and contagious behaviour. However, keeping the assumption that the jumps arise from a compounded Poisson process we get four new free variables, whose estimation using market data is far from trivial.

Conclusion

In this thesis, we introduce an extension of the multi-name model proposed in [2]. In particular, we include a jump-diffusion similar to [1] and an endogenous contagion mechanism like presented in [4]. We derive some of the theoretical properties of the model from a SPDE point of view and study its ability to fit prices for credit derivatives, while we are mainly interested in the contribution of the contagion feature.

We first introduce a finite particle system, where each particle X_t^i, $i = 1, ..., N$, represents the distance-to-default of a firm i at time t. Then by taking the limit in the population size of the system, we observe that the limiting empirical measure satisfies a SPDE on the positive half-line with absorbing origin. In addition, if we suppose that the distances-to-default of the companies in the finite system are given by a jump-diffusion with a positive feedback effect arising from defaults, it turns out that in the limit this leads to a conditional McKean-Vlasov problem. Moreover, the law of this problem indeed satisfies the same SPDE as the limiting empirical measure. Using this results, we are able to find an explicit expression for the density function of the large basket system.

Since we consider an idiosyncratic Brownian motion for each particle, it is natural to also think about including idiosyncratic jumps. However, one can heuristically argue that in the limit they turn into a drift term, which cancels out with the compensator arising from the risk neutral setting. Therefore, they do not have any effect in the large population case.

Finally, we want to study how the model can fit market spreads of a CDO index. We start by estimating the initial state of the density function using balance sheet statements of the regarding index companies. Moreover, we exclude the correlation parameter from the least-square-type calibration problem, but estimating it using the empirical correlation observed in the market. We observe that the parameters, resulting from the optimisation problem, indicate a more unstable market compared to results from a pre-crisis analysis. Moreover, the error measures show that the jump-diffusion better fits the prices than the diffusion. At this point it is important to mention that this thesis does not aim to derive efficient numerical methods for the pricing and calibration exercise. Therefore, due to huge numerical costs, the number of simulations for the Monte Carlo routines are kept fairly low compared to [1]. This could be one reason for the errors of the fits to be not smaller.

The main observation from the numerical implementation is that the parameter α adds a noteable contribution to the model in terms of leading to a better fit. This gives rise to the hypothesis that contagion is a estimable feature and it is worthy of analysing its impact in more detail as part of future research. Moreover, one could try to find a robust way of estimating the parameter from market observables.

Appendix A

A.1 Weak Derivatives and Sobolev Spaces

As proposed in [4], one can also consider the more general case that the impact kernel k lies in the Sobolev space $\mathcal{W}_0^{1,1}(\mathbb{R}_+)$. In the following, we recall the definition of weak derivatives and the derived Sobolev spaces. Thereby we focus only on the definitions on $\mathbb{R}$. The explanations are highly inspired by [30]. We start by considering the space given by

$$C_c^\infty(U) := \{\phi : U \to \mathbb{R} \mid \phi \in C^\infty(U),\ \operatorname{supp}(\phi) \subseteq U \text{ compact}\}, \tag{A.1}$$

for an open subset $U \subseteq \mathbb{R}$ and $C^\infty(U)$ denoting the set of all smooth functions mapping from U into $\mathbb{R}$. The upper set of function is often referred to as the space of so-called test functions. In case, one assumes another function $f \in C^1(U)$ and takes a arbitrary $\phi \in C_c^\infty(U)$, we get by integration by parts

$$\int_U f(x)\phi'(x)dx = [f(x)\phi(x)]_{x\in U} - \int_U f'(x)\phi(x)dx = -\int_U f'(x)\phi(x)dx,$$

where the second equality follows from the fact that ϕ has a compact support lying in the open set U. Following this intuition we can define the weak derivative of a function $f \in L^1(U)$.

Definition A.1 (Weak Derivative)**.** Consider, for $1 \le p \le \infty$, a function $f \in L^p(U)$, where $U \subseteq \mathbb{R}$ is an open set. Then we call $g \in L^p(U)$ a weak derivative of order $k \in \mathbb{N}_0$ of f if and only if it holds

$$\int_U f(x)\phi^{(k)}(x)dx = (-1)^k \int_U g(x)\phi(x)dx \tag{A.2}$$

for all test functions $\phi \in C_c^\infty(U)$, where $\phi^{(k)}$ denotes the k-th derivative of ϕ.

We often write $g = f^{(k)}$, although the derivative is meant in the weak sense. Based on the idea of weak derivatives, we can now introduce the so called *Sobolev spaces*. It contains all functions $f \in L^p(U)$ so that f itself and its weak derivative up to a certain order $k \in \mathbb{N}_0$, if it exists, have a finite L^p norm. This yields the following more rigorous defintion.

Definition A.2 (Sobolev Space)**.** For any $1 \le p \le \infty$, we define for an open $U \subseteq \mathbb{R}$ and $k \in \mathbb{N}_0$ the Sobolev space

$$\mathcal{W}^{k,p}(U) := \left\{ f \in L^p(U) \mid \text{For all } i \le k,\ f^{(i)} \text{ exists in the weak sense and } f^{(i)} \in L^P(U) \right\}.$$

One can equip the space $\mathcal{W}^{k,p}(U)$ with a natural norm defined by

$$\|f\|_{\mathcal{W}^{k,p}(U)} := \begin{cases} \left(\sum_{i=0}^k \int_U |f^{(i)}(x)|^p dx\right)^{\frac{1}{p}} & \text{, if } p \in [1,\infty), \\ \max_{i=0,\dots,k} \|f^{(i)}\|_\infty & \text{, if } p = \infty. \end{cases} \tag{A.3}$$

Using this norm, we state the last result of this section regarding Sobolev spaces. Firstly, we introduce the space $\mathcal{W}_0^{k,p} := \overline{C_c^\infty}^{\|\cdot\|_{\mathcal{W}^{k,p}(U)}}$. Therefore, $\mathcal{W}_0^{k,p}$ is the closure of $C_c^\infty(U)$ with respect to the norm $\|\cdot\|_{\mathcal{W}^{k,p}(U)}$. One can even show that the latter space is dense in $\mathcal{W}^{k,p}(U)$. In particular, for an arbitrary $f \in \mathcal{W}_0^{k,p}(U)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(U)$, such that $\lim_{n\to\infty}\|f_n - f\|_{\mathcal{W}^{k,p}(U)} = 0$. The space $\mathcal{W}_0^{k,p}$ is a special case of spaces defined by a linear operator called trace. The latter can be characterised by the following theorem, which we state without proof.

Theorem A.3 (Trace Theorem)**.** *Let $1 \leq p < \infty$ and $U \subseteq \mathbb{R}$ be open and bounded. Assume that ∂U is C^1. Then there exists a bounded linear operator $T : \mathcal{W}^{1,p}(U) \to L^p(\partial U)$ such that*

1. $Tf = f|_{\partial U}$ *if* $f \in \mathcal{W}^{1,p}(U) \cap C(\overline{U})$,
2. $\|Tf\|_{L^p(\partial U)} \leq C\|f\|_{\mathcal{W}^{1,p}(U)}$ *with* $C(p, U) > 0$ *constant.*

In the notation of the theorem, we call Tf the trace of f. The name gets clear by the first point of the theorem. It describes the behavior of the function f on the boundary ∂U, which makes sense, since we demand $f \in C(\overline{U})$. Using the notation of traces, we can characterise the space $\mathcal{W}_0^{1,p}(U)$ introduced above. Indeed, it holds

$$f \in \mathcal{W}_0^{1,p}(U) \iff Tf = 0. \tag{A.4}$$

One can easily show that the first implication is true. For this, take an arbitrary $f \in \mathcal{W}_0^{1,p}(U)$. Then we know by the definition of the space, that there exists a sequence $(f_m)_{m\in\mathbb{N}} \subseteq C_c^\infty(U)$ such that $f_m \to f$ with respect to $\|\cdot\|_{\mathcal{W}^{1,p}}$ as m tends to infinity. Clearly, we have $Tf_m = 0$, since $f_m \in C_c^\infty(U)$ with U open and so $f_m|_{\partial U} = 0$. Lastly, by the trace theorem we know that T is a linear bounded operator and hence continuous. Therefore, $f_m \to f$ in $\mathcal{W}^{1,p}(U)$ implies $Tf_m \to Tf$ in $L^p(\partial U)$. By this, the desired implication follows.

A.2 Supplementary Plots and Tables

We use this section to display three plots of the processes used for the illustration presented in Section 3.6. In Figure A.1, we see the realisation of the underlying common diffusion process W^0 and the respective common Poisson process J^0. Lastly, we include a plot of the initial density considered for the illustrations. One can easily observe the two parts within the system represented by the peaks in the function.

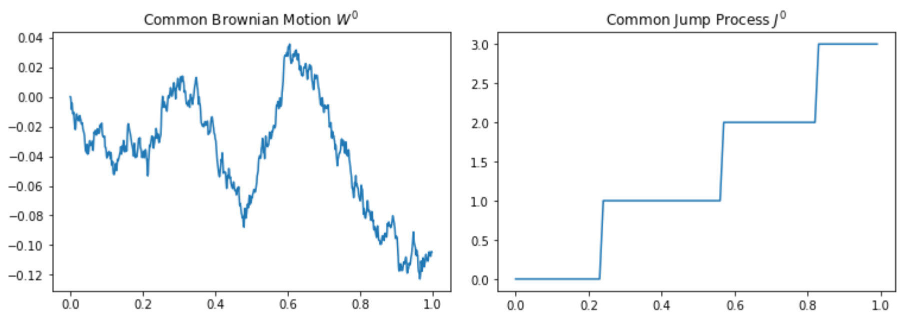

Figure A.1: The plot shows the realisation of the common Brownian motion and Poisson process over the time horizon $[0, 1]$ with drift parameter $\mu = 0$, volatility $\sigma = 0.08$ and intensity $\lambda = 0.75$.

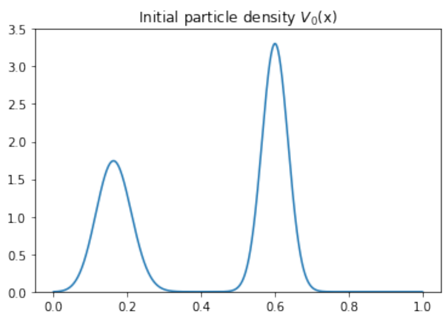

Figure A.2: The plot shows the initial density of the particles V_0, which is assumed to be given by $V_0(x) = \frac{1}{2}x(2 - x)\exp(-50(x - 0.3)^2) + \frac{1}{2}\exp(-100(x - 1.2)^2)$.

A.3 Calibration Results for Comparison

The following table, we display the parameters estimates as presented in [1]. As explained in Section 4.5, we use them for a comparison with our results to deduce how the market stability is seen differently from a post-crisis point of view.

Parameters	Diffusion	Jump-Diffusion
σ	0.18	0.16
ρ	0.22	0.11
λ	-	0.04
μ_Y	-	-0.07
σ_Y	-	0.01

Table A.1: Table of the estimated parameters for the 22^{nd} February 2007 as presented in [1].

A.4 Code

In this section, we present the core parts of the code used simulate the SPDE of the limiting measure ν and to calibrate the model to market data.

```
### IMPORT PACKAGES ###

import numpy as np
import scipy
import pandas as pd
from scipy.interpolate import interp1d as interp
from scipy.integrate import quadrature as quad
from scipy.optimize import least_squares
from sklearn.neighbors import KernelDensity
```

```
### DEFINE CLASS FOR PARAMETERS OF THE SPDE ###

class Parameters:
    def __init__(self, time_steps, space_steps, total_time, upper_space_limit,
      sigma, rho, gauss_steps, lamb, mu_jump, sigma_jump, alpha):
        self.time_steps = time_steps
        self.space_steps = space_steps
        self.total_time = total_time
        self.upper_space_limit = upper_space_limit
        self.sigma = sigma
        self.rho = rho
        self.gauss_steps = gauss_steps
        self.lamb = lamb
        self.mu_jump = mu_jump
        self.sigma_jump = sigma_jump
        self.time_delta = self.total_time / self.time_steps
        self.space_delta = self.upper_space_limit / self.space_steps
        self.alpha = alpha
```

```
### DEFINE CLASS TO RUN SIMULATION OF SPDE ###

class SPDE:
    def __init__(self, V0, parameters):
        BM = generate_BM(parameters.total_time, parameters.time_steps)
        shifts = [parameters.rho * parameters.sigma * (BM[i+1] - BM[i])
            for i in range(parameters.time_steps)]
        jumps = generate_Poisson(parameters.total_time, parameters.time_steps,
            parameters.lamb)
        self.para = parameters
        self.jumps = jumps
        self.space_grid = np.array([i * self.para.space_delta
```

```
            for i in range(self.para.space_steps + 1)])
        self.time_grid = np.array([i * self.para.time_delta
            for i in range(self.para.time_steps + 1)])
        self.data = [np.array([V0(x) for x in self.space_grid]) /
            quad(V0,0.0,self.para.upper_space_limit)[0]]
        self.loss = [0.0]
        self.shifts = shifts
        self.jumps = jumps

    def simulate(self):
        for i in range(self.para.time_steps):
            f = self.data[i]
            kernel_dis = np.zeros_like(self.time_grid)
            kernel_dis[1] = 2.0 / self.para.time_delta
            kernel = interp(self.time_grid, kernel_dis, bounds_error = False,
                fill_value = 0.0)
            no_losses = len(self.loss)
            extended_losses = np.zeros_like(self.time_grid)
            extended_losses[:no_losses] = self.loss
            L = interp(self.time_grid, extended_losses, bounds_error = False,
                fill_value = 0.0)
            d_L = L(self.time_grid[i]) - L(self.time_grid[i - 1])
            cont = self.para.alpha * d_L
            f_next = push_forward(f, self.space_grid, self.para.time_delta,
                self.shifts[i], self.para.sigma, self.para.rho,
                self.para.mu_jump, self.para.sigma_jump, self.jumps, i, cont)
            self.data.append(f_next)
            self.loss.append(1.0 - integrate.trapz(f_next, self.space_grid))
```

```
 ### DEFINE HELPING FUNCTIONS FOR SPDE SIMULATION ###

def push_forward(fvals, grid, dt, shift, sigma, rho, mu_jump, sigma_jump,
  jumps, i, cont) :
    f = interp(grid, fvals, bounds_error=False, fill_value=0.0)
    x0 = np.linspace(-5, 5, 400)
    gauss = GAUSS_CONST * np.exp( -0.5 * x0 * x0 )
    height = np.random.normal(mu_jump, sigma_jump)
    SHIFT = shift + sigma * np.sqrt(1 - rho ** 2) * np.sqrt(dt) * x0 +
        height * jumps[i] - cont
    shifted_f = f(grid[:,None] - SHIFT)
    f_conv = integrate.trapz(shifted_f * gauss, x0, axis = 1)
    return f_conv

def generate_BM(time_horizon, no_steps):
    dt = time_horizon / no_steps
    discrete_BM = np.random.normal(0.0, np.sqrt(dt), no_steps + 1)
    discrete_BM[0] = 0.0
    discrete_BM = np.cumsum(discrete_BM)
    return discrete_BM

def generate_Poisson(time_horizon, no_steps, lamb):
    dt = time_horizon / no_steps
    pois = np.random.poisson(lamb * dt, no_steps)
    return pois
```

```
 ### DEFINE PRICING FUNCTION FOR CDO INDEX ###

def CDO_index_price(V0, parameters, r, Rec, freq, mat, NSim):
    spreads = []
    payment_delta = 1.0 / freq
    payment_dates = [i * payment_delta for i in range(1, freq * max(mat) + 1)]
    bank_acc = np.exp([p * r for p in payment_dates])
    averaging = np.zeros(parameters.time_steps)
    for i in range(NSim):
```

```
        LM = LinearModel(V0, parameters)
        LM.simulate()
        L = LM.loss
        Z = [1 - l for l in L]
        averaging += Z[1:]
    averaging /= NSim
    MC_loss = averaging[int((parameters.time_steps / parameters.total_time)
        // freq) - 1 :: int((parameters.time_steps / parameters.total_time)
        // freq)]
    for T in mat:
        temp = MC_loss[ : freq * T]
        temp2 = np.abs(np.diff(temp))
        btemp = bank_acc[1 : freq * T]
        fee_leg = payment_delta * np.sum(temp[1:] / btemp)
        prot_leg = np.sum(temp2 / btemp)
        spreads.append( (1.0 - Rec) * prot_leg / fee_leg * 10 ** 4 )
    return spreads
```

```
### DEFINE CLASS FOR CALIBRATION PROBLEM ###

class Calibration_Index:
    def __init__(self, prices, r, Rec, mat, freq, NSim, V0):
        self.prices = prices
        self.r = r
        self.Rec = Rec
        self.mat = mat
        self.freq = freq
        self.NSim = NSim
        self.bounds = ...
        self.V0 = V0

    def value_function(self, theta):
        sigma, lamb, mu_jump, sigma_jump =
            theta[0], theta[1], theta[2], theta[3]
        parameters = Parameters(TIME_STEPS, SPACE_STEPS, TOTAL_TIME,
            UPPER_SPACE_LIMIT, sigma, RHO, 20, lamb,
            mu_jump, sigma_jump, ALPHA)
        model_prices = CDO_index_price(V0, parameters, self.r, self.Rec,
            self.freq, self.mat, self.NSim)
        return np.sum((np.array(model_prices) - np.array(self.prices))**2)

    def calibrate(self):
        self.calibration_result = optimize.minimize(self.value_function,
            x0 = np.array(...), method = 'powell', bounds = self.bounds)
```

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