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# Unconstrained Utility Maximization Problem via Four Methods

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# Declaration

The work contained in this thesis is my own work unless otherwise stated.

**Abstract**

In this paper, we study the utility maximization problem with different utility functions and stochastic factor models. There are four methods to solve the utility maximization problem, such as primal HJB, dual HJB, primal FBSDE and dual FBSDE. Our goal in this paper is to prove that these four methods have the exact same solutions for the utility maximization problem. We first solve the utility maximization problem under geometric brownian motion assumption for power utility function and non-HARA utility function. Closed formula solutions can be found showing that we can get the exact same solution by these methods. Then we study this problem under stochastic factor models. We can not get the closed formula solution in this case. So we use numerical method to plot all the wealth processes from these four methods and compare them. We check results with different time step size and calculate the mean square error to compare all methods precisely. We conclude that we can have the same solution by these four methods.

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# Contents

# List of Figures

# List of Tables

# Chapter 1

# Introduction

The utility maximization is a basic problem in mathematical finance and the axiomatic foundation of it can be tracked back to Von Neumann and Morgenstern[24]. The goal of this problem is to maximize the agent's expected utility of the wealth at the end of time period by constructing investment strategy in the market. Utility maximization problem is essentially optimal investment problem. The trading strategies include short selling, borrowing and other restrictions. See [12] for details.

Stochastic theory has a significant influence on solving financial problems. In 1951, Kiyosi Itô established the theory of stochastic differential equation of Brownian motion in [11], which opened a new way to study Markov process. J.L Doob published Stochastic processes [7] in 1953, where the basic theory of stochastic processes were systematically defined. With the development of stochastic theory and dynamic portfolio optimization, Merton published two landmark papers [19][20] about Hamilton-Jacobi-Bellman equation in a Markovian context that introduced optimal portfolio selection problem in continuous time. By stochastic control methods, Merton got the optimal investment strategies and a closed formula for the value function. Pliska[23], Cox and Huang[4][5], Karatzas et al.[13] had further researches and solved the optimal investment problem in a non-Markov setting.

By solving the primal HJB equation, we can get the solution of utility maximization problem. However, it is hard to solve the HJB equation for most utility functions. That led to emergence of dual control method which provides an powerful tool to solve the utility maximization problem. Xu and Shreve[25] firstly employed the stochastic duality theory of Bismut[2] to study the no-short-selling constrained optimal investment problem. The dual approach to the utility maximization problem was initial formulated in a complete market by Pliska[23], Cox and Huang[4] and Karatzas, Lehoczky and Shreve[13]. Karatzas et al.[14], He and Pearson[9][10] and Kramkov and Schachermayer[15][16] then used this method in the incomplete market. Cvitanić and Karatzas[6] considered the case of constrained strategies. This approach is to convert constrained problem into a family of unconstrained problems and find a optimal one. Dual problem can be transformed into primal problem by derivation. However, it is sometimes difficult to obtain the corresponding dual problem. Labbé and Heunis[17] introduced a simply method to construct the corresponding dual problem. Although dual problem are often easier to solve than primal method, sometimes we still can't find the closed formula solution for dual problem. In this case, numerical method should be used to solve primal problem and dual problem.

Bismut[3] introduced backward stochastic differential equations(BSDEs) in the linear case in 1976. Pardoux and Peng studied the general nonlinear case in the paper [21]. Their connections with mathematical finance, stochastic control and partial differential equations make BSDEs popular. El Karoui, Peng and Quenez published the first paper[8] about applications of BSDEs in mathematical finance. BSDEs provides a probabilistic representation for nonlinear PDEs, extending the Feymann-Kac formula for linear PDEs. Thus this representation creates a possibility to use numerical method to solve nonlinear PDEs. Øksendal and Sulem[26] proved that the relationship between optimal primal wealth processes, optimal strategy processes and optimal adjoint processes of dual problem obtained from forward and backward stochastic differential equations(FBSDEs).

Li and Zheng[18] constructed the necessary and sufficient conditions for both the primal and dual problems in terms of forward and backward stochastic differential equations. By this formula, we can get the results of primal problem by solving the corresponding dual FBSDEs and vice versa.

In this paper, we study the utility maximization problem with different utility function. Four methods are used to solve this problem such as primal HJB, dual HJB, primal FBSDEs and dual FBSDEs. The goal of this paper is to prove we can get the exact same solution for this problem by four methods. The market model setting and theorems used in this paper are mainly referred to the paper written by Li and Zheng[18] and this paper is mainly divided into two parts. In the first part, we set up the market model. Here we assume that market has only two assets, one risk-free asset and one risky asset satisfying geometric Brownian motion(GBM). We solve the utility maximization problem with all coefficients are constant and control set $K = \mathbb{R}$. We first use Dynamic Programming Principle to get the HJB equation and solve the primal problem. One example about power utility function is given to show that the primal HJB method works. However, primal HJB method has some shortcomings in solving complex utility functions such as non-HARA utility function. Then we construct the dual problem and find the dual process. We can find that the dual problem can be converted into the primal problem. Then necessary and sufficient conditions theorems and dynamic relations of primal and dual problems are introduced. We compare results of dual HJB and dual FBSDEs for power utility function and non-HARA utiliy function to show that two methods have the exact same solutions. In the second part, we try to solve the utility maximization under the stochastic factor model and control set is still the whole space. We divide the second part into three subsections. In the first subsection, the drift term of risky asset price is replaced by CIR a affine process. We have semi-linear PDEs from primal method and dual method which can be represented by BSDEs [22]. Further research on how to solve the BSDEs in this problem can be discussed in the future. The only difference between the second subsection and the first subsection is the utility function. We use log utility function in this subsection. The value functions can be solved from primal HJB and dual HJB equations by ansatzs. Primal FBSDEs and dual FBSDEs approaches can be solved numerically. We plot sample paths of optimal wealth processes and optimal control processes for four methods to compare the solution of each method. In addition, we compute the mean squared errors for each method. We do the same work in the third subsection as we do in the second subsection. The only difference is that the drift term of risky asset price goes from $H$ to $\sqrt{H}$ and we consider two special cases.

The rest of the paper is organized as follows. Chapter 2 and Chapter 3 correspond to the first part and the second part mentioned above. Chapter 4 concludes the paper.

# Chapter 2

# Unconstrained Utility Maximization Problem under Geometric Brownian Motion

## 2.1 Market Model Set up

Let $W = (W_t)_{0 \le t \le T}$ be a standard 1-dimension Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t), t \in [0,T]$ is the natural filtration induced by $W$, and $T > 0$ is a fixed termial time.

We denote by $\mathcal{P}(0,T;\mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valued progressively measurable processes on $[0,T] \times \Omega$, by $\mathcal{S}^2(0,T;\mathbb{R}^N)$ the set of processes $Y$ in $\mathcal{P}(0,T;\mathbb{R}^N)$ such that

$$E[\sup_{0 \le t \le T} |Y_t|^2 dt] < \infty$$

and by $\mathcal{H}^2(0,T;\mathbb{R}^N)$ the set of processes $Z$ in $\mathcal{P}(0,T;\mathbb{R}^N)$ such that

$$E[\int_0^T |Z_t|^2 dt] < \infty$$

Assume market has two assets, risk-free asset(saving account) $S_0$, risky asset $S$, satisfying SDE:

$$\begin{cases} dS_0(t) = rS_0(t)dt \\ dS(t) = S(t)(\mu dt + \sigma dW(t)) \end{cases} \tag{2.1.1}$$

with $S_0(0) = 1, S(0) = S > 0$, where $r, \mu, \sigma$ are all constant, $W$ is a standard Brownian motion. We also assume that $\sigma > 0$.

Assume $\pi_t$ is an $\mathcal{F}_t-$adapted process, $\int_0^T \pi_t^2 dt < \infty$ a.s. and $\pi_t$ is proportional portfolio process. Define the set of admissible portfolio strategies by

$$\mathcal{A} := \{\pi \in \mathcal{H}^2(0,T;\mathbb{R}) : \pi(t) \in K = \mathbb{R} \quad \text{for} \quad t \in [0,T] \quad \text{a.e.}\}$$

Given any $\pi \in \mathcal{A}$, we define $X_t^\pi$ is the investor's total wealth at time t. Then $\pi_t X_t^\pi$ is amount of money invested in $S$ and $(1-\pi_t)X_t^\pi$ is amount of money in saving account $S_0$. The wealth process $X^\pi$ satisfies SDE:

$$dX^\pi(t) = X^\pi(t)[(r + \pi(t)\sigma\theta)dt + \pi(t)\sigma dW(t)] \tag{2.1.2}$$

with initial wealth $X^\pi(0) = x$, where $\theta = \frac{\mu - r}{\sigma}$ is the market price of risk.

Let $U : \mathbb{R}_+ \triangleq [0,\infty) \to \mathbb{R}$ be a given utility function that is twice continuously differentiable, strictly increasing, strictly concave and satisfy the following conditions:

$$U(0) = \lim_{x \to 0} U(x) > -\infty, \lim_{x \to 0} U^{'}(x) = \infty, \lim_{x \to \infty} U^{'}(x) = 0$$

We set $U(x) = -\infty$ if $x < 0$.

Define the value function of the expected utility maximization problem as

$$V \triangleq \sup_{\pi \in \mathcal{A}} E[U(X^\pi(T))] \tag{2.1.3}$$

$$V(t,x) = \sup_{\pi \in \mathcal{A}} E[U(X^\pi(T))|X^\pi(t) = x] \tag{2.1.4}$$

To avoid trivialities, we assume that

$$-\infty < V < +\infty$$

## 2.2 Dynamic Programming Principle and Primal HJB Method

From Bellman[1], we have the following theorem.

**Theorem 2.1.1 (Dynamic Programming Principle)** For any h $\geq 0$

$$V(t,x) = \sup_{\pi \in \mathcal{A}} E[V(t+h, X^\pi_{t+h})|X^\pi_t = x] \tag{2.2.1}$$

This theorem has a significant influence on solving stochastic control problem. Assume $V \in C^{1,2}$. By Ito's formula, we have

$$\begin{aligned} V(t+h, X^\pi_{t+h}) &= V(t,x) + \int_t^{t+h} (\frac{\partial V(s, X^\pi_s)}{\partial s} ds + \frac{\partial V(s, X^\pi_s)}{\partial x} dX^\pi_s + \frac{1}{2}\frac{\partial^2 V(s, X^\pi_s)}{\partial x^2} d[X^\pi, X^\pi]_s) \\ &= V(t,x) + \int_t^{t+h} (\frac{\partial V}{\partial s} + \frac{\partial V}{\partial x}(rX^\pi_s + \pi_s X^\pi_s \theta\sigma) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi_s^2 X^{\pi 2}_s \sigma^2) ds \\ &+ \int_t^{t+h} \frac{\partial V}{\partial x} \pi_s X_s \sigma dW_s \end{aligned} \tag{2.2.2}$$

Substituting (2.2.2) to DPP(2.2.1), then canceling $V(t,x)$, we can get:

$$0 = \sup_\pi E[\int_t^{t+h} (\frac{\partial V}{\partial s} + \frac{\partial V}{\partial x}(rX^\pi_s + \pi_s X^\pi_s \theta\sigma) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\pi_s^2 X^{\pi 2}_s \sigma^2) ds | X^\pi_t = x] \tag{2.2.3}$$

divide by $h > 0$ on both sides of the equation(2.2.3) and then let $h \to 0$. By Mean Value Theorem, $s \to t$, $X^\pi_s \to X^\pi_t = x$, we can get

$$\frac{\partial V(t,x)}{\partial t} + rx\frac{\partial V}{\partial x} + \sup_\pi(\pi x \theta\sigma \frac{\partial V}{\partial x} + \frac{1}{2}\pi^2 x^2 \sigma^2 \frac{\partial^2 V}{\partial x^2}) = 0 \tag{2.2.4}$$

called Hamilton-Jacobi-Bellman(HJB) equation. Terminal condition is given by

$$V(T,x) = \sup_\pi E[U(X^\pi(T))|X^\pi(T) = x] = U(x)$$

Value function $V(t,x)$ satisfies HJB equation, thus we need to solve PDE(2.2.4). If V is strictly concave in $x$, then $V_{xx} < 0$. The maximum point $\pi^*$ in (2.2.4) satisfies

$$x\theta\sigma V_x + \pi^* x^2 \sigma^2 V_{xx} = 0$$

which gives

$$\pi^*_{t,x} = -\frac{\theta}{\sigma}\frac{V_x}{xV_{xx}} \tag{2.2.5}$$

Substituting $\pi^*$ into HJB and simplifying the expression, we get

$$V_t + rxV_x - \frac{1}{2}\theta^2 \frac{V_x^2}{V_{xx}} = 0 \tag{2.2.6}$$

with terminal condition $V(T, x) = U(x)$. (2.2.6) is hard to solve because it is a nonlinear PDE. We need to guess an ansatz for it. Here is an example where we can solve the primal HJB equation directly to get the solution of utility maximization problem.

**Example 2.2.1(power utility function)** $U$ is a power utility function defined by

$$U(x) = \frac{1}{\beta} x^{\beta}, \quad x \in (0, \infty)$$

where $\beta \in (0, 1)$ is a constant. In this case, we know that $V(t, x) = U(x)f(t)$. Substituting it into (2.2.6), we can get

$$\frac{1}{\beta} x^{\beta} \frac{\partial f}{\partial t} + rxx^{\beta-1} f - \frac{1}{2}\theta^2 \frac{x^{2\beta-2} f^2}{(\beta - 1)x^{\beta-2} f} = 0$$

Simplifying this equation to get a ODE for $f$

$$\frac{1}{\beta} f_t + rf - \frac{\theta^2}{2(\beta - 1)} f = 0$$

with the terminal condition $f(T) = 1$. By solving this ODE, we can get $f$

$$f(t) = \exp(\beta(r + \frac{\theta^2}{2(1 - \beta)})(T - t))$$

So the value function $V(t, x)$ is

$$V(t, x) = U(x)f(t) = U(x) \exp(\beta(r + \frac{\theta^2}{2(1 - \beta)})(T - t))$$

The maximum of the Hamiltonian in the HJB equation is achieved at

$$\pi^*(t, x) = -\frac{\theta}{\sigma} \frac{V_x}{xV_{xx}} = \frac{\theta}{(1 - \beta)\sigma}$$

Substituting $\pi^*(t, x) = \frac{\theta}{(1-\beta)\sigma}$ into the wealth equation we get the wealth process $X_t^{\pi}$ satisfying a linear SDE

$$dX_t^{\pi^*} = X_t^{\pi^*} \left( (r + \frac{\theta^2}{1 - \beta})dt + \frac{\theta}{1 - \beta} dW_t \right)$$

with initial wealth $X^{\pi}(0) = x$

The optimal wealth process is given by

$$X_t^{\pi^*} = x \exp((r + \frac{(1 - 2\beta)\theta^2}{2(1 - \beta)^2})t + \frac{\theta}{1 - \beta} W_t)$$

In this example, the utility function is power utility so we can solve the HJB PDE easily, which means primal HJB method works. However, if $U$ is not power(or log) such as non-HARA utility function, it is difficult to solve the HJB PDE. In next section, we will introduce Dual Control Method to solve Utility Maximization problem.

## 2.3 Dual problem and Dual HJB Method

First of all, we define the dual function of $U$ as

$$\tilde{U}(y) = \sup_{x>0} (U(x) - xy) \tag{2.3.1}$$

If $y < 0$, we can easily have $\tilde{U}(y) = \infty$ . $\tilde{U}$ is twice continuously differentiable, strictly decreasing and strictly convex on $(0, \infty)$.

The dual process $Y$ is a strictly positive and has the following semi-martingale decomposition:

$$\begin{aligned} dY(t) &= Y(t)(\alpha dt + \beta dW(t)), \quad 0 \le t \le T \\ Y(0) &= y \end{aligned} \tag{2.3.2}$$

We need to choose $\alpha$ and $\beta$ such that $X^\pi Y$ is a super-martingale for all admissible control process $\pi \in \mathcal{A}$.

By applying Ito's lemma, we have

$$\begin{aligned} d(X^\pi(t)Y(t)) &= X^\pi(t)dY(t) + Y(t)dX^\pi(t) + d[X^\pi, Y]_t \\ &= X^\pi(t)Y(t)[(\alpha + r + \pi(t)\theta\sigma + \pi(t)\beta\sigma)dt + (\pi(t)\sigma + \beta)dW(t)] \end{aligned}$$

$$X^\pi(0)Y(0) = xy$$

$X^\pi Y$ is a super-martingale if and only if

$$\alpha + r + \pi(t)\theta\sigma + \pi(t)\beta\sigma \leq 0$$

for all $\pi \in K$ a.s. for a.e. $t \in [0, T]$. Thus we can have

$$\alpha + r + \delta_K(-\sigma(\theta + \beta)) \leq 0$$

where $\delta_K(z) = \sup_{\pi \in K}\{-\pi z\}$ is the support function of the set $-K$.

Define $v = -\sigma(\theta + \beta)$. We have

$$\alpha \leq -(r + \delta_K(v)), \quad \beta = -(\theta^{-1}v + \theta)$$

According to the assumption, we know $K = \mathbb{R}$ and $\sigma > 0$. Then we can get:

$$\begin{aligned} &\theta + \beta = 0, \quad \beta = -\theta \\ &\alpha + r \leq 0, \quad \alpha \leq -r \end{aligned} \tag{2.3.3}$$

Recall the definition of dual function of $U$:

$$\tilde{U}(y) = \sup_{x>0}(U(x) - xy)$$

We have

$$U(x) \leq \tilde{U}(y) + xy, \forall x, y > 0$$

and the equality holds if and only if $U^{'}(x) = y$. Therefore,

$$E[U(X^\pi(T))] \leq E[\tilde{U}(Y(T))] + E[X^\pi(T)Y(T)] \leq E[\tilde{U}(Y(T))] + xy$$

which leads to

$$\sup_\pi E[U(X^\pi(T))] \leq \inf_{y,\alpha}(E[\tilde{U}(Y(T))] + xy)$$

For any fixed $y$, the solution $Y$ of SDE(2.3.2) satisfying condition(2.3.3) is bounded above by the process $Y^{(y)}$ satisfying the SDE:

$$\begin{aligned} &dY^{(y)}(t) = -Y^{(y)}(t)(rdt + \theta dW(t)), \quad 0 \leq t \leq T \\ &Y^{(y)}(0) = y \end{aligned} \tag{2.3.4}$$

That is equivalent to $Y(t) \leq Y^{(y)}(t)$ a.s. for $0 \leq t \leq T$. Then $E[\tilde{U}(Y(T))] \geq E[\tilde{U}(Y^{(y)}(T))]$ for any fixed $y$ because $\tilde{U}$ is a strictly decreasing function. We can get the optimal $\alpha$ is $-r$. The dual process $Y$ satisfies

$$\begin{aligned} &dY(t) = -Y(t)(rdt + \theta dW(t)), \quad 0 \leq t \leq T \\ &Y(0) = y \end{aligned} \tag{2.3.5}$$

The optimal value of the dual minimization problem is defined by

$$\tilde{V} \triangleq \inf_{y \in (0,\infty)}(xy + E[\tilde{U}(Y(T))]) \tag{2.3.6}$$

Define $\hat{V}(t, y) = E[\tilde{U}(Y(T))|Y(t) = y]$ and

$$\tilde{V}(t, x) = \inf_{y \in (0,\infty)}(xy + \hat{V}(t, y)) \tag{2.3.7}$$

Any $\hat{y} \in (0,\infty)$ satisfying $x\hat{y} + E[\tilde{U}(Y(T))] = \tilde{V}(x)$ is called the optimal dual control and the corresponding $Y$ is called the fgoptimal dual process.

Then we will solve the dual minimization problem. For $0 \leq t \leq T$, minimum points is obtained by solving

$$\frac{\partial \hat{V}(t,y)}{\partial y} + x = 0 \tag{2.3.8}$$

Since $\hat{V}(t,\cdot)$ is strictly convex, $\hat{V}_y(t,\cdot)$ is strictly increasing. That means there exists unique $y$ solving (2.3.8), write it $\tilde{y} = y(t,x)$. Since the process start from time 0, by setting $t = 0$, we have $\hat{y} = y(0,x)$. Then we can get:

$$\tilde{V}(t,x) = \hat{V}(t, y(t,x)) + xy(t,x) \tag{2.3.9}$$

By (2.3.9):

$$\begin{aligned}\tilde{V}_t &= \hat{V}_t + \hat{V}_y\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial t} \\ &= \hat{V}_t + (\hat{V}_y + x)\frac{\partial y}{\partial t} \\ &= \hat{V}_t \end{aligned}\tag{2.3.10}$$

$$\tilde{V}_x = \hat{V}_y\frac{\partial y}{\partial x} + y + x\frac{\partial y}{\partial x} = y \tag{2.3.11}$$

$$\tilde{V}_{xx} = \frac{\partial y}{\partial x} \tag{2.3.12}$$

By (2.3.8):

$$\frac{\partial(\hat{V}_y + x)}{\partial x} = \hat{V}_{yy}\frac{\partial y}{\partial x} + 1 = 0$$

$\Rightarrow$

$$\frac{\partial y}{\partial x} = -\frac{1}{\hat{V}_{yy}}$$

$\therefore$

$$\tilde{V}_{xx} = -\frac{1}{\hat{V}_{yy}}$$

Recall

$$\hat{V}(t,y) = E[\tilde{U}(Y(T))|Y(t) = y]$$

$$dY(t) = -Y(t)(rdt + \theta dW(t)), \quad 0 \leq t \leq T$$

By Feynman-Kac Theorem[22], $\hat{V}$ satisfies a linear PDE(called dual HJB equation):

$$\hat{V}_t - ry\hat{V}_y + \frac{1}{2}\theta^2 y^2 \hat{V}_{yy} = 0, \hat{V}(T,y) = \tilde{U}(y) \tag{2.3.13}$$

Substitute $y, \hat{V}_y, \hat{V}_{yy}, \hat{V}_t$ into (2.3.13)

$$\tilde{V}_t - r\tilde{V}_x(-x) + \frac{1}{2}\theta^2\tilde{V}_x^2(-\frac{1}{\tilde{V}_{xx}}) = 0$$

$$\tilde{V}_t + rx\tilde{V}_x - \frac{1}{2}\theta^2\frac{\tilde{V}_x^2}{\tilde{V}_{xx}} = 0 \tag{2.3.14}$$

that is exactly the HJB equation

$$\tilde{V}(T,x) = \inf_{y\in(0,\infty)}(xy + \hat{V}(T,y)) = \inf_{y\in(0,\infty)}(xy + \tilde{U}(y)) = U(x)$$

We have shown that $\tilde{V}$ is a classical solution to the HJB equation and satisfies the terminal condition. The significance of this result is that we no longer to guess a solution form of the HJB equation, which is almost impossible for general utility function except for power or log utility functions. That means we can find a representation of the classical solution to the HJB equation via two simple convex dual operations and solution of a linear PDE.

Here are two examples of solving Utility Maximization problem by this method.

**Example 2.3.1(power utility function)** $U$ is a power utility function defined by $U(x) = \frac{1}{\beta}x^{\beta}, x \in (0,\infty)$, where $\beta \in (0,1)$ is a constant. In this case, the dual problem can be written as

$$\tilde{V}(t,x) = \inf_{y\in(0,\infty)}(xy + E[\tilde{U}(Y(T))|Y(t)=y] = \inf_{y\in(0,\infty)}(xy + \hat{V}(t,y))$$

The dual function of $U$ is

$$\tilde{U}(y) = \sup_{x>0}(\frac{1}{\beta}x^{\beta} - xy)$$

Taking derivative with respect to $x$ of $\frac{1}{\beta}x^{\beta} - xy$ and set it equal to be 0, we have

$$x^{\beta-1} - y = 0, \quad x = y^{\frac{1}{\beta-1}}$$

Then the dual function can be written as $\tilde{U}(y) = -\frac{1}{\alpha}y^{\alpha}$ where $\alpha = \frac{\beta}{\beta-1}$ is a negative constant. And the dual value function is given by

$$\hat{V}(t,y) = E[\tilde{U}(Y(T))|Y(t)=y]$$

where $Y$ satisfies the SDE(2.3.5). By calculating, we have

$$\hat{V}(t,y) = \tilde{U}(y)\exp((\frac{1}{2}\alpha(\alpha-1)\theta^2 - \alpha r)(T-t))$$

where $\theta = \frac{\mu-r}{\sigma}$.

To solve

$$\inf_{y\in(0,\infty)}(xy + \hat{V}(t,y))$$

We can take derivative with respect to $y$ and then set it to be 0. We have:

$$x - y^{\alpha-1}\exp((\frac{1}{2}\alpha(\alpha-1)\theta^2 - \alpha r)(T-t)) = 0$$

Then we can get $\tilde{y} = y(t,x)$

$$y(t,x) = x^{\frac{1}{\alpha-1}}\exp((-\frac{1}{2}\alpha\theta^2 + \frac{\alpha r}{\alpha-1})(T-t))$$

$$\hat{y} = y(0,x) = x^{\frac{1}{\alpha-1}}\exp((-\frac{1}{2}\alpha\theta^2 + \frac{\alpha r}{\alpha-1})T)$$

$\Rightarrow$

$$V(t,x) = \tilde{V}(t,x) = xy(t,x) + \hat{V}(t,y(t,x)) = U(x)\exp(\beta(r + \frac{1}{2}\theta^2\frac{1}{1-\beta})(T-t))$$

Recall that $\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{V_x}{xV_{xx}}$. By calculating $V_x = x^{\beta-1}\exp(\beta(r+\frac{1}{2}\theta^2\frac{1}{1-\beta})(T-t))$ and $V_{xx} = (\beta-1)x^{\beta-2}\exp(\beta(r+\frac{1}{2}\theta^2\frac{1}{1-\beta})(T-t))$ and then substituting them into this equation, we have

$$\pi^*(t,x) = \frac{\theta}{(1-\beta)\sigma}$$

Substituting $\pi^*(t,x) = \frac{\theta}{(1-\beta)\sigma}$ into the wealth equation we can get

$$dX_t^{\pi^*} = X_t^{\pi^*}\left((r + \frac{\theta^2}{1-\beta})dt + \frac{\theta}{1-\beta}dW_t\right)$$

with initial wealth $X^\pi(0) = x$

The optimal wealth process is given by

$$X_t^{\pi^*} = x\exp((r + \frac{(1-2\beta)\theta^2}{2(1-\beta)^2})t + \frac{\theta}{1-\beta}W_t)$$

Compared with **example 2.2.1**, **example 2.3.1** get the exact same solution of optimal wealth process by using dual HJB method.

**Example 2.3.2(non-HARA utility function)** Another example is the non-HARA utility maximization. $U$ is a non-HARA utility function defined by $U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$ for $x > 0$, where $H(x) = \sqrt{2}(-1 + \sqrt{1+4x})^{-1/2}$. In this case, the dual function of $U$ is

$$\tilde{U}(y) = \sup_{x>0}(\frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x) - xy)$$

Taking derivative with respect to $x$ of $\frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x) - xy$ and set it equal to be 0, we have

$$-H(x)^{-4}H^{'}(x) - H(x)^{-2}H^{'}(x) + H(x) + xH^{'}(x) = y$$

By calculating, we can get $-H(x)^{-4}H^{'}(x) - H(x)^{-2}H^{'}(x) + xH^{'}(x) = 0$, which means $H(x) = y$.

Then the dual function can be written as $\tilde{U}(y) = \frac{1}{3}y^{-3} + y^{-1} + xy - xy = \frac{1}{3}y^{-3} + y^{-1}$. And the dual value function is given by

$$\hat{V}(t,y) = E[\tilde{U}(Y(T))|Y(t) = y]$$

where $Y$ satifies the SDE(2.3.5). So we have $Y(T) = Y(t)\exp(-(r+\frac{\theta^2}{2})(T-t) - \theta W_{T-t})$

Therefore,

$$\begin{aligned}
\hat{V}(t,y) &= E\left[\frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t) + 3\theta W_{T-t}) + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t) + \theta W_{T-t})\right] \\
&= \frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t))E[\exp(3\theta W_{T-t})] + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t))E[\exp(\theta W_{T-t})] \\
&= \frac{1}{3}y^{-3}\exp(3(r+\frac{\theta^2}{2})(T-t))\exp(\frac{9\theta^2(T-t)}{2}) + y^{-1}\exp((r+\frac{\theta^2}{2})(T-t))\exp(\frac{\theta^2(T-t)}{2}) \\
&= \frac{1}{3}y^{-3}\exp((3r+6\theta^2)(T-t)) + y^{-1}\exp((r+\theta^2)(T-t))
\end{aligned}$$

Same as the example 2.3.1, we also need to solve

$$\inf_{y\in(0,\infty)}(xy + \hat{V}(t,y))$$

Taking derivative with respect to $y$ and then set it to be 0. We have:

$$x = y^{-4}\exp((3r+6\theta^2)(T-t)) - y^{-2}\exp((r+\theta^2)(T-t))$$

$$xy^4 - \exp((3r+6\theta^2)(T-t)) - y^2\exp((r+\theta^2)(T-t)) = 0$$

$$y^2 = \frac{\exp((r+\theta^2)(T-t)) + \sqrt{\exp((2r+2\theta^2)(T-t)) + 4x\exp((3r+6\theta^2)(T-t))}}{2x}$$

$$\tilde{y} = y(t,x) = \frac{1}{\sqrt{2x}}\left[\exp((r+\theta^2)(T-t)) + \sqrt{\exp((2r+2\theta^2)(T-t)) + 4x\exp((3r+6\theta^2)(T-t))}\right]^{\frac{1}{2}}$$

And $\hat{y} = y(0,x)$. Then

$$\begin{aligned}
V(t,x) = \tilde{V}(t,x) &= xy(t,x) + \hat{V}(t,y(t,x)) \\
&= x\tilde{y} + \frac{1}{3}\tilde{y}^{-3}\exp((3r+6\theta^2)(T-t)) + \tilde{y}^{-1}\exp((r+\theta^2)(T-t))
\end{aligned}$$

Recall that $\pi^*(t,x) = -\frac{\theta}{\sigma}\frac{V_x}{xV_{xx}}$, $V_x = \tilde{V}_x = \hat{y}$ and $V_{xx} = \tilde{V}_{xx} = -\frac{1}{\hat{V}_{yy}}$, we have

$$\pi^*(t,x) = \frac{\theta}{\sigma}\frac{\hat{y}\hat{V}_{yy}}{x}$$

We can get $\hat{V}_{yy}$ from previous equation. Then

$$\pi^*(t,x) = \frac{\theta}{\sigma}\frac{4\hat{y}^{-4}\exp((3r+6\theta^2)(T-t)) + 2\hat{y}^{-2}\exp((r+\theta^2)(T-t))}{x}$$

Substituting $\pi^*(t,x) = \frac{\theta}{\sigma}\frac{4\hat{y}^{-4}\exp((3r+6\theta^2)(T-t))+2\hat{y}^{-2}\exp((r+\theta^2)(T-t))}{x}$ into the wealth equation we have

$$\begin{aligned} dX_t^{\pi^*} =& \left[\theta^2\left(4\hat{y}^{-4}\exp((3r+6\theta^2)(T-t)) + 2\hat{y}^{-2}\exp((r+\theta^2)(T-t))\right) + rX_t^{\pi^*}\right]dt \\ &+ \theta\left(4\hat{y}^{-4}\exp((3r+6\theta^2)(T-t)) + 2\hat{y}^{-2}\exp((r+\theta^2)(T-t))\right)dW_t \end{aligned}$$

Equation 2.3.8 holds for all $\pi \in \mathcal{A}$. Thus we have

$$X_t^{\pi^*} = Y_t^{-4}\exp((3r+6\theta^2)(T-t)) - Y_t^{-2}\exp((r+\theta^2)(T-t))$$

where $Y_t$ satisfies

$$\begin{aligned} &dY(t) = -Y(t)(rdt + \theta dW(t)), \quad 0 \le t \le T \\ &Y(0) = \hat{y} \end{aligned}$$

We have $Y_t = \hat{y}\exp(-(r+\frac{\theta^2}{2})t - \theta W_t)$ and

$$\begin{aligned} X_t^{\pi^*} &= Y_t^{-4}\exp((3r+6\theta^2)(T-t)) - Y_t^{-2}\exp((r+\theta^2)(T-t)) \\ &= \hat{y}^{-4}\exp(4(r+\frac{\theta^2}{2})t + 4\theta W_t)\exp((3r+6\theta^2)(T-t)) \\ &\quad + \hat{y}^{-2}\exp(2(r+\frac{\theta^2}{2})t + 2\theta W_t)\exp((r+\theta^2)(T-t)) \\ &= \hat{y}^{-4}e^{3(r+2\theta^2)T}e^{(r-4\theta^2)t+4\theta W_t} + \hat{y}^{-2}e^{(r+\theta^2)T}e^{rt+2\theta W_t} \end{aligned}$$

## 2.4 Necessary and sufficient conditions for primal problems

In this part, all the setting, lemmas and theorems are from Li and Zheng[18]. There will be N risky assets and all coefficients are processes.

Recall the wealth process $X_t^\pi$ satisfying SDE:

$$dX_t^\pi = X_t^\pi[(r(t) + \pi^T(t)\sigma(t)\theta(t))dt + \pi^T(t)\sigma(t)dW_t]$$

with initial wealth $x$, where $\theta(t) = \frac{\mu(t)-r(t)}{\sigma(t)}$ and $W$ is a standard Brownian motion. The value function is

$$V \triangleq \sup_{\pi\in\mathcal{A}} E[U(X^\pi(T))]$$

$$V(t,x) = \sup_{\pi\in\mathcal{A}} E[U(X^\pi(T))|X^\pi(t) = x]$$

Given an admissble control $\pi \in \mathcal{A}$ and a solution $X^\pi$, the associated adjoint equation is the following linear BSDE in the unknown processes $p_1 \in \mathcal{H}^2(0,T;\mathbb{R})$ and $q_1 \in \mathcal{H}^2(0,T;\mathbb{R}^N)$

$$\begin{aligned} &dp_1(t) = -[(r + \pi^T(t)\sigma(t)\theta(t))p_1(t) + q_1^T(t)\sigma^T(t)\pi(t)]dt + q_1^T(t)dW(t) \\ &p_1(T) = -U'(X^\pi(T)) \end{aligned} \tag{2.4.1}$$

Define the Hamiltonian function $\mathcal{H}$ by

$$\mathcal{H}(t,x,\pi,p_1,q_1) \triangleq x(r(t) + \pi^T\sigma(t)\theta(t))p_1 + x\pi^T\sigma(t)q_1 \tag{2.4.2}$$

Then the adjoint process is a pair of processes $(p_1, q_1)$ satisfying the following BSDE

$$dp_1(t) = -\frac{\partial}{\partial x}\mathcal{H}(t, X^\pi(t), \pi(t), p_1(t), q_1(t))dt + q_1^T(t)dW(t)$$

with the terminal condition $p_1(T) = -U^{'}(X^\pi(T))$, which is the BSDE(2.4.1).

**Lemma 2.4.1** Let $\pi^* \in \mathcal{A}$ and strictly positive, adapted process $X^{\pi^*}$ satisfy the SDE(2.1.2). Then there exists a unique solution $(\hat{p_1}, \hat{q_1})$ to the adjoint BSDE(2.4.1).

**Theorem 2.4.2 (Primal problem and associated FBSDE)** Let $\pi^* \in \mathcal{A}$. Then $\pi^*$ is optimal for the primal problem if and only if the solution $(X^{\pi^*}, \hat{p_1}, \hat{q_1})$ of FBSDE

$$\begin{aligned}
&dX^{\pi^*}(t) = X^{\pi^*}[(r(t) + \pi^{*T}(t)\sigma(t)\theta(t))dt + \pi^{*T}(t)\sigma(t)dW(t)] \\
&X^{\pi^*}(0) = x \\
&d\hat{p_1}(t) = -[(r(t) + \pi^{*T}(t)^T\sigma(t)\theta(t))\hat{p_1}(t) + \hat{q_1}^T(t)\sigma^T(t)\pi^*(t)]dt + \hat{q_1}^T(t)dW(t) \\
&\hat{p_1}(T) = -U^{'}(X^{\pi^*}(T))
\end{aligned} \tag{2.4.3}$$

satisfies the condition

$$-X^{\pi^*}(t)\sigma(t)[\theta(t)\hat{p_1}(t) + \hat{q_1}(t)] \in N_K(\pi^*(t)), \quad \forall t \in [0, T], \mathbb{P} - a.s. \tag{2.4.4}$$

where $N_K(x)$ is the normal cone of the closed convex set $K$ at $x \in K$, defined as

$$N_K(x) \triangleq \{y \in \mathbb{R}^N : \forall x^* \in K, y(x^* - x) \leq 0\}$$

According to the assumption, we know all coefficients are constant, $\sigma > 0$ and $K = \mathbb{R}$. Then we have

$$\theta\hat{p_1}(t) + \hat{q_1}(t) = 0 \tag{2.4.5}$$

Substituting (2.4.5) into (2.4.3) we have

$$\begin{aligned}
&dX^{\pi^*}(t) = X^{\pi^*}[(r + \pi^*(t)\sigma\theta)dt + \pi^*(t)\sigma dW(t)] \\
&X^{\pi^*} = x \\
&d\hat{p_1}(t) = -r\hat{p_1}(t)dt - \theta\hat{p_1}(t)dW(t) \\
&\hat{p_1}(T) = -U^{'}(X^{\pi^*}(T))
\end{aligned} \tag{2.4.6}$$

## 2.5 Necessary and sufficient conditions for dual problems

In this part, all the setting, lemmas and theorems are from Li and Zheng[18]. There will be N risky assets and all coefficients are processes. We address the dual problem. We assume that for any $(y, v) \in (0, \infty) \times \mathcal{D}$, $E[\tilde{U}(Y^{(y,v)}(T))^2] < \infty$ to ensure the existence of an optimal solution. Given an admissible dual control $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ with the dual process $Y^{(\hat{y},\hat{v})}$, the associated adjoint equation for dual problem is the following linear BSDE in the unknown processes $\hat{p_2} \in \mathcal{H}^2(0, T; \mathbb{R})$ and $\hat{q_2} \in \mathcal{H}^2(0, T; \mathbb{R}^N)$

$$\begin{aligned}
&d\hat{p_2}(t) = \{[r(t) + \delta_K(\hat{v}(t))]\hat{p_2}(t) + \hat{q_2}^T(t)[\theta(t) + \sigma^{-1}(t)\hat{v}(t)]\}dt + \hat{q_2}^T(t)dW(t) \\
&\hat{p_2}(T) = -\tilde{U}^{'}(Y^{(\hat{y},\hat{v})}(T))
\end{aligned} \tag{2.5.1}$$

Since $\hat{p_2}Y^{(\hat{y},\hat{v})}$ is a martingale, we can find $\hat{p_2}(t), 0 \leq t \leq T$ from the relation

$$\hat{p_2}(t)Y^{(\hat{y},\hat{v})}(t) = E[\hat{p_2}(T)Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t] = -E[\tilde{U}^{'}(Y^{(\hat{y},\hat{v})}(T))Y^{(\hat{y},\hat{v})}(T)|\mathcal{F}_t] \tag{2.5.2}$$

**Lemma 2.5.1** Let $(y, v) \in (0, \infty) \times \mathcal{D}$ and $Y^{(y,v)}$ be the corresponding state process satisfying the SDE(2.3.4). Then the random variable $Y^{(y,v)}(T)\tilde{U}^{'}(Y^{(y,v)}(T))$ is square integrable and there exists a solution to the adjoint BSDE(2.5.1)

**Theorem 2.5.2 (Dual problem and associated FBSDE)** Let $(\hat{y}, \hat{v}) \in (0,\infty)\times\mathcal{D}$. Then $(\hat{y}, \hat{v})$ is optimal for the dual problem if and only if the solution $(Y^{(\hat{y},\hat{v})}, \hat{p_2}, \hat{q_2})$ of FBSDE

$$\begin{aligned}
&dY^{(\hat{y},\hat{v})}(t) = -Y^{(\hat{y},\hat{v})}(t)\{[r(t) + \delta_K(\hat{v}(t))]dt + [\theta(t) + \sigma^{-1}(t)\hat{v}(t)]dW(t)\} \\
&Y^{(\hat{y},\hat{v})}(0) = \hat{y} \\
&d\hat{p_2}(t) = \{[r(t) + \delta_K(\hat{v}(t))]^T\hat{p_2}(t) + \hat{q_2}^T(t)[\theta(t) + \sigma^{-1}(t)]\}dt + \hat{q_2}^T(t)dW(t) \\
&\hat{p_2}(T) = -\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))
\end{aligned} \tag{2.5.3}$$

satisfies the condition

$$\begin{aligned}
&\hat{p_2}(0) = x \\
&\hat{p_2}(t)^{-1}[\sigma(t)]^{-1}\hat{q_2}(t) \in K \\
&\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}(t)\sigma^{-1}(t)\hat{v}(t) = 0, \forall t \in [0,T]\mathbb{P} - a.s.
\end{aligned} \tag{2.5.4}$$

## 2.6 Dynamic relations of primal and dual problems

We state the dynamic relations of the optimal portfolio and wealth processes of the primal problem and the adjoint processes of the dual problem and vice versa from Li and Zheng[18].

**Theorem 2.6.1 (From dual problem to primal problem)** Suppose that $(\hat{y}, \hat{v}) \in (0,\infty)\times\mathcal{D}$ is optimal for the dual problem. Let $(Y^{(\hat{y},\hat{v})}, \hat{p_2}, \hat{q_2})$ be the associated process that solve the FBSDE(2.5.3) and satisfies condition (2.5.4). Define

$$\pi^*(t) \triangleq \frac{[\sigma(t)]^{-1}\hat{q_2}(t)}{\hat{p_2}(t)}, \quad t \in [0,T] \tag{2.6.1}$$

Then $\pi^*$ is the optimal control for the primal problem with initial wealth $x$. The optimal wealth process and associated adjoint process are given by

$$\begin{aligned}
&X^{\pi^*}(t) = \hat{p_2}(t) \\
&\hat{p_1}(t) = -Y^{(\hat{y},\hat{v})}(t) \\
&\hat{q_1}(t) = Y^{(\hat{y},\hat{v})}(t)(\sigma^{-1}(t)\hat{v}(t) + \theta(t))
\end{aligned} \tag{2.6.2}$$

**Theorem 2.6.3 (From primal problem to dual problem)** Suppose that $\pi^* \in \mathcal{A}$ is optimal for primal problem with initial wealth $x$. Let $(X^{\pi^*}, \hat{p_1}, \hat{q_1})$ be the associated process that satisfies the FBSDE(2.4.3) and condition (2.4.4). Define

$$\begin{aligned}
&\hat{y} \triangleq -\hat{p_1}(0) \\
&\hat{v}(t) \triangleq -\sigma(t)[\frac{\hat{q_1}(t)}{\hat{p_1}(t)} + \theta(t)], \quad \forall t \in [0,T]
\end{aligned} \tag{2.6.3}$$

Then $(\hat{y}, \hat{v})$ is the optimal control for the dual problem. The optimal dual process and associated adjoint process are given by

$$\begin{aligned}
&Y^{(\hat{y},\hat{v})}(t) = -\hat{p_1}(t), \\
&\hat{p_2}(t) = X^{\hat{\pi}}(t), \\
&\hat{q_2}(t) = \sigma^T(t)\hat{\pi}(t)X^{\hat{\pi}}(t).
\end{aligned} \tag{2.6.4}$$

Here are two examples that we use primal and dual FBSDE method to solve the maximization problem.

**Example 2.6.1(power utility function)** Recall that power utility function $U(x) = \frac{1}{\beta}x^\beta$, $\tilde{U}(y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}$, $\tilde{U}'(y) = -y^{\frac{1}{\beta-1}}$. We have calculated before that $Y(T) = y\exp(-(r+\frac{\theta^2}{2})T - \theta W_T)$. Then we can get

$$Y^{(\hat{y})}(T) = \hat{y}\exp(-(r + \frac{\theta^2}{2})T - \theta W_T)$$

Since $\hat{p_2}Y^{(\hat{y})}$ is a martingale, we have

$$\begin{aligned}\hat{p_2}(t)Y^{(\hat{y})}(t) &= E[\hat{p_2}(T)Y^{(\hat{y})}(T)|\mathcal{F}_t] = -E[\tilde{U}'(Y^{(\hat{y})}(T))Y^{(\hat{y})}(T)|\mathcal{F}_t] \\ &= E[Y^{(\hat{y})}(T)^{\frac{\beta}{\beta-1}}|\mathcal{F}_t] \\ &= \hat{y}^{\frac{\beta}{\beta-1}} \exp(-\frac{\beta}{\beta-1}(r+\frac{\theta^2}{2})T - \frac{\theta\beta}{\beta-1}W_t)\exp(\frac{1}{2}\theta^2(\frac{\beta}{\beta-1})^2(T-t))\end{aligned}$$

So we can get $\hat{p_2}(t)$

$$\begin{aligned}\hat{p_2}(t) = \hat{y}^{\frac{\beta}{\beta-1}} \exp(-\frac{\beta}{\beta-1}(r+\frac{\theta^2}{2})T - \frac{\theta\beta}{\beta-1}W_t)\exp(\frac{1}{2}\theta^2(\frac{\beta}{\beta-1})^2(T-t)) \\ \hat{y}^{-1}\exp((r+\frac{\theta^2}{2})t+\theta W_t) \\ = \hat{y}^{\frac{1}{\beta-1}} \exp(-\frac{\beta}{\beta-1}rT + \frac{1}{2}\frac{\beta\theta^2}{(\beta-1)^2}T)\exp(-\frac{\theta}{\beta-1}W_t)\exp(rt+\frac{1}{2}\frac{(1-2\beta)\theta^2}{(\beta-1)^2}t)\end{aligned}$$

By Theorem 2.5.2, we need to satisfy three conditions. The second and third conditions always hold by our assumption. We need to check the first condition.

$$\hat{p_2}(0) = \hat{y}^{\frac{1}{\beta-1}} \exp(-\frac{\beta}{\beta-1}rT + \frac{1}{2}\frac{\beta\theta^2}{(\beta-1)^2}T) = x$$

Only when $\hat{y} = x^{\beta-1}\exp(\beta rT - \frac{1}{2}\frac{\beta\theta^2}{\beta-1}T)$, the first condition holds. Therefore, we can get

$$\hat{p_2}(t) = x\exp(rt + \frac{1}{2}\frac{(1-2\beta)\theta^2}{(\beta-1)^2}t + \frac{\theta}{\beta-1}W_t)$$

Applying Ito's lemma on $\hat{p_2}$, we have

$$d\hat{p_2}(t) = \hat{p_2}\left[\left(r + \frac{(1-2\beta)\theta^2}{2(\beta-1)^2} + \frac{1}{2}(\frac{\theta}{1-\beta})^2\right)dt + \frac{\theta}{1-\beta}dW_t\right]$$

Comparing with FBSDE(2.5.3), we have

$$\hat{q_2}(t) = \frac{\theta}{1-\beta}\hat{p_2}(t)$$

Applying Theorem 2.6.1, we have

$$\pi^* = \frac{\hat{q_2}(t)}{\sigma\hat{p_2}(t)} = \frac{\theta}{(1-\beta)\sigma}$$

$$X^{\pi^*}(t) = \hat{p_2}(t) = x\exp(rt + \frac{(1-2\beta)\theta^2}{2(1-\beta)^2}t + \frac{\theta}{1-\beta}W(t))$$

Compared with **example 2.3.1**, **example 2.6.1** get the exact same solution of optimal wealth by using dual FBSDE method.

Then we need to verify the optimal control and wealth process are correct. Define $\hat{y} = -\hat{p_1}(0)$, $\hat{v}(t) = -\sigma[\frac{\hat{q_1}(t)}{\hat{p_1}(t)} + \theta] = 0$

We know that $dX^{\pi^*}(t) = X^{\pi^*}[(r + \pi^*(t)\sigma\theta)dt + \pi^*(t)\sigma dW(t)]$, then we have

$$dX^{\pi^*}(t) = X^{\pi^*}(t)((r + \frac{\theta^2}{1-\beta})dt + \frac{\theta}{1-\beta}dW_t)$$

$$\mathcal{H}(t,x,a,p_1,q_1) = x(r + \frac{\theta^2}{1-\beta})p_1 + x\frac{\theta}{1-\beta}q_1$$

Then we have

$$\begin{aligned}-d\hat{p_1}(t) &= ((r + \frac{\theta^2}{1-\beta})\hat{p_1}(t) + \frac{\theta}{1-\beta}\hat{q_1}(t))dt - \hat{q_1}(t)dW_t \\ &= r\hat{p_1}(t)dt + \theta\hat{p_1}(t)dW_t\end{aligned}$$

$\Rightarrow$

$$\hat{p_1}(t) = \hat{p_1}(0)\exp(-(r+\frac{\theta^2}{2})t - \theta W_t)$$

So we have

$$Y^{(\hat{y})}(t) = -\hat{p_1}(t)$$

We can also get

$$\hat{p_2} = X^{\hat{\pi}}(t)$$

$$\hat{q_2} = \sigma^T(t)\hat{\pi}(t)X^{\hat{\pi}}(t)$$

which means the optimal control and wealth process are correct.

**Example 1.5(non-HARA utility function)** Recall that non-HARA utility function $U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$ for $x > 0$, where $H(x) = \sqrt{2}(-1+\sqrt{1+4x})^{-1/2}$. $\tilde{U}(y) = \frac{1}{3}y^{-3} + y^{-1}$, $\tilde{U}'(y) = -y^{-4} - y^{-2}$ and $\hat{y} = \frac{1}{\sqrt{2x}}\left[\exp((r+\theta^2)T) + \sqrt{\exp((2r+2\theta^2)T) + 4x\exp((3r+6\theta^2)T)}\right]^{\frac{1}{2}}$. We have calculated before that $Y(T) = y\exp(-(r+\frac{\theta^2}{2})T - \theta W_T)$. Then we can get

$$Y^{(\hat{y})}(T) = \hat{y}\exp(-(r+\frac{\theta^2}{2})T - \theta W_T)$$

$$\begin{aligned}\hat{p_2}(t)Y^{(y)}(t) &= E[\hat{p_2}(T)Y^{(y)}(T)|\mathcal{F}_t] = -E[\tilde{U}'(Y^{(\hat{y})}(T))Y^{(y)}(T)|\mathcal{F}_t] \\ &= E[Y^{(\hat{y})}(T)^{-3} + Y^{(\hat{y})}(T)^{-1}|\mathcal{F}_t] \\ &= \hat{y}^{-3}e^{3(r+\frac{\theta^2}{2})T}e^{3\theta W(t)}e^{\frac{9\theta^2}{2}(T-t)} + \hat{y}^{-1}e^{(r+\frac{\theta^2}{2})T}e^{\theta W(t)}e^{\frac{\theta^2}{2}(T-t)}\end{aligned}$$

So we can get $\hat{p_2}(t)$

$$\hat{p_2}(t) = \hat{y}^{-4}e^{3(r+2\theta^2)T}e^{(r-4\theta^2)t+4\theta W(t)} + \hat{y}^{-2}e^{(r+\theta^2)T}e^{rt+2\theta W(t)}$$

We need to satisfy the first condition. Thus we have

$$\begin{aligned}&\hat{p_2}(0) = x = \hat{y}^{-4}e^{3(r+2\theta^2)T} + \hat{y}^{-2}e^{(r+\theta^2)T} \\ &\hat{y} = \frac{1}{\sqrt{2x}}\left[\exp((r+\theta^2)T) + \sqrt{\exp((2r+2\theta^2)T) + 4x\exp((3r+6\theta^2)T)}\right]^{\frac{1}{2}}\end{aligned}$$

Applying Ito's lemma on $\hat{p_2}$, we get

$$\hat{q_2}(t) = \left[4\hat{y}^{-4}e^{3(r+2\theta^2)T}e^{(r-4\theta^2)t+4\theta W(t)} + 2\hat{y}^{-2}e^{(r+\theta^2)T}e^{rt+2\theta W(t)}\right]\theta$$

Applying Theorem 2.6.1, we have

$$\pi^* = \frac{\hat{q_2}(t)}{\sigma\hat{p_2}(t)}$$

$$X^{\pi^*}(t) = \hat{p_2}(t) = \hat{y}^{-4}e^{3(r+2\theta^2)T}e^{(r-4\theta^2)t+4\theta W(t)} + \hat{y}^{-2}e^{(r+\theta^2)T}e^{rt+2\theta W(t)}$$

Compared with **exaple 2.3.2**, **example 2.6.2** get the exact same solution of optimal wealth by using dual FBSDE method.

We also need to do verification. Define $\hat{y} = -\hat{p_1}(0)$, $\hat{v}(t) = -\sigma[\frac{\hat{q_1}(t)}{\hat{p_1}(t)} + \theta] = 0$

We know that $dX^{\pi^*}(t) = X^{\pi^*}[(r + \pi^*(t)\sigma\theta)dt + \pi^*(t)\sigma dW(t)]$, then we have

$$dX^{\pi^*}(t) = X^{\pi^*}(t)\{[r+\theta\frac{\hat{q_2}(t)}{\hat{p_2}(t)}]dt + \frac{\hat{q_2}(t)}{\hat{p_2}(t)}dW_t\}$$

$$\mathcal{H}(t,x,a,p_1,q_1) = x(r+\theta\frac{\hat{q_2}(t)}{\hat{p_2}(t)})p_1) + x\frac{\hat{q_2}(t)}{\hat{p_2}(t)}q_1$$

Then we have

$$
\begin{aligned}
-d\hat{p_1}(t) &= ((r+\theta\frac{\hat{q_2}(t)}{\hat{p_2}(t)})\hat{p_1}(t)+\frac{\hat{q_2}(t)}{\hat{p_2}(t)}\hat{q_1}(t))dt-\hat{q_1}(t)dW_t \\
&= r\hat{p_1}(t)dt+\theta\hat{p_1}(t)dW_t
\end{aligned}
$$

$\Rightarrow$

$$
\hat{p_1}(t)=\hat{p_1}(0)\exp(-(r+\frac{\theta^2}{2})t-\theta W_t)
$$

So we have

$$
Y^{(\hat{y})}(t)=-\hat{p_1}(t)
$$

We can also get

$$
\hat{p_2}=X^{\hat{\pi}}(t)
$$

$$
\hat{q_2}=\sigma^T(t)\hat{\pi}(t)X^{\hat{\pi}}(t)
$$

which means the optimal control and wealth process are correct.

# Chapter 3

# Unconstrained Utility Maximization Problem under Factor Models

In this chapter, we study utility maximization problem for different utility functions under different stochastic factor models. The volatility term of risky asset price is still constant and the drift term of risky asset price becomes a process.

## 3.1 Primal HJB Method

### 3.1.1 Power utility function under Stochastic Factor Model 1

In this part, the drift term of risky asset price process will be replaced by CIR affine process, not constant vector. $K = \mathbb{R}$ and we will consider power utility function.

Same as before, we assume market has two assets, risky-free asset $S_0$, risky asset $S$, satisfying SDE:

$$\begin{cases} dS_0(t) = rS_0(t)dt \\ dS(t) = H(t)S(t)dt + \sigma S(t)dW(t) \\ dH(t) = k(c - H(t))dt + \sigma_1\sqrt{H(t)}dW(t) \end{cases} \tag{3.1.1}$$

with $S_0(0) = 1, S(0) = S, H(0) = h$, where $r, \sigma, k, c, \sigma_1$ are all constant, $W$ is a standard Brownian motion.

The wealth process $X^\pi$ satisfies SDE

$$\begin{aligned} dX^\pi(t) &= (1 - \pi(t))X^\pi(t)rdt + \pi(t)X^\pi(t)\frac{dS(t)}{S(t)} \\ &= (1 - \pi(t))X^\pi(t)rdt + \pi(t)X^\pi(t)(H(t)dt + \sigma dW(t)) \\ &= X^\pi(t)[(r + \pi(t)(H(t) - r))dt + \pi(t)\sigma dW(t)] \end{aligned} \tag{3.1.2}$$

The value function is

$$V(t, x, h) = \sup_{\pi\in\mathcal{A}} E[U(X^\pi(T))|X^\pi(t) = x, H(t) = h] \tag{3.1.3}$$

with terminal condition $V(T, x, h) = \frac{1}{\beta}x^\beta$ where $U$ is power utility function.

We can get the HJB equation:

$$\partial_t V + k(c-h)\partial_h V + \frac{1}{2}\sigma_1^2 h\partial_{hh}V + \sup_{\pi\in\mathcal{A}}\{x(r + \pi(h-r))\partial_x V + \frac{1}{2}(x\pi\sigma)^2\partial_{xx}V + x\pi\sigma\sigma_1\sqrt{h}\partial_{xh}V\} = 0 \tag{3.1.4}$$

Now we need to solve (2.4).

Firstly, we need to find the optimal $\pi^*$ for

$$\sup_{\pi\in\mathcal{A}}\{x(r+\pi(h-r))\partial_x V+\frac{1}{2}(x\pi\sigma)^2\partial_{xx}V+x\pi\sigma\sigma_1\sqrt{h}\partial_{xh}V\}$$

Taking derivative with respect to $\pi$ and then make it to be 0, we can get

$$\pi^*=-\frac{(h-r)V_x+\sigma\sigma_1\sqrt{h}V_{xh}}{\sigma^2 xV_{xx}} \tag{3.1.5}$$

Substituting $\pi^*$ into HJB, we have

$$\partial_t V+k(c-h)\partial_h V+\frac{1}{2}\sigma_1^2 h\partial_{hh}V+xr\partial_x V-\frac{[(h-r)\partial_x V+\sigma\sigma_1\sqrt{h}V_{xh}]^2}{2\sigma^2\partial_{xx}V}=0 \tag{3.1.6}$$

Assume that $V(t,x,h)=U(x)f(t,h)$. Then we can get

$$\begin{aligned}
\partial_t V&=U(x)\partial_t f\\
\partial_h V&=U(x)\partial_h f\\
\partial_{hh}V&=U(x)\partial_{hh}f\\
\partial_x V&=\frac{\beta}{x}V\\
\partial_{xx}V&=\frac{\beta(\beta-1)}{x^2}V\\
\partial_{xh}V&=\frac{\beta}{x}U(x)\partial_h f
\end{aligned}$$

Substituting them into HJB and then canceling $U$, we have

$$\partial_t f+k(c-h)\partial_h f+\frac{1}{2}\sigma_1^2 h\partial_{hh}f+\beta rf-\frac{\beta[(h-r)f+\sigma\sigma_1\sqrt{h}\partial_h f]^2}{2\sigma^2(\beta-1)f}=0$$

$$\pi^*=-\frac{(h-r)f+\sigma\sigma_1\sqrt{h}\partial_h f}{\sigma^2(\beta-1)f}$$

with the terminal condition $f(T,h)=1$ and $dH(t)=k(c-H(t))dt+\sigma_1\sqrt{H(t)}dW(t)$.

This is a semilinear PDE in the form

$$-\partial_t f-\mathcal{L}f-g(t,h,f,\sigma_1\sqrt{h}\partial_h f)=0$$

where $\mathcal{L}f=k(c-h)\partial_h f+\frac{1}{2}\sigma_1^2 h\partial_{hh}f$ and $g(t,h,f,\sigma_1\sqrt{h}\partial_h f)=\beta rf-\frac{\beta[(h-r)f+\sigma\sigma_1\sqrt{h}\partial_h f]^2}{2\sigma^2(\beta-1)f}$. We shall represent the solution to this PDE by means of BSDE

$$-dY_t=g(t,H_t,Y_t,Z_t)dt-Z_t dW_t,\quad Y_T=f(T,h)=1 \tag{3.1.7}$$

where $Y_t=f(t,H_t), Z_t=\sigma_1\sqrt{H_t}\partial_h f(t,H_t)$. The solution to the BSDE provides a solution to the semilinear PDE.

### 3.1.2 Log utility function under Stochastic Factor Model 1

In this part, the only difference from part one is that we use log utility $U(x)=\log(x)$ and we know that $\lim\limits_{\beta\to 0}\frac{1}{\beta}(x^\beta-1)=\log(x)$

The value function is

$$V(t,x,h)=\sup_{\pi\in\mathcal{A}}E[U(X^\pi(T))|X^\pi(t)=x,H(t)=h] \tag{3.1.8}$$

with terminal condition $V(T,x,h) = \log(x)$ where $U$ is log utility function.

We will get the same HJB equation,

$$\partial_t V + k(c-h)\partial_h V + \frac{1}{2}\sigma_1^2 h \partial_{hh} V + xr\partial_x V - \frac{[(h-r)\partial_x V + \sigma\sigma_1\sqrt{h}V_{xh}]^2}{2\sigma^2\partial_{xx}V} = 0 \tag{3.1.9}$$

Assume that $V(t,x,h) = \log(x) + f(t,h)$. Then we can get

$$\begin{aligned}
\partial_t V &= \partial_t f \\
\partial_h V &= \partial_h f \\
\partial_{hh} V &= \partial_{hh} f \\
\partial_x V &= \frac{1}{x} \\
\partial_{xx} V &= -\frac{1}{x^2} \\
\partial_{xh} V &= 0
\end{aligned}$$

Substituting them into HJB, then we have

$$\partial_t f + k(c-h)\partial_h f + \frac{1}{2}\sigma_1^2 h\partial_{hh} f + r + \frac{(h-r)^2}{2\sigma^2} = 0 \tag{3.1.10}$$

$$\pi^* = \frac{h-r}{\sigma^2} \tag{3.1.11}$$

with the terminal condition $f(T,h) = 0$ and $dH(t) = k(c-H(t))dt + \sigma_1\sqrt{H(t)}dW(t)$. Using the Feynman-Kac formula for $f$, we can get the Feynman-Kac representation for $f$

$$f(t,h) = E[\int_t^T -r - \frac{(H_s-r)^2}{2\sigma^2} ds | H_t = h] \tag{3.1.12}$$

Then we can get the value function $V(t,x,h) = \log(x) + E[\int_t^T -r - \frac{(H_s-r)^2}{2\sigma^2} ds|H_t = h]$

### 3.1.3 Power utility function under Stochastic Factor Model 2

In this part, we will use the power utility function and we assume market has two assets, risky-free asset $S_0$, risky asset $S$, satisfying SDE:

$$\begin{cases}
dS_0(t) = rS_0(t)dt \\
dS(t) = \sqrt{H(t)}S(t)dt + \sigma S(t)dW(t) \\
dH(t) = k(c-H(t))dt + \sigma_1\sqrt{H(t)}dW(t)
\end{cases} \tag{3.1.13}$$

with $S_0(0) = 1, S(0) = S, H(0) = h$, where $r, \sigma, k, c, \sigma_1$ are all constant, $W$ is a standard Brownian motion.

The wealth process $X^\pi$ satisfies SDE

$$\begin{aligned}
dX^\pi(t) &= (1-\pi(t))X^\pi(t)rdt + \pi(t)X^\pi(t)\frac{dS(t)}{S(t)} \\
&= (1-\pi(t))X^\pi(t)rdt + \pi(t)X^\pi(t)(\sqrt{H(t)}dt + \sigma dW(t)) \\
&= X^\pi(t)[(r + \pi(t)(\sqrt{H(t)} - r))dt + \pi(t)\sigma dW(t)]
\end{aligned} \tag{3.1.14}$$

The value function is

$$V(t,x,h) = \sup_{\pi\in\mathcal{A}} E[U(X^\pi(T))|X^\pi(t) = x, H(t) = h] \tag{3.1.15}$$

with terminal condition $V(T,x,h) = \frac{1}{\beta}x^\beta$ where $U$ is power utility function.

Then we can get the HJB equation:

$$\partial_t V + k(c-h)\partial_h V + \frac{1}{2}\sigma_1^2 h \partial_{hh} V + \sup_{\pi\in\mathcal{A}}\{x(r+\pi(\sqrt{h}-r))\partial_x V + \frac{1}{2}(x\pi\sigma)^2\partial_{xx}V + x\pi\sigma\sigma_1\sqrt{h}\partial_{xh}V\} = 0 \tag{3.1.16}$$

Firstly, we need to find the optimal $\pi^*$ for

$$\sup_{\pi\in\mathcal{A}}\{x(r+\pi(\sqrt{h}-r))\partial_x V + \frac{1}{2}(x\pi\sigma)^2\partial_{xx}V + x\pi\sigma\sigma_1\sqrt{h}\partial_{xh}V\}$$

Taking derivative with respect to $\pi$ and then make it to be 0, we can get

$$\pi^* = -\frac{(\sqrt{h}-r)V_x + \sigma\sigma_1\sqrt{h}V_{xh}}{\sigma^2 x V_{xx}} \tag{3.1.17}$$

Substituting $\pi^*$ into HJB, we have

$$\partial_t V + k(c-h)\partial_h V + \frac{1}{2}\sigma_1^2 h\partial_{hh}V + xr\partial_x V - \frac{[(\sqrt{h}-r)\partial_x V + \sigma\sigma_1\sqrt{h}\partial_{xh}V]^2}{2\sigma^2\partial_{xx}V} = 0 \tag{3.1.18}$$

Assume that $V(t,x,h) = U(x)f(t,h)$. Then we can get

$$\begin{aligned}
\partial_t V &= U(x)\partial_t f\\
\partial_h V &= U(x)\partial_h f\\
\partial_{hh} V &= U(x)\partial_{hh} f\\
\partial_x V &= \frac{\beta}{x}V\\
\partial_{xx} V &= \frac{\beta(\beta-1)}{x^2}V\\
\partial_{xh} V &= \frac{\beta}{x}U(x)\partial_h f
\end{aligned}$$

Substituting them into HJB and then cancel $U$, we have

$$\partial_t f + k(c-h)\partial_h f + \frac{1}{2}\sigma_1^2 h\partial_{hh} f + \beta r f - \frac{\beta[(\sqrt{h}-r)f + \sigma\sigma_1\sqrt{h}\partial_h f]^2}{2\sigma^2(\beta-1)f} = 0$$

$$\pi^* = -\frac{(\sqrt{h}-r)f + \sigma\sigma_1\sqrt{h}\partial_h f}{\sigma^2(\beta-1)f}$$

with the terminal condition $f(T,h) = 1$ and $dH(t) = k(c-H(t))dt + \sigma_1\sqrt{H(t)}dW(t)$.

This is a semilinear PDE in the form

$$-\partial_t f - \mathcal{L}f - g(t,h,f,\sigma_1\sqrt{h}\partial_h f) = 0$$

where $\mathcal{L}f = k(c-h)\partial_h f + \frac{1}{2}\sigma_1^2 h\partial_{hh} f$ and $g(t,h,f,\sigma_1\sqrt{h}\partial_h f) = \beta r f - \frac{\beta[(\sqrt{h}-r)f+\sigma\sigma_1\sqrt{h}\partial_h f]^2}{2\sigma^2(\beta-1)f}$. We shall represent the solution to this PDE by means of BSDE

$$-dY_t = g(t,H_t,Y_t,Z_t)dt - Z_t dW_t, \quad Y_T = f(T,h) = 1$$

where $Y_t = f(t,H_t), Z_t = \sigma_1\sqrt{H_t}\partial_h f(t,H_t)$. The solution to the BSDE provides a solution to the semilinear PDE.

Ansatz:$V(t,x,h) = U(x)f(t,h)$, $f(t,h) = \exp A(t) + B(t)\sqrt{h} + C(t)h$. Then we can get:

$$\begin{aligned}
\partial_t V &= V(A'(t) + B'(t)\sqrt{h} + C'(t)h) \\
\partial_h V &= V(\frac{1}{2}B(t)h^{-\frac{1}{2}} + C(t)) \\
\partial_{hh} V &= V[(\frac{1}{2}B(t)h^{-\frac{1}{2}} + C(t))^2 + (-\frac{1}{4}B(t)h^{-\frac{3}{2}})] \\
\partial_x V &= \frac{\beta}{x}V \\
\partial_{xx} V &= \frac{\beta(\beta-1)}{x^2}V \\
\partial_{xh} V &= \frac{\beta}{x}V(\frac{1}{2}B(t)h^{-\frac{1}{2}} + C(t))
\end{aligned}$$

Substituting them into HJB and then cancel $V$, we have

$$\begin{aligned}
&(A' + B'\sqrt{h} + C'h) + k(c-h)(\frac{1}{2}Bh^{-\frac{1}{2}} + C) + \frac{1}{2}\sigma_1^2 h[(\frac{1}{2}Bh^{-\frac{1}{2}} + C)^2 + (-\frac{1}{4}Bh^{-\frac{3}{2}})] + \beta r \\
&- \frac{[\sqrt{h} - r + \sigma\sigma_1\sqrt{h}(\frac{1}{2}Bh^{-\frac{1}{2}} + C)]^2\beta}{2\sigma^2(\beta-1)} = 0
\end{aligned} \tag{3.1.19}$$

$$\pi^* = -\frac{(\sqrt{h} - r) + \sigma\sigma_1(\frac{1}{2}B(t) + C(t)\sqrt{h})}{\sigma^2(\beta-1)} \tag{3.1.20}$$

Let coefficients of $h^{-\frac{1}{2}}, 1, \sqrt{h}, h$ be 0. We can get

$$\begin{cases}
h^{-\frac{1}{2}}: & 0 = \frac{1}{2}kcB(t) - \frac{1}{8}\sigma_1^2 B(t) \\
1: & 0 = A'(t) + kcC(t) + \frac{1}{8}\sigma_1^2 B^2(t) + \beta r - \frac{\beta(\frac{1}{2}\sigma\sigma_1 B(t) - r)^2}{2\sigma^2(\beta-1)} \\
\sqrt{h}: & 0 = B'(t) - \frac{1}{2}kB(t) + \frac{1}{2}\sigma_1^2 B(t)C(t) - \frac{\beta(\sigma\sigma_1 C(t) + 1)(\frac{1}{2}\sigma\sigma_1 B(t) - r)}{\sigma^2(\beta-1)} \\
h: & 0 = C'(t) - kC(t) + \frac{1}{2}\sigma_1^2 C^2(t) - \frac{\beta(\sigma\sigma_1 C(t) + 1)^2}{2\sigma^2(\beta-1)}
\end{cases} \tag{3.1.21}$$

We first consider $0 = \frac{1}{2}kcB(t) - \frac{1}{8}\sigma_1^2 B(t)$ There are two cases: $B(t) = 0$ or $\frac{1}{2}kc - \frac{1}{8}\sigma_1^2 = 0$.

**Case1**: $B(t) = 0$. Under this condition, we also need one assumption that $r = 0$ to make the equation $\sqrt{h}: \quad 0 = B'(t) - \frac{1}{2}kB(t) + \frac{1}{2}\sigma_1^2 B(t)C(t) - \frac{\beta(\sigma\sigma_1 C(t)+1)(\frac{1}{2}\sigma\sigma_1 B(t) - r)}{\sigma^2(\beta-1)}$ hold. Then we need to solve $A(t), C(t)$.

Let's consider the equation $0 = C'(t) - kC(t) + \frac{1}{2}\sigma_1^2 C^2(t) - \frac{\beta(\sigma\sigma_1 C(t)+1)^2}{2\sigma^2(\beta-1)}$. We have

$$C'(t) - (k + \frac{\sigma_1\beta}{\sigma(\beta-1)})C(t) + (-\frac{\sigma_1^2}{2(\beta-1)})C^2(t) - \frac{\beta}{2\sigma^2(\beta-1)} = 0$$

Let $a_1 = -\frac{\sigma_1^2}{2(\beta-1)}$, $b_1 = k + \frac{\sigma_1\beta}{\sigma(\beta-1)}$, and $c_1 = \frac{\beta}{2\sigma^2(\beta-1)}$. Then we have,

$$C'(t) + a_1(C(t) - \frac{b_1}{2a_1})^2 - (\frac{b_1^2}{4a_1} + c_1) = 0$$

Let $C(t) = \chi(t) + \frac{b_1}{2a_1}$, $\phi = \frac{b_1^2}{4a_1} + c_1$, and $a_1 = \frac{1}{k_1}$, we can get $\chi(T) = -\frac{b_1}{2a_1}$

$$\frac{\partial_t \chi}{k_1\phi - \chi^2} = \frac{1}{k_1}$$

This is Riccati equation and we can get the solution

$$\chi(t) = \sqrt{k_1\phi}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}}$$

where

$$\gamma = \sqrt{\frac{\phi}{k_1}} \quad \text{and} \quad \zeta = \frac{\frac{b_1}{2a_1} + \sqrt{\phi k_1}}{\frac{b_1}{2a_1} - \sqrt{\phi k_1}}$$

We have already known $C(t)$ and $B(t) = 0$, then we can get the solution of $A(t)$ by $0 = A'(t) + kcC(t) + \frac{1}{8}\sigma_1^2 B^2(t) + \beta r - \frac{\beta(\frac{1}{2}\sigma\sigma_1 B(t)-r)^2}{2\sigma^2(\beta-1)} = A'(t) + kcC(t)$

$$A(t) = \int -kcC(t)dt + constant$$

we can get the value of constant by terminal conditon $A(T) = 0$. After that, we get the solution for the HJB equation.

**Case2:** $\frac{1}{2}kc - \frac{1}{8}\sigma_1^2 = 0$. Under this condition, we need to solve $A(t), B(t)$ and $C(t)$.

Let's first consider the equation $0 = C'(t) - kC(t) + \frac{1}{2}\sigma_1^2 C^2(t) - \frac{\beta(\sigma\sigma_1 C(t)+1)^2}{2\sigma^2(\beta-1)}$. Same as Case1, we can have $C(t) = \chi(t) + \frac{b_1}{2a_1}$

$$\chi(t) = \sqrt{k_1\phi}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}}$$

where

$$\gamma = \sqrt{\frac{\phi}{k_1}} \quad \text{and} \quad \zeta = \frac{\frac{b_1}{2a_1} + \sqrt{\phi k_1}}{\frac{b_1}{2a_1} - \sqrt{\phi k_1}}$$

$$a_1 = -\frac{\sigma_1^2}{2(\beta-1)}, b_1 = k + \frac{\sigma_1\beta}{\sigma(\beta-1)}, c_1 = \frac{\beta}{2\sigma^2(\beta-1)}, \phi = \frac{b_1^2}{4a_1} + c_1, a_1 = \frac{1}{k_1}$$

Then we need to solve $B(t)$ by $0 = B'(t) - \frac{1}{2}kB(t) + \frac{1}{2}\sigma_1^2 B(t)C(t) - \frac{\beta(\sigma\sigma_1 C(t)+1)(\frac{1}{2}\sigma\sigma_1 B(t)-r)}{\sigma^2(\beta-1)}$. We have

$$B'(t) + (\frac{1}{2}\sigma_1^2 C(t) - \frac{1}{2}k - \frac{\beta\sigma_1^2}{2(\beta-1)}C(t) - \frac{\beta\sigma_1}{2\sigma(\beta-1)})B(t) = -\frac{\beta r}{\sigma^2(\beta-1)} - \frac{\beta r\sigma_1}{\sigma(\beta-1)}C(t)$$

Let $\frac{1}{2}\sigma_1^2 C(t) - \frac{1}{2}k - \frac{\beta\sigma_1^2}{2(\beta-1)}C(t) - \frac{\beta\sigma_1}{2\sigma(\beta-1)} = P(t)$, $-\frac{\beta r}{\sigma^2(\beta-1)} - \frac{\beta r\sigma_1}{\sigma(\beta-1)}C(t) = Q(t)$. Then we can get

$$B'(t) + P(t)B(t) = Q(t)$$

The solution is

$$B(t) = \frac{\int \mu(t)Q(t)dt + constant}{\mu(t)}$$
$$\mu(t) = e^{\int P(t)dt}$$

We can get the value of constant by terminal conditon $B(T) = 0$. After solving $C(t)$ and $B(t)$, we can get the solution of $A(t)$ by $0 = A'(t) + kcC(t) + \frac{1}{8}\sigma_1^2 B^2(t) + \beta r - \frac{\beta(\frac{1}{2}\sigma\sigma_1 B(t)-r)^2}{2\sigma^2(\beta-1)}$

$$A(t) = \int -kcC(t) - \frac{1}{8}\sigma_1^2 B^2(t) - \beta r + \frac{\beta(\frac{1}{2}\sigma\sigma_1 B(t) - r)^2}{2\sigma^2(\beta-1)}dt + constant$$

we can get the value of constant by terminal conditon $A(T) = 0$. After that, we get the solution for the HJB equation.

## 3.2 Dual HJB Method

### 3.2.1 Power utility function under Stochastic Factor Model 1

Define the dual function of $U$ by

$$\tilde{U}(y) = \sup_{x>0}(U(x) - xy)$$

For power utility function $U(x) = \frac{1}{\beta}x^\beta$, we can get $\tilde{U}(y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}$.

The dual process $Y$ is a strictly positive and has the following semi-martingale decomposition:

$$\begin{aligned} &dY(t) = Y(t)(\alpha(t)dt + \beta(t)dW(t)), \quad 0 \le t \le T \\ &Y(0) = y \end{aligned} \tag{3.2.1}$$

We need to choose $\alpha$ and $\beta$ such that $X^\pi Y$ is a super-martingale for all admissible control process $\pi \in \mathcal{A}$.

By Ito's formula, we have

$$\begin{aligned} d(X^\pi(t)Y(t)) &= X^\pi(t)dY(t) + Y(t)dX^\pi(t) + d[X^\pi, Y]_t \\ &= X^\pi(t)Y(t)[(\alpha(t) + r + \pi(t)(H(t) - r) + \pi(t)\beta(t)\sigma)dt \\ &+ (\pi(t)\sigma + \beta(t))dW(t)] \end{aligned}$$

$$X^\pi(0)Y(0) = xy$$

$X^\pi Y$ is a super-martingale if and only if

$$\alpha(t) + r + \pi(t)(H(t) - r) + \pi(t)\beta(t)\sigma \le 0$$

for all $\pi \in K$ a.s. for a.e. $t \in [0, T]$. So we can have

$$\alpha(t) + r + \delta_K(-(H(t) - r) - \beta(t)\sigma) \le 0$$

where $\delta_K(z) = \sup_{\pi \in K}\{-\pi z\}$ is the support function of the set $-K$.

Define $v(t) = -(H(t) - r) - \beta(t)\sigma$. We have

$$\alpha(t) \le -(r + \delta_K(v(t))), \quad \beta(t) = -(\sigma^{-1}v(t) + \sigma^{-1}(H(t) - r))$$

According to the assumption, we know $K = \mathbb{R}$ and $\sigma > 0$. So we have $v(t) = 0$. Then we can get:

$$\begin{aligned} &\alpha(t) + r \le 0, \quad \alpha(t) \le -r \\ &\beta(t) = -\frac{H(t) - r}{\sigma} \end{aligned} \tag{3.2.2}$$

So we can get the dual process:

$$\begin{aligned} &dY(t) = -Y(t)(rdt + \frac{H(t) - r}{\sigma}dW(t)), \quad 0 \le t \le T \\ &Y(0) = y \end{aligned} \tag{3.2.3}$$

The optimal value of the dual minimization problem is defined by

$$\tilde{V} = \inf_{y \in (0,\infty)}(xy + E[\tilde{U}(Y(T))]) \tag{3.2.4}$$

Define the dual value function

$$\hat{V}(t, y, h) = E[\tilde{U}(Y(T))|Y(t) = y,\ H(t) = h] \tag{3.2.5}$$

with the terminal condition $\hat{V}(T, y, h) = \tilde{U}(y)$.

So we can get the HJB:

$$\partial_t\hat{V} + k(c - h)\partial_h\hat{V} + \frac{1}{2}\sigma_1^2 h\partial_{hh}\hat{V} + [-ry\partial_y\hat{V} + \frac{(h - r)^2y^2}{2\sigma^2}\partial_{yy}\hat{V} - \frac{(h - r)y\sigma_1\sqrt{h}}{\sigma}\partial_{yh}\hat{V}] = 0 \tag{3.2.6}$$

Assume that $\hat{V}(t,y,h) = \tilde{U}(y)\tilde{f}(t,h)$. Then we can get

$$\begin{aligned}
\partial_t \hat{V} &= \tilde{U}(y)\partial_t \tilde{f} \\
\partial_h \hat{V} &= \tilde{U}(y)\partial_h \tilde{f} \\
\partial_{hh} \hat{V} &= \tilde{U}(y)\partial_{hh} \tilde{f} \\
\partial_y \hat{V} &= -\frac{\beta}{1-\beta} y^{-1}\hat{V} \\
\partial_{yy} \hat{V} &= \frac{\beta}{1-\beta} y^{-2}\hat{V} + (\frac{\beta}{1-\beta})^2 y^{-2}\hat{V} \\
\partial_{yh} \hat{V} &= -\frac{\beta}{1-\beta} y^{-1}\tilde{U}(y)\partial_h \tilde{f}
\end{aligned}$$

Substituting them into HJB and then cancel $\tilde{U}$, we have

$$\partial_t \tilde{f} + k(c-h)\partial_h \tilde{f} + \frac{1}{2}\sigma_1^2 h \partial_{hh}\tilde{f} + \frac{\beta r}{1-\beta}\tilde{f} + \frac{(h-r)^2\beta}{2\sigma^2(1-\beta)^2}\tilde{f} + \frac{(h-r)\sigma_1\sqrt{h}\beta}{\sigma(1-\beta)}\partial_h \tilde{f} = 0 \qquad (3.2.7)$$

with the terminal condition $\tilde{f}(T,h) = 1$ and $dH(t) = k(c - H(t))dt + \sigma_1\sqrt{H(t)}dW(t)$.

This is a semilinear PDE in the form

$$-\partial_t \tilde{f} - \mathcal{L}\tilde{f} - g(t,h,\tilde{f},\sigma_1\sqrt{h}\partial_h \tilde{f}) = 0$$

where $\mathcal{L}\tilde{f} = k(c-h)\partial_h \tilde{f} + \frac{1}{2}\sigma_1^2 h\partial_{hh}\tilde{f}$ and $g(t,h,\tilde{f},\sigma_1\sqrt{h}\partial_h\tilde{f}) = \frac{\beta r}{1-\beta}\tilde{f} + \frac{(h-r)^2\beta}{2\sigma^2(1-\beta)^2}\tilde{f} + \frac{(h-r)\sigma_1\sqrt{h}\beta}{\sigma(1-\beta)}\partial_h\tilde{f}$. We shall represent the solution to this PDE by means of BSDE

$$-dY_t = g(t,H_t,Y_t,Z_t)dt - Z_t dW_t, \quad Y_T = f(T,h) = 1$$

where $Y_t = \tilde{f}(t,H_t), Z_t = \sigma_1\sqrt{H_t}\partial_h\tilde{f}(t,H_t)$. The solution to the BSDE provide a solution to the semilinear PDE.

### 3.2.2 Log utility function under Stochastic Factor Model 1

Same as before, the only difference from part one is that we use log utility $U(x) = \log(x)$. The dual function of log utility function is $\tilde{U}(y) = -\log(y) - 1$

We will get the same HJB equation,

$$\partial_t \hat{V} + k(c-h)\partial_h \hat{V} + \frac{1}{2}\sigma_1^2 h\partial_{hh}\hat{V} + [-ry\partial_y\hat{V} + \frac{(h-r)^2y^2}{2\sigma^2}\partial_{yy}\hat{V} - \frac{(h-r)y\sigma_1\sqrt{h}}{\sigma}\partial_{yh}\hat{V}] = 0 \quad (3.2.8)$$

Then we need to guess the solution of $\hat{V}$.

Ansatz:$\hat{V}(t,y,h) = \tilde{U}(y) + \tilde{f}(t,h)$ with terminal condition $\tilde{f}(T,h) = 0$. Then we can get:

$$\begin{aligned}
\partial_t \hat{V} &= \partial_t \tilde{f} \\
\partial_h \hat{V} &= \partial_h \tilde{f} \\
\partial_{hh} \hat{V} &= \partial_{hh} \tilde{f} \\
\partial_y \hat{V} &= -\frac{1}{y} \\
\partial_{yy} \hat{V} &= \frac{1}{y^2} \\
\partial_{yh} \hat{V} &= 0
\end{aligned}$$

Substituting them into HJB, then we have

$$\partial_t \tilde{f} + k(c-h)\partial_h \tilde{f} + \frac{1}{2}\sigma_1^2 h\partial_{hh}\tilde{f} + r + \frac{(h-r)^2}{2\sigma^2} = 0 \qquad (3.2.9)$$

with the terminal condition $\tilde{f}(T,h)=0$ and $dH(t)=k(c-H(t))dt+\sigma_1\sqrt{H(t)}dW(t)$. Using the Feynman-Kac formula for $f$, we can get the Feynman-Kac representation for $\tilde{f}$

$$\tilde{f}(t,h)=E[\int_t^T -r-\frac{(H_s-r)^2}{2\sigma^2}ds|H_t=h] \tag{3.2.10}$$

Then we get $\hat{V}(t,y,h)=\tilde{U}(y)+\tilde{f}(t,h)=-\log(y)-1+E[\int_t^T -r-\frac{(H_s-r)^2}{2\sigma^2}ds|H_t=h]$.

By $\tilde{V}=\inf_{y\in(0,\infty)}(xy+E[\tilde{U}(Y(T))])$, we need to take derivative with respect to $y$ and then set it to be 0. Then we get

$$-\frac{1}{y}+x=0$$

We can get the optimal $y$ and then we have

$$\begin{aligned}\tilde{V}(t,x,h)&=x*\frac{1}{x}+\log(x)-1+E[\int_t^T -r-\frac{(H_s-r)^2}{2\sigma^2}ds|H_t=h]\\&=\log(x)+E[\int_t^T -r-\frac{(H_s-r)^2}{2\sigma^2}ds|H_t=h]\end{aligned} \tag{3.2.11}$$

Compared with primal HJB method, we get the same solution by dual method.

### 3.2.3 Power utility function under Stochastic Factor Model 2

In this part, we will use the power utility function and we assume market has two assets, risky-free asset $S_0$, risky asset $S$, satisfying SDE:

$$\begin{cases}dS_0(t)=rS_0(t)dt\\ dS(t)=\sqrt{H(t)}S(t)dt+\sigma S(t)dW(t)\\ dH(t)=k(c-H(t))dt+\sigma_1\sqrt{H(t)}dW(t)\end{cases} \tag{3.2.12}$$

The wealth process $X^\pi$ satisfies SDE

$$dX^\pi(t)=X^\pi(t)[(r+\pi(t)(\sqrt{H(t)}-r))dt+\pi(t)\sigma dW(t)] \tag{3.2.13}$$

The dual process satisfies SDE:

$$\begin{aligned}&dY(t)=-Y(t)(rdt+\frac{\sqrt{H(t)}-r}{\sigma}dW(t)),\quad 0\le t\le T\\&Y(0)=y\end{aligned} \tag{3.2.14}$$

The optimal value of the dual minimization problem is defined by

$$\tilde{V}=\inf_{y\in(0,\infty)}(xy+E[\tilde{U}(Y(T))]) \tag{3.2.15}$$

Define the dual value function

$$\hat{V}(t,y,h)=E[\tilde{U}(Y(T))|Y(t)=y,\ H(t)=h] \tag{3.2.16}$$

with the terminal condition $\hat{V}(T,y,h)=\tilde{U}(y)$.

Then we can get the HJB equation:

$$\partial_t\hat{V}+k(c-h)\partial_h\hat{V}+\frac{1}{2}\sigma_1^2h\partial_{hh}\hat{V}+[-ry\partial_y\hat{V}+\frac{(\sqrt{h}-r)^2y^2}{2\sigma^2}\partial_{yy}\hat{V}-\frac{(\sqrt{h}-r)y\sigma_1\sqrt{h}}{\sigma}\partial_{yh}\hat{V}]=0 \tag{3.2.17}$$

Ansatz:$\hat{V}(t,y,h) = \tilde{U}(y)\tilde{f}(t,h), \tilde{f}(t,h) = \exp A(t) + B(t)\sqrt{h} + C(t)h$. Then we can get:

$$\begin{aligned}
\partial_t \hat{V} &= \hat{V}(A'(t) + B'(t)\sqrt{h} + C'(t)h) \\
\partial_h \hat{V} &= \hat{V}(\frac{1}{2}B(t)h^{-\frac{1}{2}} + C(t)) \\
\partial_{hh} \hat{V} &= \hat{V}[(\frac{1}{2}B(t)h^{-\frac{1}{2}} + C(t))^2 + (-\frac{1}{4}B(t)h^{-\frac{3}{2}})] \\
\partial_y \hat{V} &= \frac{\beta}{y(\beta-1)}\hat{V} \\
\partial_{yy} \hat{V} &= \frac{\beta}{y^2(\beta-1)^2}\hat{V} \\
\partial_{yh} \hat{V} &= \frac{\beta}{y(\beta-1)}\hat{V}(\frac{1}{2}B(t)h^{-\frac{1}{2}} + C(t))
\end{aligned}$$

Substituting them into HJB and then canceling $\hat{V}$, we have

$$\begin{aligned}
&(A' + B'\sqrt{h} + C'h) + k(c-h)(\frac{1}{2}Bh^{-\frac{1}{2}} + C) + \frac{1}{2}\sigma_1^2 h[(\frac{1}{2}Bh^{-\frac{1}{2}} + C)^2 + (-\frac{1}{4}Bh^{-\frac{3}{2}})] \\
&- \frac{\beta r}{\beta-1} + \frac{(\sqrt{h}-r)^2\beta}{2\sigma^2(\beta-1)^2} - \frac{(\sqrt{h}-r)\sigma_1\sqrt{h}\beta}{\sigma(\beta-1)}(\frac{1}{2}Bh^{-\frac{1}{2}} + C) = 0 \qquad (3.2.18)
\end{aligned}$$

Let coefficients of $h^{-\frac{1}{2}}, 1, \sqrt{h}, h$ be 0. We can get

$$\begin{cases}
h^{-\frac{1}{2}}: \quad 0 = \frac{1}{2}kcB(t) - \frac{1}{8}\sigma_1^2 B(t) \\
1: \quad 0 = A'(t) + kcC(t) + \frac{1}{8}\sigma_1^2 B^2(t) - \frac{\beta r}{\beta-1} + \frac{\beta r^2}{2\sigma^2(\beta-1)^2} + \frac{\sigma_1\beta r}{2\sigma(\beta-1)}B(t) \\
\sqrt{h}: \quad 0 = B'(t) - \frac{1}{2}kB(t) + \frac{1}{2}\sigma_1^2 B(t)C(t) - \frac{\beta r}{\sigma^2(\beta-1)^2} - \frac{\sigma_1\beta}{\sigma(\beta-1)}(\frac{1}{2}B(t) - rC(t)) \\
h: \quad 0 = C'(t) - kC(t) + \frac{1}{2}\sigma_1^2 C^2(t) + \frac{\beta}{2\sigma^2(\beta-1)^2} - \frac{\sigma_1\beta}{\sigma(\beta-1)}C(t)
\end{cases} \qquad (3.2.19)$$

Similarly, we first consider $0 = \frac{1}{2}kcB(t) - \frac{1}{8}\sigma_1^2 B(t)$. There are also two cases: $B(t) = 0$ or $\frac{1}{2}kc - \frac{1}{8}\sigma_1^2 = 0$.

**Case1**: $B(t) = 0$ and $r = 0$. In this case, we need to solve $A(t)$ and $C(t)$. Let's consider the equation $0 = C'(t) - kC(t) + \frac{1}{2}\sigma_1^2 C^2(t) + \frac{\beta}{2\sigma^2(\beta-1)^2} - \frac{\sigma_1\beta}{\sigma(\beta-1)}C(t)$. We have

$$C'(t) - (k + \frac{\sigma_1\beta}{\sigma(\beta-1)})C(t) + \frac{1}{2}\sigma_1^2 C^2(t) + \frac{\beta}{2\sigma^2(\beta-1)^2} = 0$$

Let $a_2 = \frac{1}{2}\sigma_1^2$, $b_2 = k + \frac{\sigma_1\beta}{\sigma(\beta-1)}$, and $c_2 = -\frac{\beta}{2\sigma^2(\beta-1)^2}$. Then we have

$$C'(t) + a_2(C(t) - \frac{b_2}{2a_2})^2 - (\frac{b_2^2}{4a_2} + c_2) = 0$$

Let $C(t) = \chi(t) + \frac{b_2}{2a_2}$, $\phi = \frac{b_2^2}{4a_2} + c_2$, and $a_2 = \frac{1}{k_2}$, we can get $\chi(T) = -\frac{b_2}{2a_2}$

$$\frac{\partial_t \chi}{k_2\phi - \chi^2} = \frac{1}{k_2}$$

This is Riccati equation and we can get the solution

$$\chi(t) = \sqrt{k_2\phi}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}}$$

where

$$\gamma = \sqrt{\frac{\phi}{k_2}} \quad \text{and} \quad \zeta = \frac{\frac{b_2}{2a_2} + \sqrt{\phi k_2}}{\frac{b_2}{2a_2} - \sqrt{\phi k_2}}$$

We have known $C(t)$ and $B(t) = 0$, then we can get the solution of $A(t)$ by $\quad 0 = A'(t) + kcC(t) + \frac{1}{8}\sigma_1^2 B^2(t) - \frac{\beta r}{\beta-1} + \frac{\beta r^2}{2\sigma^2(\beta-1)^2} + \frac{\sigma_1 \beta r}{2\sigma(\beta-1)} B(t) = A'(t) + kcC(t)$

$$A(t) = \int -kcC(t)dt + constant$$

we can get the value of constant by terminal conditon $A(T) = 0$. After that, we get the solution for the HJB equation.

**Case2**: $0 = \frac{1}{2}kcB(t) - \frac{1}{8}\sigma_1^2 B(t)$. Under this condition, we need to solve $A(t), B(t)$ and $C(t)$. Let's consider the equation $0 = C'(t) - kC(t) + \frac{1}{2}\sigma_1^2 C^2(t) + \frac{\beta}{2\sigma^2(\beta-1)^2} - \frac{\sigma_1\beta}{\sigma(\beta-1)}C(t)$. Similarly, we have $C(t) = \chi(t) + \frac{b_2}{2a_2}$

$$\chi(t) = \sqrt{k_2\phi}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}$$

where

$$\gamma = \sqrt{\frac{\phi}{k_2}} \quad \text{and} \quad \zeta = \frac{\frac{b_2}{2a_2} + \sqrt{\phi k_2}}{\frac{b_2}{2a_2} - \sqrt{\phi k_2}}$$

$$a_2 = \frac{1}{2}\sigma_1^2, b_2 = k + \frac{\sigma_1\beta}{\sigma(\beta-1)}, c_2 = -\frac{\beta}{2\sigma^2(\beta-1)^2}, \phi = \frac{b_2^2}{4a_2} + c_2, a_2 = \frac{1}{k_2}$$

Then we need to solve $B(t)$ by $0 = B'(t) - \frac{1}{2}kB(t) + \frac{1}{2}\sigma_1^2 B(t)C(t) - \frac{\beta r}{\sigma^2(\beta-1)^2} - \frac{\sigma_1\beta}{\sigma(\beta-1)}(\frac{1}{2}B(t) - rC(t))$. We have

$$B'(t) + (\frac{1}{2}\sigma_1^2 C(t) - \frac{1}{2}k - \frac{\beta\sigma_1}{2\sigma(\beta-1)})B(t) = \frac{\beta r}{\sigma^2(\beta-1)^2} - \frac{\beta r\sigma_1}{\sigma(\beta-1)}C(t)$$

Let $\frac{1}{2}\sigma_1^2 C(t) - \frac{1}{2}k - \frac{\beta\sigma_1}{2\sigma(\beta-1)} = P(t)$, $\frac{\beta r}{\sigma^2(\beta-1)^2} - \frac{\beta r\sigma_1}{\sigma(\beta-1)}C(t) = Q(t)$. Then we have

$$B'(t) + P(t)B(t) = Q(t)$$

The solution is

$$B(t) = \frac{\int \mu(t)Q(t)dt + constant}{\mu(t)}$$
$$\mu(t) = e^{\int P(t)dt}$$

we can get the value of constant by terminal conditon $B(T) = 0$. After solving $C(t)$ and $B(t)$, we can get the solution of $A(t)$ by $\quad 0 = A'(t) + kcC(t) + \frac{1}{8}\sigma_1^2 B^2(t) - \frac{\beta r}{\beta-1} + \frac{\beta r^2}{2\sigma^2(\beta-1)^2} + \frac{\sigma_1\beta r}{2\sigma(\beta-1)}B(t)$

$$A(t) = \int -kcC(t) - \frac{1}{8}\sigma_1^2 B^2(t) + \frac{\beta r}{\beta-1} - \frac{\beta r^2}{2\sigma^2(\beta-1)^2} - \frac{\sigma_1\beta r}{2\sigma(\beta-1)}B(t)dt + constant$$

we can get the value of constant by terminal conditon $A(T) = 0$. After that, we get the solution for the HJB equation.

## 3.3 FBSDE and dual FBSDE Method

Recall the theorem 2.4.2 and 2.5.2:

**Theorem 2.4.2 (Primal problem and associated FBSDE)** Let $\pi^* \in \mathcal{A}$. Then $\pi^*$ is optimal for the primal problem if and only if the solution $(X^{\pi^*}, \hat{p_1}, \hat{q_1})$ of FBSDE

$$\begin{aligned}
&dX^{\pi^*}(t) = X^{\pi^*}[(r(t) + {\pi^*}^T(t)\sigma(t)\theta(t))dt + {\pi^*}^T(t)\sigma(t)dW(t)] \\
&X^{\pi^*}(0) = x \\
&d\hat{p_1}(t) = -[(r(t) + {\pi^*}^T(t)^T\sigma(t)\theta(t))\hat{p_1}(t) + \hat{q_1}^T(t)\sigma^T(t)\pi^*(t)]dt + \hat{q_1}^T(t)dW(t) \\
&\hat{p_1}(T) = -U^{'}(X^{\pi^*}(T))
\end{aligned} \tag{3.3.1}$$

satisfies the condition

$$-X^{\pi^*}(t)\sigma(t)[\theta(t)\hat{p_1}(t)+\hat{q_1}(t)]\in N_K(\pi^*(t)),\quad \forall t\in[0,T],\mathbb{P}-a.s. \tag{3.3.2}$$

where $N_K(x)$ is the normal cone of the closed convex set $K$ at $x\in K$, defined as

$$N_K(x)\triangleq\{y\in\mathbb{R}^N:\forall x^*\in K, y(x^*-x)\le 0\}$$

**Theorem 2.5.2 (Dual problem and associated FBSDE)** Let $(\hat{y},\hat{v})\in(0,\infty)\times\mathcal{D}$. Then $(\hat{y},\hat{v})$ is optimal for the dual problem if and only if the solution $(Y^{(\hat{y},\hat{v})},\hat{p_2},\hat{q_2})$ of FBSDE

$$\begin{aligned}
&dY^{(\hat{y},\hat{v})}(t)=-Y^{(\hat{y},\hat{v})}(t)\{[r(t)+\delta_K(\hat{v}(t))]dt+[\theta(t)+\sigma^{-1}(t)\hat{v}(t)]dW(t)\}\\
&Y^{(\hat{y},\hat{v})}(0)=\hat{y}\\
&d\hat{p_2}(t)=\{[r(t)+\delta_K(\hat{v}(t))]^T\hat{p_2}(t)+\hat{q_2}^T(t)[\theta(t)+\sigma^{-1}(t)]\}dt+\hat{q_2}^T(t)dW(t)\\
&\hat{p_2}(T)=-\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))
\end{aligned} \tag{3.3.3}$$

satisfies the condition

$$\begin{aligned}
&\hat{p_2}(0)=x\\
&\hat{p_2}(t)^{-1}[\sigma(t)]^{-1}\hat{q_2}(t)\in K\\
&\hat{p_2}(t)\delta_K(\hat{v}(t))+\hat{q_2}(t)\sigma^{-1}(t)\hat{v}(t)=0,\forall t\in[0,T]\mathbb{P}-a.s.
\end{aligned} \tag{3.3.4}$$

### 3.3.1 Log utility function under Stochastic Factor Medol 1

In this part, we have

$$\begin{cases}dS_0(t)=rS_0(t)dt\\ dS(t)=H(t)S(t)dt+\sigma S(t)dW(t)\\ dH(t)=k(c-H(t))dt+\sigma_1\sqrt{H(t)}dW(t)\end{cases} \tag{3.3.5}$$

$K=\mathbb{R}$ and we will consider the log utility function.

For primal FBSDE, since $K=\mathbb{R}$, we have $\hat{q_1}(t)=-\frac{H_t-r}{\sigma}\hat{p_1}(t)$. By theorem 2.4.2, we have

$$d\hat{p_1}(t)=-r\hat{p_1}(t)dt-\frac{H_t-r}{\sigma}\hat{p_1}(t)dW_t$$

$$\hat{p_1}(T)=-U'(X^{\pi^*}(T))=-\frac{1}{X(T)}$$

We need to find $p_0$ such that $\hat{p_1}(T)=-\frac{1}{X(T)}$. Now consider the optimal control problem $\min\limits_{p_0,\pi}E[(\hat{p_1}(T)+\frac{1}{X(T)})^2]$. If we can manage to find $p_0$ and $\pi$ such that the minimum value is zero, we are done. However, if we seek a numerical solution, there is no possibility we can get the minimum value is 0 and we may be satisfied as long as the minimum value is sufficiently close to 0. To solve the optimal control problem numerically, we divide interval $[0,T]$ by $n$ intervals with step size $\Delta=T/n$ and gird points $t_0=0,t_i=\Delta i,i=1,...,n$. Assume on subinterval $[t_i,t_{i+1})$, control $\pi^*(t_i)=\alpha(t_i)+\beta(t_i)H_{t_i}$ is taken constant, where $\alpha(t_i)$ and $\beta(t_i)$ are piecewise constant within each subinterval. Using Euler scheme, we have

$$\begin{aligned}
&H_{t_{i+1}}=H_{t_i}+k(c-H_{t_i})\Delta+\sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i}\\
&X_{t_{i+1}}=X_{t_i}+X_{t_i}(r+\pi_{t_i}(H_{t_i}-r))\Delta+\pi_{t_i}X_{t_i}\sigma\sqrt{\Delta}Z_{t_i}\\
&\hat{p_1}(t_{i+1})=\hat{p_1}(t_i)-r\hat{p_1}(t_i)\Delta-\frac{H_{t_i}-r}{\sigma}\hat{p_1}(t_i)\sqrt{\Delta}Z_{t_i}
\end{aligned}$$

For dual FBSDE, we can write it as following by theorem 2.5.2

$$\begin{aligned}
&dY(t)=-Y(t)\{rdt+\frac{H_t-r}{\sigma}dW(t)\}\\
&Y(0)=y\\
&d\hat{p_2}(t)=\{r\hat{p_2}(t)+\hat{q_2}(t)\frac{H_t-r}{\sigma}\}dt+\hat{q_2}(t)dW_t\\
&\hat{p_2}(T)=-\tilde{U}'(Y^{\hat{y}}(T))=\frac{1}{Y(T)}
\end{aligned}$$

satisfies the condition

$$\hat{p_2}(0) = x_0$$
$$\hat{p_2}(t)^{-1}\sigma^{-1}\hat{q_2}(t) \in \mathbb{R}$$
$$\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}^T(t)\sigma^{-1}\hat{v}(t) = 0, \forall t \in [0,T]\mathbb{P} - a.s.$$

Same as before, we consider the optimal control problem $\min_{y_0,q_2} E[(\hat{p_2}(T) - \frac{1}{Y(T)})^2]$. To solve the optimal control problem numerically, we divide interval $[0,T]$ by $n$ intervals with step size $\Delta = T/n$ and gird points $t_0 = 0, t_i = \Delta i, i = 1,...,n$. Assume on subinterval $[t_i, t_{i+1})$, control $\hat{q_2}(t_i)$ is taken constant. Using Euler scheme, we have

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i}$$
$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i}r\Delta - Y_{t_i}\frac{H_{t_i} - r}{\sigma}\sqrt{\Delta}Z_{t_i}$$
$$\hat{p_2}(t_{i+1}) = \hat{p_2}(t_i) + \{r\hat{p_2}(t_i) + \hat{q_2}(t_i)\frac{H_{t_i} - r}{\sigma}\}\Delta + \hat{q_2}(t_i)\sqrt{\Delta}Z_{t_i}$$

with initial condition $\hat{p_2}(0) = x_0$ and terminal condition $\hat{p_2}(T) = \frac{1}{Y(T)}$

We set the parameters as $r = 0.05, k = 1, c = 1, \sigma = 1, \sigma_1 = 0.5, \beta = 0.1, h_0 = 0.5, x_0 = 10, T = 1, \Delta = dt = 0.01$. By simulation method, we have the optimal parameters $\alpha(t)$ and $\beta(t)$ as follows:

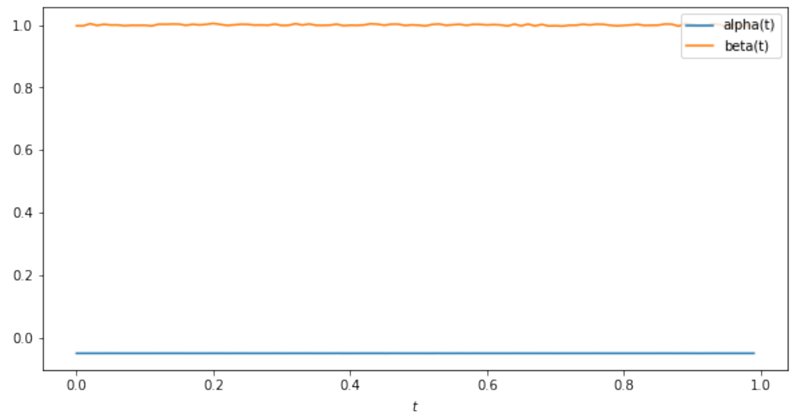


Figure 3.1: Optimal $\alpha(t)$ and $\beta(t)$ from Primal FBSDE, dt=0.01

Using the optimal parameters $\alpha(t)$ and $\beta(t)$, we can get $\pi^*(t) = \alpha(t) + \beta(t)H(t)$ from primal FBSDE. Similarly, we can get $\pi^*(t)$ from dual FBSDE by using the optimal parameter $q_2(t)$ and theorem 2.6.3. Plot them on graph, we can get

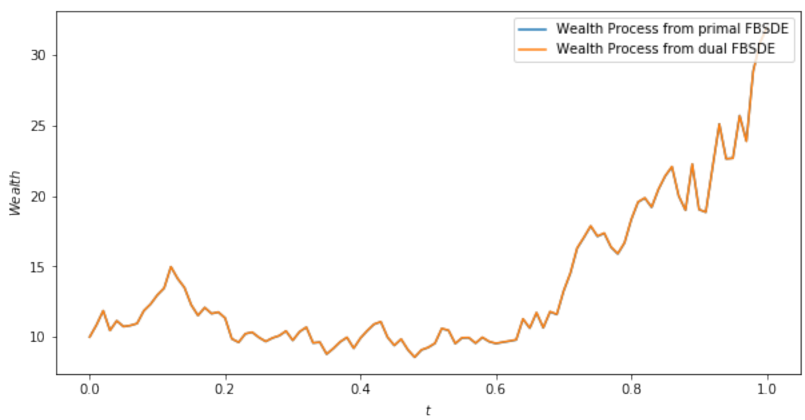


Figure 3.2: Optimal Pi from Primal FBSDE and Dual FBSDE,dt=0.01

We can see that in figure 3.2, the two optimal control processes overlap completely, which means the optimal control should be the same from primal FBSDE and dual FBSDE.

We can also get the wealth processes from primal FBSDE and dual FBSDE:

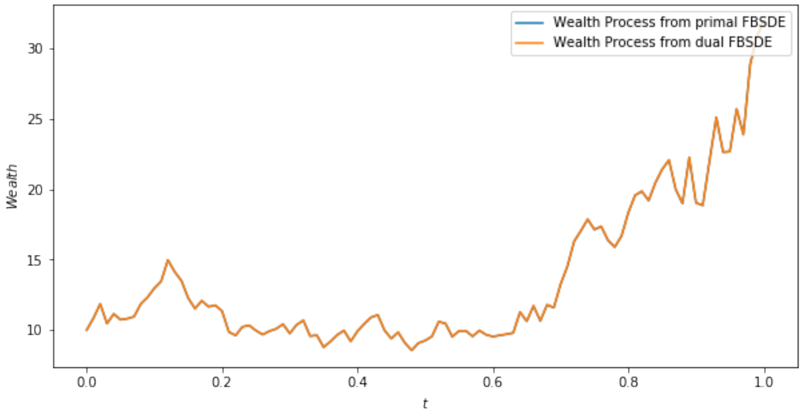


Figure 3.3: Wealth Processes from Primal FBSDE and Dual FBSDE dt=0.01

We can see that the wealth processes from two method almost completely overlap, which means we can get almost the same wealth processes by both methods and both methods are effective.

### 3.3.2 Power utility function under Stochastic Factor Model 2: Case 2

In this part, we have

$$\begin{cases} dS_0(t) = rS_0(t)dt \\ dS(t) = \sqrt{H(t)}S(t)dt + \sigma S(t)dW(t) \\ dH(t) = k(c - H(t))dt + \sigma_1\sqrt{H(t)}dW(t) \end{cases} \tag{3.3.6}$$

$K = \mathbb{R}$ and we will consider the power utility function.

For primal FBSDE, since $K = \mathbb{R}$, we have $\hat{q}_1(t) = -\frac{\sqrt{H_t}-r}{\sigma}\hat{p}_1(t)$. By theorem 2.4.2, we have

$$d\hat{p}_1(t) = -r\hat{p}_1(t)dt - \frac{\sqrt{H_t} - r}{\sigma}\hat{p}_1(t)dW_t$$

$$\hat{p}_1(T) = -U^{'}(X^{\pi^*}(T)) = -X(T)^{\beta-1}$$

We need to find $p_0$ such that $\hat{p}_1(T) = -X(T)^{\beta-1}$. Now consider the optimal control problem $\min_{p_0,\pi} E[(\hat{p}_1(T) + X(T)^{\beta-1})^2]$. If we can manage to find $p_0$ and $\pi$ such that the minimum value is zero, we are done. However, if we seek a numerical solution, there is no possibility we can get the minimum value is 0 and we may be satisfied as long as the minimum value is sufficiently close to 0. To solve the optimal control problem numerically, we divide interval $[0, T]$ by $n$ intervals with step size $\Delta = T/n$ and gird points $t_0 = 0, t_i = \Delta i, i = 1, ..., n$. Assume on subinterval $[t_i, t_{i+1})$, control $\pi^*(t_i) = \alpha(t_i) + \beta(t_i)\sqrt{H_{t_i}}$ is taken constant, where $\alpha(t_i)$ and $\beta(t_i)$ are piecewise constant within each subinterval. Using Euler scheme, we have

$$\begin{aligned} H_{t_{i+1}} &= H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i} \\ X_{t_{i+1}} &= X_{t_i} + X_{t_i}(r + \pi_{t_i}(\sqrt{H_{t_i}} - r))\Delta + \pi_{t_i}X_{t_i}\sigma\sqrt{\Delta}Z_{t_i} \\ \hat{p}_1(t_{i+1}) &= \hat{p}_1(t_i) - r\hat{p}_1(t_i)\Delta - \frac{\sqrt{H_{t_i}} - r}{\sigma}\hat{p}_1(t_i)\sqrt{\Delta}Z_{t_i} \end{aligned}$$

For dual FBSDE, we can write it as following by theorem 2.5.2

$$
\begin{aligned}
dY(t) &= -Y(t)\{rdt + \frac{\sqrt{H_t} - r}{\sigma} dW(t)\} \\
Y(0) &= y \\
d\hat{p_2}(t) &= \{r\hat{p_2}(t) + \hat{q_2}(t)\frac{\sqrt{H_t} - r}{\sigma}\}dt + \hat{q_2}(t)dW_t \\
\hat{p_2}(T) &= -\tilde{U}'(Y^{\hat{y}}(T)) = Y(T)^{\frac{1}{\beta-1}}
\end{aligned}
$$

satisfies the condition

$$
\begin{aligned}
&\hat{p_2}(0) = x_0 \\
&\hat{p_2}(t)^{-1}\sigma^{-1}\hat{q_2}(t) \in \mathbb{R} \\
&\hat{p_2}(t)\delta_K(\hat{v}(t)) + \hat{q_2}^T(t)\sigma^{-1}\hat{v}(t) = 0, \forall t \in [0,T]\mathbb{P} - a.s.
\end{aligned}
$$

Same as before, we consider the optimal control problem $\min_{y_0, q_2} E[(\hat{p_2}(T) - Y(T)^{\frac{1}{\beta-1}})^2]$. To solve the optimal control problem numerically, we divide interval $[0,T]$ by $n$ intervals with step size $\Delta = T/n$ and gird points $t_0 = 0, t_i = \Delta i, i = 1, ..., n$. Assume on subinterval $[t_i, t_{i+1})$, control $\hat{q_2}(t_i)$ is taken constant. Using Euler scheme, we have

$$
\begin{aligned}
H_{t_{i+1}} &= H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i} \\
Y_{t_{i+1}} &= Y_{t_i} - Y_{t_i}r\Delta - Y_{t_i}\frac{\sqrt{H_{t_i}} - r}{\sigma}\sqrt{\Delta}Z_{t_i} \\
\hat{p_2}(t_{i+1}) &= \hat{p_2}(t_i) + \{r\hat{p_2}(t_i) + \hat{q_2}(t_i)\frac{\sqrt{H_{t_i}} - r}{\sigma}\}\Delta + \hat{q_2}(t_i)\sqrt{\Delta}Z_{t_i}
\end{aligned}
$$

with initial condition $\hat{p_2}(0) = x_0$ and terminal condition $\hat{p_2}(T) = Y(T)^{\frac{1}{\beta-1}}$

For **case 2**, we set the parameters as $r = 0.05, k = 1, c = 1, \sigma = 1, \sigma_1 = 2, \beta = 0.1, h_0 = 0.5, x_0 = 10, T = 1, \Delta = dt = 0.01$ to match the condition $4kc = \sigma_1^2$. By simulation method, we have the optimal parameters $\alpha(t)$ and $\beta(t)$ as follows:

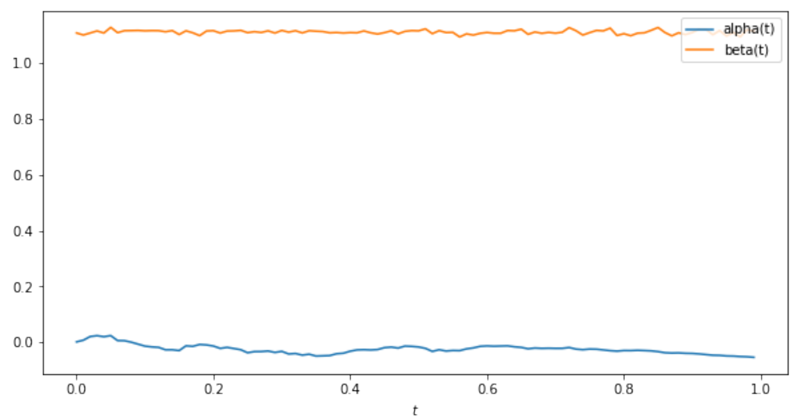


Figure 3.4: Optimal $\alpha(t)$ and $\beta(t)$ from Primal FBSDE, dt=0.01

Using the formula $\pi^*(t) = \alpha(t) + \beta(t)\sqrt{H(t)}$, optimal parameter $q_2(t)$ and theorem 2.6.3, we can get the optimal control processes from primal FBSDE and dual FBSDE.

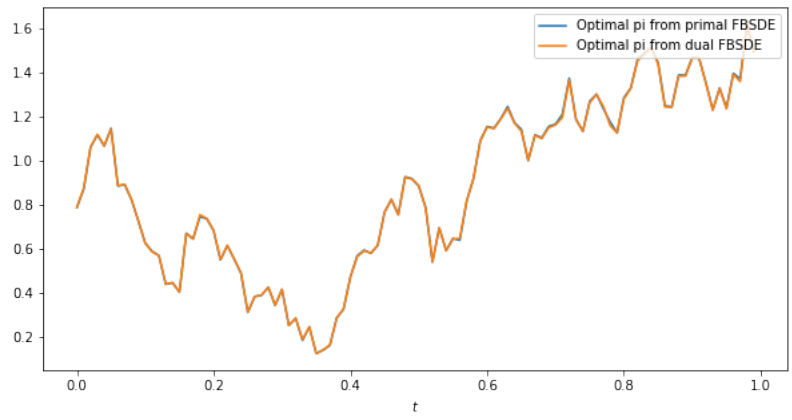


Figure 3.5: Optimal Pi from Primal FBSDE and Dual FBSDE,dt=0.01

Similarly, we have the two almost coincide lines. This is same as the result we get in stochastic factor model 1.

The wealth processes from primal FBSDE and dual FBSDE are shown as following:

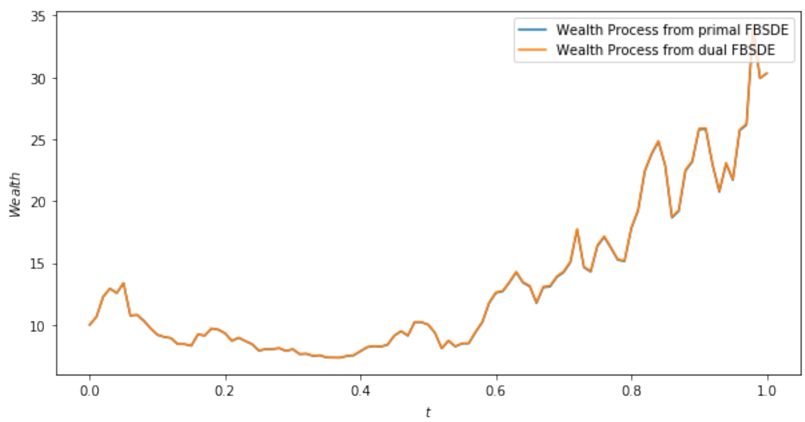


Figure 3.6: Wealth Processes from Primal FBSDE and Dual FBSDE dt=0.01

Same as before, we can get the almost same results from primal FBSDE and dual FBSDE.

## 3.4 Numerical Verification

In this section, we will use numerical and simulation methods to show that we can get the exact same **wealth process** by primal HJB, dual HJB, primal FBSDE and dual FBSDE for **Log Utility function under Stochastic Factor Model 1** and **Power Utility function under Stochastic Factor Model 2: Case 2**.

### 3.4.1 Log Utility function under Stochastic Factor Model 1

In this part, we consider the log utility function we have

$$dS(t) = H(t)S(t)dt + \sigma S(t)dW(t)$$

$$dH(t) = k(c - H(t))dt + \sigma_1\sqrt{H(t)}dW(t)$$

$$dX^\pi(t) = X^\pi(t)[(r + \pi(t)(H(t) - r))dt + \pi(t)\sigma dW(t)]$$

For primal method, using Euler scheme, we can get

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i}$$

$$X_{t_{i+1}} = X_{t_i} + X_{t_i}(r + \pi_{t_i}(H_{t_i} - r))\Delta + \pi_{t_i} X_{t_i} \sigma \sqrt{\Delta} Z_{t_i}$$

$$\pi^*(t_i) = \frac{H_{t_i} - r}{\sigma^2}$$

For dual method, we know that

$$X^{\pi^*(t)} = -\frac{\partial \hat{V}(t, \hat{Y}(t))}{\partial \hat{Y}(t)} = \frac{1}{\hat{Y}(t)}$$

$$dY(t) = -Y(t)\{rdt + \frac{H_t - r}{\sigma} dW(t)\}$$

By using Euler scheme, we can get

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1 \sqrt{H_{t_i}} \sqrt{\Delta} Z_{t_i}$$

$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i} r\Delta - Y_{t_i} \frac{H_t - r}{\sigma} \sqrt{\Delta} Z_{t_i}$$

We set the same parameters as before in the FBSDE methods and by simulation method, we can get the wealth processes from primal HJB and dual HJB as following:

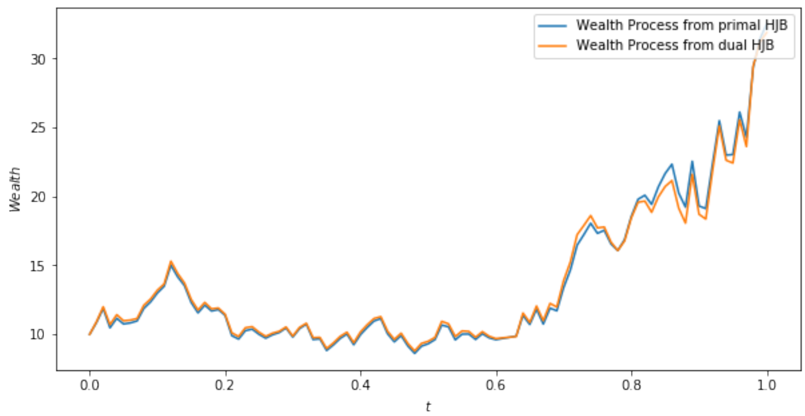


Figure 3.7: Wealth Processes from Primal and Dual HJB, dt=0.01

We plot wealth processes and optimal control processes from four methods in one figure to compare the results.

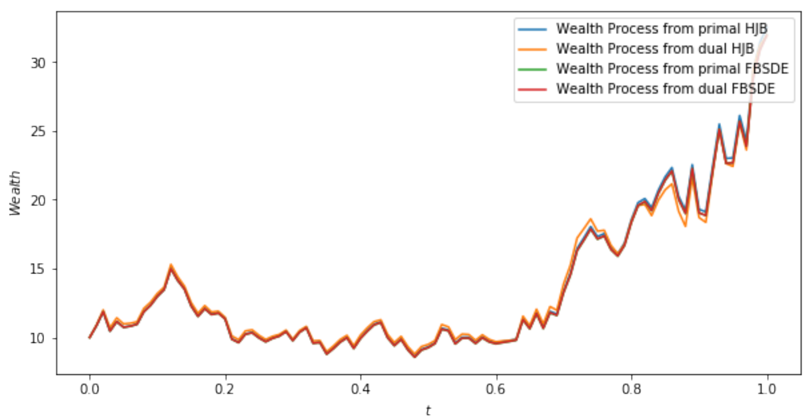


Figure 3.8: Wealth Processes from Four Methods, dt=0.01

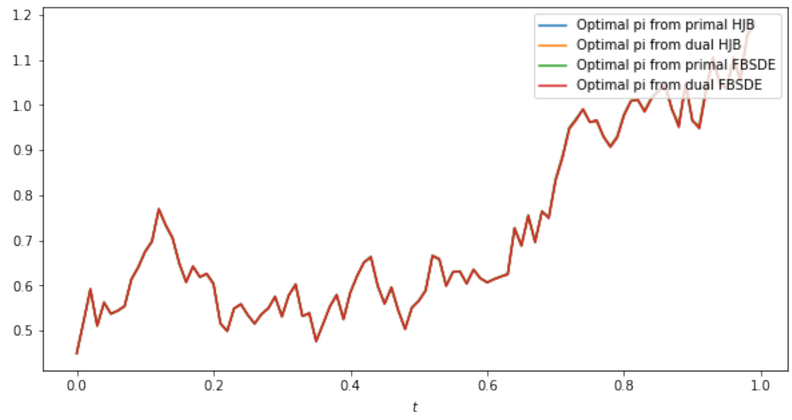


Figure 3.9: Optimal Pi from Four Methods ,dt=0.01

From figure 3.8 and 3.9, we can see that four wealth processes have same trend all the time and slightly difference. As time increase, the difference in values of four wealth processes becomes larger. In addition, the optimal control processes from four methods are almost the same. Then We calculate the mean square error of wealth process from other three methods in comparison to the primal HJB. We have the mean square errors are 0.14153 for dual HJB, 0.02870 for primal FBSDE and 0.02742 for dual FBSDE. MSEs are too small compared with the values of wealth process. Thus we can consider the wealth processes from these four methods to be the same.

Then we use different time step size to see the effect of time step size. Other parameters won't be changed. We have the wealth processes and optimal control processes for $dt = 0.02$ and $dt = 0.05$ and we also calculate the MSEs for $dt = 0.02$ and $dt = 0.05$.

| Method | MSE dt = 0.01 | MSE dt = 0.02 | MSE dt = 0.05 |
|---|---|---|---|
| Primal HJB | 0 | 0 | 0 |
| Dual HJB | 0.14153 | 1.01434 | 3.49741 |
| Primal FBSDE | 0.02870 | 0.25264 | 3.48545 |
| Dual FBSDE | 0.02742 | 0.19643 | 1.03490 |

Table 3.1: Mean Square Error

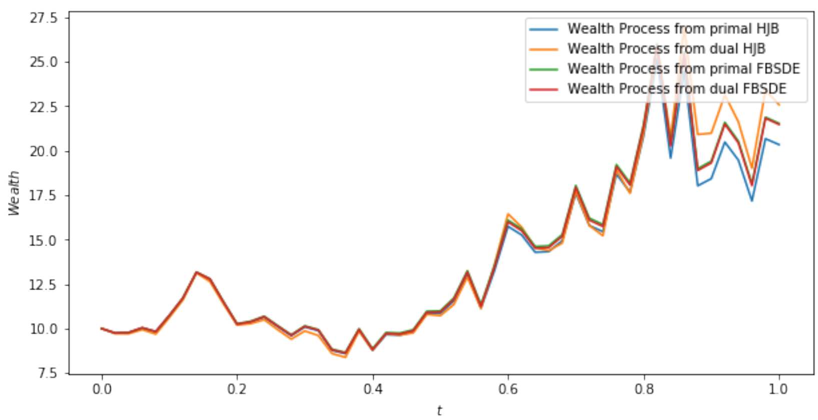


Figure 3.10: Wealth Processes from Four Methods, dt=0.02

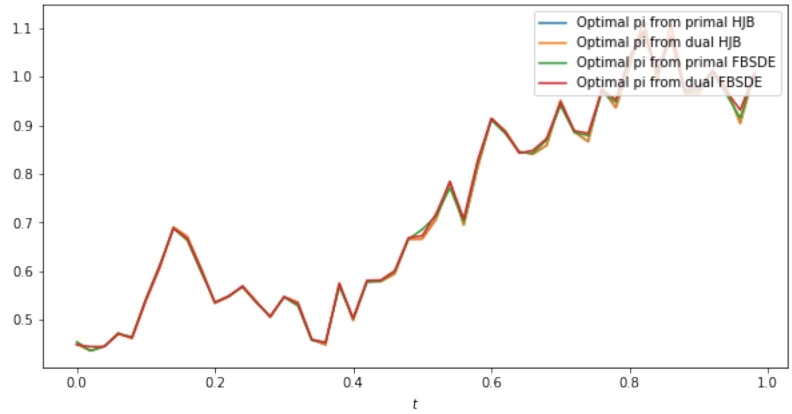


Figure 3.11: Optimal Pi from Four Methods, dt=0.02

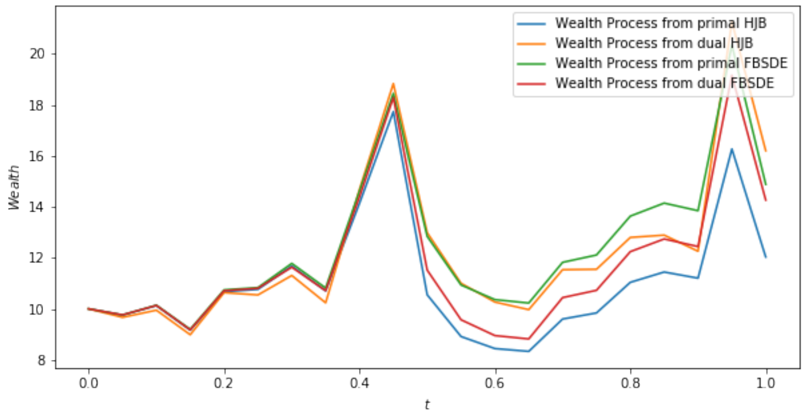


Figure 3.12: Wealth Processes from Four Methods, dt=0.05

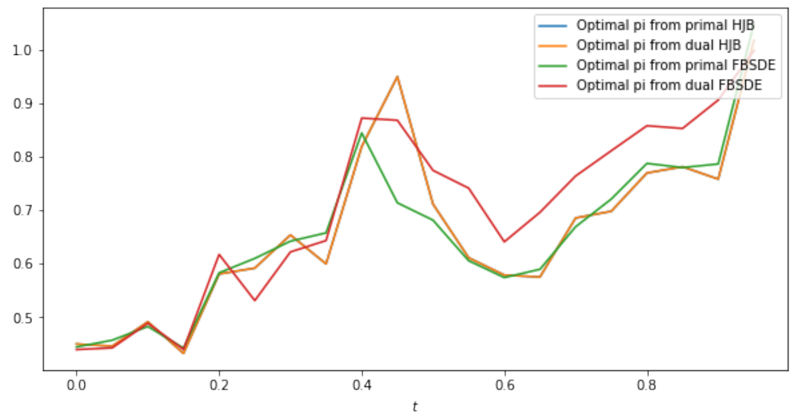


Figure 3.13: Optimal Pi from Four Methods, dt=0.05

As can be seen in figure 3.8, 3.10 and 3.12, the simulation results become worse from four methods as the time step size become larger. From $t = 0$ to $t = 0.5$, difference among four methods is slight. But from $t = 0.5$ to $t = 1$, the difference among these wealth processes can be seen obviously. The reason is that as time step size become larger, the simulation results become worse and error could be accumulated as $t$ increases. In addition, we know that as time step size increases, the error becomes larger from table 3.1.

### 3.4.2 Power utility function under Stochastic Factor Model 2: Case 2

In this part, we consider the power utility function we have

$$dS(t) = \sqrt{H(t)}S(t)dt + \sigma S(t)dW(t)$$

$$dH(t) = k(c - H(t))dt + \sigma_1\sqrt{H(t)}dW(t)$$

$$dX^{\pi}(t) = X^{\pi}(t)[(r + \pi(t)(\sqrt{H(t)} - r))dt + \pi(t)\sigma dW(t)]$$

For primal method, using Euler scheme, we can get

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i}$$

$$X_{t_{i+1}} = X_{t_i} + X_{t_i}(r + \pi_{t_i}(\sqrt{H_{t_i}} - r))\Delta + \pi_{t_i}X_{t_i}\sigma\sqrt{\Delta}Z_{t_i}$$

$$\pi^*(t) = -\frac{(\sqrt{h} - r) + \sigma\sigma_1(\frac{1}{2}B(t) + C(t)\sqrt{h})}{\sigma^2(\beta - 1)}$$

For dual method, we know that

$$X^{\pi^*(t)} = -\frac{\partial\hat{V}(t, \hat{Y}(t))}{\partial\hat{Y}(t)}$$

We have already known that $\hat{V}(t, y, h) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}\exp A(t) + B(t)\sqrt{h} + C(t)h$, by taking derivative of $\tilde{V}$ with respect to y and setting it to be 0, we get $\hat{y}(t) = x^{\beta-1}\exp A(t) + B(t)\sqrt{h} + C(t)h^{1-\beta}$, and thus $\hat{y}(0) = x^{\beta-1}\exp A(0) + B(0)\sqrt{h} + C(0)h^{1-\beta}$.

So we have

$$X(t) = Y(t)^{\frac{1}{\beta-1}}\exp\left(A(t) + B(t)\sqrt{H_t} + C(t)H_t\right)$$

$$dY(t) = -Y(t)\{rdt + \frac{\sqrt{H_t} - r}{\sigma}dW(t)\}$$

By using Euler scheme, we can get

$$H_{t_{i+1}} = H_{t_i} + k(c - H_{t_i})\Delta + \sigma_1\sqrt{H_{t_i}}\sqrt{\Delta}Z_{t_i}$$

$$Y_{t_{i+1}} = Y_{t_i} - Y_{t_i}r\Delta - Y_{t_i}\frac{\sqrt{H_t} - r}{\sigma}\sqrt{\Delta}Z_{t_i}$$

We set the same parameters as befor to match the condition $\frac{1}{2}kc - \frac{1}{8}\sigma_1^2 = 0$. We first use numerical method to find $A(t), B(t)$ and $C(t)$ for primal HJB and dual HJB. Then we use simulation method to get the wealth processes. The results are shown below:

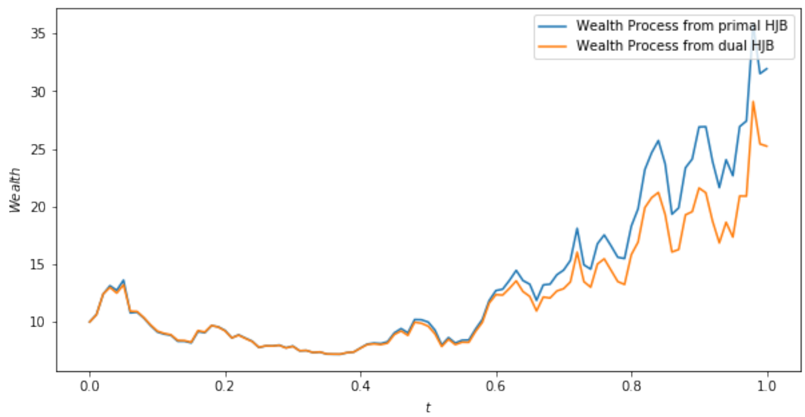


Figure 3.14: Wealth Processes from Primal and Dual HJB, dt=0.01

Then we plot wealth processes and optimal control processes from four methods in one figure to compare the results.

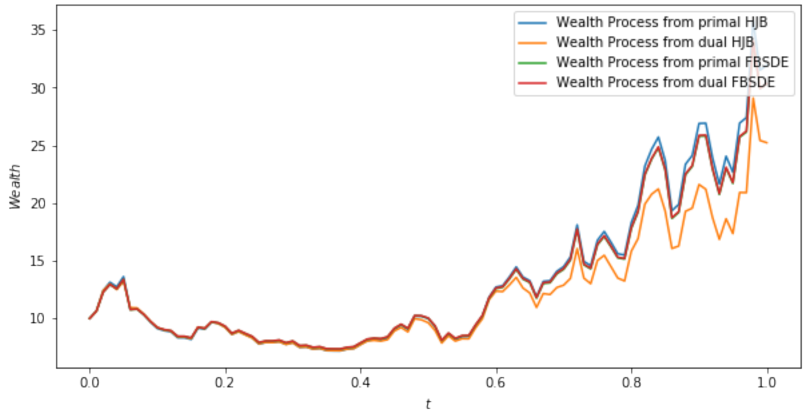


Figure 3.15: Wealth Processes from Four Methods, dt=0.01

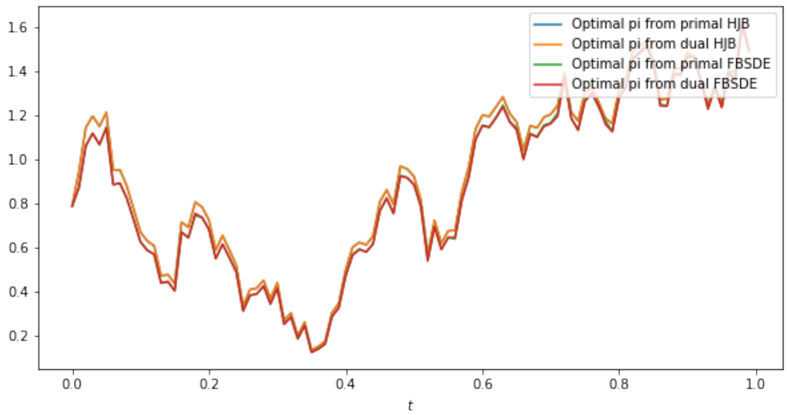


Figure 3.16: Optimal Pi from Four Methods ,dt=0.01

We also calculate the mean square error of wealth process from other three methods in comparison to the primal HJB. They are 5.55117, 0.25668 and 0.22774 separately for dual HJB, primal FBSDE and dual FBSDE. We can get the same result that four methods will have the exact same solution for utility maximization problem. There will the same effect of time step size that as the time step size increases, the error becomes larger.

By numerical verification in two stochastic factor models and different utility functions, we have proved that we have the exact same wealth process by primal HJB, dual HJB, primal FBSDE and dual FBSDE, which means we can get same solution for utility maximization problem by these four methods.

# Chapter 4

# Conclusion

In this paper, we study the utility maximization problem under different models using four approaches. Under the simple assumption model, we can get the closed formula solution for this problem with power utility function and non-HARA utility function by dual HJB and FBSDE. It is clear that different methods have the exact same solution for the utility maximization problem. However, we can not find the closed formula solution under complex models such as stochastic factor models. In this case, we use numerical method to plot the wealth processes and optimal control processes in one graph from primal HJB, dual HJB,primal FBSDE and dual FBSDE and then compare the results. We also calculate the mean square error from other three methods in comparison to the primal HJB. We can conclude that the wealth processes from these four methods to be the same. Thus we can conclude that all the four methods can get the exact same solution for the utility maximization problem given in this paper.

For further reseach, we can consider the utility maximization problem under constrained to find whether we can get the same solution by primal HJB, dual HJB, primal FBSDE and dual FBSDE. In addition, we use the BSDEs representation to provide a solution to the semilinear PDE. Can we solve the BSDEs directly or we need to find a numerical solution to the BSDEs. It is also a problem that we can consider in the future.

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