Imperial College, London
June 2018

Controllability and Observability

Fritz Colonius
Universität Augsburg

Introduction

We will consider control systems in continuous or discrete time on a state space M given by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in \Omega \text{ for } t \in \mathbb{R}\} \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

or

$$\begin{aligned} x_{k+1} &= f(x_k, u_k), u \in \mathcal{U} = \{u : \mathbb{Z} \to \mathbb{R}^m \,|u_k \in \Omega \text{ for } k \in \mathbb{Z}\} \\ y_k &= h(x_k, u_k). \end{aligned}$$

We assume that for very initial state $x(0) = x_0 \in M$ and every control function u there is a unique solution $\varphi(t, x_0, u)$, $t \in \mathbb{R}$.
The reachable set from x_0 is

$$\mathcal{R}(x_0) = \{\varphi(t, x_0, u) \,|t > 0, u \in \mathcal{U}\}$$

and the controllable set for x_0 is

$$\mathcal{C}(x_0) = \{x \,|x_0 = \varphi(t, x, u), t > 0, u \in \mathcal{U}\}$$

Controls versus perturbations

The term $u(\cdot)$ in the system equations above may be interpreted as a control function, which may be chosen in order to achieve a desired behavior of the system.

A “dual” interpretation is obtained by considering $u(\cdot)$ as a **time dependent (deterministic) perturbation** acting on the system. Then a typical question is:
What is the “worst” behavior that may occur under such perturbations?

Later we will also see that there are relations to stochastic perturbations acting on the system.

Kalman Criterion

The system is called controllable if $\mathcal{R}(x_0) = M$ for all $x_0 \in M$
$(\Leftrightarrow \mathcal{C}(x_0) = M$ for all $x_0 \in M)$

Theorem (Kalman). A linear control system without control constraints

$$\dot{x} = Ax + Bu \qquad \text{where } A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times m}$$

is controllable in $\mathbb{R}^d$ iff $\operatorname{rank}[B, AB, \ldots, A^{d-1}B] = d$.
Then we also say that (A, B) is controllable.

Example (the linear oscillator is controllable)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - kx_2 + u \end{aligned}$$

Here $d = 2$, $A = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $[B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$.

Proof of Kalman's Criterion

For a subspace V the smallest A-invariant subspace $\supset V$ is given by

$$\langle A\,|V\,\rangle = V + AV + ... + A^{d-1}V.$$

The matrix

$$W_t := \int_0^t e^{As}BB^\top e^{A^\top s}ds$$

is positive semi-definite and for every $t > 0$

$$\langle A\,|\mathrm{im}B\,\rangle = \mathrm{im}W_t.$$

This follows (for the orthogonal complements) using the series expansion of e^{At}.

Furthermore, the reachable subspace from 0 satisfies

$$\mathcal{R}(0) = \langle A\,|\mathrm{im}B\,\rangle = \mathrm{im}[B\ AB\ \ldots\ A^{d-1}B].$$

Here "$\supset$" follows by choosing for $\langle A\,|\mathrm{im}B\,\rangle \ni x = W_t z$

$$u(\tau) = B^\top e^{A^\top \tau}z, \tau \in [0, t].$$

Example. $A = \begin{bmatrix} -2 & -6 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $[B, AB] = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ has rank 1.

Not controllable!

Observability

A control system is observable if for all $u \in \mathcal{U}$

$$h(\varphi(t, x_1, u)) = h(\varphi(t, x_2, u)) \text{ for } t \geq 0 \implies x_1 = x_2.$$

Theorem. A linear control system

$$\dot{x} = Ax + Bu, \;\; y = Cx,$$

is observable (on arbitrarily short time intervals) iff

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ ... \\ CA^{d-1} \end{bmatrix} = d.$$

Proof. By linearity, the control does not play a role. Hence observability means

$$Ce^{At}x_1 = Ce^{At}x_2, t \geq 0.$$

Local controllability by linearization

Let x^* be an equilibrium for a control system

$$\dot{x}(t) = f(x(t), u(t))$$

for a control $u^* \in \mathbb{R}^m$, i.e., $0 = f(x^*, u^*)$.
If f is C^1 and the linearized (autonomous) control system

$$\dot{y}(t) = \frac{\partial f(x^*, u^*)}{\partial x} y(t) + \frac{\partial f(x^*, u^*)}{\partial u} v(t)$$

is controllable, then the nonlinear control system is locally controllable near x^*.

Proof: Use the implicit function theorem.

A controllability result for nonlinear systems

Consider a control system

$$\dot{x} = f(x, u) \text{ with } u(t) = (u_i(t))_{i=1,\ldots,m} \in \Omega$$

with smooth vector fields $f(\cdot, u), u \in \Omega \subset \mathbb{R}^m$.
The Lie bracket of vector fields is the vector field

$$[g, h](x) = \frac{\partial h(x)}{\partial x} g(x) - \frac{\partial g(x)}{\partial x} h(x).$$

The Lie algebra generated by $F = \{f(\cdot, u) \,|\, u \in \Omega\}$ is the smallest vector space containing F closed under Lie brackets denoted by

$$\mathcal{LA}(F).$$

A controllability result for nonlinear systems

Theorem. Consider

$$\dot{x} = f(x, u), u \in \Omega,$$

where Ω is a neighborhood of $0 \in \mathbb{R}^m$. Let $F = \{f(\cdot, u) \,|u \in \Omega\}$. Assume that for some $x \in \mathbb{R}^d$

$$\{h(x) \,|h \in \mathcal{LA}(F)\} = \mathbb{R}^d.$$

Then the system is **locally accessible**, i.e. for every $T > 0$

$$\mathrm{int}\mathcal{R}_{\leq T}(x) \neq \varnothing \text{ and } \mathrm{int}\mathcal{C}_{\leq T}(x) \neq \varnothing.$$

An Example

$$\dot{x}_1 = u(t),\ \dot{x}_2 = (x_1)^2 \text{ in } \mathbb{R}^2.$$

Then $\mathcal{R}(0,0) = \mathbb{R} \times \mathbb{R}_+$.
"$\subset$" is clear. For "$\supset$" let $u(t) \equiv \varepsilon^{-1}$ on $[0, t_0)$ and $u(t) \equiv 0$ for $t \geq t_0$,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t/\varepsilon \\ t^3/\left(3\varepsilon^2\right) \end{bmatrix}, t \in [0, t_0] \text{ and } \dot{x}_1 = 0 \text{ for } t > t_0.$$

Here $f_0 = \begin{bmatrix} 0 \\ (x_1)^2 \end{bmatrix}, f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f_{0*} = \frac{\partial f_0}{\partial x} = \begin{bmatrix} 0 & 0 \\ 2x_1 & 0 \end{bmatrix}, f_{1*} = 0$.

Thus

$$[f_0, f_1] = \begin{bmatrix} 0 \\ -2x_1 \end{bmatrix}, [[f_0, f_1], f_1] = -[f_0, f_1]_* f_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Hence $\{h(0) \,|\, h \in \mathcal{LA}(f_0, f_1)\} = \mathbb{R}^2$.
Observe that linearization in $(0,0)$ yields the non-controllable system

$$A = \frac{\partial f(0,0)}{\partial x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \frac{\partial f(0,0)}{\partial u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Sketch of the proof I

It is convenient to denote by $e^{th}x_0$ the solution of

$$\dot{x} = h(x), x(0) = x_0.$$

The condition

$$\{h(x)\,|h \in \mathcal{LA}(F)\} = \mathbb{R}^d.$$

implies that there are vector fields $h_1, \ldots, h_d \in F$ and a neighborhood of $0 \in \mathbb{R}^d$ such that for the map

$$\Phi_{h_0,\ldots,h_d} : \mathbb{R}^d \ni (t_1, \ldots, t_d) \mapsto e^{t_d h_d} \cdots e^{t_1 h_1} x$$

the Jacobi matrix satisfies: For every $\varepsilon > 0$ there are $t^0 = (t_1^0, \ldots, t_d^0)$ with $0 < t_i^0 < \varepsilon$ such that

$$\mathrm{rank}\,(\Phi_{h_0,\ldots,h_d})_*\,(t^0) = d.$$

This follows since for vector fields $h_0, \ldots, h_k$ and a slice $\Phi_{h_0,\ldots,h_k} : W \to \mathbb{R}^d$, $\mathrm{rank}\,(\Phi_{h_0,\ldots,h_k})_*\,(t^0) = k$, it follows that all vector fields in $\mathcal{LA}(h_0, \ldots, h_k)$ are tangential to this slice.

Sketch of the proof II

Let $h_i = f(\cdot, u_i) \in F$ with

$$\operatorname{rank} (\Phi_{h_0,\dots,h_d})_* (t^0) = d.$$

Then one defines a control u by $u(t) = u_1$ on $[0, t_1], u(t) = u_2$ on $(t_1, t_1 + t_2]$ etc.

The inverse function theorem guarantees that the reachable set (by varying the t_i) has nonvoid interior.

Driftless control systems

A control-affine system is called driftless if it has the form

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)),\; u(t) = (u_i(t))_i \in \Omega \subset \mathbb{R}^m.$$

Corollary. A driftless control system with control range Ω being a neighborhood of 0 satisfies $\mathcal{R}(x) = \mathbb{R}^d$ for all $x \in \mathbb{R}^d$ if

$$\{ h(x) \,|\, h \in \mathcal{LA}(f_1, \ldots, f_m) \} = \mathbb{R}^d \text{ for all } x \in \mathbb{R}^d.$$

The **proof** uses that

$$\mathcal{LA}(f_1, \ldots, f_m) = \mathcal{LA}\left(\sum_{i=1}^{m} u_i f_i(\cdot) \,|\, (u_i) \in \Omega \right)$$

and that one can go forward "and backward" in time.

Example

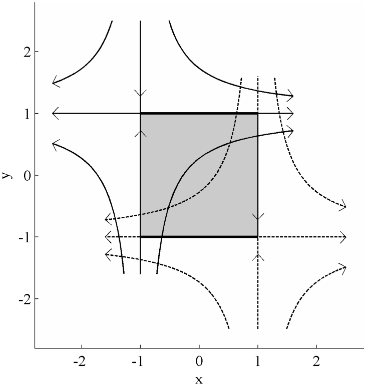
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),\ \ u(t) \in \Omega = [-1,1].$$

The phase portrait for $u \equiv 1$ is obtained from the one for $u \equiv 0$ by

$$\tilde{x} = x + 1, \tilde{y} = y - 1.$$

Hence the equilibrium for $u \equiv 1$ is $(-1,1)$.
Then a maximal set of approximate controllability is

$$D = (-1,1) \times [-1,1].$$



Definition. A set $\varnothing \neq D \subset \mathbb{R}^d$ is a **control set** if it is maximal with
(i) for all $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$;
(ii) for all $x \in D$ one has $D \subset \overline{\mathcal{R}(x)}$.
It is called an invariant control set if for all $x \in D$ one has $\overline{D} = \overline{\mathcal{R}(x)}$.

Note. Local accessibiliy implies $\mathrm{int}D \subset \mathcal{R}(x)$ for all $x \in D$.

The linear case

Theorem. Consider $\dot{x} = Ax + Bu,\ u(t) \in \Omega$, with (A, B) controllable and Ω a compact neighborhood of $0 \in \mathbb{R}^m$.
(i) There is a unique control set D with $\mathrm{int}D \neq \varnothing$, D is convex and $0 \in \mathrm{int}D$.
(ii) D is closed iff $\mathcal{R}(x) \subset D$ for all $x \in D$ iff D is an invariant control set.
(iii) D is bounded iff A is hyperbolic (i.e., $\mathrm{spec}(A) \cap \imath\mathbb{R} = \varnothing$) and $D = \mathbb{R}^d$ iff $\mathrm{spec}(A) \subset \imath\mathbb{R}$.

- Convexity of D follows by Lyapunov's convexity theorem.
- Null controllability holds iff $\mathrm{spec}(A) \subset \overline{\mathbb{C}_-}$.
- (iii) shows that complete controllability with bounded controls is exceptional.

Existence of invariant control sets

Proposition. Let M be a compact positively invariant set (i.e. $\varphi(t,x,u) \in M$ for all $x \in M, u \in \mathcal{U}$).
Then M contains an invariant control set.
If every point in M is locally accessible, then the number of invariant control sets in M is finite and each of them has nonvoid interior.

Proof: Existence follows using Zorn's lemma.

Note that the number of control sets in M with nonvoid interior may be infinite.

If $\mathrm{int} D \neq \varnothing$, then

$$D = \overline{\mathcal{R}(x)} \cap \mathcal{C}(x) \text{ for every } x \in \mathrm{int} D.$$

Continuous stirred tank reactor

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 - a(x_1 - x_c) + B\alpha(1 - x_2)e^{x_1} \\ -x_2 + \alpha(1 - x_2)e^{x_1} \end{bmatrix} + u(t) \begin{bmatrix} x_c - x_1 \\ 0 \end{bmatrix},$$

where x_1 is the temperature and x_2 is the product concentration, x_c is the coolant temperature and the control affects the heat transfer coefficient with parameters

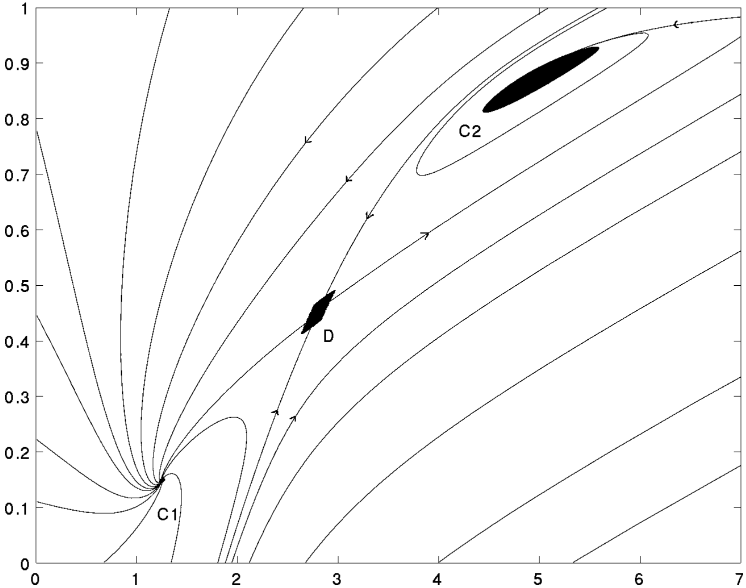
$$a = 0.95, \ \alpha = 0.05, \ B = 10.0, \ c_c = 1.0$$

and control range

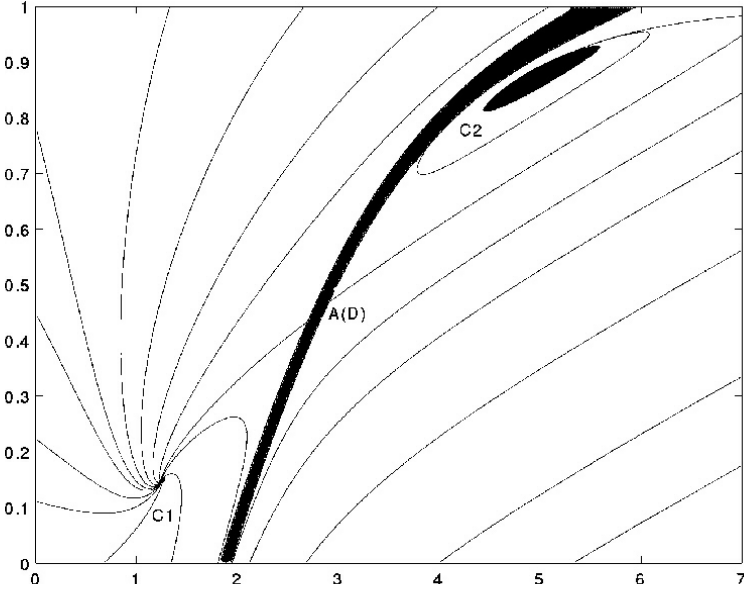
$$\Omega = [-0.15, 0.15].$$

The uncontrolled system has an unstable fixed point at

$$(x_1^*, x_2^*) \sim (2.8, 0.45) \in D$$



C1
C2
D


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
C1
C2
A(D)

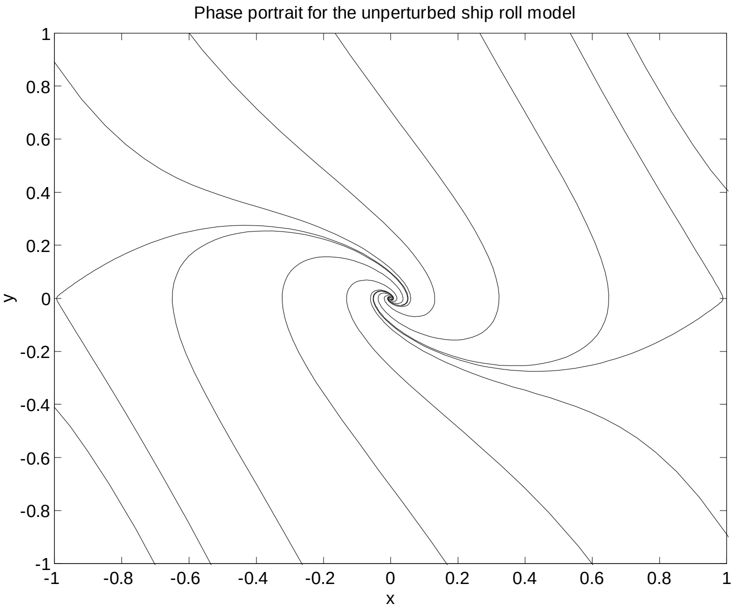
Ship roll motion

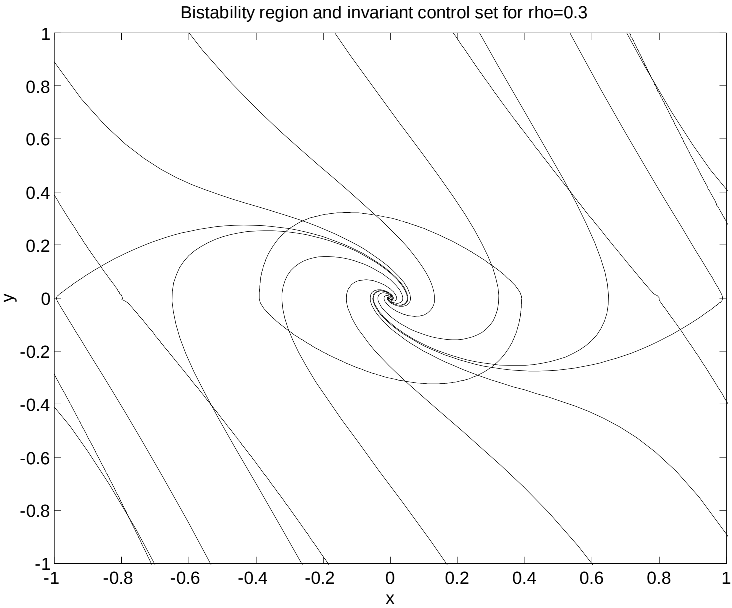
Thompson et al.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \alpha x^3 - \delta_1 y - \delta_2 y\,|y| + u(t) \text{ with } u(t) \in [-\rho, \rho]. \end{aligned}$$

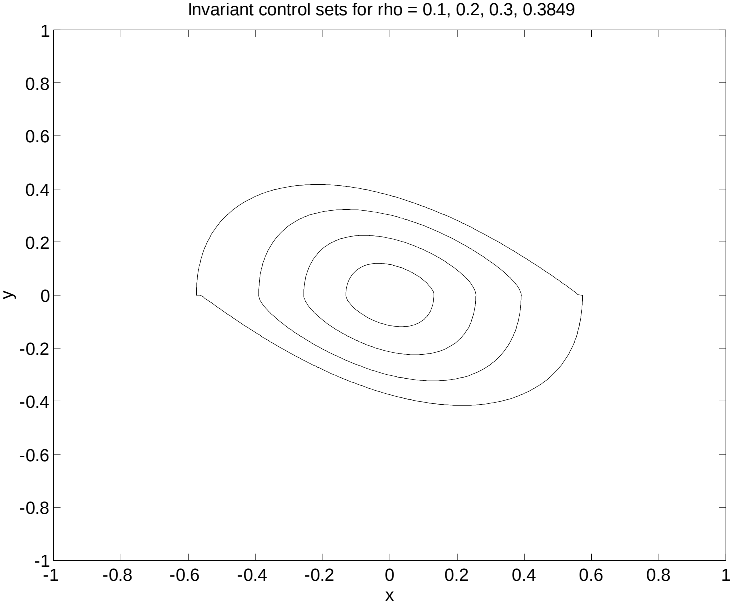
nonlinear oscillator with linear ($\delta_1 > 0$) and quadratic ($\delta_2 > 0$) viscous damping
capsizing occurs for $|x| \geq 1/\sqrt{\alpha}$
choose $\alpha = 1.0, \delta_1 = \delta_2 = 1.0$
For $u \in (-\frac{2}{3}, \frac{2}{3})$ there are three fixed points, one is stable and two are unstable.

We consider the controlled (perturbed!) dynamics with control range $U = [-\rho, \rho]$.
Note the behavior of the control sets for $\rho > 0.3849 < \frac{2}{3}$.


Phase portrait for the unperturbed ship roll model
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x



Bistability region and invariant control set for rho=0.3
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Invariant control sets for rho = 0.1, 0.2, 0.3, 0.3849
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

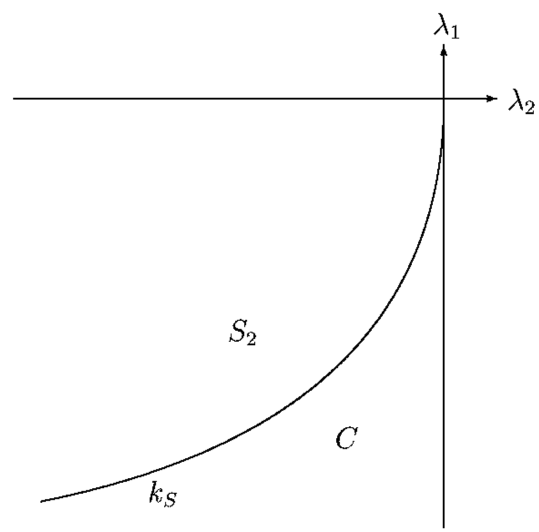
Takens-Bogdanov System

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \lambda_1 + \lambda_2 x + x^2 + xy \end{aligned}$$

Universal unfolding of a singularity with zero as eigenvalue that is geometrically simple and algebraically double
A model for
- motion of a thin panel in a flow field (Holmes),
- nonlinear behavior of solar gravity modes (Merryfield et al.),
- shock wave phenomena (Keyfitz),
- competing species in population dynamics (Burchard)

Depending on λ_1, λ_2 there are two fixed points: a saddle and a stable focus (S_2), or a stable focus with homoclinic orbit (k_S), or two fixed points with unstable limit cycle (C).

bifurcation diagram

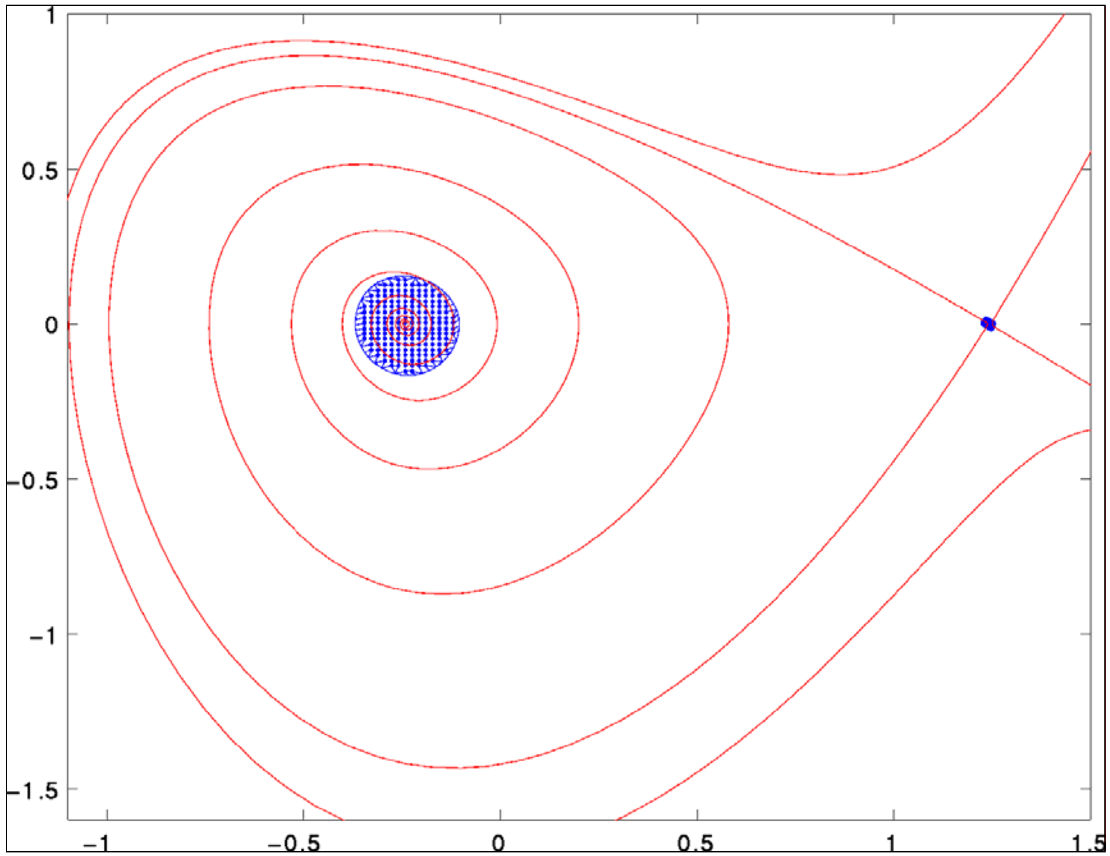


S_2 two fixed points: saddle and stable focus
k_S stable focus, homoclinic orbit
C two fixed points, unstable limit cycle

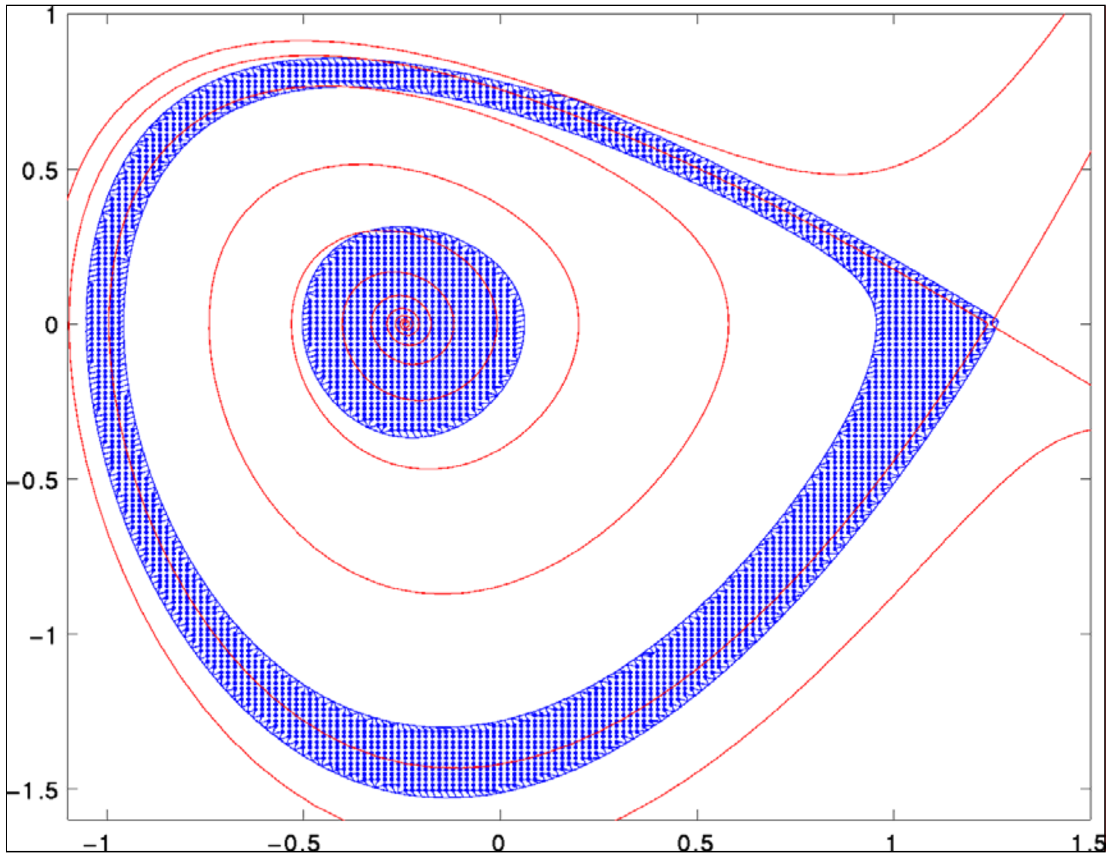
Controlled Takens-Bogdanov system

In area S_2 with $\lambda_1 = -0.3,\ \lambda_2 = -1.0$

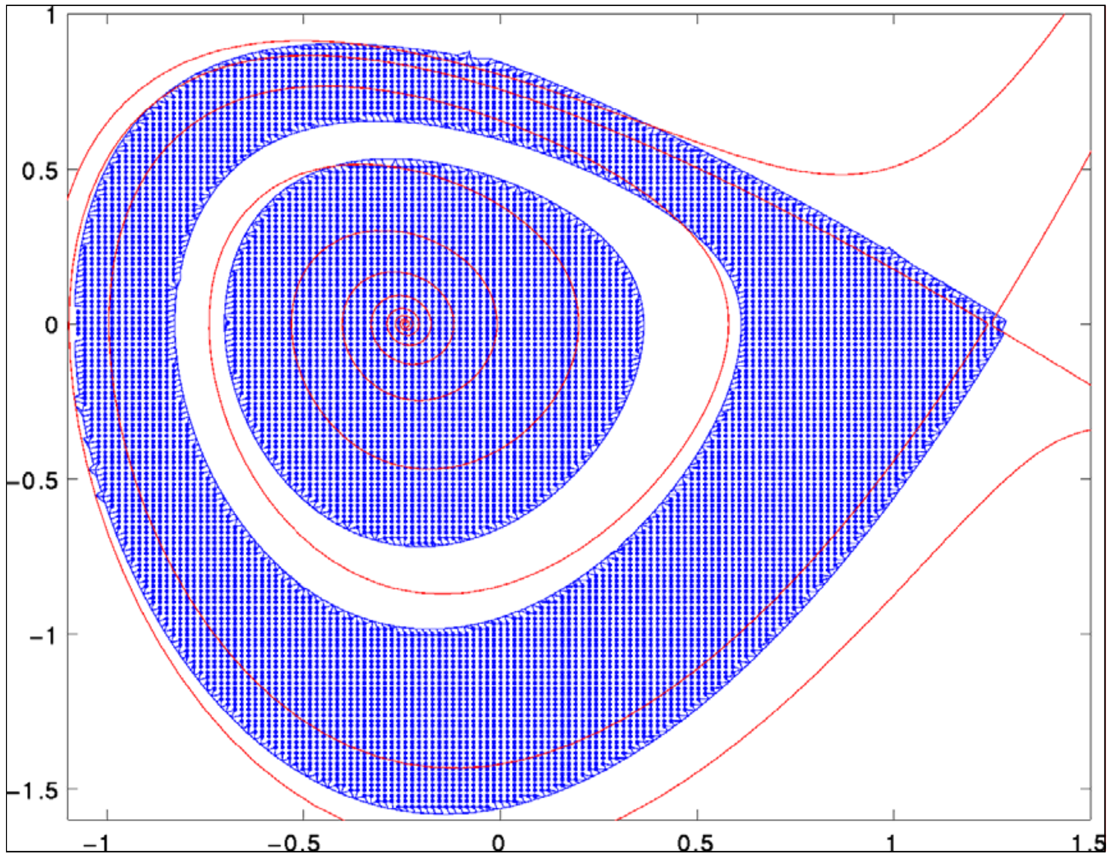
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \lambda_1 + \lambda_2 x + x^2 + xy + u(t) \text{ with } u(t) \in [-\rho, \rho]. \end{aligned}$$



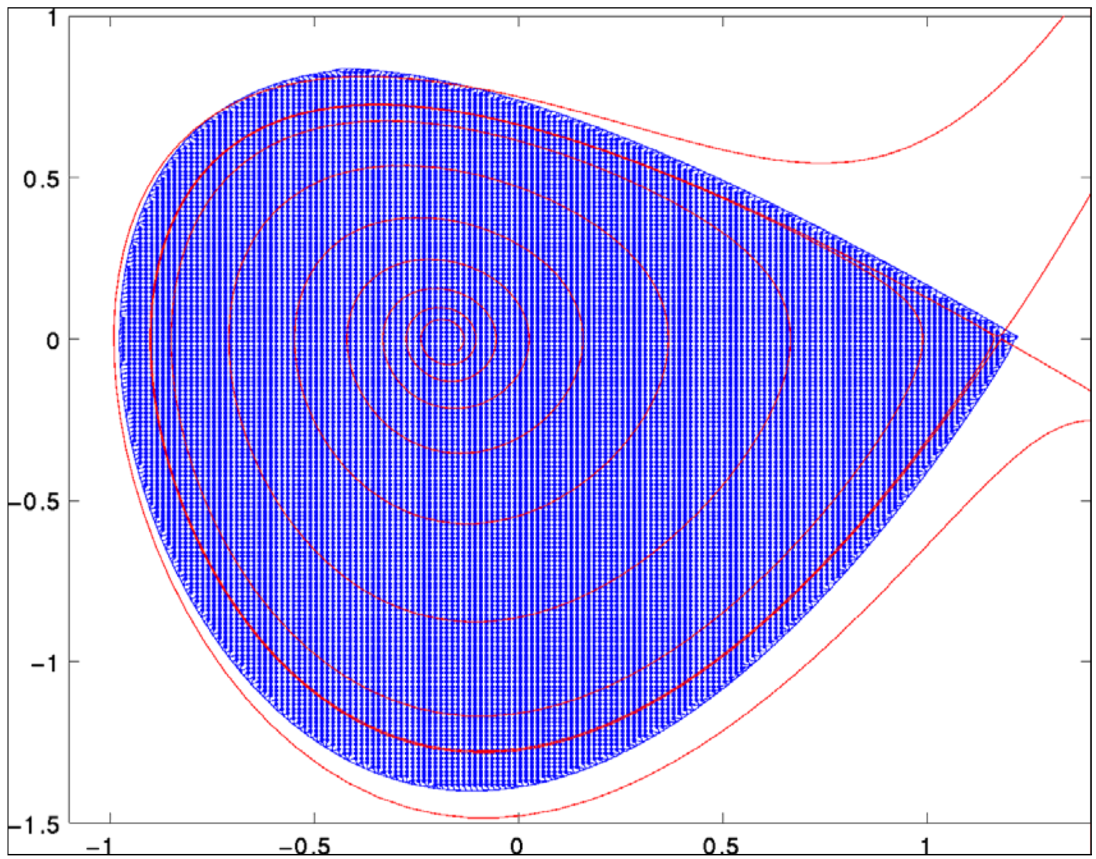
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5



1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5



1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5


1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

Concluding remarks

For linear control systems without control constraints, controllability and observability are determined by linear algebra.

In the presence of control constraints, controllability is exceptional, instead subsets of controllability (control sets) play an important role for linear systems.

For nonlinear systems, Lie-algebraic arguments are used to guarantee local accessibility, which can be used together with control sets.