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Constrained Quadratic Risk Minimization Problem via Four Different Approaches

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Declaration

The work contained in this thesis is my own work unless otherwise stated.

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Abstract

In this paper, we study the continuous-time stochastic linear quadratic control problem with financial applications, and we aim to minimize the constrained quadratic risk function, a convex cost function, in both the wealth process and portfolio strategy in an incomplete market. By the convex duality method, we construct the associated dual problem satisfied the necessary and sufficient optimality conditions. There are four main approaches used to solve the stochastic linear quadratic control problem in this paper, such as primal HJB, dual HJB, primal FBSDE and dual FBSDE, and the goal is to prove all four approaches have the same solutions of the problem. Then, we discuss the quadratic risk minimization problem with both no control constraints and cone-constraints and derive numerical solutions of each approach. For no control constraints and cone-constraints problem, we model the asset dynamics with constant coefficients and deterministic coefficients, respectively. We mainly use the numerical methods, the Runge-Kutta method and the Euler method, to solve quadratic risk minimization problems. We also compare results of all approaches intuitively and analytically by plotting paths of optimal processes and calculating the mean squared errors and variances of differences between results. To compare all methods more accurately and precisely, we check results with different values of coefficients. In the end, we study how to solve the quadratic risk minimization problem under the stochastic factor model by primal and dual HJB approaches.

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Chapter 1

Introduction

In the 1960s, the stochastic control was first introduced in papers to study the stochastic linear regulators with engineering applications, for example, Davis [5] introduced this field. For the financial applications, the stochastic control was developed enormously from the 1970s, especially Robert Merton [20] [21] published two landmark papers about the Hamilton-Jacobi-Bellman equation and the requirement of an underlying Markov state process. Based on papers of Merton, Zariphopoulou [26], Davis and Norman [6], Øksendal and Sulem [27] and many scholars had extensive research on this topic. Karatzas and Shreve [12] also discussed the application of stochastic control to financial problems in their monograph.

For stochastic linear quadratic control problem, its applications on mean-variance portfolio selection problems were studied in many papers, see, e.g. Schweizer [23] and Yong and Zhou [25]. Without portfolio constraints, the stochastic maximum principle is used to solve the stochastic linear quadratic problem and obtain the optimal control as linear feedback control of the wealth process, where linear feedback control includes a solution of corresponding stochastic Ricatti equation. Also, the admissibility of optimal control depends on whether there are control constraints. If there are no control constraints, linear feedback control is straightforwardly admissible; Otherwise, it is more difficult to solve the stochastic linear quadratic control problem, and optimal control is not a linear feedback control anymore. Lim and Zhou [19] and Hu and Zhou [10] introduced approaches used to solve unconstrained and constrained stochastic linear quadratic control problem respectively.

Shreve and Xu [24] firstly adopted the stochastic duality theory of Bismut [2] to solve the constrained optimal investment problem. After that, the convex duality method was more used to deal with incomplete market models, see, e.g. Karatzas et al. [11], Pearson and He [8] [9], Cvitanić and Karatzas [3], but sometimes, it is difficult to obtain the associated dual problem. Labbé and Heunis[15] introduced a simple and elegant method to construct the corresponding dual problem without a prior hypothesis of the market. For convex stochastic linear quadratic control problems, the convex duality method was often used to solve the utility maximization problems, for example, Kramkov and Schachermayer [13] [14]. If the filtration is generated by standard Brownian motions without control constraints, the dynamic optimization problem can be reformulated as a static dual problem, and the optimal wealth process and the optimal control process can be solved by the dual relation, the martingale property and the martingale representation theorem; Otherwise, the convex duality method cannot solve the problem straightforwardly. There are vast literatures on using the convex duality method to solve financial problems, see, e.g. Labbé and Heunis[16], Czichowsky and Schweizer [4]. Moreover, the convex duality method can usually solve the stochastic linear quadratic control problem more convenient since the dual problem is often solved explicitly.

For forward backward stochastic differential equations, Øksendal and Sulem [28] demonstrated that the optimal wealth process and the optimal control process are related to the adjoint optimal adjoint processes from forward backward stochastic differential equations. Li and Zheng [18] stated the necessary and sufficient optimality conditions for primal and dual stochastic linear quadratic control problems and indicated the relationship between the optimal solutions through their corresponding forward backward stochastic differential equations.

In this paper, we study the continuous-time stochastic linear quadratic control problem with financial applications, and we aim to minimize the constrained quadratic risk function, a convex cost function, in both the wealth process and portfolio strategy in an incomplete market. Here we assume that portfolio strategy must take values in the closed convex set which is applicable to include short selling, borrowing, and other trading restrictions, see [12]. This is mainly referred to the article "Constrained Quadratic Risk Minimization via Forward and Backward Stochastic Differential Equations" wrote by Li and Zheng [18]. Not similar with Li and Zheng [18], there are four main approaches used to solve the stochastic linear quadratic control problem in this paper, such as primal HJB, dual HJB, primal FBSDE and dual FBSDE, and the goal is to prove all four approaches have the exact same solutions of the problem.

Following Li and Zheng [18], we construct primal and dual problems with necessary and sufficient conditions. When there are no control constraints, we solve the quadratic risk minimization problem with constant coefficients by all approaches, and we consider there is only one risky asset in the portfolio. For stochastic Ricatti equations, we use the Runge-Kutta method, inspired by File and Bullo [7], to find numerical solutions. Primal FBSDE and dual FBSDE approaches are also solved numerically; we first use the stochastic maximum principle obtain fully-coupled FBSDEs, and convert them into terminal quadratic error minimization problems with piecewise constant parameters to find numerical solutions. To compare all approaches, we plot sample paths of optimal wealth processes and optimal control processes for each method on a graph to see if they are close to each other and also compute the mean squared errors and variances of differences between each method. Then, we check results with different values of coefficients to compare all methods more accurately and precisely. When there are cone-constraints (no short selling), we solve the quadratic risk minimization problem with deterministic coefficients by all approaches, and we consider there are two risky assets in the portfolio. In this case, the convex duality method becomes much more complicated. Especially for the dual HJB approach, we have to consider four different situations of the quadratic risk minimization problem to find all solutions. In primal FBSDE and dual FBSDE approaches, we cannot use the stochastic maximum principle to replace the optimal controls by other parameters anymore, so we assume optimal controls are also piecewise constant and use numerical minimization to find the solutions. Instead of plotting the sample path of optimal control processes for each method, since the dimension becomes two, we plot the paths of errors of two controls for each approach.

Furthermore, we try to solve the quadratic risk minimization problem under the stochastic factor model, where asset price has a random drift term and drift term follows the OU process. For this topic, Alghalith [1] introduced general explicit solutions to the portfolio optimization problem. Stochastic Ricatti equations become semi-linear PDEs, and we can find the associated BSDE representations, viscosity solutions, inspired by Pham [22]. Due to the limited time, we only solved the problem by primal HJB and dual HJB approaches, other approaches will be discussed in the future.

The paper is organized as follows: In Chapter 2, we set up the model and formulate the quadratic risk minimization problem. In Chapter 3, we discuss the quadratic risk minimization problem with constant coefficients and no control constraints under all approaches and compare results. In Chapter 4, we discuss the quadratic risk minimization problem with deterministic coefficients and cone-constraints under all approaches and compare results. In Chapter 5, we study how to solve the quadratic risk minimization problem under stochastic factor model by primal and dual HJB approaches.

Chapter 2

Theoretical Framework

2.1 The Market Model and Optimization Problems

In this paper, we have the following settings:

- $T > 0$ denotes a fixed terminal time,
- $\{W(t); t \in [0, T]\}$ denotes a $\mathbb{R}^N$-valued standard Brownian motion with scalar entries $W_m(t), m = 1, ..., N$,
- On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
- $\{\mathcal{F}_t\}$ denotes the $\mathbb{P}$-augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \le s \le t)$ generated by W,
- $\mathcal{P}(0, T; \mathbb{R}^N)$ denotes the set of all $\mathbb{R}^N$ -valued progressively measurable processes on $[0, T] \times \Omega$,
- $\mathcal{H}^2(0, T; \mathbb{R}^N)$ denotes the set of processes x in $\mathcal{P}(0, T; \mathbb{R}^N)$ satisfying $E[\int_0^T |x(t)|^2 dt] < \infty$,
- $\mathcal{S}^2(0, T; \mathbb{R}^N)$ denotes the set of processes x in $\mathcal{P}(0, T; \mathbb{R}^N)$ satisfying $E[\sup_{0 \le t \le T} |x_t^2|] < \infty$
- SDE denotes stochastic differential equation,
- BSDE denotes backward stochastic differential equation,
- FBSDE denotes forward and backward stochastic differential equation,
- w is suppressed in SDEs and integrals, except in places where an explicit w is needed.

For the market model, we consider that the market is consisted of a bank account and N risky assets, where

- Bank account has price $\{S_0(t)\}$ given by

$$dS_0(t) = r(t)S_0(t)dt, \quad 0 \le t \le T, \quad S_0(0) = 1,$$

- N risky assets have prices $\{S_n(t)\}$, $n = 1, ..., N$ given by

$$dS_n(t) = S_n(t) \left[b_n(t)dt + \sum_{m=1}^{N} \sigma_{nm}(t)dW_m(t) \right], \quad 0 \le t \le T, \quad S_n(0) > 0,$$

- $r \in \mathcal{P}(0, T; \mathbb{R})$ denotes scalar interest rate,
- $b \in \mathcal{P}(0, T; \mathbb{R}^N)$ denotes vector of appreciation rates,
- $\sigma \in \mathcal{P}(0, T; \mathbb{R}^{N \times N})$ denotes volatility matrix, and σ are uniformly bounded.

Especially for the volatility matrix $\sigma(t)$, there exists a positive constant k such that

$$z^\top \sigma(t)\sigma^\top(t)z \geq k|z|^2$$

for all $(z, w, t) \in \mathbb{R}^N \times \Omega \times [0, T]$, where $z^\top$ is the transpose of z. This strong non-degeneracy condition ensures that matrices $\sigma(t)$, $\sigma^\top(t)$ are invertible and uniformly bounded.

In the rest of the paper, we consider that there is a small investor with initial wealth $x_0 > 0$ and a self-financing strategy. Define the set of admissible portfolio strategies by

$$\mathcal{A} := \{\pi \in \mathcal{H}^2(0, T; \mathbb{R}^N) : \pi(t) \in K \; for \; t \in [0, T] \; a.e.\}$$

where $K \subseteq \mathbb{R}^N$ is a closed convex set containing 0 and $\pi(t)$ is a portfolio process with each entry $\pi_n(t)$, n for $n = 1, ..., N$. $\pi(t)$ represents the amounts invested in the risky assets. Given any $\pi \in \mathcal{A}$, the investor's total wealth X^π should satisfy the following SDE

$$\begin{cases} dX^\pi(t) = [r(t)X^\pi(t) + \pi^\top(t)\sigma(t)\theta(t)]dt + \pi^\top(t)\sigma(t)dW(t), \quad 0 \leq t \leq T \\ X^\pi(0) = x_0, \end{cases} \tag{2.1.1}$$

where $\theta(t) = \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}]$ represents the market price of risk at time t, which is uniformly bounded, and $\mathbf{1} \in \mathbb{R}^N$ has all unit entries. A pair (X, π) is admissible when $\pi(t) \in \mathcal{A}$ and X is a strong solution to SDE (2.1.1) with control process $\pi(t)$.

Define the funtional $J : \mathcal{A} \to \mathbb{R}$ by

$$J(\pi) := E\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\right],$$

where $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$f(w, t, x, \pi) := \frac{1}{2}[Q(t)x^2 + 2S^\top(t)x\pi + \pi^\top R(t)\pi],$$

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$g(w, x) := \frac{1}{2}[ax^2 + 2cx].$$

Following Li and Zheng [18], to ensure J is a convex functional of π, the random variables $a, c \in L^\infty_{\mathcal{F}_T}(\mathbb{R})$ should satisfy

$$0 < \inf_{w \in \Omega} a(w) \leq \sup_{w \in \Omega} a(w) < \infty.$$

The processes $Q \in \mathcal{P}(0, T; \mathbb{R})$, $S \in \mathcal{P}(0, T; \mathbb{R}^N)$ and $R \in \mathcal{P}(0, T; \mathbb{R}^{N \times N})$, shown in $f(w, t, x, \pi)$, are uniformly bounded, and $R(t)$ is a symmetric matrix. Also, the matrix

$$\begin{pmatrix} Q(t) & S^\top(t) \\ S(t) & R(t) \end{pmatrix}$$

is non-negative definite for all $(w, t) \in \Omega \times [0, T]$.

The quadratic risk minimization problem discussed in this paper is

$$\text{Minimize } J(\pi) \text{ subject to } (X, \pi) \text{ admissible.}$$

The optimal admissible control $\hat{\pi}$ is obtain when $J(\hat{\pi}) \leq J(\pi)$ for all $\pi \in \mathcal{A}$.

Inspired by the work of Chantal and Andrew [17], we can construct the associated dual problem. Let $\mathbb{B}$ denote

$$\mathbb{B} := \mathbb{R} \times \mathcal{H}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^N),$$

we say $X \in \mathbb{B}$ if and only if

$$X(t) = x_0 + \int_0^t \dot{X}(\tau)d\tau + \int_0^t \Lambda_X^\top(\tau)dW(\tau), \; 0 \leq t \leq T,$$

where some $(x_0, \dot{X}, \Lambda_X) \in \mathbb{B}$. We then convert the quadratic risk minimization problem into a primal optimization problem over the whole set $\mathbb{B}$. For each $X \equiv (x_0, \dot{X}, \Lambda_X) \in \mathbb{B}$, define

$$\mathcal{U}(X) := \{\pi \in \mathcal{A} \text{ such that } \dot{X}(t) = rX(t) + \pi^\top(t)\sigma\theta \text{ and } \Lambda_X(t) = \sigma^\top \pi(t) \text{ for } \forall t \in [0,T],\ \mathcal{P} - a.e.\}.$$

The set $\mathcal{U}(X)$ contains all admissible controls $\pi \in \mathcal{A}$, which associated wealth process X is admissible. Note that $\mathcal{U}(X) \neq \emptyset$ if and only if $(\dot{X}, \Lambda_X) \in \mathcal{S}(t, X(t))$ for $(\mathbb{P} \otimes Leb)$-a.e. $(w,t) \in \Omega \times [0,T]$, where $\mathcal{S}$ is a set valued function defined by

$$\mathcal{S}(w,t,X(t)) := \{(v,s) : v = rx + \xi^\top\theta \text{ and } [\sigma^\top]^{-1}\xi \in K\}.$$

Define the penalty function $L : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty]$ by

$$L(w,t,x,v,\xi) = f(w,t,x,[\sigma^\top]^{-1}\xi) + \Psi_{\mathcal{S}(w,t,x)}(v,\xi)$$

and the penalty function $l_0 : \mathbb{R} \to [0,\infty]$ by

$$l_0(x) = \Psi_{\{x_0\}}(x),$$

where $\Psi_U(u)$ is a penalty function which equals 0 if u is in set U and $+\infty$ otherwise.

For $X \in \mathbb{B}$, we define the cost functional as

$$\Phi(X) := l_0(x_0) + E[g(X_T)] + E\left[\int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t)dt\right],$$

where $\Phi(X) = \infty$ if $X(0) \neq x_0$ or $\mathcal{U}(X) = \emptyset$. The quadratic risk minimization problem can be written as

$$\text{Minimize } \Phi(X) \text{ subject to } X \in \mathbb{B}.$$

By the convex duality method, we construct the dual problem over the set $\mathbb{B}$. Define the following convex conjugate functions

$$m_0(y) := \sup_{x\in\mathbb{R}}\{xy - l_0(x)\},$$

$$m_T(w,y) := \sup_{x\in\mathbb{R}}\{-xy - g(w,x)\},$$

$$M(w,t,y,s,\gamma) := \sup_{x,v\in\mathbb{R},\xi\in\mathbb{R}^N}\{xs + vy + \xi^\top\gamma - L(w,t,x,v,\xi)\},$$

for all $(w,t,y,s,\gamma) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$. For each $Y \equiv (y, \dot{Y}, \Lambda_Y) \in \mathbb{B}$, we define

$$\Psi(Y) := m_0(y) + E[m_T(Y(T))] + E\left[\int_0^T M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))dt\right].$$

Then, the associated dual problem is given by

$$\text{Minimize } \Psi(Y) \text{ subject to } Y \in \mathbb{B}.$$

The associated dual problem also can be reformulated as a stochastic control problem. We can find that

$$m_0 = x_0 y$$

$$m_T(w,y) = \frac{(y+c)^2}{2a}$$

$$M(w,t,y,s,\gamma) = \phi(t, s + ry, \sigma[\theta y + \gamma]),$$

where ϕ is the conjugate function of $\tilde{f}(w,t,x,\pi) = f(w,t,x,\pi) + \Psi_K(\pi)$, namely,

$$\phi(w,t,\alpha,\beta) := \sup_{x\in\mathbb{R},\pi\in K}\{x\alpha + \pi^\top\beta - f(w,t,x,\pi)\}.$$

The dual control problem is therefore given by

$$\text{Minimize } \tilde{\Psi}(y, \alpha, \beta) := m_0(y) + E[m_T(Y(T))] + E\left[\int_0^T \phi(t, \alpha(t), \beta(t))dt\right],$$

where Y satisfies

$$\begin{cases} dY(t) = [\alpha(t) - rY(t)]dt + [\sigma^{-1}\beta(t) - \theta Y(t)]^\top dW(t), 0 < t < T \\ Y(0) = y. \end{cases} \tag{2.1.2}$$

The dual control process for Y is $(y, \alpha, \beta) \in \mathbb{B}$, and $Y^{(y,\alpha,\beta)} \in \mathcal{S}^2(0, T; \mathbb{R})$. Note that the control constraint is implicit for the dual problem, this is the reason why the convex duality method can usually solve the stochastic linear quadratic control problem more convenient.

Chapter 3

Quadratic Risk Minimization without Control Constraints

In this chapter, we study the quadratic risk minimization problem without control constraints. We assume that all coefficients are constant, $K = \mathbb{R}^N$, $Q = 0$ and $S = 0$.

3.1 HJB Method

The funtional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$J(\pi) := E\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\right],$$

where $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$f(w, t, x, \pi) := \frac{1}{2}\pi^\top R(t)\pi,$$

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$g(w, x) := \frac{1}{2}[ax^2 + 2cx].$$

The quadratic risk minimization problem discussed in this paper is

$$\text{Minimize } J(\pi) \text{ subject to } (X, \pi) \text{ admissible},$$

where X satisfies

$$\begin{cases} dX^\pi(t) = [rX^\pi(t) + \pi^\top(t)\sigma\theta]dt + \pi^\top(t)\sigma dW(t), \quad 0 \le t \le T, \\ X^\pi(0) = x_0. \end{cases} \tag{3.1.1}$$

Define value function $V(t, x)$ by

$$V(t, x) = \inf_\pi E\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\Big| X^\pi(0) = x_0\right].$$

By Dynamic Programming Principle, we can drive the HJB equation,

$$\begin{cases} \frac{\partial V}{\partial t} + \inf_\pi\{\mathcal{L}^\pi V + f(t, x, \pi)\} = 0, \\ V(T, x) = g(x), \end{cases}$$

where $\mathcal{L}^\pi$ is the operator defined by

$$\mathcal{L}^\pi V = (rx + \pi^\top\sigma\theta)V_x + \frac{1}{2}\pi^\top\sigma\sigma^\top\pi V_{xx}.$$

Substituting $f(t,x,\pi)$ and $g(x)$ into the HJB equation above, we obtain that

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x + \inf_\pi\{\pi^\top \sigma\theta V_x + \frac{1}{2}\pi^\top \sigma\sigma^\top \pi V_{xx} + \frac{1}{2}\pi^\top R\pi\} = 0, \\ V(T,x) = \frac{1}{2}[ax^2 + 2cx]. \end{cases}$$

To find the optimal $\hat{\pi}$, we obtain the first-order condition is

$$(\sigma\theta)^\top V_x + \pi^\top \sigma\sigma^\top V_{xx} + \pi^\top R = 0.$$

Since R is a symmetric matrix, we can obtain that $(\sigma\sigma^\top V_{xx} + R) = (\sigma\sigma^\top V_{xx} + R)^\top$. Assume that $(\sigma\sigma^\top V_{xx} + R)$ is invertible, so we know that $((\sigma\sigma^\top V_{xx} + R)^{-1})^\top = ((\sigma\sigma^\top V_{xx} + R)^\top)^{-1} = (\sigma\sigma^\top V_{xx} + R)^{-1}$. Thus, the optimal control $\hat{\pi}$ is equal to $-(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x$.

Then the value function V satisfies:

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x - \frac{1}{2}(\sigma\theta)^\top(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x^2 = 0, \\ V(T,x) = \frac{1}{2}[ax^2 + 2cx]. \end{cases} \tag{3.1.2}$$

To solve this nonlinear HJB PDE, we assume that $V(t,x) = v_0(t) + v_1(t)x + v_2(t)x^2$. Substituting $V(t,x)$ into the HJB equation (3.1.2), we obtain that

$$\begin{cases} \partial_t v_0(t) - \frac{1}{2}(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_1^2(t) = 0, \\ \partial_t v_1(t) + rv_1(t) - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_1(t)v_2(t) = 0, \\ \partial_t v_2(t) + 2rv_2(t) - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_2^2(t) = 0, \end{cases}$$

with terminal conditions $v_0(T) = 0$, $v_1(T) = c$ and $v_2(T) = \frac{1}{2}a$.

We can find that the ODE of $v_2(t)$ satisfies a Riccati equation, and cannot get closed-form solution. It has to be solved numerically by using the Runge-Kutta method. First of all, we reformulate the ODE as an initial condition problem. Let $\tau = T - t$, then the ODE becomes

$$\begin{cases} -\partial_\tau v_2(\tau) + 2rv_2(\tau) - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(\tau) + R)^{-1}(\sigma\theta)v_2^2(\tau) = 0, \\ v_2(0) = \frac{1}{2}a. \end{cases}$$

Following the approach introduced in File and Bullo [7], we divide the interval $[0,T]$ into N equal subintervals, where $\tau_i = 0 + is, i = 0, ..., n$. Thus, the general numerical solution of the ODE is

$$v_2(\tau_{i+1}) = v_2(\tau_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{cases} k_1 = sf(\tau_i, v_2(t_i)), \\ k_2 = sf(\tau_i + \frac{s}{2}, v_2(\tau_i) + \frac{k_1}{2}), \\ k_3 = sf(\tau_i + \frac{s}{2}, v_2(\tau_i) + \frac{k_2}{2}), \\ k_4 = sf(\tau_i + s, v_2(\tau_i) + k_3), \\ f(\tau, v_2) = (2r - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(\tau) + R)^{-1}(\sigma\theta)v_2(\tau))v_2(\tau). \end{cases}$$

Then, we use the solution above to solve the ODEs of $v_1(t)$ and $v_0(t)$, recall that

$$\begin{cases} \partial_t v_1(t) + rv_1(t) - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_1(t)v_2(t) = 0, \\ v_1(T) = c, \\ \partial_t v_0(t) - \frac{1}{2}(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_1^2(t) = 0, \ v_0(T) = 0, \end{cases}$$

and we obtain that

$$\begin{cases} v_1(0) = ce^{\int_0^T r - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_2(t)dt}, \\ v_0(0) = -\frac{1}{2}\int_0^T(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_1^2(t)dt. \end{cases}$$

Recall that

$$\hat{\pi} = -(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x$$

and

$$\begin{cases} V_{xx} = 2v_2(t), \\ V_x = v_1(t) + 2v_2(t)x, \end{cases}$$

so

$$\hat{\pi}(t) = -(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)(v_1(t) + 2v_2(t)X^{\hat{\pi}}(t)).$$

We insert the optimal $\hat{\pi}$ into the SDE of total wealth X^π (3.1.1),

$$\begin{aligned} dX^{\hat{\pi}}(t) &= [rX^{\hat{\pi}}(t) - (\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)(v_1(t) + 2v_2(t)X^{\hat{\pi}}(t))]dt \\ &\quad - (\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}\sigma(v_1(t) + 2v_2(t)X^{\hat{\pi}}(t))dW(t) \\ &= [rX^{\hat{\pi}}(t) - (\sigma\theta)^\top(\sigma\sigma^\top + (2v_2(t))^{-1}R)^{-1}(\sigma\theta)Z(t)]dt \\ &\quad - (\sigma\theta)^\top(\sigma\sigma^\top + (2v_2(t))^{-1}R)^{-1}\sigma Z(t)dW(t), \end{aligned}$$

where $Z(t) = X^{\hat{\pi}}(t) + h(t), 0 < t < T$. Define $h(t) = \frac{v_1(t)}{2v_2(t)}$ with terminal condition $h(T) = \frac{c}{a}$. Then, we can find that

$$h_t(t) - rh(t) = 0$$

and

$$h(t) = \frac{v_1(t)}{2v_2(t)} = \frac{c}{a}e^{-r(T-t)}.$$

Also, the process $Z(t) = X^{\hat{\pi}}(t) + h(t)$ with initial condition $Z(0) = x_0 + \frac{c}{a}e^{-rT}$. By Ito's lemma, we obtain that

$$dZ(t) = [r - A(t)\theta]Z(t)dt - A(t)Z(t)dW(t),$$

where $A(t) = (\sigma\theta)^\top(\sigma\sigma^\top + (2v_2(t))^{-1}R)^{-1}\sigma$, which is the expression for a geometric Brownian motion, with solution

$$Z(t) = Z(0)e^{\int_0^t r - A(u)\theta - \frac{1}{2}A(u)A^\top(u)du - \int_0^t A(u)dW(u)}.$$

Therefore, we can find the optimal wealth process $X^{\hat{\pi}}(t)$,

$$X^{\hat{\pi}}(t) = (x_0 + \frac{c}{a}e^{-rT})e^{\int_0^t r - A(u)\theta - \frac{1}{2}A(u)A^\top(u)du - \int_0^t A(u)dW(u)} - \frac{c}{a}e^{-r(T-t)}. \tag{3.1.3}$$

3.2 Dual HJB Method

The dual control problem is given by

$$\text{Minimize } \tilde{\Psi}(y, \alpha, \beta) := m_0(y) + E[m_T(Y(T))] + E\left[\int_0^T \phi(t, \alpha(t), \beta(t))dt\right],$$

where Y satisfies

$$\begin{cases} dY(t) = [\alpha(t) - rY(t)]dt + [\sigma^{-1}\beta(t) - \theta Y(t)]^\top dW(t), 0 < t < T, \\ Y(0) = y. \end{cases} \tag{3.2.1}$$

Define dual value function $\tilde{V}(t, y)$ by

$$\tilde{V}(t, y) = \inf_{\alpha,\beta} E\left[m_T(Y(T)) + \int_0^T \phi(t, \alpha(t), \beta(t))dt\Big|Y(0) = y\right],$$

and we can find the relationship between the primal value function and dual value function is

$$\tilde{V}(t,y) = \sup_{x\in\mathbb{R}}\{-V(t,x) - xy\},$$

$$V(t,x) = \inf_{y\in\mathbb{R}}\{-\tilde{V}(t,y) - xy\}.$$

The minimum point is obtained by solving

$$0 = -x - \frac{\partial}{\partial y}\tilde{V}(t,y). \tag{3.2.2}$$

There exists an unique y solving the equation above, write it $y = y(t,x)$. Therefore, we have

$$V(t,x) = -xy(t,x) - \tilde{V}(t, y(t,x))$$

and then, we obtain that

$$\begin{aligned} V_t &= -x\frac{\partial y}{\partial t} - \tilde{V}_t - \tilde{V}_y\frac{\partial y}{\partial t} \\ &= (-x - \tilde{V}_y)\frac{\partial y}{\partial t} - \tilde{V}_t \\ &= -\tilde{V}_t, \end{aligned}$$

$$\begin{aligned} V_x &= -y - x\frac{\partial y}{\partial x} - \tilde{V}_y\frac{\partial y}{\partial x} \\ &= (-x - \tilde{V}_y)\frac{\partial y}{\partial x} - y \\ &= -y, \end{aligned}$$

and

$$V_{xx} = -\frac{\partial y}{\partial x}.$$

From the first-order condition (3.2.2), we know that

$$\frac{\partial}{\partial x}(-x - \frac{\partial}{\partial y}\tilde{V}(t,y)) = -1 - \tilde{V}_{yy}\frac{\partial y}{\partial x},$$

$\Rightarrow$

$$\frac{\partial y}{\partial x} = -\frac{1}{\tilde{V}_{yy}}.$$

By Dynamic Programming Principle, we can drive the dual HJB equation,

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + \inf_{\alpha,\beta}\{\mathcal{L}^{\alpha,\beta}\tilde{V} + \phi(t,\alpha,\beta)\} = 0, \\ \tilde{V}(T,y) = m_T(Y(T)), \end{cases}$$

where $\mathcal{L}^{\alpha,\beta}$ is the operator defined by

$$\mathcal{L}^{\alpha,\beta}\tilde{V} = (\alpha - ry)\tilde{V}_y + \frac{1}{2}[\sigma^{-1}\beta - \theta y]^\top[\sigma^{-1}\beta - \theta y]\tilde{V}_{yy}.$$

Substituting $\phi(t,\alpha,\beta)$ and $m_T(Y(T))$ into the HJB equation above, we obtain that

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + \inf_{\alpha,\beta}\{(\alpha - ry)\tilde{V}_y + \\ \quad \frac{1}{2}[\sigma^{-1}\beta - \theta y]^\top[\sigma^{-1}\beta - \theta y]\tilde{V}_{yy} + \sup_{x,\pi}\{x\alpha + \pi^\top\beta - \frac{1}{2}\pi^\top R\pi\}\} = 0, \\ \tilde{V}(T,y) = \frac{(y+c)^2}{2a}. \end{cases}$$

Then, we know that α must be 0, otherwise, $\sup_x\{x\alpha\} = \infty$. The optimal $\hat{\pi}$ can be find by the first order condition,

$$\beta^\top - \pi^\top R = 0.$$

Therefore $\hat{\pi} = R^{-1}\beta$, and the HJB equation becomes

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top \theta y^2 \tilde{V}_{yy} \\ \qquad + \inf_\beta\{\frac{1}{2}[(\sigma^{-1}\beta)^\top(\sigma^{-1}\beta) - (\sigma^{-1}\beta)^\top \theta y - \theta^\top(\sigma^{-1}\beta)y]\tilde{V}_{yy} + \frac{1}{2}\beta^\top R^{-1}\beta\} = 0, \\ \tilde{V}(T,y) = \frac{(y+c)^2}{2a}. \end{cases}$$

To find the optimal $\hat{\beta}$, we obtain the first-order condition is

$$\beta^\top {\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} - \theta^\top \sigma^{-1} y\tilde{V}_{yy} + \beta^\top R^{-1} = 0.$$

Assume that $({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})$ is invertible, so we know that $(({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1})^\top = (({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^\top)^{-1} = ({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}$. Since R is a symmetric matrix and invertible, we can obtain that $({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1}) = ({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^\top$. Thus, optimal control $\hat{\beta}$ is equal to $({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}({\sigma^{-1}}^\top\theta)y\tilde{V}_{yy}$.

Dual value function $\tilde{V}$ satisfies:

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top \theta y^2 \tilde{V}_{yy} - \frac{1}{2}({\sigma^{-1}}^\top\theta)^\top({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}({\sigma^{-1}}^\top\theta)y^2\tilde{V}_{yy}^2 = 0, \\ \tilde{V}(T,y) = \frac{(y+c)^2}{2a}. \end{cases} \tag{3.2.3}$$

Substituting $y, \tilde{V}_y, \tilde{V}_{yy}, \tilde{V}_t$ into the equation above, we can find that

$$\begin{aligned} -\frac{\partial V}{\partial t} &- r(-V_x)(-x) + \frac{1}{2}\theta^\top\theta(-V_x)^2V_{xx}^{-1} \\ &\qquad - \frac{1}{2}({\sigma^{-1}}^\top\theta)^\top({\sigma^{-1}}^\top\sigma^{-1}V_{xx}^{-1} + R^{-1})^{-1}({\sigma^{-1}}^\top\theta)(-V_x)^2V_{xx}^{-2} = 0 \end{aligned}$$

$\Rightarrow$

$$\frac{\partial V}{\partial t} + rxV_x - \frac{1}{2}(\sigma\theta)^\top(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x^2 = 0 \textit{ (primal HJB equation)}$$

and

$$\begin{aligned} V(T,x) &= \inf_{y\in\mathbb{R}}\{-xy - \tilde{V}(T,y)\} \\ &= \inf_{y\in\mathbb{R}}\{-xy - \frac{(y+c)^2}{2a}\} \\ &= -x(-ax-c) - \frac{(-ax-c+c)^2}{2a} \\ &= \frac{1}{2}ax^2 + cx. \end{aligned}$$

These prove the relationship between the primal and dual problems. Next, to solve the nonlinear dual HJB PDE, we assume that $\tilde{V}(t,y) = \tilde{v}_0(t) + \tilde{v}_1(t)y + \tilde{v}_2(t)y^2$. Substituting $\tilde{V}(t,y)$ into the dual HJB equation (3.2.3), we obtain that

$$\begin{cases} \partial_t\tilde{v}_0(t) = 0, \\ \partial_t\tilde{v}_1(t) - r\tilde{v}_1(t) = 0, \\ \partial_t\tilde{v}_2(t) - 2r\tilde{v}_2(t) + (\sigma\theta)^\top(2R\tilde{v}_2(t) + \sigma\sigma^\top)^{-1}(\sigma\theta)\tilde{v}_2(t) = 0, \end{cases}$$

with terminal conditions $\tilde{v}_0(T) = \frac{c^2}{2a}$, $\tilde{v}_1(T) = \frac{c}{a}$ and $\tilde{v}_2(T) = \frac{1}{2a}$.

Straightforwardly, the corresponding solutions of first two ODES are

$$\begin{cases} \tilde{v}_0(0) = \frac{c^2}{2a}, \\ \tilde{v}_1(0) = \frac{c}{a}e^{-rT}. \end{cases}$$

We can find that the ODE of $\tilde{v}_2(t)$ satisfies a Riccati equation, and cannot get closed-form solution. It has to be solved numerically by using the Runge-Kutta method. Similar with Section 3.1, we reformulate the ODE as an initial condition problem. Let $\tau = T - t$, then the ODE becomes

$$\begin{cases} -\partial_\tau \tilde{v}_2(\tau) - 2r\tilde{v}_2(\tau) + (\sigma\theta)^\top (2R\tilde{v}_2(\tau) + \sigma\sigma^\top)^{-1}(\sigma\theta)\tilde{v}_2(\tau) = 0, \\ \tilde{v}_2(0) = \frac{1}{2a}, \end{cases}$$

and we divide the interval $[0, T]$ into N equal subintervals, where $\tau_i = 0 + is, i = 0, ..., n$. Thus, the general numerical solution of the ODE is

$$\tilde{v}_2(\tau_{i+1}) = \tilde{v}_2(\tau_i) + \frac{1}{6}(\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4),$$

where

$$\begin{cases} \tilde{k}_1 = s\tilde{f}(\tau_i, \tilde{v}_2(\tau_i)), \\ \tilde{k}_2 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_1}{2}), \\ \tilde{k}_3 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_2}{2}), \\ \tilde{k}_4 = s\tilde{f}(\tau_i + s, \tilde{v}_2(\tau_i) + \tilde{k}_3), \\ \tilde{f}(\tau, \tilde{v}_2) = ((\sigma\theta)^\top (2R\tilde{v}_2(\tau) + \sigma\sigma^\top)^{-1}(\sigma\theta) - 2r)\tilde{v}_2(\tau). \end{cases}$$

Recall that

$$\hat{\beta} = (\sigma^{-1^\top}\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}(\sigma^{-1^\top}\theta)y\tilde{V}_{yy},$$

$$\hat{\alpha} = 0,$$

and

$$\tilde{V}_{yy} = 2\tilde{v}_2(t),$$

so

$$\hat{\beta}(t) = 2(2\sigma^{-1^\top}\sigma^{-1}\tilde{v}_2(t) + R^{-1})^{-1}(\sigma^{-1^\top}\theta)Y^{(y,\hat{\alpha},\hat{\beta})}(t)\tilde{v}_2(t).$$

We insert the optimal $\hat{\beta}$ and $\hat{\alpha}$ into the SDE of $Y^{(y,\alpha,\beta)}$ (3.2.1),

$$dY^{(y,\hat{\alpha},\hat{\beta})}(t) = -rY^{(y,\hat{\alpha},\hat{\beta})}(t)dt - \tilde{A}^\top(t)Y^{(y,\hat{\alpha},\hat{\beta})}(t)dW(t),$$

where $\tilde{A}(t) = \sigma^\top(2R\tilde{v}_2(t)+\sigma\sigma^\top)^{-1}\sigma\theta$ with initial condition $Y^{(y,\hat{\alpha},\hat{\beta})}(0) = y$, which is the expression for a geometric Brownian motion, with solution

$$Y^{(y,\hat{\alpha},\hat{\beta})}(t) = ye^{-\int_0^t (r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}.$$

Recall that

$$X^{\hat{\pi}}(t) = -\tilde{V}_y(t, Y^{(y,\hat{\alpha},\hat{\beta})}(t)),$$

$$V_{xx} = 2v_2(t) = \frac{1}{\tilde{V}_{yy}} = \frac{1}{2\tilde{v}_2(t)},$$

$$y = -V_x = -(v_1(t) + 2xv_2(t)),$$

$$h(t) = \frac{v_1(t)}{2v_2(t)} = \frac{c}{a}e^{-r(T-t)},$$

$$A(t) = (\sigma\theta)^\top(\sigma\sigma^\top + (2v_2(t))^{-1}R)^{-1}\sigma,$$

$$\tilde{v}_1(0) = \frac{c}{a}e^{-rT},$$

and

$$\tilde{V}_y = \tilde{v}_1(t) + 2y\tilde{v}_2(t),$$

so

$$\begin{aligned} X^{\hat{\pi}}(t) &= -(\frac{c}{a}e^{-r(T-t)} + 2\tilde{v}_2(t)ye^{-\int_0^t (r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}) \\ &= -\frac{c}{a}e^{-r(T-t)} + (x_0 + \frac{v_1(0)}{2v_2(0)})\frac{v_2(0)}{v_2(t)}e^{-\int_0^t (r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}. \end{aligned}$$

From the ODE of $v_2(t)$,

$$\partial_t v_2(t) + 2rv_2(t) - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_2^2(t) = 0,$$

we can know that

$$\frac{v_2(0)}{v_2(t)} = e^{\int_0^t 2r - A(u)\theta du},$$

and we also find that $A(t) = \tilde{A}^\top(t)$. Thus,

$$X^{\hat{\pi}}(t) = (x_0 + \frac{c}{a}e^{-rT})e^{\int_0^t r - A(u)\theta - \frac{1}{2}A(u)A^\top(u)du - \int_0^t A(u)dW(u)} - \frac{c}{a}e^{-r(T-t)}. \tag{3.2.4}$$

Finally, we obtain the exact same solution with primal HJB approach.

3.3 FBSDE Method

Given any admissible control $\pi \in \mathcal{A}$ and solution X^π to the SDE (3.1.1), the associated adjoint equation in unknown processes $p_1 \in \mathcal{S}^2(0,T;\mathbb{R})$ and $q_1 \in \mathcal{H}^2(0,t;\mathbb{R}^N)$ is the following linear BSDE

$$\begin{cases} dp_1(t) = -rp_1(t)dt + q_1^\top(t)dW(t), \\ p_1(T) = -aX^\pi(T) - c. \end{cases}$$

From Pham [22], we know that there exists a unique solution (p_1, q_1) to the BSDE above. Also, from Li and Zheng [18], we have the following theorem.

Theorem 3.3.1 (Primal problem and associated FBSDE)**.** *let $\hat{\pi} \in \mathcal{A}$. Then $\hat{\pi}$ is optimal for the primal problem if and only if the solution $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ of FBSDE*

$$\begin{cases} dX^{\hat{\pi}}(t) = [r(t)X^{\hat{\pi}}(t) + \hat{\pi}^\top(t)\sigma(t)\theta(t)]dt + \hat{\pi}^\top(t)\sigma(t)dW(t) \\ X^{\hat{\pi}}(0) = x_0 \\ d\hat{p}_1(t) = [-r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t) + S^\top(t)\hat{\pi}(t)]dt + \hat{q}_1^\top(t)dW(t) \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c \end{cases}$$

satisfies the condition

$$[\hat{\pi}^\top - \pi^\top]\left[\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) + S(t)X^{\hat{\pi}}(t) + R(t)\hat{\pi}(t)\right] \geq 0$$

for $(\mathbb{P} \otimes Leb)$-a.e. $(w,t) \in \Omega \times [0,T]$ and $\pi \in K$.

For the assumptions in this chapter, the FBSDE then becomes

$$\begin{cases} dX^{\hat{\pi}}(t) = [rX^{\hat{\pi}}(t) + \hat{\pi}^\top(t)\sigma\theta]dt + \hat{\pi}^\top(t)\sigma dW(t), \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{p}_1(t) = -r\hat{p}_1(t)dt + \hat{q}_1^\top(t)dW(t), \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c, \end{cases} \tag{3.3.1}$$

and the condition becomes

$$\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1(t) + R\hat{\pi}(t) = 0.$$

Therefore, we can find the optimal control $\hat{\pi}$ is

$$\hat{\pi}(t) = -R^{-1}(\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1(t)).$$

Substituting $\hat{\pi}(t)$ into the FBSDE (3.3.1), we can get a fully-coupled linear FBSDE,

$$\begin{cases} dX^{\hat{\pi}}(t) = [rX^{\hat{\pi}}(t) - (\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1(t))^\top R^{-1}\sigma\theta]dt - (\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1(t))^\top R^{-1}\sigma dW(t), \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{p}_1(t) = -r\hat{p}_1(t)dt + \hat{q}_1^\top(t)dW(t), \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c. \end{cases}$$

To solve this fully-coupled linear FBSDE, we assume $\hat{p}_1(0) = p_0$ and $\hat{q}_1(t) = \hat{q}_1$ are piecewise constant and given, then we solve the following system

$$\begin{cases} dX^{\hat{\pi}}(t) = [rX^{\hat{\pi}}(t) - (\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1)^\top R^{-1}\sigma\theta]dt - (\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1)^\top R^{-1}\sigma dW(t), \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{p}_1(t) = -r\hat{p}_1(t)dt + \hat{q}_1^\top dW(t), \\ \hat{p}_1(0) = p_0. \end{cases} \tag{3.3.2}$$

Then, the equations above become linear SDEs. Also, we can find that the terminal condition

$$\hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c$$

is equivalent to

$$E[(\hat{p}_1(T) + aX^{\hat{\pi}}(T) + c)^2] = 0.$$

Now consider the following optimal control problem:

$$\min_{p_0, \hat{q}_1} J(p_0, \hat{q}_1) := E[(\hat{p}_1(T) + aX^{\hat{\pi}}(T) + c)^2].$$

To solve this optimal control problem numerically, we divide interval $[0, T]$ by m intervals with step size $h = T/m$ and grid points $t_i = hi, i = 0, 1, ..., m$. Assume on the subinterval $[t_i, t_{i+1})$, control $\hat{q}_1$ is taken constant, say that $\hat{q}_{1i}$, for $i = 0, 1, ..., m-1$. Use the Euler method to discretize SDEs (3.3.2) as

$$\begin{cases} X^{\hat{\pi}}(t_{i+1}) = X^{\hat{\pi}}(t_i) + [rX^{\hat{\pi}}(t_i) - (\hat{p}_1(t_i)\sigma\theta + \sigma\hat{q}_{1i})^\top R^{-1}\sigma\theta]h \\ \qquad\qquad -(\hat{p}_1(t_i)\sigma\theta + \sigma\hat{q}_{1i})^\top R^{-1}\sigma\sqrt{h}\xi_i, \\ X^{\hat{\pi}}(0) = x_0, \\ \hat{p}_1(t_{i+1}) = \hat{p}_1(t_i) - r\hat{p}_1(t_i)h + \hat{q}_{1i}^\top\sqrt{h}\xi_i, \\ \hat{p}_1(0) = p_0, \end{cases} \tag{3.3.3}$$

for $i = 0, 1, ..., m-1$, where $\xi_i, i = 0, 1, ..., m-1$, are independent standard normal variables in $\mathbb{R}^N$. We now solve the optimal control problem as follows:

$$\min_{p_0, \hat{q}_{10}, \cdots, \hat{q}_{1m-1}} J(p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}) := E[(\hat{p}_1(t_m) + aX^{\hat{\pi}}(t_m) + c)^2]$$

subject to the discretized SDE above.

After we obtain the optimal values of $p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}$, we can find the optimal control $\hat{\pi}(t)$ numerically,

$$\hat{\pi}(t_i) = -R^{-1}(\hat{p}_1(t_i)\sigma\theta + \sigma\hat{q}_{1i}), \quad \text{for } i = 0, 1, ..., m-1,$$

where $\hat{p}_1(t_i)$ follows the discretized SDE (3.3.3). The corresponding optimal wealth process $X^{\hat{\pi}}(t)$ can be obtained by the discretized SDE (3.3.3).

3.4 Dual FBSDE Method

Given any admissible control $(y, \alpha, \beta) \in \mathbb{B}$ and solution $Y^{(y,\alpha,\beta)}$ to the SDE (3.2.1), the associated adjoint equation in unknown processes $p_2 \in \mathcal{S}^2(0, T; \mathbb{R})$ and $q_2 \in \mathcal{H}^2(0, t; \mathbb{R}^N)$ is the following linear BSDE

$$\begin{cases} dp_2(t) = [rp_2(t) + q_2^\top(t)\theta]dt + q_2^\top(t)dW(t) \\ p_2(T) = -\frac{Y^{(y,\alpha,\beta)}(T)+c}{a} \end{cases}$$

From Pham [22], we know that there exists a unique solution (p_2, q_2) to the BSDE above. Also, from Li and Zheng [18], we have the following assumption and theorem.

Assumption 3.4.1. let $(\hat{\alpha}, \hat{\beta})$ be given and α, β be any admissible control. Then there exists a $Z \in \mathcal{P}(0, T; \mathbb{R})$ satisfying $E[\int_0^T |Z(t)|dt] < \infty$ and

$$Z(t) \geq \frac{\phi(t, \hat{\alpha}(t) + \varepsilon\alpha(t), \hat{\beta}(t) + \varepsilon\beta(t)) - \phi(t, \hat{\alpha}(t), \hat{\beta}(t))}{\varepsilon}$$

for $(\mathbb{P} \otimes Leb)$-a.e. $(w, t) \in \Omega \times [0, T]$ and $\varepsilon \in (0, 1]$.

Theorem 3.4.2 (Dual problem and associated FBSDE)**.** *let $(\hat{y}, \hat{\alpha}, \hat{\beta}) \in \mathbb{B}$ satisfy assumption above. Then $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is optimal for the dual problem if and only if the solution $(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2)$ of FBSDE*

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \left[\hat{\alpha}(t) - r(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]dt + \left[\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t) \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y} \\ d\hat{p}_2(t) = [r(t)\hat{p}_2(t) + \hat{q}_2^\top(t)\theta(t)]dt + \hat{q}_2^\top(t)dW(t) \\ \hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T)+c}{a} \end{cases}$$

satisfies the conditions

$$\begin{cases} \hat{p}_2(0) = x_0 \\ [\sigma^\top]^{-1}(t)\hat{q}_2(t) \in K \\ \left(\hat{p}_2(t), [\sigma^\top]^{-1}(t)\hat{q}_2(t)\right) \in \partial\phi(\hat{\alpha}(t), \hat{\beta}(t)) \end{cases}$$

for $(\mathbb{P} \otimes Leb)$-a.e. $(w, t) \in \Omega \times [0, T]$.

For the assumptions in this chapter, we can know that $\alpha(t) = 0$ and $\phi(t, \beta) = \frac{1}{2}\beta^\top R^{-1}\beta$ for the dual problem. The FBSDE then becomes

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = -rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)dt + \left[\sigma^{-1}\hat{\beta}(t) - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t), \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y}, \\ d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\top(t)\theta]dt + \hat{q}_2^\top(t)dW(t), \\ \hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T)+c}{a}. \end{cases} \tag{3.4.1}$$

We can solve the problem by the stochastic maximum principle. In this case, the Hamiltonian takes the form

$$\mathcal{H}(y, \beta, p_2, q_2) = -ryp_2(t) - [\sigma^{-1}\beta(t) - \theta y]^\top q_2(t) + \frac{1}{2}\beta^\top(t)R^{-1}\beta(t).$$

Let $\hat{\beta} \in \mathcal{A}$ be a candidate for the optimal control, and $Y^{(\hat{y},\hat{\alpha},\hat{\beta})}$ and $(\hat{p}_2, \hat{q}_2)$ be the corresponding processes. Then,

$$\mathcal{H}(y, \beta, \hat{p}_2, \hat{q}_2) = -ry\hat{p}_2(t) + y^\top\theta^\top\hat{q}_2(t) - \beta^\top(t)(\sigma^{-1})^\top\hat{q}_2(t) + \frac{1}{2}\beta^\top(t)R^{-1}\beta(t).$$

We see that $\hat{\beta}$ is optimal if and only if

$$-\hat{q}_2^\top(t)\sigma^{-1} + \hat{\beta}^\top(t)R^{-1} = 0, \ 0 \leq t \leq T, \ a.s.,$$

$\Rightarrow$

$$\hat{\beta}(t) = R(\sigma^\top)^{-1}\hat{q}_2(t).$$

Substituting $\hat{\beta}(t)$ into the FBSDE (3.4.1), we can get a fully-coupled linear FBSDE,

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = -rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)dt + \left[\sigma^{-1}R(\sigma^\top)^{-1}\hat{q}_2(t) - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t), \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y}, \\ d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\top(t)\theta]dt + \hat{q}_2^\top(t)dW(t), \\ \hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T)+c}{a}. \end{cases}$$

Recall that $\hat{p}_2(0) = x_0$, to solve this fully-coupled linear FBSDE, we assume $\hat{y} = y_0$ and $\hat{q}_2(t) = \hat{q}_2$ are piecewise constant and given, then we solve the following system

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = -rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)dt + \left[\sigma^{-1}R(\sigma^\top)^{-1}\hat{q}_2 - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t), \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = y_0, \\ d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\top\theta]dt + \hat{q}_2^\top dW(t), \\ \hat{p}_2(0) = x_0. \end{cases} \tag{3.4.2}$$

Then, the equations above become linear SDEs. Also, the terminal condition

$$\hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a}$$

is equivalent to

$$E[(\hat{p}_2(T) + \frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a})^2] = 0.$$

Now consider the following optimal control problem:

$$\min_{y_0,\hat{q}_2} J(y_0,\hat{q}_2) := E[(\hat{p}_2(T) + \frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a})^2].$$

To solve this optimal control problem numerically, we divide interval $[0,T]$ by m intervals with step size $h = T/m$ and grid points $t_i = hi, i = 0,1,...,m$. Assume on the subinterval $[t_i, t_{i+1})$, control $\hat{q}_2$ is taken constant, say that $\hat{q}_{2i}$, for $i = 0,1,...,m-1$. Use the Euler method to discretize SDEs (3.4.2) as

$$\begin{cases} \hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_{i+1}) = \hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_i) - rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_i)h \\ \qquad\qquad + \left[\sigma^{-1}R(\sigma^\top)^{-1}\hat{q}_{2i} - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_i)\right]^\top \sqrt{h}\xi_i, \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = y_0, \\ \hat{p}_2(t_{i+1}) = \hat{p}_2(t_i) + [r\hat{p}_2(t_i) + \hat{q}_{2i}^\top\theta]h + \hat{q}_{2i}^\top\sqrt{h}\xi_i, \\ \hat{p}_2(0) = x_0, \end{cases} \tag{3.4.3}$$

for $i = 0,1,...,m-1$, where $\xi_i, i = 0,1,...,m-1$, are independent standard normal variables in $\mathbb{R}^N$. We now solve the optimal control problem as follows:

$$\min_{y_0,\hat{q}_{20},...,\hat{q}_{2m-1}} J(y_0,\hat{q}_{20},...,\hat{q}_{2m-1}) := E[(\hat{p}_2(t_m) + \frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_m) + c}{a})^2]$$

subject to the discretized SDE above.

After we obtain the optimal values of $y_0, \hat{q}_{20}, ..., \hat{q}_{2m-1}$, we can find the optimal controls $\hat{y}$ and $\hat{\beta}(t)$ respectively,

$$\hat{y} = y_0,$$
$$\hat{\beta}(t_i) = R(\sigma^\top)^{-1}\hat{q}_{2i}, \quad \text{for } i = 0,1,...,m-1.$$

From Li and Zheng [18], we have another theorem to help us find the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$.

Theorem 3.4.3 (From dual problem to primal problem)**.** *Suppose that* $(\hat{y}, \hat{\alpha}, \hat{\beta})$ *is optimal for the dual problem. Let* $(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2)$ *be the associated process that satisfies the FBSDE and condition above. Define*

$$\hat{\pi}(t) := (\sigma^\top)^{-1}\hat{q}_2,\ t \in [0, T].$$

Then $\hat{\pi}$ *is the optimal control for the primal problem with initial wealth* x_0*. The optimal wealth process and associated adjoint processes are given by*

$$\begin{cases} X^{\hat{\pi}}(t) = \hat{p}_2(t) \\ \hat{p}_1(t) = Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) \\ \hat{q}_1(t) = \sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\ \textit{for}\ \forall t \in [0, T]. \end{cases}$$

Therefore, we can find the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$ numerically,

$$\hat{\pi}(t_i) = (\sigma^\top)^{-1}\hat{q}_{2i}, \quad \text{for } i = 0, 1, ..., m-1,$$

$$X^{\hat{\pi}}(t_i) = \hat{p}_2(t_i), \quad \text{for } i = 0, 1, ..., m,$$

where $\hat{p}_2(t_i)$ follows the discretized SDE (3.4.3).

3.5 Comparison

In the sections above, we discuss solving the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$ by primal HJB method, dual HJB method, primal FBSDE method and dual FBSDE method respectively. This section will prove that all methods have same results.

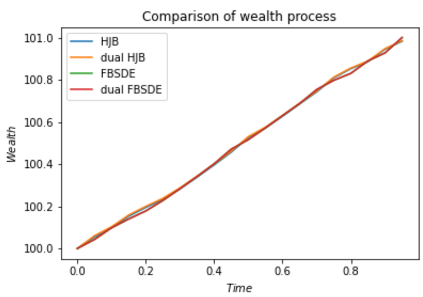
To compare their results, we can firstly solve the Riccati equations of $v_2(t)$ and $\tilde{v}_2(t)$ numerically at the same grid points as the FBSDE method and the dual FBSDE method. Then, we solve numerical optimization problems to find $p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}$ in primal FBSDE and $y_0, \hat{q}_{20}, ..., \hat{q}_{2m-1}$ in dual FBSDE respectively. To solve these numerical optimization problems, we need to generate sample paths M times using the same parameters, then compute the expected value of the objective cost function, and update to get new parameters. The parameters are optimal solutions when the expected value of the objective cost function is close to 0. After these calculations, we can determine the optimal wealth process $X^{\hat{\pi}}(t)$ and optimal control $\hat{\pi}(t)$ under each approach by generating a set of independent standard normal variables on each subinterval and using parameters we obtained from previous calculations.

To compare the results intuitively, we assume that $r = 0.01, b = 0.04, \sigma = 0.5, R = 50, a = 2, c = 10, x_0 = 100, T = 1$ and there is only one stock. Then, we can plot sample paths of the optimal wealth process $X^{\hat{\pi}}(t)$ and optimal control $\hat{\pi}(t)$ under each method on a graph to see if they are close to each other, and also compute the mean squared errors and variances of differences between results of each method. In all comparisons, we let the the primal HJB approach be benchmark approach, and compare the result of chosen approach with the result of the primal HJB approach. Figure 3.1, 3.2 and 3.3 illustrate the optimal wealth process $X^{\hat{\pi}}(t)$ and control $\hat{\pi}(t)$ under different approaches with different step sizes. Table 3.1, 3.2 and 3.3 record the statistical information of differences between results of each methods with different step sizes.

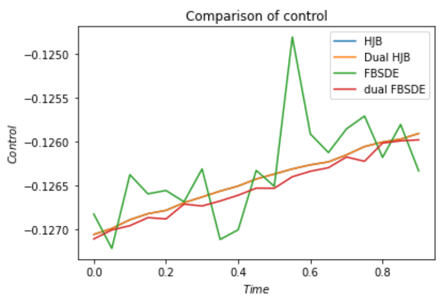
Firstly, we can straightforwardly find that the differences between results of each approach are very close to 0 from plots and statistical information. For example, the mean squared errors of difference between optimal wealth processes $X^{\hat{\pi}}(t)$ from primal HJB and FBSDE with step size $h = 0.025$ is 1.070×10^{-4}, and there is only few gap between optimal wealth processes $X^{\hat{\pi}}(t)$ in Figure 3.3(a). In Figure 3.1(b) 3.2(b) and 3.3(b), we cannot see the optimal controls $\hat{\pi}(t)$ under primal HJB method. The reason is that the results of primal and dual HJB methods are almost identical, so the path under dual HJB method overlaps on the path under primal HJB method. For primal and dual FBSDE methods, Figure 3.2(b) and 3.3(b), the optimal control $\hat{\pi}(t)$ fluctuates more than the optimal control $\hat{\pi}(t)$ under primal and dual HJB methods, since we use the numerical optimization to solve the whole problem in primal and dual FBSDE methods, but only use the numerical method to solve part of the problem in primal and dual HJB methods.

Furthermore, we can easily find that the differences between results of each method are gradually smaller when the step size is decreasing and the number of grid points is increasing, for example, the mean squared errors of difference between optimal wealth processes $X^{\hat{\pi}}(t)$ from primal HJB approach and FBSDE approach reduces 1.6223×10^{-4} when the step size $h = 0.05$ decreases to $h = 0.0125$.

To compare results more carefully, we check results with different values of coefficients $r = 0.03, b = 0.1, \sigma = 0.7, R = 10, a = 1, c = 15$. Figure 3.4, 3.5 and 3.6 show the optimal wealth process $X^{\hat{\pi}}(t)$ and control $\hat{\pi}(t)$ under different approaches with new coefficients and different step sizes. Table 3.4, 3.5 and 3.6 record the statistical information of differences between results of each method with different step sizes and new coefficients. After we change the values of coefficients, we can find that the mean squared errors and variances of differences between results of all methods have obvious increases, for example, the mean squared errors of optimal wealth processes $X^{\hat{\pi}}(t)$ from primal HJB and FBSDE increase 3.20763×10^{3} with the step size $h = 0.0125$. It also can be seen from Figure 3.4, 3.5 and 3.6. For the optimal wealth processes $X^{\hat{\pi}}(t)$ under all approaches, they are not always very close to each other anymore, but they have the same upward trend in total. From the statistical information, Table 3.4, 3.5 and 3.6, the mean squared errors and variances are still quite small, lower than 10^{-2}, and decreasing when the step size becomes smaller. In sum, four approaches will give us the same results of the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$.

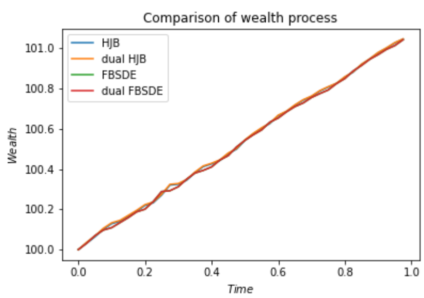


(a) Optimal wealth process $X^{\hat{\pi}}(t)$

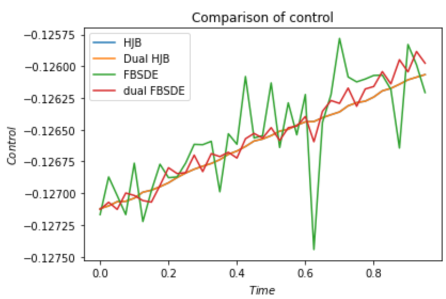


(b) Optimal control $\hat{\pi}(t)$

Figure 3.1: Comparison without control constraints at step size $h = 0.05$

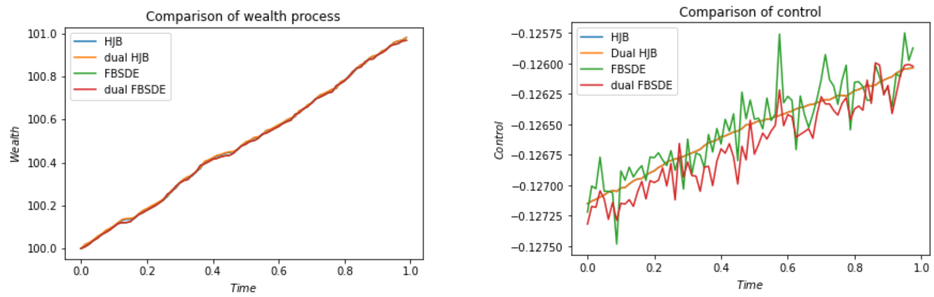


(a) Optimal wealth process $X^{\hat{\pi}}(t)$



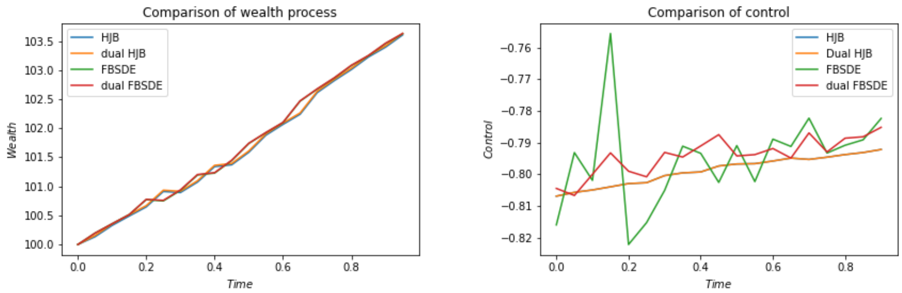
(b) Optimal control $\hat{\pi}(t)$

Figure 3.2: Comparison without control constraints at step size $h = 0.025$



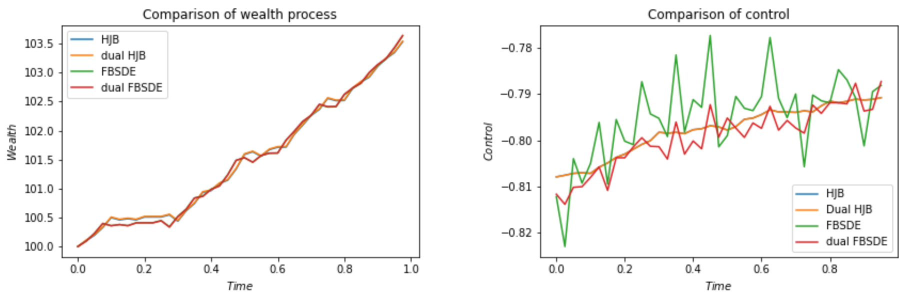
(a) Optimal wealth process $X^{\hat{\pi}}(t)$ (b) Optimal control $\hat{\pi}(t)$

Figure 3.3: Comparison without control constraints at step size $h = 0.0125$



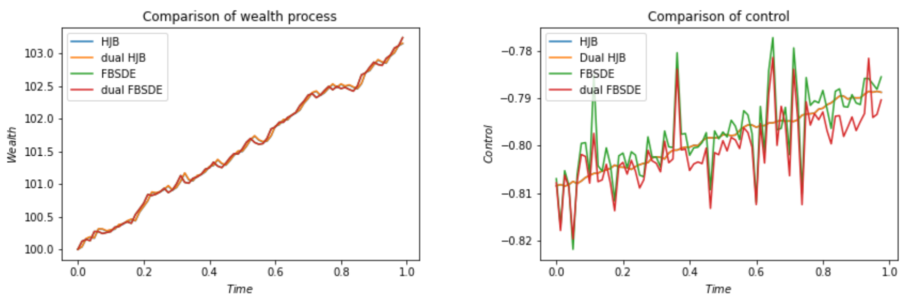
(a) Optimal wealth process $X^{\hat{\pi}}(t)$ (b) Optimal control $\hat{\pi}(t)$

Figure 3.4: Comparison without control constraints with new coefficients ($h = 0.05$)



(a) Optimal wealth process $X^{\hat{\pi}}(t)$ (b) Optimal control $\hat{\pi}(t)$

Figure 3.5: Comparison without control constraints with new coefficients ($h = 0.025$)



(a) Optimal wealth process $X^{\hat{\pi}}(t)$ (b) Optimal control $\hat{\pi}(t)$

Figure 3.6: Comparison without control constraints with new coefficients ($h = 0.0125$)

	Optimal wealth process $X^{\hat{\pi}}(t)$		Optimal control $\hat{\pi}(t)$	
Approach	MSE	Variance	MSE	Variance
HJB	0	0	0	0
Dual HJB	1.767×10^{-5}	1.834×10^{-6}	7.545×10^{-16}	1.311×10^{-16}
FBSDE	2.176×10^{-4}	2.154×10^{-4}	1.583×10^{-7}	1.362×10^{-7}
Dual FBSDE	2.181×10^{-4}	2.163×10^{-4}	4.095×10^{-9}	4.055×10^{-9}

Table 3.1: MSE and variances of differences without control constraints at step size $h = 0.05$

	Optimal wealth process $X^{\hat{\pi}}(t)$		Optimal control $\hat{\pi}(t)$	
Approach	MSE	Variance	MSE	Variance
HJB	0	0	0	0
Dual HJB	1.010×10^{-5}	1.319×10^{-6}	7.092×10^{-16}	1.598×10^{-16}
FBSDE	1.070×10^{-4}	1.505×10^{-4}	5.765×10^{-8}	5.460×10^{-7}
Dual FBSDE	1.060×10^{-4}	1.505×10^{-4}	9.647×10^{-9}	5.134×10^{-9}

Table 3.2: MSE and variances of differences without control constraints at step size $h = 0.025$

	Optimal wealth process $X^{\hat{\pi}}(t)$		Optimal control $\hat{\pi}(t)$	
Approach	MSE	Variance	MSE	Variance
HJB	0	0	0	0
Dual HJB	7.135×10^{-6}	1.216×10^{-6}	7.420×10^{-16}	1.666×10^{-16}
FBSDE	5.537×10^{-5}	4.361×10^{-5}	2.880×10^{-8}	2.708×10^{-8}
Dual FBSDE	5.567×10^{-5}	4.368×10^{-5}	2.022×10^{-8}	1.355×10^{-8}

Table 3.3: MSE and variances of differences without control constraints at step size $h = 0.0125$

	Optimal wealth process $X^{\hat{\pi}}(t)$		Optimal control $\hat{\pi}(t)$	
Approach	MSE	Variance	MSE	Variance
HJB	0	0	0	0
Dual HJB	4.717×10^{-4}	2.364×10^{-5}	1.213×10^{-12}	2.305×10^{-13}
FBSDE	8.632×10^{-3}	6.497×10^{-3}	1.942×10^{-4}	1.809×10^{-4}
Dual FBSDE	9.009×10^{-3}	6.421×10^{-4}	3.214×10^{-5}	9.850×10^{-6}

Table 3.4: MSE and variances of differences with no constraints and new coefficients ($h = 0.05$)

	Optimal wealth process $X^{\hat{\pi}}(t)$		Optimal control $\hat{\pi}(t)$	
Approach	MSE	Variance	MSE	Variance
HJB	0	0	0	0
Dual HJB	1.205×10^{-4}	3.025×10^{-6}	3.996×10^{-13}	4.580×10^{-14}
FBSDE	7.831×10^{-3}	7.752×10^{-3}	5.459×10^{-5}	4.813×10^{-5}
Dual FBSDE	8.007×10^{-3}	7.951×10^{-3}	9.142×10^{-6}	7.077×10^{-6}

Table 3.5: MSE and variances of differences with no constraints and new coefficients ($h = 0.025$)

	Optimal wealth process $X^{\hat{\pi}}(t)$		Optimal control $\hat{\pi}(t)$	
Approach	MSE	Variance	MSE	Variance
HJB	0	0	0	0
Dual HJB	3.051×10^{-5}	3.842×10^{-7}	3.455×10^{-13}	4.449×10^{-14}
FBSDE	3.263×10^{-3}	3.231×10^{-3}	4.012×10^{-5}	3.895×10^{-5}
Dual FBSDE	3.280×10^{-3}	3.249×10^{-3}	3.693×10^{-5}	3.289×10^{-5}

Table 3.6: MSE and variances of differences with no constraints and new coefficients ($h = 0.0125$)

Chapter 4

Quadratic Risk Minimization with Cone-Constraints

In this chapter, we study the quadratic risk minimization problem with cone-constraints (no short selling). We assume that all coefficients are deterministic, $K = \mathbb{R}^2_+$.

4.1 HJB Method

The funtional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$J(\pi) := E\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\right],$$

where $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$f(w, t, x, \pi) := \frac{1}{2}[Qx^2 + 2S^\top x\pi + \pi^\top R\pi],$$

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$g(w, x) := \frac{1}{2}[ax^2 + 2cx].$$

The quadratic risk minimization problem discussed in this paper is

$$\text{Minimize } J(\pi) \text{ subject to } (X, \pi) \text{ admissible},$$

where X satisfies

$$\begin{cases} dX^\pi(t) = [rX^\pi(t) + \pi^\top(t)\sigma\theta]dt + \pi^\top(t)\sigma dW(t), & 0 \le t \le T, \\ X^\pi(0) = x_0. \end{cases} \tag{4.1.1}$$

Define value function $V(t, x)$ by

$$V(t, x) = \inf_\pi E\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\Big|X^\pi(0) = x_0\right].$$

By Dynamic Programming Principle, we can drive the HJB equation,

$$\begin{cases} \frac{\partial V}{\partial t} + \inf_\pi\{\mathcal{L}^\pi V + f(t, x, \pi)\} = 0, \\ V(T, x) = g(x), \end{cases}$$

where $\mathcal{L}^\pi$ is the operator defined by

$$\mathcal{L}^\pi V = (rx + \pi^\top\sigma\theta)V_x + \frac{1}{2}\pi^\top\sigma\sigma^\top\pi V_{xx}.$$

Substituting $f(t,x,\pi)$ and $g(x)$ into the HJB equation above, we obtain that

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x + \frac{1}{2}Qx^2 + \inf_\pi\{\pi^\top \sigma\theta V_x + \frac{1}{2}\pi^\top \sigma\sigma^\top \pi V_{xx} + S^\top x\pi + \frac{1}{2}\pi^\top R\pi\} = 0, \\ V(T,x) = \frac{1}{2}[ax^2 + 2cx]. \end{cases}$$

To find the optimal $\hat{\pi}$, we obtain the first-order condition is

$$(\sigma\theta)^\top V_x + \pi^\top \sigma\sigma^\top V_{xx} + S^\top x + \pi^\top R = 0.$$

Since R is a symmetric matrix, we can obtain that $(\sigma\sigma^\top V_{xx} + R) = (\sigma\sigma^\top V_{xx} + R)^\top$. Assume that $(\sigma\sigma^\top V_{xx}+R)$ is invertible, so $((\sigma\sigma^\top V_{xx}+R)^{-1})^\top = ((\sigma\sigma^\top V_{xx}+R)^\top)^{-1} = (\sigma\sigma^\top V_{xx}+R)^{-1}$. Due to $K = \mathbb{R}^2_+$, the optimal control cannot take negative values, so $\hat{\pi} = (-(\sigma\sigma^\top V_{xx}+R)^{-1}(\sigma\theta V_x+Sx))^+$.

Then, the value function V satisfies:

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x + \frac{1}{2}Qx^2 + \frac{1}{2}(-(\sigma\theta V_x + Sx)^\top(\sigma\sigma^\top V_{xx} + R)^{-1})^+(\sigma\theta V_x + Sx) = 0, \\ V(T,x) = \frac{1}{2}[ax^2 + 2cx]. \end{cases} \tag{4.1.2}$$

The value function (4.1.2) is much more complicated to be solved by assuming $V(t,x)$ is in the general case, $V(t,x) = v_0(t) + v_1(t)x + v_2(t)x^2$. Therefore, to simplify this function, we assume that $c = 0$ and $V(t,x) = v_2(t)x^2$ to solve this nonlinear HJB PDE. Substituting $V(t,x)$ into the HJB equation (4.1.2), we can obtain that

$$\partial_t v_2(t) + 2rv_2(t) + \frac{1}{2}Q + \frac{1}{2}(-(2\sigma\theta v_2(t) + S)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1})^+(2\sigma\theta v_2(t) + S) = 0$$

with terminal conditions $v_2(T) = \frac{1}{2}a$.

We can find that the ODE of $v_2(t)$ satisfies a Riccati equation, and cannot get closed-form solution. It has to be solved numerically by using the Runge-Kutta method. First of all, we reformulate the ODE as an initial condition problem. Let $\tau = T - t$, then the ODE becomes

$$\begin{cases} -\partial_t v_2(\tau) + 2rv_2(\tau) + \frac{1}{2}Q + \frac{1}{2}(-(2\sigma\theta v_2(\tau) + S)^\top(2\sigma\sigma^\top v_2(\tau) + R)^{-1})^+(2\sigma\theta v_2(\tau) + S) = 0 \\ v_2(0) = \frac{1}{2}a \end{cases}$$

Following the approach introduced in File and Bullo [7], we divide the interval $[0,T]$ into N equal subintervals, where $\tau_i = 0 + is, i = 0, ..., n$. Thus, the general numerical solution of the ODE is

$$v_2(\tau_{i+1}) = v_2(\tau_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{cases} k_1 = sf(\tau_i, v_2(t_i)), \\ k_2 = sf(\tau_i + \frac{s}{2}, v_2(\tau_i) + \frac{k_1}{2}), \\ k_3 = sf(\tau_i + \frac{s}{2}, v_2(\tau_i) + \frac{k_2}{2}), \\ k_4 = sf(\tau_i + s, v_2(\tau_i) + k_3), \\ f(\tau, v_2) = 2rv_2(\tau) + \frac{1}{2}Q + \frac{1}{2}(-(2\sigma\theta v_2(\tau) + S)^\top(2\sigma\sigma^\top v_2(\tau) + R)^{-1})^+(2\sigma\theta v_2(\tau) + S). \end{cases}$$

Recall that

$$\hat{\pi} = (-(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta V_x + Sx))^+$$

and

$$\begin{cases} V_{xx} = 2v_2(t), \\ V_x = 2v_2(t)x, \end{cases}$$

so

$$\hat{\pi}(t) = (-(2\sigma\sigma^\top v_2(t) + R)^{-1}(2\sigma\theta v_2(t) + S)X^{\hat{\pi}}(t))^+.$$

We insert the optimal $\hat{\pi}$ into the SDE of total wealth X^{π} (4.1.1),

$$\begin{aligned} dX^{\hat{\pi}}(t) &= [rX^{\hat{\pi}}(t) + (-X^{\hat{\pi}}(t)(2\sigma\theta v_2(t)+S)^{\top}(2\sigma\sigma^{\top}v_2(t)+R)^{-1})^{+}(\sigma\theta)]dt \\ &\quad + (-X^{\hat{\pi}}(t)(2\sigma\theta v_2(t)+S)^{\top}(2\sigma\sigma^{\top}v_2(t)+R)^{-1})^{+}\sigma dW(t), \end{aligned}$$

and we can find that the optimal wealth process $X^{\hat{\pi}}(t)$ follows a geometric Brownian motion. Assuming the initial value of optimal wealth process x_0 is positive, then the optimal wealth process $X^{\hat{\pi}}(t)$ is positive for all time t, $0 < t < T$. Therefore, the optimal control $\hat{\pi}$ can be written in the form,

$$\hat{\pi}(t) = (-(2\sigma\sigma^{\top}v_2(t)+R)^{-1}(2\sigma\theta v_2(t)+S))^{+}X^{\hat{\pi}}(t).$$

The SDE of the optimal wealth process $X^{\hat{\pi}}(t)$ becomes

$$\begin{aligned} dX^{\hat{\pi}}(t) &= [r + (-(2\sigma\theta v_2(t)+S)^{\top}(2\sigma\sigma^{\top}v_2(t)+R)^{-1})^{+}(\sigma\theta)]X^{\hat{\pi}}(t)dt \\ &\quad + (-(2\sigma\theta v_2(t)+S)^{\top}(2\sigma\sigma^{\top}v_2(t)+R)^{-1})^{+}\sigma X^{\hat{\pi}}(t)dW(t) \\ &= [r + A(t)\theta]X^{\hat{\pi}}(t)dt + A(t)X^{\hat{\pi}}(t)dW(t), \end{aligned}$$

where $A(t) = (-(2\sigma\theta v_2(t)+S)^{\top}(2\sigma\sigma^{\top}v_2(t)+R)^{-1})^{+}\sigma$, and the corresponding solution is given by

$$X^{\hat{\pi}}(t) = x_0 e^{\int_0^t r + A(u)\theta - \frac{1}{2}A(u)A^{\top}(u)du + \int_0^t A(u)dW(u)}. \tag{4.1.3}$$

4.2 Dual HJB Method

The dual control problem is given by

$$\text{Minimize } \tilde{\Psi}(y,\alpha,\beta) := m_0(y) + E[m_T(Y(T))] + E\left[\int_0^T \phi(t,\alpha(t),\beta(t))dt\right],$$

where Y satisfies

$$\begin{cases} dY(t) = [\alpha(t) - rY(t)]dt + [\sigma^{-1}\beta(t) - \theta Y(t)]^{\top}dW(t), \\ Y(0) = y. \end{cases} \tag{4.2.1}$$

Define dual value function $\tilde{V}(t,y)$ by

$$\tilde{V}(t,y) = \inf_{\alpha,\beta} E\left[m_T(Y(T)) + \int_0^T \phi(t,\alpha(t),\beta(t))dt \middle| Y(0) = y\right],$$

and we can find the relationship between the primal value function and dual value function is

$$\tilde{V}(t,y) = \sup_{x\in\mathbb{R}}\{-V(t,x) - xy\},$$

$$V(t,x) = \inf_{y\in\mathbb{R}}\{-\tilde{V}(t,y) - xy\}.$$

The minimum point is obtained by solving

$$0 = -x - \frac{\partial}{\partial y}\tilde{V}(t,y). \tag{4.2.2}$$

There exists a unique y solving the equation above, write it $y = y(t,x)$. Therefore, we have

$$V(t,x) = -xy(t,x) - \tilde{V}(t,y(t,x))$$

and then, we obtain that

$$\begin{aligned} V_t &= -x\frac{\partial y}{\partial t} - \tilde{V}_t - \tilde{V}_y\frac{\partial y}{\partial t} \\ &= (-x - \tilde{V}_y)\frac{\partial y}{\partial t} - \tilde{V}_t \\ &= -\tilde{V}_t, \end{aligned}$$

$$\begin{aligned} V_x &= -y - x\frac{\partial y}{\partial x} - \tilde{V}_y\frac{\partial y}{\partial x} \\ &= (-x - \tilde{V}_y)\frac{\partial y}{\partial x} - y \\ &= -y, \end{aligned}$$

and

$$V_{xx} = -\frac{\partial y}{\partial x}.$$

From the first-order condition (4.2.2), we know that

$$\frac{\partial}{\partial x}(-x - \frac{\partial}{\partial y}\tilde{V}(t,y)) = -1 - \tilde{V}_{yy}\frac{\partial y}{\partial x},$$

$\Rightarrow$

$$\frac{\partial y}{\partial x} = -\frac{1}{\tilde{V}_{yy}}.$$

By Dynamic Programming Principle, we can drive the dual HJB equation,

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + \inf_{\alpha,\beta}\{\mathcal{L}^{\alpha.\beta}\tilde{V} + \phi(t,\alpha,\beta)\} = 0, \\ \tilde{V}(T,y) = m_T(Y(T)), \end{cases}$$

where $\mathcal{L}^{\alpha,\beta}$ is the operator defined by

$$\mathcal{L}^{\alpha,\beta}\tilde{V} = (\alpha - ry)\tilde{V}_y + \frac{1}{2}[\sigma^{-1}\beta - \theta y]^\top[\sigma^{-1}\beta - \theta y]\tilde{V}_{yy}.$$

Substituting $\phi(t,\alpha,\beta)$ and $m_T(Y(T))$ into the HJB equation above, we obtain that

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + \inf_{\alpha,\beta}\{(\alpha - ry)\tilde{V}_y + \frac{1}{2}[\sigma^{-1}\beta - \theta y]^\top[\sigma^{-1}\beta - \theta y]\tilde{V}_{yy} + \\ \qquad \sup_{x,\pi}\{x\alpha + \pi^\top\beta - \frac{1}{2}[Qx^2 + 2S^\top x\pi + \pi^\top R\pi]\}\} = 0, \\ \tilde{V}(T,y) = \frac{(y+c)^2}{2a}. \end{cases} \tag{4.2.3}$$

Then, let $\hat{\pi}$ be the optimal value of π, then the sup term in HJB equation becomes

$$\sup_x\{x\alpha + \hat{\pi}^\top\beta - \frac{1}{2}[Qx^2 + 2S^\top x\hat{\pi} + \hat{\pi}^\top R\hat{\pi}]\},$$

and the optimal $\hat{x}$ can be found by the first order condition,

$$\alpha - Qx - S^\top\hat{\pi} = 0.$$

Therefore, $\hat{x} = Q^{-1}(\alpha - S^\top\hat{\pi})$. To find the optimal $\hat{\pi}$, we substitute the optimal $\hat{x}$ back to the equation above,

$$\sup_\pi\{\pi^\top\beta - \frac{1}{2}\pi^\top R\pi + \frac{1}{2}Q^{-1}(\alpha - S^\top\pi)^2\},$$

and the first order condition is

$$\beta^\top - \pi^\top R - Q^{-1}(\alpha - S^\top\pi)S^\top = 0.$$

Therefore $\hat{\pi} = ((R - Q^{-1}SS^\top)^{-1}(\beta - Q^{-1}\alpha S))^+$, and the HJB equation (4.2.3) becomes

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top \theta y^2 \tilde{V}_{yy} + \inf_{\alpha,\beta} \{\alpha \tilde{V}_y + \frac{1}{2}[(\sigma^{-1}\beta)^\top(\sigma^{-1}\beta) - (\sigma^{-1}\beta)^\top \theta y - \theta^\top(\sigma^{-1}\beta)y]\tilde{V}_{yy} \\ \quad + \frac{1}{2}Q^{-1}\alpha^2 + \frac{1}{2}((\beta - Q^{-1}\alpha S)^\top (R - Q^{-1}SS^\top)^{-1})^+(\beta - Q^{-1}\alpha S)\} = 0, \\ \tilde{V}(T, y) = \frac{(y+c)^2}{2a}. \end{cases}$$

To simplify the dual HJB equation, we assume that $Q = 0$ and $S = 0$, therefore $\hat{\alpha} = 0$, and then

$$\hat{\pi} = (R^{-1}\beta)^+,$$

and the HJB equation (4.2.3) becomes

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top \theta y^2 \tilde{V}_{yy} + \inf_{\beta} \{\frac{1}{2}[(\sigma^{-1}\beta)^\top(\sigma^{-1}\beta) - (\sigma^{-1}\beta)^\top \theta y - \theta^\top(\sigma^{-1}\beta)y]\tilde{V}_{yy} \\ \quad + \frac{1}{2}(\beta^\top R^{-1})^+\beta\} = 0, \\ \tilde{V}(T, y) = \frac{(y+c)^2}{2a}. \end{cases}$$

The first order condition for $\hat{\beta}$ is

$$\beta^\top {\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} - \theta^\top \sigma^{-1} y \tilde{V}_{yy} + (\beta^\top R^{-1})^+ = 0.$$

In this case, let $\beta^\top R^{-1} = (\beta^\top R_1^{-1}, \beta^\top R_2^{-1})$, where R_i^{-1} represent the ith column of R^{-1} for $i = 1, 2$. Then, we need to consider four different cases, $(\beta^\top R^{-1})^+ = \beta^\top R^{-1}$, $(\beta^\top R^{-1})^+ = \mathbf{0}$, $(\beta^\top R^{-1})^+ = (\beta^\top R_1^{-1}, 0)$ and $(\beta^\top R^{-1})^+ = (0, \beta^\top R_2^{-1})$. Therefore, we can know that

$$\begin{cases} \hat{\beta} = ({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}({\sigma^{-1}}^\top \theta) y \tilde{V}_{yy} & \text{for} \quad (\beta^\top R^{-1})^+ = \beta^\top R^{-1}, \\ \hat{\beta} = \sigma\theta y & \text{for} \quad (\beta^\top R^{-1})^+ = \mathbf{0}, \\ \hat{\beta} = \left(({\sigma^{-1}}^\top \sigma^{-1})_1 \tilde{V}_{yy} + R_1^{-1}, ({\sigma^{-1}}^\top \sigma^{-1})_2 \tilde{V}_{yy}\right)^{-1}({\sigma^{-1}}^\top \theta) y \tilde{V}_{yy} & \text{for} \quad (\beta^\top R^{-1})^+ = (\beta^\top R_1^{-1}, 0), \\ \hat{\beta} = \left(({\sigma^{-1}}^\top \sigma^{-1})_1 \tilde{V}_{yy}, ({\sigma^{-1}}^\top \sigma^{-1})_2 \tilde{V}_{yy} + R_2^{-1}\right)^{-1}({\sigma^{-1}}^\top \theta) y \tilde{V}_{yy} & \text{for} \quad (\beta^\top R^{-1})^+ = (0, \beta^\top R_2^{-1}), \end{cases}$$

where $({\sigma^{-1}}^\top \sigma^{-1})_i$ represents the ith column of $({\sigma^{-1}}^\top \sigma^{-1})$ for $i = 1, 2$.

For the first case, $(\beta^\top R^{-1})^+ = \beta^\top R^{-1}$, the dual value function $\tilde{V}$ satisfies:

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top \theta y^2 \tilde{V}_{yy} - \frac{1}{2}({\sigma^{-1}}^\top \theta)^\top ({\sigma^{-1}}^\top \sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}({\sigma^{-1}}^\top \theta) y^2 \tilde{V}_{yy}^2 = 0, \\ \tilde{V}(T, y) = \frac{(y+c)^2}{2a}. \end{cases}$$

We can find that it is the exact same with the dual HJB equation at Section 3.2 with similar conditions, so we know the dual HJB equation can be converted into the primal HJB equation. Next, similar with Section 4.1, the dual value function is hard to be solved by assuming $\tilde{V}(t, y)$ is in the general case. We assume that $c = 0$ and $\tilde{V}(t, y) = \tilde{v}_2(t)y^2$ to simplify the equation. Substituting $\tilde{V}(t, y)$ into the dual HJB equation above, we obtain that

$$\partial_t \tilde{v}_2(t) - 2r\tilde{v}_2(t) + (\sigma\theta)^\top (2R\tilde{v}_2(t) + \sigma\sigma^\top)^{-1}(\sigma\theta)\tilde{v}_2(t) = 0,$$

with terminal conditions $\tilde{v}_2(T) = \frac{1}{2a}$.

The ODE of $\tilde{v}_2(t)$ satisfies a Riccati equation, and cannot get closed-form solution. It has to be solved numerically by using the Runge-Kutta method. Similar with Section 3.2, we reformulate the ODE as an initial condition problem. Let $\tau = T - t$, then the ODE becomes

$$\begin{cases} -\partial_\tau \tilde{v}_2(\tau) - 2r\tilde{v}_2(\tau) + (\sigma\theta)^\top (2R\tilde{v}_2(\tau) + \sigma\sigma^\top)^{-1}(\sigma\theta)\tilde{v}_2(\tau) = 0, \\ \tilde{v}_2(0) = \frac{1}{2a}, \end{cases}$$

and we divide the interval $[0, T]$ into N equal subintervals, where $\tau_i = 0 + is, i = 0, ..., n$. Thus, the general numerical solution of the ODE is

$$\tilde{v}_2(\tau_{i+1}) = \tilde{v}_2(\tau_i) + \frac{1}{6}(\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4),$$

where

$$\begin{cases} \tilde{k}_1 = s\tilde{f}(\tau_i, \tilde{v}_2(\tau_i)), \\ \tilde{k}_2 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_1}{2}), \\ \tilde{k}_3 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_2}{2}), \\ \tilde{k}_4 = s\tilde{f}(\tau_i + s, \tilde{v}_2(\tau_i) + \tilde{k}_3), \\ \tilde{f}(\tau, \tilde{v}_2) = ((\sigma\theta)^\top (2R\tilde{v}_2(\tau) + \sigma\sigma^\top)^{-1}(\sigma\theta) - 2r)\tilde{v}_2(\tau). \end{cases}$$

Recall that

$$\hat{\beta} = (\sigma^{-1^\top}\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}(\sigma^{-1^\top}\theta)y\tilde{V}_{yy},$$
$$\hat{\alpha} = 0,$$

and

$$\tilde{V}_{yy} = 2\tilde{v}_2(t),$$

so

$$\hat{\beta}(t) = 2(2\sigma^{-1^\top}\sigma^{-1}\tilde{v}_2(t) + R^{-1})^{-1}(\sigma^{-1^\top}\theta)Y^{(y,\hat{\alpha},\hat{\beta})}(t)\tilde{v}_2(t).$$

We insert the optimal $\hat{\beta}$ and $\hat{\alpha}$ into the SDE of $Y^{(y,\alpha,\beta)}$ (4.2.1),

$$dY^{(y,\hat{\alpha},\hat{\beta})}(t) = -rY^{(y,\hat{\alpha},\hat{\beta})}(t)dt - \tilde{A}^\top(t)Y^{(y,\hat{\alpha},\hat{\beta})}(t)dW(t),$$

where $\tilde{A}(t) = \sigma^\top(2R\tilde{v}_2(t)+\sigma\sigma^\top)^{-1}\sigma\theta$ with initial condition $Y^{(y,\hat{\alpha},\hat{\beta})}(0) = y$, which is the expression for a geometric Brownian motion, with solution

$$Y^{(y,\hat{\alpha},\hat{\beta})}(t) = ye^{-\int_0^t (r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}.$$

Recall that

$$X^{\hat{\pi}}(t) = -\tilde{V}_y(t, Y^{(y,\hat{\alpha},\hat{\beta})}(t)),$$
$$V_{xx} = 2v_2(t) = \frac{1}{\tilde{V}_{yy}} = \frac{1}{2\tilde{v}_2(t)},$$
$$y = -V_x = -(2xv_2(t)),$$
$$A(t) = -(\sigma\theta)^\top(\sigma\sigma^\top + (2v_2(t))^{-1}R)^{-1}\sigma \quad \text{for} \quad Q = 0, S = 0 \text{ and } \hat{\pi} > 0,$$

and

$$\tilde{V}_y = 2y\tilde{v}_2(t),$$

so

$$\begin{aligned} X^{\hat{\pi}}(t) &= -(2v_2(t))^{-1}(-2v_2(0)x_0)e^{-\int_0^t (r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)} \\ &= \frac{v_2(0)}{v_2(t)}x_0e^{-\int_0^t (r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}. \end{aligned}$$

From the ODE of $v_2(t)$, when $Q = 0, S = 0$ and $\hat{\pi} > 0$,

$$\partial_t v_2(t) + 2rv_2(t) - 2(\sigma\theta)^\top(2\sigma\sigma^\top v_2(t) + R)^{-1}(\sigma\theta)v_2^2(t) = 0,$$

we can know that

$$\frac{v_2(0)}{v_2(t)} = e^{\int_0^t 2r + A(u)\theta du},$$

and we also find that $A(t) = -\tilde{A}^\top(t)$. Thus,

$$X^{\hat{\pi}}(t) = x_0e^{\int_0^t r + A(u)\theta - \frac{1}{2}A(u)A^\top(u)du + \int_0^t A(u)dW(u)}. \tag{4.2.4}$$

We obtain the exact same solution with primal HJB approach for $Q = 0, S = 0$ and $\hat{\pi} > 0$.

For the second case, $(\beta^\top R^{-1})^+ = \mathbf{0}$, the dual value function $\tilde{V}$ satisfies:

$$\begin{cases} \frac{\partial\tilde{V}}{\partial t} - ry\tilde{V}_y = 0, \\ \tilde{V}(T, y) = \frac{(y+c)^2}{2a}. \end{cases}$$

Substituting $y, \tilde{V}_y, \tilde{V}_t$ into the equation above, we can find that

$$\begin{cases} -\frac{\partial V}{\partial t} - r(-V_x)(-x) = 0 \\ \tilde{V}(T,y) = \frac{(y+c)^2}{2a} \end{cases} \tag{4.2.5}$$

and

$$\begin{aligned} V(T,x) &= \inf_{y\in\mathbb{R}}\{-xy - \tilde{V}(T,y)\} \\ &= \inf_{y\in\mathbb{R}}\{-xy - \frac{(y+c)^2}{2a}\} \\ &= -x(-ax-c) - \frac{(-ax-c+c)^2}{2a} \\ &= \frac{1}{2}ax^2 + cx \end{aligned}$$

$\Rightarrow$

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x = 0, \\ V(T,x) = \frac{1}{2}[ax^2 + 2cx]. \end{cases}$$

which is the HJB equation for primal problems when $Q = 0$, $S = 0$ and $\hat{\pi} = \mathbf{0}$. Next, to solve the dual HJB PDE, if $c = 0$, we assume that $\tilde{V}(t,y) = \tilde{v}_2(t)y^2$. Substituting $\tilde{V}(t,y)$ into the dual HJB equation (4.2.5), we obtain that

$$\partial_t\tilde{v}_2(t) - 2r\tilde{v}_2(t) = 0,$$

with terminal condition $\tilde{v}_2(T) = \frac{1}{2a}$. Therefore, we can obtain that $\tilde{v}_2(0) = \frac{1}{2a}e^{-2rT}$.

Recall that

$$\hat{\beta} = \sigma\theta Y^{(y,\hat{\alpha},\hat{\beta})}(t)$$

and

$$\hat{\alpha} = 0,$$

we insert the optimal $\hat{\beta}$ and $\hat{\alpha}$ into the SDE of $Y^{(y,\alpha,\beta)}$ (4.2.1),

$$dY^{(y,\hat{\alpha},\hat{\beta})}(t) = -rY^{(y,\hat{\alpha},\hat{\beta})}(t)dt,$$

with initial condition $Y^{(y,\hat{\alpha},\hat{\beta})}(0) = y$. It is straightforward to obtain the solution,

$$Y^{(y,\hat{\alpha},\hat{\beta})}(t) = ye^{-rt}.$$

Recall that

$$\begin{aligned} X^{\hat{\pi}}(t) &= -\tilde{V}_y(t, Y^{(y,\hat{\alpha},\hat{\beta})}(t)), \\ V_{xx} = 2v_2(t) &= \frac{1}{\tilde{V}_{yy}} = \frac{1}{2\tilde{v}_2(t)}, \\ y = -V_x &= -(2xv_2(t)), \end{aligned}$$

and

$$\tilde{V}_y = 2y\tilde{v}_2(t),$$

so

$$\begin{aligned} X^{\hat{\pi}}(t) &= -(2v_2(t))^{-1}(-2v_2(0)x_0)e^{-rt} \\ &= \frac{v_2(0)}{v_2(t)}x_0e^{-rt}. \end{aligned}$$

From the ODE of $v_2(t)$ when $Q = 0, S = 0$ and $\hat{\pi} = \mathbf{0}$,

$$\partial_t v_2(t) + 2rv_2(t) = 0,$$

we can know that

$$\frac{v_2(0)}{v_2(t)} = e^{2rt}.$$

Thus,

$$X^{\hat{\pi}}(t) = x_0 e^{rt}. \tag{4.2.6}$$

We obtain the exact same solution with primal HJB approach for $Q = 0, S = 0$ and $\hat{\pi} = \mathbf{0}$.

For the third case, $(\beta^\top R^{-1})^+ = (\beta^\top R_1^{-1}, 0)$, the dual value function $\tilde{V}$ satisfies:

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top\theta y^2 \tilde{V}_{yy} \\ \quad -\frac{1}{2}({\sigma^{-1}}^\top\theta)^\top \left(({\sigma^{-1}}^\top\sigma^{-1})_1\tilde{V}_{yy} + R_1^{-1}, ({\sigma^{-1}}^\top\sigma^{-1})_2\tilde{V}_{yy}\right)^{-1}({\sigma^{-1}}^\top\theta)y^2\tilde{V}_{yy}^2 = 0, \\ \tilde{V}(T,y) = \frac{(y+c)^2}{2a}. \end{cases} \tag{4.2.7}$$

Substituting $y, \tilde{V}_y, \tilde{V}_{yy}, \tilde{V}_t$ into the equation above, we can find that

$$\begin{aligned} &-\frac{\partial V}{\partial t} - r(-V_x)(-x) + \frac{1}{2}\theta^\top\theta(-V_x)^2 V_{xx}^{-1} \\ &\quad - \frac{1}{2}({\sigma^{-1}}^\top\theta)^\top \left(({\sigma^{-1}}^\top\sigma^{-1})_1\tilde{V}_{xx}^{-1} + R_1^{-1}, ({\sigma^{-1}}^\top\sigma^{-1})_2\tilde{V}_{xx}^{-1}\right)^{-1}({\sigma^{-1}}^\top\theta)(-V_x)^2\tilde{V}_{xx}^{-2} = 0 \end{aligned}$$

$\Rightarrow$

$$\frac{\partial V}{\partial t} + rxV_x - \frac{1}{2}((\sigma\theta)_1, 0)\left((\sigma\sigma^\top)_1 V_{xx} + R_1, (\sigma\sigma^\top)_2 V_{xx}\right)^{-1}(\sigma\theta)V_x^2 = 0$$

(primal HJB equation)

and

$$\begin{aligned} V(T,x) &= \inf_{y\in\mathbb{R}}\{-xy - \tilde{V}(T,y)\} \\ &= \inf_{y\in\mathbb{R}}\{-xy - \frac{(y+c)^2}{2a}\} \\ &= -x(-ax-c) - \frac{(-ax-c+c)^2}{2a} \\ &= \frac{1}{2}ax^2 + cx. \end{aligned}$$

These prove the relationship between the primal and dual problems. Next, to solve the nonlinear dual HJB PDE, we assume that $c = 0$ and $\tilde{V}(t,y) = \tilde{v}_2(t)y^2$. Substituting $\tilde{V}(t,y)$ into the dual HJB equation (4.2.7), we obtain that

$$\partial_t\tilde{v}_2(t) - 2r\tilde{v}_2(t) + ((\sigma\theta)_1, 0)\left(2R_1\tilde{v}_2(t) + (\sigma\sigma^\top)_1, (\sigma\sigma^\top)_2\right)^{-1}(\sigma\theta)\tilde{v}_2(t) = 0,$$

with terminal conditions $\tilde{v}_2(T) = \frac{1}{2a}$, where $(\sigma\sigma^\top)_i$ and R_i represent the ith column of $(\sigma\sigma^\top)$ and R respectively for $i = 1, 2$.

We can find that the ODE of $\tilde{v}_2(t)$ satisfies a Riccati equation, and cannot get closed-form solution. It has to be solved numerically by using the Runge-Kutta method. We reformulate the ODE as an initial condition problem. Let $\tau = T - t$, then the ODE becomes

$$\begin{cases} -\partial_\tau\tilde{v}_2(\tau) - 2r\tilde{v}_2(\tau) + ((\sigma\theta)_1, 0)\left(2R_1\tilde{v}_2(\tau) + (\sigma\sigma^\top)_1, (\sigma\sigma^\top)_2\right)^{-1}(\sigma\theta)\tilde{v}_2(\tau) = 0, \\ \tilde{v}_2(0) = \frac{1}{2a}, \end{cases}$$

and we divide the interval $[0, T]$ into N equal subintervals, where $\tau_i = 0 + is, i = 0, ..., n$. Thus, the general numerical solution of the ODE is

$$\tilde{v}_2(\tau_{i+1}) = \tilde{v}_2(\tau_i) + \frac{1}{6}(\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4),$$

where

$$\begin{cases} \tilde{k}_1 = s\tilde{f}(\tau_i, \tilde{v}_2(\tau_i)), \\ \tilde{k}_2 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_1}{2}), \\ \tilde{k}_3 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_2}{2}), \\ \tilde{k}_4 = s\tilde{f}(\tau_i + s, \tilde{v}_2(\tau_i) + \tilde{k}_3), \\ \tilde{f}(\tau, \tilde{v}_2) = (((\sigma\theta)_1, 0) \left(2R_1\tilde{v}_2(\tau) + (\sigma\sigma^\top)_1, (\sigma\sigma^\top)_2\right)^{-1} (\sigma\theta) - 2r)\tilde{v}_2(\tau). \end{cases}$$

Recall that

$$\hat{\beta} = \left((\sigma^{-1}{}^\top \sigma^{-1})_1 \tilde{V}_{yy} + R_1^{-1}, (\sigma^{-1}{}^\top \sigma^{-1})_2 \tilde{V}_{yy}\right)^{-1} (\sigma^{-1}{}^\top \theta) y \tilde{V}_{yy},$$

$$\hat{\alpha} = 0,$$

and

$$\tilde{V}_{yy} = 2\tilde{v}_2(t),$$

so

$$\hat{\beta}(t) = 2\left(2(\sigma^{-1}{}^\top \sigma^{-1})_1 \tilde{v}_2(t) + R_1^{-1}, 2(\sigma^{-1}{}^\top \sigma^{-1})_2 \tilde{v}_2(t)\right)^{-1} (\sigma^{-1}{}^\top \theta) Y^{(y,\hat{\alpha},\hat{\beta})}(t)\tilde{v}_2(t).$$

We insert the optimal $\hat{\beta}$ and $\hat{\alpha}$ into the SDE of $Y^{(y,\alpha,\beta)}$ (4.2.1),

$$dY^{(y,\hat{\alpha},\hat{\beta})}(t) = -rY^{(y,\hat{\alpha},\hat{\beta})}(t)dt - \tilde{A}^\top(t)Y^{(y,\hat{\alpha},\hat{\beta})}(t)dW(t),$$

where $\tilde{A}(t) = \sigma^\top \left(2R_1\tilde{v}_2(t) + (\sigma\sigma^\top)_1, (\sigma\sigma^\top)_2\right)^{-1} ((\sigma\theta)_1, 0)^\top$ with initial condition $Y^{(y,\hat{\alpha},\hat{\beta})}(0) = y$, which is the expression for a geometric Brownian motion, with solution

$$Y^{(y,\hat{\alpha},\hat{\beta})}(t) = ye^{-\int_0^t (r + \frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}.$$

Recall that

$$X^{\hat{\pi}}(t) = -\tilde{V}_y(t, Y^{(y,\hat{\alpha},\hat{\beta})}(t)),$$

$$V_{xx} = 2v_2(t) = \frac{1}{\tilde{V}_{yy}} = \frac{1}{2\tilde{v}_2(t)},$$

$$y = -V_x = -(2xv_2(t)),$$

$$A(t) = (-(2\sigma\theta v_2(t))^\top (2\sigma\sigma^\top v_2(t) + R)^{-1})^+ \sigma, \quad \text{for} \quad Q = 0 \text{ and } S = 0,$$

and

$$\tilde{V}_y = 2y\tilde{v}_2(t),$$

so

$$\begin{aligned} X^{\hat{\pi}}(t) &= -(2v_2(t))^{-1}(-2v_2(0)x_0)e^{-\int_0^t (r + \frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)} \\ &= \frac{v_2(0)}{v_2(t)} x_0 e^{-\int_0^t (r + \frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t \tilde{A}^\top(u)dW(u)}. \end{aligned}$$

From the ODE of $v_2(t)$, when $Q = 0$ and $S = 0$,

$$\partial_t v_2(t) + 2rv_2(t) + \frac{1}{2}(-(2\sigma\theta v_2(t))^\top (2\sigma\sigma^\top v_2(t) + R)^{-1})^+ (2\sigma\theta v_2(t)) = 0,$$

we can know that

$$\frac{v_2(0)}{v_2(t)} = e^{\int_0^t 2r + A(u)\theta du},$$

and we also find that $A(t) = -\tilde{A}^\top(t)$ when $\hat{\pi} = (\hat{\pi}_1, 0)^\top$. Thus,

$$X^{\hat{\pi}}(t) = x_0 e^{\int_0^t r + A(u)\theta - \frac{1}{2}A(u)A^\top(u)du + \int_0^t A(u)dW(u)}. \tag{4.2.8}$$

We obtain the exact same solution with primal HJB approach for $Q = 0, S = 0$ and $\hat{\pi} = (\hat{\pi}_1, 0)^\top$.

For the last case, $(\beta^\top R^{-1})^+ = (0, \beta^\top R_2^{-1})$, the dual value function $\tilde{V}$ satisfies:

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top\theta y^2 \tilde{V}_{yy} \\ \qquad -\frac{1}{2}({\sigma^{-1}}^\top\theta)^\top \left(({\sigma^{-1}}^\top\sigma^{-1})_1\tilde{V}_{yy}, ({\sigma^{-1}}^\top\sigma^{-1})_2\tilde{V}_{yy} + R_2^{-1}\right)^{-1} ({\sigma^{-1}}^\top\theta)y^2\tilde{V}_{yy}^2 = 0, \\ \tilde{V}(T, y) = \frac{(y+c)^2}{2a}. \end{cases} \tag{4.2.9}$$

Substituting $y, \tilde{V}_y, \tilde{V}_{yy}, \tilde{V}_t$ into the equation above, we can find that

$$\begin{aligned} &-\frac{\partial V}{\partial t} - r(-V_x)(-x) + \frac{1}{2}\theta^\top\theta(-V_x)^2 V_{xx}^{-1} \\ &\quad - \frac{1}{2}({\sigma^{-1}}^\top\theta)^\top \left(({\sigma^{-1}}^\top\sigma^{-1})_1\tilde{V}_{xx}^{-1}, ({\sigma^{-1}}^\top\sigma^{-1})_2\tilde{V}_{xx}^{-1} + R_2^{-1}\right)^{-1} ({\sigma^{-1}}^\top\theta)(-V_x)^2\tilde{V}_{xx}^{-2} = 0 \end{aligned}$$

$\Rightarrow$

$$\frac{\partial V}{\partial t} + rxV_x - \frac{1}{2}(0, (\sigma\theta)_2)\left((\sigma\sigma^\top)_1 V_{xx}, (\sigma\sigma^\top)_2 V_{xx} + R_2\right)^{-1}(\sigma\theta)V_x^2 = 0$$

(primal HJB equation)

and

$$\begin{aligned} V(T, x) &= \inf_{y\in\mathbb{R}}\{-xy - \tilde{V}(T, y)\} \\ &= \inf_{y\in\mathbb{R}}\{-xy - \frac{(y+c)^2}{2a}\} \\ &= -x(-ax - c) - \frac{(-ax - c + c)^2}{2a} \\ &= \frac{1}{2}ax^2 + cx. \end{aligned}$$

These prove the relationship between the primal and dual problems. Next, to solve the nonlinear dual HJB PDE, we assume that $c = 0$ and $\tilde{V}(t, y) = \tilde{v}_2(t)y^2$. Substituting $\tilde{V}(t, y)$ into the dual HJB equation (4.2.9), we obtain that

$$\partial_t\tilde{v}_2(t) - 2r\tilde{v}_2(t) + (0, (\sigma\theta)_2)\left((\sigma\sigma^\top)_1, 2R_2\tilde{v}_2(t) + (\sigma\sigma^\top)_2\right)^{-1}(\sigma\theta)\tilde{v}_2(t) = 0,$$

with terminal conditions $\tilde{v}_2(T) = \frac{1}{2a}$, where $(\sigma\sigma^\top)_i$ and R_i represent the ith column of $(\sigma\sigma^\top)$ and R respectively for $i = 1, 2$.

We can find that the ODE of $\tilde{v}_2(t)$ satisfies a Riccati equation, and cannot get closed-form solution. It has to be solved numerically by using the Runge-Kutta method. We reformulate the ODE as an initial condition problem. Let $\tau = T - t$, then the ODE becomes

$$\begin{cases} -\partial_\tau\tilde{v}_2(\tau) - 2r\tilde{v}_2(\tau) + (0, (\sigma\theta)_2)\left((\sigma\sigma^\top)_1, 2R_2\tilde{v}_2(\tau) + (\sigma\sigma^\top)_2\right)^{-1}(\sigma\theta)\tilde{v}_2(\tau) = 0, \\ \tilde{v}_2(0) = \frac{1}{2a}, \end{cases}$$

and we divide the interval $[0, T]$ into N equal subintervals, where $\tau_i = 0 + is, i = 0, ..., n$. Thus, the general numerical solution of the ODE is

$$\tilde{v}_2(\tau_{i+1}) = \tilde{v}_2(\tau_i) + \frac{1}{6}(\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4),$$

where

$$\begin{cases} \tilde{k}_1 = s\tilde{f}(\tau_i, \tilde{v}_2(\tau_i)), \\ \tilde{k}_2 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_1}{2}), \\ \tilde{k}_3 = s\tilde{f}(\tau_i + \frac{s}{2}, \tilde{v}_2(\tau_i) + \frac{\tilde{k}_2}{2}), \\ \tilde{k}_4 = s\tilde{f}(\tau_i + s, \tilde{v}_2(\tau_i) + \tilde{k}_3), \\ \tilde{f}(\tau, \tilde{v}_2) = ((0, (\sigma\theta)_2)\left((\sigma\sigma^\top)_1, 2R_2\tilde{v}_2(\tau) + (\sigma\sigma^\top)_2\right)^{-1}(\sigma\theta) - 2r)\tilde{v}_2(\tau). \end{cases}$$

Recall that

$$\hat{\beta} = \left((\sigma^{-1^\top}\sigma^{-1})_1\tilde{V}_{yy}, (\sigma^{-1^\top}\sigma^{-1})_2\tilde{V}_{yy} + R_2^{-1}\right)^{-1}(\sigma^{-1^\top}\theta)y\tilde{V}_{yy},$$

$$\hat{\alpha} = 0,$$

and

$$\tilde{V}_{yy} = 2\tilde{v}_2(t),$$

so

$$\hat{\beta}(t) = 2\left(2(\sigma^{-1^\top}\sigma^{-1})_1\tilde{v}_2(t), 2(\sigma^{-1^\top}\sigma^{-1})_2\tilde{v}_2(t) + R_2^{-1}\right)^{-1}(\sigma^{-1^\top}\theta)Y^{(y,\hat{\alpha},\hat{\beta})}(t)\tilde{v}_2(t).$$

We insert the optimal $\hat{\beta}$ and $\hat{\alpha}$ into the SDE of $Y^{(y,\alpha,\beta)}$ (4.2.1),

$$dY^{(y,\hat{\alpha},\hat{\beta})}(t) = -rY^{(y,\hat{\alpha},\hat{\beta})}(t)dt - \tilde{A}^\top(t)Y^{(y,\hat{\alpha},\hat{\beta})}(t)dW(t),$$

where $\tilde{A}(t) = \sigma^\top\left((\sigma\sigma^\top)_1, 2R_2\tilde{v}_2(t) + (\sigma\sigma^\top)_2\right)^{-1}(0,(\sigma\theta)_2)^\top$ with initial condition $Y^{(y,\hat{\alpha},\hat{\beta})}(0) = y$, which is the expression for a geometric Brownian motion, with solution

$$Y^{(y,\hat{\alpha},\hat{\beta})}(t) = ye^{-\int_0^t(r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t\tilde{A}^\top(u)dW(u)}.$$

Recall that

$$X^{\hat{\pi}}(t) = -\tilde{V}_y(t, Y^{(y,\hat{\alpha},\hat{\beta})}(t)),$$

$$V_{xx} = 2v_2(t) = \frac{1}{\tilde{V}_{yy}} = \frac{1}{2\tilde{v}_2(t)},$$

$$y = -V_x = -(2xv_2(t)),$$

$$A(t) = (-(2\sigma\theta v_2(t))^\top(2\sigma\sigma^\top v_2(t) + R)^{-1})^+\sigma \quad \text{for} \quad Q = 0 \text{ and } S = 0,$$

and

$$\tilde{V}_y = 2y\tilde{v}_2(t),$$

so

$$\begin{aligned} X^{\hat{\pi}}(t) &= -(2v_2(t))^{-1}(-2v_2(0)x_0)e^{-\int_0^t(r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t\tilde{A}^\top(u)dW(u)} \\ &= \frac{v_2(0)}{v_2(t)}x_0e^{-\int_0^t(r+\frac{1}{2}\tilde{A}^\top(u)\tilde{A}(u))du - \int_0^t\tilde{A}^\top(u)dW(u)}. \end{aligned}$$

From the ODE of $v_2(t)$, when $Q = 0$ and $S = 0$,

$$\partial_t v_2(t) + 2rv_2(t) + \frac{1}{2}(-(2\sigma\theta v_2(t))^\top(2\sigma\sigma^\top v_2(t) + R)^{-1})^+(2\sigma\theta v_2(t)) = 0,$$

we can know that

$$\frac{v_2(0)}{v_2(t)} = e^{\int_0^t 2r + A(u)\theta du},$$

and we also find that $A(t) = -\tilde{A}^\top(t)$ when $\hat{\pi} = (0, \hat{\pi}_2)^\top$. Thus,

$$X^{\hat{\pi}}(t) = x_0e^{\int_0^t r + A(u)\theta - \frac{1}{2}A(u)A^\top(u)du + \int_0^t A(u)dW(u)}. \tag{4.2.10}$$

We obtain the exact same solution with primal HJB approach for $Q = 0, S = 0$ and $\hat{\pi} = (0, \hat{\pi}_2)^\top$. In the problem with cone-constraints, we can find that solving the dual problem is not more convenient than solving the primal problem. As we can see from this chapter, we have to discuss four different cases of the dual problem due to $K = \mathbb{R}_+^2$. If the dimension is higher, $N > 2$, there will be more different cases need to be studied. To sum up, the results of dual HJB method are same with the results of primal HJB method in all different cases.

4.3 FBSDE Method

For the conditions in this chapter, the FBSDE then becomes

$$\begin{cases} dX^{\hat{\pi}}(t) = [rX^{\hat{\pi}}(t) + \hat{\pi}^\top(t)\sigma\theta]dt + \hat{\pi}^\top(t)\sigma dW(t), \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{p}_1(t) = [-r\hat{p}_1(t) + QX^{\hat{\pi}}(t) + S^\top\hat{\pi}(t)]dt + \hat{q}_1^\top(t)dW(t), \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c, \end{cases} \tag{4.3.1}$$

and the condition in Theorem 3.3.1 becomes

$$\hat{\pi}^\top[\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1(t) + SX^{\hat{\pi}}(t) + R\hat{\pi}(t)] = 0.$$

To solve this FBSDE, we assume $\hat{\pi}(t) = \hat{\pi}$, $\hat{p}_1(0) = p_0$ and $\hat{q}_1(t) = \hat{q}_1$, where $\hat{\pi}(t)$ and $\hat{q}_1(t)$ are piecewise constant and given. Then, we solve the following system

$$\begin{cases} dX^{\hat{\pi}}(t) = [rX^{\hat{\pi}}(t) + \hat{\pi}^\top\sigma\theta]dt + \hat{\pi}^\top\sigma dW(t), \\ X^{\hat{\pi}}(0) = x_0, \\ d\hat{p}_1(t) = [-r\hat{p}_1(t) + QX^{\hat{\pi}}(t) + S^\top\hat{\pi}]dt + \hat{q}_1^\top dW(t), \\ \hat{p}_1(0) = p_0, \end{cases} \tag{4.3.2}$$

with condition

$$\hat{\pi}^\top[\hat{p}_1(t)\sigma\theta + \sigma\hat{q}_1 + SX^{\hat{\pi}}(t) + R\hat{\pi}] = 0.$$

The equations above become linear SDEs. Also, we can find that the terminal condition

$$\hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c$$

is equivalent to

$$E[(\hat{p}_1(T) + aX^{\hat{\pi}}(T) + c)^2] = 0.$$

Now consider the following optimal control problem:

$$\min_{\hat{\pi}, p_0, \hat{q}_1} J(\hat{\pi}, p_0, \hat{q}_1) := E[(\hat{p}_1(T) + aX^{\hat{\pi}}(T) + c)^2].$$

To solve this optimal control problem numerically, we divide interval $[0, T]$ by m intervals with step size $h = T/m$ and grid points $t_i = hi, i = 0, 1, ..., m$. Assume on the subinterval $[t_i, t_{i+1})$, $\hat{\pi}$ and $\hat{q}_1$ are taken constant, say that $\hat{\pi}_i$ and $\hat{q}_{1i}$, for $i = 0, 1, ..., m-1$. Use the Euler method to discretize SDEs (4.3.2) as

$$\begin{cases} X^{\hat{\pi}}(t_{i+1}) = X^{\hat{\pi}}(t_i) + [rX^{\hat{\pi}}(t_i) + \hat{\pi}_i^\top\sigma\theta]h + \hat{\pi}_i^\top\sigma\sqrt{h}\xi_i, \\ X^{\hat{\pi}}(0) = x_0, \\ \hat{p}_1(t_{i+1}) = \hat{p}_1(t_i) - [r\hat{p}_1(t_i) - QX^{\hat{\pi}}(t_i) - S^\top\hat{\pi}_i]h + \hat{q}_{1i}^\top\sqrt{h}\xi_i, \\ \hat{p}_1(0) = p_0, \end{cases} \tag{4.3.3}$$

with condition

$$\hat{\pi}_i^\top[\hat{p}_1(t_i)\sigma\theta + \sigma\hat{q}_{1i} + SX^{\hat{\pi}}(t_i) + R\hat{\pi}_i] = 0,$$

for $i = 0, 1, ..., m-1$, where $\xi_i, i = 0, 1, ..., m-1$, are independent standard normal variables in $\mathbb{R}^N$. We now solve the optimal control problem as follows:

$$\min_{\hat{\pi}_0, ..., \hat{\pi}_{m-1}, p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}} J(\hat{\pi}_0, ..., \hat{\pi}_{m-1}, p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}) := E[(\hat{p}_1(t_m) + aX^{\hat{\pi}}(t_m) + c)^2]$$

subject to the discretized SDE above.

After we obtain the optimal values of $\hat{\pi}_0, ..., \hat{\pi}_{m-1}, p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}$, we can find the corresponding discrete optimal wealth process $X^{\hat{\pi}}(t)$ by the discretized SDE (4.3.3).

4.4 Dual FBSDE Method

For the conditions in this chapter, the dual FBSDE becomes

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \left[\hat{\alpha}(t) - rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right] dt + \left[\sigma^{-1}\hat{\beta}(t) - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t), \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y}, \\ d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\top(t)\theta]dt + \hat{q}_2^\top(t)dW(t), \\ \hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T)+c}{a}. \end{cases} \tag{4.4.1}$$

Now there are two controls in the SDE of dual wealth process $\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)$ (4.2.1), therefore we cannot use the same method with Section 3.4. From the second condition in Theorem 3.4.2, we can know that $[\sigma^\top]^{-1}\hat{q}_2(t)$ should be non-negative for all time t, $0 < t < T$. To simplify the dual FBSDE (4.4.1), we assume $Q = 0$ and $S = 0$, then $\hat{\alpha} = 0$ and the dual FBSDE (4.4.1) becomes

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = -rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)dt + \left[\sigma^{-1}\hat{\beta}(t) - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t), \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y}, \\ d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\top(t)\theta]dt + \hat{q}_2^\top(t)dW(t), \\ \hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T)+c}{a}. \end{cases}$$

To satisfy the third condition in Theorem 3.4.2, we solve the problem by the stochastic maximum principle. In this case, the Hamiltonian takes the form

$$\mathcal{H}(y, \beta, p_2, q_2) = -ryp_2(t) - [\sigma^{-1}\beta(t) - \theta y]^\top q_2(t) + \frac{1}{2}(\beta^\top(t)R^{-1})^+\beta(t).$$

Let $\hat{\beta} \in \mathcal{A}$ be a candidate for the optimal control, and $Y^{(\hat{y},\hat{\alpha},\hat{\beta})}$ and $(\hat{p}_2, \hat{q}_2)$ be the corresponding processes. Then,

$$\mathcal{H}(y, \beta, \hat{p}_2, \hat{q}_2) = -ry\hat{p}_2(t) + y^\top\theta^\top\hat{q}_2(t) - \beta^\top(t)(\sigma^{-1})^\top\hat{q}_2(t) + \frac{1}{2}(\beta^\top(t)R^{-1})^+\beta(t).$$

We see that $\hat{\beta}$ is optimal if and only if

$$-\hat{q}_2^\top(t)\sigma^{-1} + (\hat{\beta}^\top(t)R^{-1})^+ = 0, \ 0 \le t \le T, \ a.s. .$$

To solve this dual FBSDE, let $\hat{p}_2(0) = x_0$, we assume $\hat{\beta}(t) = \hat{\beta}$, $\hat{y} = y_0$ and $\hat{q}_2(t) = \hat{q}_2$, where $\hat{\beta}(t)$ and $\hat{q}_2(t)$ are piecewise constant and given. Then, we solve the following system

$$\begin{cases} d\hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = -rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)dt + \left[\sigma^{-1}\hat{\beta} - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^\top dW(t), \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = y_0, \\ d\hat{p}_2(t) = [r\hat{p}_2(t) + \hat{q}_2^\top\theta]dt + \hat{q}_2^\top dW(t), \\ \hat{p}_2(0) = x_0, \end{cases} \tag{4.4.2}$$

with conditions $[\sigma^\top]^{-1}\hat{q}_2 \ge 0$ and $-\hat{q}_2^\top\sigma^{-1} + (\hat{\beta}^\top R^{-1})^+ = 0$. Then, the equations above become linear SDEs. Also, the terminal condition

$$\hat{p}_2(T) = -\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a}$$

is equivalent to

$$E[(\hat{p}_2(T) + \frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a})^2] = 0.$$

Now consider the following optimal control problem:

$$\min_{\hat{\beta}, y_0, \hat{q}_2} J(\hat{\beta}, y_0, \hat{q}_2) := E[(\hat{p}_2(T) + \frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a})^2].$$

To solve this optimal control problem numerically, we divide interval $[0, T]$ by m intervals with step size $h = T/m$ and grid points $t_i = hi, i = 0, 1, ..., m$. Assume on the subinterval $[t_i, t_{i+1})$, controls $\hat{\beta}$, $\hat{q}_2$ are taken constant, say that $\hat{\beta}_i$, $\hat{q}_{2i}$, for $i = 0, 1, ..., m-1$. Use the Euler method to discretize SDEs (4.4.2) as

$$\begin{cases} \hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_{i+1}) = \hat{Y}^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_i) - rY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_i)h + \left[\sigma^{-1}\hat{\beta}_i - \theta Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_i)\right]^\top \sqrt{h}\xi_i, \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = y_0, \\ \hat{p}_2(t_{i+1}) = \hat{p}_2(t_i) + [r\hat{p}_2(t_i) + \hat{q}_{2i}^\top\theta]h + \hat{q}_{2i}^\top\sqrt{h}\xi_i, \\ \hat{p}_2(0) = x_0, \end{cases} \tag{4.4.3}$$

with conditions $[\sigma^\top]^{-1}\hat{q}_{2i} \geq 0$ and $-\hat{q}_{2i}^\top\sigma^{-1} + (\hat{\beta}_i^\top R^{-1})^+ = 0$, for $i = 0, 1, ..., m-1$, where $\xi_i, i = 0, 1, ..., m-1$, are independent standard normal variables in $\mathbb{R}^N$. We now solve the optimal control problem as follows:

$$\min_{\hat{\beta}_0,...,\hat{\beta}_{m-1},y_0,\hat{q}_{20},...,\hat{q}_{2m-1}} J(\hat{\beta}_0, ..., \hat{\beta}_{m-1}, y_0, \hat{q}_{20}, ..., \hat{q}_{2m-1}) := E[(\hat{p}_2(t_m) + \frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t_m) + c}{a})^2]$$

subject to the discretized SDE (4.4.3).

After we obtain the optimal values of $\hat{\beta}_0, ..., \hat{\beta}_{m-1}, y_0, \hat{q}_{20}, ..., \hat{q}_{2m-1}$, we can find the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$ numerically, from Theorem 3.4.3,

$$\hat{\pi}(t_i) = (\sigma^\top)^{-1}\hat{q}_{2i}, \quad \text{for } i = 0, 1, ..., m-1,$$

$$X^{\hat{\pi}}(t_i) = \hat{p}_2(t_i), \quad \text{for } i = 0, 1, ..., m,$$

where $\hat{p}_2(t_i)$ follows the discretized SDE (4.4.3).

4.5 Comparison

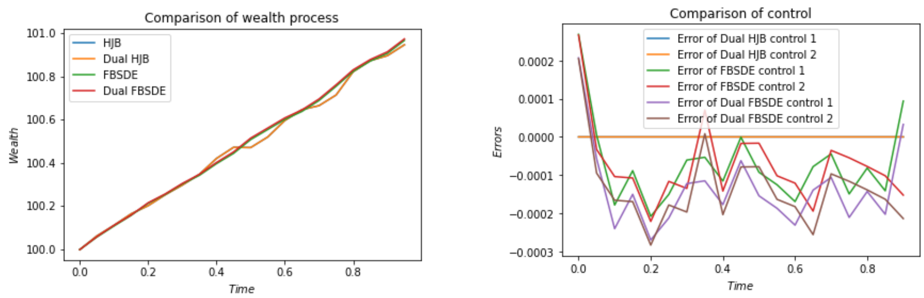
In the sections above, we discuss solving the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$ by primal HJB method, dual HJB method, primal FBSDE method and dual FBSDE method respectively. This section will prove that all methods have same results.

To compare their results, we can firstly solve the Riccati equations of $v_2(t)$ and $\tilde{v}_2(t)$ numerically at same grid points as the FBSDE method and the dual FBSDE method. Then, we solve numerical optimization problems to find $\hat{\pi}_0, ..., \hat{\pi}_{m-1}, p_0, \hat{q}_{10}, ..., \hat{q}_{1m-1}$ in primal FBSDE and $\hat{\beta}_0, ..., \hat{\beta}_{m-1}, y_0, \hat{q}_{20}, ..., \hat{q}_{2m-1}$ in dual FBSDE respectively. To solve these numerical optimization problems, we need to generate sample paths M times using the same parameters, then compute the expected value of the objective cost function, and update to get new parameters. The parameters are optimal solutions when the expected value of the objective cost function is close to 0. After these calculations, we can determine the optimal wealth process $X^{\hat{\pi}}(t)$ and optimal control $\hat{\pi}(t)$ under each approach by generating a set of independent standard normal variables on each subinterval and using parameters we obtained from previous calculations.

To compare the results intuitively, we assume that $r = 0.01, b = \begin{pmatrix} 0.04 \\ 0.03 \end{pmatrix}, \sigma = \begin{pmatrix} 0.5 & 0.01 \\ 0.01 & 0.4 \end{pmatrix}$, $R = \begin{pmatrix} 100 & -10 \\ -20 & 80 \end{pmatrix}, a = -2, c = 0, x_0 = 100, T = 1$ and there are two stocks. Then, we can plot sample paths of the optimal wealth process $X^{\hat{\pi}}(t)$ under each method on a graph to see if they are close to each other, and also compute the mean squared errors and variances of differences between results of each method. For the optimal control $\hat{\pi}(t)$, we plot the paths of errors of two controls for each method. In all comparisons, we let the the primal HJB approach be benchmark approach, and compare the result of chosen approach with the result of the primal HJB approach. Figure 4.1, 4.2 and 4.3 show the optimal wealth process $X^{\hat{\pi}}(t)$ and errors of optimal controls $\hat{\pi}(t)$ under different approaches with different step sizes. Table 4.1, 4.2 and 4.3 record the statistical information of differences between results of each methods with different step sizes.

Firstly, it is straightforward to find that the optimal wealth processes $X^{\hat{\pi}}(t)$ under all approaches are very close in Figure 4.1(a) and the mean squared errors of differences between results are almost smaller than 6×10^{-4}, shown in Table 4.1. Also, from statistical information, the mean squared errors and variances of differences between errors of optimal control $\hat{\pi}(t)$ are all close to 0, such as the mean squared errors from primal HJB and FBSDE with step size $h = 0.025$ are 2.736×10^{-9} and 1.110×10^{-8} respectively. As we can see from Figure 4.1(b), 4.2(b) and 4.3(b), the path of the error of optimal control $\hat{\pi}_1(t)$ under dual HJB method is not shown, and the error of optimal control $\hat{\pi}_2(t)$ under dual HJB method is almost straight line around 0. The reason is that the errors of optimal control $\hat{\pi}(t)$ under dual HJB method are too small, so the path of the error of optimal control $\hat{\pi}_2(t)$ overlaps on the path of the error of optimal control $\hat{\pi}_1(t)$, and looks like a straight line. Similar with the results at Section 3.5, for primal and dual FBSDE methods, we find that the errors of optimal control $\hat{\pi}(t)$ fluctuates more, shown in Figure 4.1(b), 4.2(b) and 4.3(b), since we use the numerical optimization to solve the whole problem, but only use the numerical method to solve part of the problem in primal and dual HJB methods. Furthermore, we can easily find that the mean squared errors and variances of differences between results of each method are gradually decreasing when the step size is decreasing and the number of grid points is increasing. For example, the mean squared errors of difference between optimal wealth processes $X^{\hat{\pi}}(t)$ from primal HJB approach and dual HJB approach reduces 4.4694×10^{-14} when the step size $h = 0.05$ decreases to $h = 0.0125$.

To compare results more accurately and precisely, we check results with different values of coefficients $r = 0.03, b = \begin{pmatrix} 0.1 \\ 0.08 \end{pmatrix}, \sigma = \begin{pmatrix} 0.7 & 0.03 \\ 0.03 & 0.1 \end{pmatrix}, R = \begin{pmatrix} 50 & -30 \\ -10 & 70 \end{pmatrix}, a = -4$. Figure 4.4, 4.5 and 4.6 show the optimal wealth process $X^{\hat{\pi}}(t)$ and the errors of optimal control $\hat{\pi}(t)$ with different step sizes and new coefficients. Table 4.4, 4.5 and 4.6 record the statistical information of differences between results of each method with different step sizes and new coefficients. We can find that the differences between the errors of optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$ are quite obvious, especially for FBSDE approach and dual FBSDE approach, in Figure 4.4, 4.5 and 4.6. For the optimal wealth process $X^{\hat{\pi}}(t)$, the mean squared errors of difference between primal HJB approach and FBSDE approach becomes 2.383×10^{-1} with step size $h = 0.0125$, which is a substantial error. Unlike Section 3.5, different values of coefficients will give us results under various degrees of accuracy in this case. Therefore, assuming parameters are piecewise constant is not the best method to solve FBSDE and dual FBSBE at Section 4.3 and 4.4. To sum up, we can demonstrate that, without any error from numerical methods, all approach will give us the same results of the optimal control $\hat{\pi}(t)$ and the optimal wealth process $X^{\hat{\pi}}(t)$.

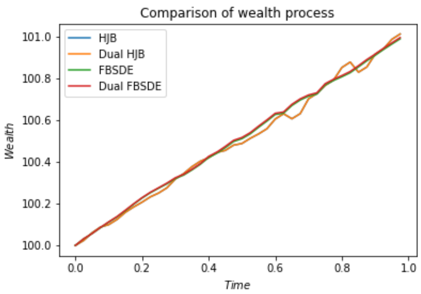


(a) Optimal wealth process $X^{\hat{\pi}}(t)$ (b) Errors of optimal control $\hat{\pi}(t)$

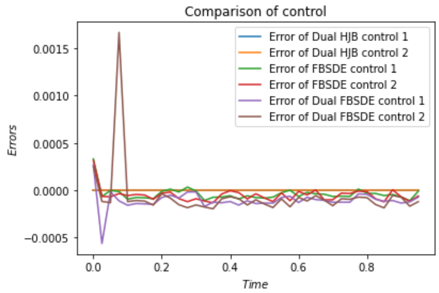
Figure 4.1: Comparison with cone-constraints at step size $h = 0.05$

	Optimal wealth process $X^{\hat{\pi}}(t)$		Error of optimal control $\hat{\pi}_1(t)$		Error of optimal control $\hat{\pi}_2(t)$	
Approach	MSE	Variance	MSE	Variance	MSE	Variance
HJB	0	0	0	0	0	0
Dual HJB	4.733×10^{-13}	2.992×10^{-13}	1.664×10^{-19}	1.086×10^{-19}	1.664×10^{-19}	1.086×10^{-19}
FBSDE	3.498×10^{-4}	3.200×10^{-4}	1.399×10^{-8}	9.499×10^{-9}	1.565×10^{-8}	9.549×10^{-9}
Dual FBSDE	4.595×10^{-4}	3.510×10^{-4}	2.402×10^{-7}	2.256×10^{-7}	4.554×10^{-8}	1.022×10^{-8}

Table 4.1: MSE and variances of differences with cone-constraints at step size $h = 0.05$

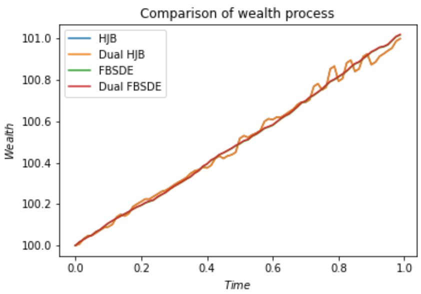


(a) Optimal wealth process $X^{\hat{\pi}}(t)$

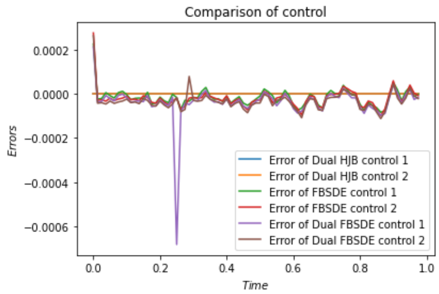


(b) Errors of optimal control $\hat{\pi}(t)$

Figure 4.2: Comparison with cone-constraints at step size $h = 0.025$

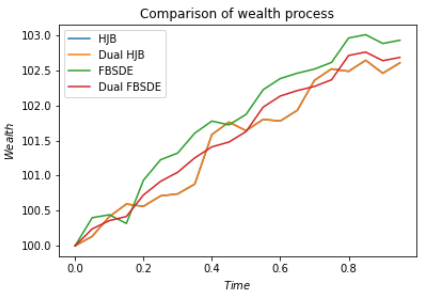


(a) Optimal wealth process $X^{\hat{\pi}}(t)$

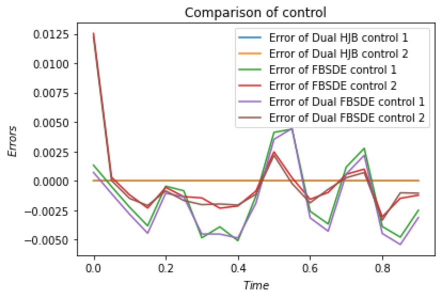


(b) Errors of optimal control $\hat{\pi}(t)$

Figure 4.3: Comparison with cone-constraints at step size $h = 0.0125$

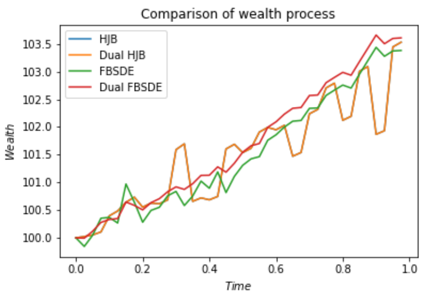


(a) Optimal wealth process $X^{\hat{\pi}}(t)$

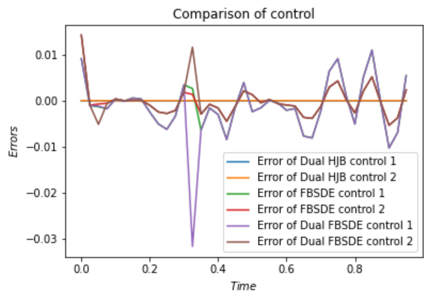


(b) Errors of optimal control $\hat{\pi}(t)$

Figure 4.4: Comparison with cone-constraints with new coefficients ($h = 0.05$)

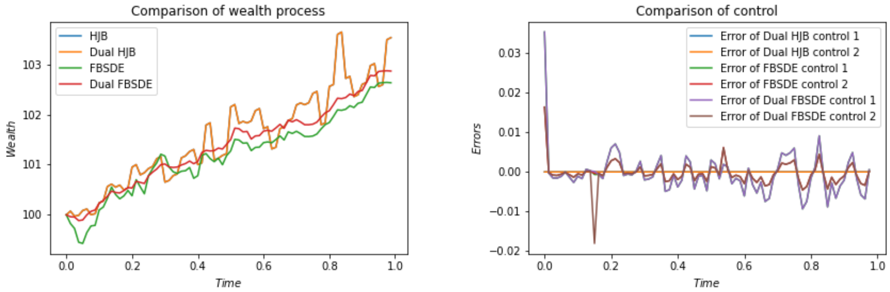


(a) Optimal wealth process $X^{\hat{\pi}}(t)$



(b) Errors of optimal control $\hat{\pi}(t)$

Figure 4.5: Comparison with cone-constraints with new coefficients ($h = 0.025$)



(a) Optimal wealth process $X^{\hat{\pi}}(t)$ (b) Errors of optimal control $\hat{\pi}(t)$

Figure 4.6: Comparison with cone-constraints with new coefficients ($h = 0.0125$)

	Optimal wealth process $X^{\hat{\pi}}(t)$		Error of optimal control $\hat{\pi}_1(t)$		Error of optimal control $\hat{\pi}_2(t)$	
Approach	MSE	Variance	MSE	Variance	MSE	Variance
HJB	0	0	0	0	0	0
Dual HJB	1.140×10^{-13}	7.341×10^{-14}	4.594×10^{-20}	3.008×10^{-20}	4.594×10^{-20}	3.008×10^{-20}
FBSDE	1.732×10^{-3}	1.040×10^{-3}	2.736×10^{-9}	1.949×10^{-9}	1.110×10^{-8}	1.037×10^{-8}
Dual FBSDE	1.860×10^{-3}	1.054×10^{-3}	4.758×10^{-9}	1.952×10^{-9}	1.998×10^{-8}	1.596×10^{-8}

Table 4.2: MSE and variances of differences with cone-constraints at step size $h = 0.025$

	Optimal wealth process $X^{\hat{\pi}}(t)$		Error of optimal control $\hat{\pi}_1(t)$		Error of optimal control $\hat{\pi}_2(t)$	
Approach	MSE	Variance	MSE	Variance	MSE	Variance
HJB	0	0	0	0	0	0
Dual HJB	2.636×10^{-14}	1.762×10^{-14}	1.135×10^{-20}	7.656×10^{-21}	1.135×10^{-20}	7.654×10^{-21}
FBSDE	5.641×10^{-4}	5.613×10^{-4}	2.184×10^{-9}	1.822×10^{-9}	7.258×10^{-9}	6.765×10^{-9}
Dual FBSDE	5.575×10^{-4}	5.576×10^{-4}	5.854×10^{-9}	4.580×10^{-9}	7.788×10^{-9}	6.723×10^{-9}

Table 4.3: MSE and variances of differences with cone-constraints at step size $h = 0.0125$

	Optimal wealth process $X^{\hat{\pi}}(t)$		Error of optimal control $\hat{\pi}_1(t)$		Error of optimal control $\hat{\pi}_2(t)$	
Approach	MSE	Variance	MSE	Variance	MSE	Variance
HJB	0	0	0	0	0	0
Dual HJB	2.567×10^{-9}	1.674×10^{-9}	1.477×10^{-13}	9.990×10^{-14}	3.479×10^{-14}	2.353×10^{-14}
FBSDE	1.508×10^{-1}	6.088×10^{-2}	1.370×10^{-5}	7.610×10^{-6}	1.071×10^{-5}	9.703×10^{-6}
Dual FBSDE	4.164×10^{-2}	3.508×10^{-2}	1.720×10^{-5}	7.759×10^{-6}	1.133×10^{-5}	9.800×10^{-6}

Table 4.4: MSE and variances of differences with constraints and new coefficients ($h = 0.05$)

	Optimal wealth process $X^{\hat{\pi}}(t)$		Error of optimal control $\hat{\pi}_1(t)$		Error of optimal control $\hat{\pi}_2(t)$	
Approach	MSE	Variance	MSE	Variance	MSE	Variance
HJB	0	0	0	0	0	0
Dual HJB	5.682×10^{-10}	3.964×10^{-10}	3.749×10^{-14}	2.665×10^{-14}	8.829×10^{-15}	6.278×10^{-15}
FBSDE	2.391×10^{-1}	2.382×10^{-1}	2.462×10^{-5}	2.394×10^{-5}	1.068×10^{-5}	1.065×10^{-5}
Dual FBSDE	2.782×10^{-1}	2.396×10^{-1}	5.027×10^{-5}	4.731×10^{-5}	1.479×10^{-5}	1.479×10^{-5}

Table 4.5: MSE and variances of differences with constraints and new coefficients ($h = 0.025$)

	Optimal wealth process $X^{\hat{\pi}}(t)$		Error of optimal control $\hat{\pi}_1(t)$		Error of optimal control $\hat{\pi}_2(t)$	
Approach	MSE	Variance	MSE	Variance	MSE	Variance
HJB	0	0	0	0	0	0
Dual HJB	1.159×10^{-10}	9.370×10^{-11}	8.227×10^{-15}	6.700×10^{-15}	1.938×10^{-15}	1.578×10^{-15}
FBSDE	2.383×10^{-1}	1.230×10^{-1}	3.011×10^{-5}	2.996×10^{-5}	7.253×10^{-6}	7.223×10^{-6}
Dual FBSDE	1.254×10^{-1}	9.953×10^{-2}	3.011×10^{-5}	2.996×10^{-5}	1.138×10^{-5}	1.122×10^{-5}

Table 4.6: MSE and variances of differences with constraints and new coefficients ($h = 0.0125$)

Chapter 5

Quadratic Risk Minimization under Stochastic Factor Model

In this chapter, we study the quadratic risk minimization problem under the stochastic factor model. The asset price $S_n(t)$ has drift term $H(t)$, and $H(t)$ follows OU process.

$$\begin{cases} dS_n(t) = S_n(t)\left[H(t)dt + \sigma(t)dW(t)\right] \\ dH(t) = k(\hat{H} - H(t))dt + \sigma_1 dW_1 \end{cases}$$

where $\hat{H}$ is a vector, k and σ_1 are square matrices, W and W_1 are independent vector standard Brownian motions. Therefore, we can know that $\theta(t) = \sigma^{-1}(t)[H(t) - r(t)\mathbf{1}]$ in this assumption. Also, we assume that all coefficients are constant, $K = \mathbb{R}^n$, $Q = 0$ and $S = 0$.

5.1 HJB Method

The funtional $J : \mathcal{A} \to \mathbb{R}$ is defined by

$$J(\pi) := E\left[\int_0^T f(t, X^\pi(t, H_t), \pi(t, H_t))dt + g(X^\pi(T, H_T))\right],$$

where $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$f(w, t, x, \pi) := \frac{1}{2}\pi^\top R\pi,$$

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$g(w, x) := \frac{1}{2}[ax^2 + 2cx].$$

The quadratic risk minimization problem discussed in this paper is

Minimize $J(\pi)$ subject to (X, π) admissible,

where X satisfies

$$\begin{cases} dX^\pi(t, H_t) = [rX^\pi(t, H_t) + \pi^\top(t, H_t)\sigma\theta(t)]dt + \pi^\top(t, H_t)\sigma dW(t), \quad 0 \le t \le T \\ X^\pi(0, h) = x_0. \end{cases} \tag{5.1.1}$$

Define value function $V(t, x, h)$ by

$$V(t, x, h) = \inf_\pi E\left[\int_0^T f(t, X^\pi(t, H_t), \pi(t, H_t))dt + g(X^\pi(T, H_T))\Big| X^\pi(0, h) = x_0, H(0) = h\right].$$

By Dynamic Programming Principle, we can drive the HJB equation,

$$\begin{cases} \frac{\partial V}{\partial t} + \inf_\pi\{\mathcal{L}^\pi V + f(t, x, \pi)\} = 0, \\ V(T, x, h) = g(x), \end{cases}$$

where $\mathcal{L}^\pi$ is the operator defined by

$$\mathcal{L}^\pi V = (rx + \pi^\top \sigma\theta)V_x + \frac{1}{2}\pi^\top \sigma\sigma^\top \pi V_{xx} + k(\hat{H} - h)V_h + \frac{1}{2}\sigma_1\sigma_1^\top V_{hh}.$$

Substituting $f(t, x, \pi)$ and $g(x)$ into the HJB equation above, we obtain that

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x + k(\hat{H} - h)V_h + \frac{1}{2}\sigma_1\sigma_1^\top V_{hh} + \inf_\pi\{\pi^\top \sigma\theta V_x + \frac{1}{2}\pi^\top \sigma\sigma^\top \pi V_{xx} + \frac{1}{2}\pi^\top R\pi\} = 0, \\ V(T, x, h) = \frac{1}{2}[ax^2 + 2cx]. \end{cases}$$

To find the optimal $\hat{\pi}$, we obtain the first-order condition is

$$(\sigma\theta)^\top V_x + \pi^\top \sigma\sigma^\top V_{xx} + \pi^\top R = 0.$$

Since R is a symmetric matrix, we can obtain that $(\sigma\sigma^\top V_{xx} + R) = (\sigma\sigma^\top V_{xx} + R)^\top$. Assume that $(\sigma\sigma^\top V_{xx} + R)$ is invertible, so we know that $((\sigma\sigma^\top V_{xx} + R)^{-1})^\top = ((\sigma\sigma^\top V_{xx} + R)^\top)^{-1} = (\sigma\sigma^\top V_{xx} + R)^{-1}$. Thus, the optimal control $\hat{\pi}$ is equal to $-(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x$.

Then the value function V satisfies:

$$\begin{cases} \frac{\partial V}{\partial t} + rxV_x + k(\hat{H} - h)V_h + \frac{1}{2}\sigma_1\sigma_1^\top V_{hh} - \frac{1}{2}(\sigma\theta)^\top (\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x^2 = 0, \\ V(T, x, h) = \frac{1}{2}[ax^2 + 2cx]. \end{cases} \quad (5.1.2)$$

It is too complicated to solve this nonlinear HJB PDE by assuming $V(t, x, h)$ is the most general case, $V(t, x, h) = v_0(t, h) + v_1(t, h)x + v_2(t, h)x^2$. Therefore, we assume that $c = 0$ and $V(t, x, h) = v_2(t, h)x^2$. Substituting $V(t, x, h)$ into the HJB equation (5.1.2), we obtain that

$$\begin{aligned} &\partial_t v_2(t, h) + 2rv_2(t, h) + k(\hat{H} - h)\partial_h v_2(t, h) + \frac{1}{2}\sigma_1\sigma_1^\top \partial_{hh} v_2(t, h) \\ &\qquad - 2(\sigma\theta(t))^\top (2\sigma\sigma^\top v_2(t, h) + R)^{-1}(\sigma\theta(t))v_2^2(t, h) = 0, \end{aligned}$$

with terminal condition $v_2(T) = \frac{1}{2}a$.

It is obvious to find that there is no the closed-form solution for $v_2(t, h)$, so we try to solve the non nonlinear PDE by the extension of Feynman-Kac formula for semi-linear PDE, inspired by Pham [22, Proposition 6.3.2, page 145]. For this method, we can obtain a BSDE representation of solution, which is the viscosity solution. As a result, more details are left to be discussed in the future.

Recall that

$$\hat{\pi} = -(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x$$

and

$$\begin{cases} V_{xx} = 2v_2(t, h) \\ V_x = 2v_2(t, h)x, \end{cases}$$

so

$$\hat{\pi}(t, H_t) = -(2\sigma\sigma^\top v_2(t, H_t) + R)^{-1}(\sigma\theta(t))2v_2(t, H_t)X^{\hat{\pi}}(t, H_t)$$

We insert the optimal $\hat{\pi}$ into the SDE of total wealth X^π (5.1.1),

$$\begin{aligned} dX^{\hat{\pi}}(t, H_t) &= [rX^{\hat{\pi}}(t, H_t) - (\sigma\theta(t))^\top (2\sigma\sigma^\top v_2(t, H_t) + R)^{-1}(\sigma\theta(t))(2v_2(t, H_t)X^{\hat{\pi}}(t, H_t))]dt \\ &\quad - (\sigma\theta(t))^\top (2\sigma\sigma^\top v_2(t, H_t) + R)^{-1}\sigma(2v_2(t, H_t)X^{\hat{\pi}}(t, H_t))dW(t) \\ &= [r - A(t, H_t)\theta(t)]X^{\hat{\pi}}(t, H_t)dt - A(t, H_t)X^{\hat{\pi}}(t, H_t)dW(t) \end{aligned}$$

where $A(t, H_t) = 2(\sigma\theta(t))^\top (2\sigma\sigma^\top v_2(t, H_t) + R)^{-1}(\sigma)v_2(t, H_t)$, and the corresponding solution is given by

$$X^{\hat{\pi}}(t, H_t) = x_0 e^{\int_0^t r - A(u, H_u)\theta(u) - \frac{1}{2}A(u, H_u)A^\top (u, H_u)du - \int_0^t A(u, H_u)dW(u)}$$

5.2 Dual HJB Method

The dual control problem is given by

$$\text{Minimize } \tilde{\Psi}(y,\alpha,\beta) := m_0(y) + E[m_T(Y(T,H_T))] + E\left[\int_0^T \phi(t,\alpha(t,H_t),\beta(t,H_t))dt\right],$$

where Y satisfies

$$\begin{cases} dY(t,H_t) = [\alpha(t,H_t) - rY(t,H_t)]dt + [\sigma^{-1}\beta(t,H_t) - \theta(t)Y(t,H_t)]^\top dW(t), \\ Y(0,h) = y. \end{cases} \tag{5.2.1}$$

Define dual value function $\tilde{V}(t,y,h)$ by

$$\tilde{V}(t,y,h) = \inf_{\alpha,\beta} E\left[m_T(Y(T,H_T)) + \int_0^T \phi(t,\alpha(t,H_t),\beta(t,H_t))dt \middle| Y(0,h) = y, H(0) = h\right],$$

and we can find the relationship between the primal value function and dual value function is

$$\tilde{V}(t,y,h) = \sup_{x\in\mathbb{R}}\{-V(t,x,h) - xy\},$$

$$V(t,x,h) = \inf_{y\in\mathbb{R}}\{-\tilde{V}(t,y,h) - xy\}.$$

The minimum point is obtained by solving

$$0 = -x - \frac{\partial}{\partial y}\tilde{V}(t,y,h). \tag{5.2.2}$$

There exists an unique y solving the equation above, write it $y = y(t,x)$. Therefore, we have

$$V(t,x,h) = -xy(t,x) - \tilde{V}(t,y(t,x),h)$$

and then, we obtain that

$$\begin{aligned} V_t &= -x\frac{\partial y}{\partial t} - \tilde{V}_t - \tilde{V}_y\frac{\partial y}{\partial t} \\ &= (-x - \tilde{V}_y)\frac{\partial y}{\partial t} - \tilde{V}_t \\ &= -\tilde{V}_t, \end{aligned}$$

$$\begin{aligned} V_x &= -y - x\frac{\partial y}{\partial x} - \tilde{V}_y\frac{\partial y}{\partial x} \\ &= (-x - \tilde{V}_y)\frac{\partial y}{\partial x} - y \\ &= -y, \end{aligned}$$

$$V_{xx} = -\frac{\partial y}{\partial x},$$

$$V_h = -\tilde{V}_h,$$

and

$$V_{hh} = -\tilde{V}_{hh}.$$

From the first-order condition (5.2.2), we know that

$$\frac{\partial}{\partial x}(-x - \frac{\partial}{\partial y}\tilde{V}(t,y,h)) = -1 - \tilde{V}_{yy}\frac{\partial y}{\partial x},$$

$\Rightarrow$

$$\frac{\partial y}{\partial x} = -\frac{1}{\tilde{V}_{yy}}.$$

By Dynamic Programming Principle, we can drive the dual HJB equation,

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + \inf_{\alpha,\beta}\{\mathcal{L}^{\alpha,\beta}\tilde{V} + \phi(t,\alpha,\beta)\} = 0, \\ \tilde{V}(T,y,h) = m_T(Y(T,H_T)), \end{cases}$$

where $\mathcal{L}^{\alpha,\beta}$ is the operator defined by

$$\mathcal{L}^{\alpha,\beta}\tilde{V} = (\alpha - ry)\tilde{V}_y + \frac{1}{2}[\sigma^{-1}\beta - \theta y]^\top[\sigma^{-1}\beta - \theta y]\tilde{V}_{yy} + k(\hat{H} - h)\tilde{V}_h + \frac{1}{2}\sigma_1\sigma_1^\top \tilde{V}_{hh}.$$

Substituting $\phi(t,\alpha,\beta)$ and $m_T(Y(T,H_T))$ into the HJB equation above, we obtain that

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + k(\hat{H} - h)\tilde{V}_h + \frac{1}{2}\sigma_1\sigma_1^\top \tilde{V}_{hh} + \inf_{\alpha,\beta}\{(\alpha - ry)\tilde{V}_y + \\ \frac{1}{2}[\sigma^{-1}\beta - \theta y]^\top[\sigma^{-1}\beta - \theta y]\tilde{V}_{yy} + \sup_{x,\pi}\{x\alpha + \pi^\top\beta - \frac{1}{2}\pi^\top R\pi\}\} = 0, \\ \tilde{V}(T,y,h) = \frac{(y+c)^2}{2a}. \end{cases}$$

Then, we know that α must be 0, otherwise, $\sup_x\{x\alpha\} = \infty$. The optimal $\hat{\pi}$ can be find by the first order condition,

$$\beta^\top - \pi^\top R = 0.$$

Therefore $\hat{\pi} = R^{-1}\beta$, and the HJB equation becomes

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top\theta y^2\tilde{V}_{yy} + k(\hat{H} - h)\tilde{V}_h + \frac{1}{2}\sigma_1\sigma_1^\top\tilde{V}_{hh} \\ \qquad + \inf_\beta\{\frac{1}{2}[(\sigma^{-1}\beta)^\top(\sigma^{-1}\beta) - (\sigma^{-1}\beta)^\top\theta y - \theta^\top(\sigma^{-1}\beta)y]\tilde{V}_{yy} + \frac{1}{2}\beta^\top R^{-1}\beta\} = 0, \\ \tilde{V}(T,y,h) = \frac{(y+c)^2}{2a}. \end{cases}$$

To find the optimal $\hat{\beta}$, we obtain the first-order condition is

$$\beta^\top {\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} - \theta^\top\sigma^{-1}y\tilde{V}_{yy} + \beta^\top R^{-1} = 0.$$

Assume that $({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})$ is invertible, so we know that $(({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1})^\top = (({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})^\top)^{-1} = ({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}$. Since R is a symmetric matrix and invertible, we can obtain that $({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1}) = ({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})^\top$. Thus, optimal control $\hat{\beta}$ is equal to $({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}({\sigma^{-1}}^\top\theta)y\tilde{V}_{yy}$.

Dual value function $\tilde{V}$ satisfies:

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} - ry\tilde{V}_y + \frac{1}{2}\theta^\top\theta y^2\tilde{V}_{yy} + k(\hat{H} - h)\tilde{V}_h + \frac{1}{2}\sigma_1\sigma_1^\top\tilde{V}_{hh} \\ \qquad -\frac{1}{2}({\sigma^{-1}}^\top\theta)^\top({\sigma^{-1}}^\top\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}({\sigma^{-1}}^\top\theta)y^2\tilde{V}_{yy}^2 = 0, \\ \tilde{V}(T,y,h) = \frac{(y+c)^2}{2a}. \end{cases} \tag{5.2.3}$$

Substituting $y, \tilde{V}_y, \tilde{V}_{yy}, \tilde{V}_t, \tilde{V}_h, \tilde{V}_{hh}$ into the equation above, we can find that

$$-\frac{\partial V}{\partial t} - r(-V_x)(-x) + \frac{1}{2}\theta^\top\theta(-V_x)^2V_{xx}^{-1} - k(\hat{H} - h)V_h - \frac{1}{2}\sigma_1\sigma_1^\top V_{hh}$$
$$-\frac{1}{2}({\sigma^{-1}}^\top\theta)^\top({\sigma^{-1}}^\top\sigma^{-1}V_{xx}^{-1} + R^{-1})^{-1}({\sigma^{-1}}^\top\theta)(-V_x)^2V_{xx}^{-2} = 0$$

$\Rightarrow$

$$\frac{\partial V}{\partial t} + rxV_x + k(\hat{H} - h)V_h + \frac{1}{2}\sigma_1\sigma_1^\top V_{hh} - \frac{1}{2}(\sigma\theta)^\top(\sigma\sigma^\top V_{xx} + R)^{-1}(\sigma\theta)V_x^2 = 0$$

(primal HJB equation)

and

$$\begin{aligned}V(T,x,h) &= \inf_{y\in\mathbb{R}}\{-xy - \tilde{V}(T,y,h)\}\\ &= \inf_{y\in\mathbb{R}}\{-xy - \frac{(y+c)^2}{2a}\}\\ &= -x(-ax-c) - \frac{(-ax-c+c)^2}{2a}\\ &= \frac{1}{2}ax^2 + cx.\end{aligned}$$

These prove the relationship between the primal and dual problems. Next, to make the nonlinear dual HJB PDE more solvable, we assume that $c = 0$ and $\tilde{V}(t,y,h) = \tilde{v}_2(t,h)y^2$. Substituting $\tilde{V}(t,y,h)$ into the dual HJB equation (5.2.3), we obtain that

$$\begin{aligned}\partial_t\tilde{v}_2(t,h) - 2r\tilde{v}_2(t,h) + k(\hat{H}-h)\partial_h\tilde{v}_2(t,h) + \frac{1}{2}\sigma_1\sigma_1^\top\partial_{hh}\tilde{v}_2(t,h)\\ +(\sigma\theta)^\top(2R\tilde{v}_2(t,h)+\sigma\sigma^\top)^{-1}(\sigma\theta)\tilde{v}_2(t,h) = 0\end{aligned}$$

with terminal condition $\tilde{v}_2(T) = \frac{1}{2a}$.

It is obvious to find that there is no the closed-form solution for $\tilde{v}_2(t,h)$, so we try to solve the non nonlinear PDE by the extension of Feynman-Kac formula for semi-linear PDE, inspired by Pham [22, Proposition 6.3.2, page 145]. For this method, we can obtain a BSDE representation of solution, which is the viscosity solution. More details leave to be discussed in the future.

Recall that

$$\hat{\beta} = (\sigma^{-1^\top}\sigma^{-1}\tilde{V}_{yy} + R^{-1})^{-1}(\sigma^{-1^\top}\theta)y\tilde{V}_{yy},$$

$$\hat{\alpha} = 0,$$

and

$$\tilde{V}_{yy} = 2\tilde{v}_2(t,h),$$

so

$$\hat{\beta}(t,H_t) = 2(2\sigma^{-1^\top}\sigma^{-1}\tilde{v}_2(t,H_t) + R^{-1})^{-1}(\sigma^{-1^\top}\theta(t))Y^{(y,\hat{\alpha},\hat{\beta})}(t,H_t)\tilde{v}_2(t,H_t).$$

We insert the optimal $\hat{\beta}$ and $\hat{\alpha}$ into the SDE of $Y^{(y,\alpha,\beta)}$ (5.2.1),

$$dY^{(y,\hat{\alpha},\hat{\beta})}(t,H_t) = -rY^{(y,\hat{\alpha},\hat{\beta})}(t,H_t)dt - \tilde{A}^\top(t,H_t)Y^{(y,\hat{\alpha},\hat{\beta})}(t,H_t)dW(t),$$

where $\tilde{A}(t,H_t) = \sigma^\top(2R\tilde{v}_2(t,H_t)+\sigma\sigma^\top)^{-1}\sigma\theta(t)$ with initial condition $Y^{(y,\hat{\alpha},\hat{\beta})}(0,h) = y$. We can find that the SDE of $Y^{(y,\hat{\alpha},\hat{\beta})}$ is the expression for a geometric Brownian motion, and the associated solution is

$$Y^{(y,\hat{\alpha},\hat{\beta})}(t,H_t) = ye^{-\int_0^t(r+\frac{1}{2}\tilde{A}^\top(u,H_u)\tilde{A}(u,H_u))du - \int_0^t\tilde{A}^\top(u,H_u)dW(u)}.$$

Recall that

$$X^{\hat{\pi}}(t,H_t) = -\tilde{V}_y(t,Y^{(y,\hat{\alpha},\hat{\beta})}(t,H_t)),$$

$$V_{xx} = 2v_2(t,H_t) = \frac{1}{\tilde{V}_{yy}} = \frac{1}{2\tilde{v}_2(t,H_t)},$$

$$y = -V_x = -2xv_2(t,H_t),$$

$$A(t,H_t) = (\sigma\theta(t))^\top(\sigma\sigma^\top + (2v_2(t,H_t))^{-1}R)^{-1}\sigma,$$

and

$$\tilde{V}_y = 2y\tilde{v}_2(t,H_t),$$

so

$$
\begin{aligned}
X^{\hat{\pi}}(t, H_t) &= -2\tilde{v}_2(t, H_t) y e^{-\int_0^t (r + \frac{1}{2}\tilde{A}^\top(u, H_u)\tilde{A}(u, H_u))du - \int_0^t \tilde{A}^\top(u, H_u)dW(u)} \\
&= x_0 \frac{v_2(0, h)}{v_2(t, H_t)} e^{-\int_0^t (r + \frac{1}{2}\tilde{A}^\top(u, H_u)\tilde{A}(u, H_u))du - \int_0^t \tilde{A}^\top(u, H_u)dW(u)}.
\end{aligned}
$$

We also find that $A(t, H_t) = \tilde{A}^\top(t, H_t)$ and $\frac{v_2(0,h)}{v_2(t,H_t)} = e^{\int_0^t 2r - A(u,H_u)\theta(u)du}$. Thus,

$$
\begin{aligned}
X^{\hat{\pi}}(t, H_t) &= x_0 e^{\int_0^t 2r - A(u,H_u)\theta(u)du} e^{-\int_0^t (r + \frac{1}{2}A(u,H_u)A^\top(u,H_u))du - \int_0^t A(u,H_u)dW(u)} \\
&= x_0 e^{\int_0^t r - A(u,H_u)\theta(u) - \frac{1}{2}A(u,H_u)A^\top(u,H_u)du - \int_0^t A(u,H_u)dW(u)}.
\end{aligned}
$$

Finally, we obtain the exact same solution with primal HJB approach.

Chapter 6

Conclusion and Open Questions

In this paper, we study the continuous-time stochastic linear quadratic control problem with financial applications, and we aim to minimize the constrained quadratic risk function, a convex cost function, in both the wealth process and portfolio strategy in an incomplete market. By the convex duality method, we construct the associated dual problem satisfied the necessary and sufficient optimality conditions. There are four main approaches used to solve the stochastic linear quadratic control problem in this paper, such as primal HJB, dual HJB, primal FBSDE and dual FBSDE, and the goal is to prove all four approaches have the same solutions of the problem. Then, we discuss the quadratic risk minimization problem with both no control constraints and cone-constraints and derive numerical solutions of each approach. For no control constraints and cone-constraints problem, we model the asset dynamics with constant coefficients and deterministic coefficients, respectively. We also compare results of all approaches intuitively and analytically by plotting paths of optimal processes and calculating the mean squared errors and variances of differences between results. We find that the differences between each approach are very close to zero from plots. The differences between each method are gradually smaller when the step size is decreasing, and the number of grid points is increasing.

Moreover, we check results with different values of coefficients to compare all methods more accurately and precisely. Since we use the numerical method and assume parameters are piecewise constant, different coefficients give us results under various degrees of accuracy. However, we demonstrate that, without any error from numerical methods, all approaches give us the same results of optimal wealth process and optimal control process. In the end, we solve the quadratic risk minimization problem under the stochastic factor model, where asset price has a random drift term and drift term follows the OU process. Due to the limited time, we only solve the problem by primal HJB and dual HJB approaches, and we prove these two approaches are equivalent by showing the same semi-analytic solution.

For the improvement of this paper, we first can continue to solve the quadratic risk minimization problem under stochastic factor model by using primal FBSDE and dual FBSDE approaches and compare results with solutions of primal and dual HJB approaches. Also, we can use more different values of coefficients to check whether solutions of each approach remain close to each other. There are still many open questions. For example, can solutions of each approach remain close when coefficients become random coefficients? Can solutions of each approach remain close when the quadratic risk minimization problem under the stochastic factor model has cone-constraints? Can solutions of each approach remain close when the quadratic risk minimization problem is under the stochastic volatility model? In the future, we will discuss these problems.

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