IMPERIAL COLLEGE LONDON

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M-THEORY AS A M(ATRIX) THEORY, NONCOMMUTATIVE GEOMETRY AND SUPERGRAVITY

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To my parents,

To my brother,

To my MSc buddies, with whom I had very intense conversations about the Nature of Physics.

”Experience is the name everyone gives to their mistakes”.

Oscar Wilde.

Abstract

This paper is a self-contained review of M-theory, with a special focus on its non-perturbative formulation, M(atrix) theory. For this to be accessible to everyone, we start with an overview of string theory and superstrings, from which we need to know the different features to understand the matrix models. We introduce some basics notions of noncommutative geometry, which is used in the construction of M(atrix) theory. We also cover the low energy limit of M-theory: 11D supergravity. Finally, we present some recent applications of the M(atrix) models: noncommutative gravity.

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1 Introduction

String theory first appeared as an attempt to describe hadronic particles, sensitive to the strong interaction. Indeed, the coupling constant of strong interactions increase with the distance between the particles. As an analogy, one can imagine two balls related by a string. As long as we take the particles away, the tension of the string increases with distance, and the balls seem to be attracted by each other, just like the strong interaction coupling constant does with two particles. So the particles were replaced by uni-dimensional objects: strings. Unfortunately, it turns out that the theory contains tachyons (particles with imaginary mass) and spin-2 particles, and that it has to live in a 26-dimensional spacetime. For these reasons, string theory was superseded by Quantum Chromodynamics (QCD), as a gauge theory for the strong interaction., based on the symmetry group $SU(2)$. However, some people noticed that the theory contained the spin-2 graviton, which is the the gauge boson of gravity. Thus, the theory called Bosonic String Theory was thought to be a good candidate as a theory unifying gravity and quantum mechanics. Two problems remained: the presence of tachyonic particles and the absence of fermions in the theory, which are very important since they are the main component of the matter (electrons, quarks...). The solutions to both problems was found in a new symmetry that relates bosons and fermions: Supersymmetry (SUSY). The bosonic string theory evolved to a superstring theory, and, in mid 80's, became the main candidate for a superunification theory, instead of 11-dimensional supergravity, which has been developed in the meantime. Both fermions and bosons were now present in the spectrum, free of tachyon. The dimension of the supersymmetric spacetime was reduced to 10. There are 5 types of such consistent superstrings theories called Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E8 \times E8$, which were thought to be all independent. However, in 1995, it was discovered that they are all related by symmetries called dualities, and are actually limits of one underlying theory living in 11 dimensions: M-theory [59]. We know very little about it, and the full theory is yet to be constructed. The only facts that we know are that it is the strong limit coupling of Type IIA, its low energy limit is the 11-dimensional Supergravity (supersymmetric theory of gravitation) and that it contains no strings but supermembranes, which are extended dimensional objects (hypersurfaces) with one-timelike dimension. M-theory is therefore not a string theory.

The attempts to describe the M-theory in a non-perturbative way can be reduced to two approaches: the AdS/CFT correspondence and the M(atrix) theory.

The AdS/CFT correspondance, where AdS stands for Anti-de Sitter and CFT stands for Conformal Field Theory, states that a 10-dimensional superstring theory involving an Anti-de Sitter spacetime and a 4-dimensional supersymmetric Yang-Mills (SYM) theory, with maximal $\mathcal{N} = 4$ supersymmetry, are equivalent. This equivalence is somehow surprising since it relates a theory which contains gravity to a theory which doesn't, and in particular non-perturbative problems in Yang-Mills theory to problems in classical superstrings or supergravity. Hence, the great advantage of this correspondence is that we might be able to relate the solution of an "easily solvable" problem on one side, to a "harder" problem on the other side.

As useful as this equivalence might be, it doesn't provide a formulation of M-theory. The main candidate for this, are the so-called matrix theories. The first model that has been developed by Banks, Fischler, Shenker and Susskind [5] is called M(atrix) theory or BFSS model. This model is based on a conjecture that grew out from the observation that the D-brane action is similar the 10-dimensional SYM action. First, the theory is compactified in a spacelike direction x^{11} with compactification radius R. The momentum p_{11} is quantized in units of $\frac{1}{R}$. Thus an integer $N = p_{11}R$ is defined. It is argued that in the $N \to \infty$, objects with vanishing and negative p_{11} decouple. Since the only objects in Type IIA which carry p_{11} are the D_0-branes, M-theory in the IMF should be a theory of N D_0-branes in the limit of large N. The exact formulation of the conjecture is:

M-theory in the infinite momentum frame (IMF) is exactly equivalent to the $N \to \infty$ limit of 0-branes supersymmetric matrix quantum mechanics, described by the 10-dimensional $U(N)$ SYM $\mathcal{N} = 1$, reduced to 0+1 dimension,

where the IMF is a frame where the physics has been highly boosted in one direction. According to them, if the conjecture is correct, this would be the first non-perturbative formulation of a quantum theory which includes gravity.

Other models were suggested: the IKKT model, which is obtained by the reduction of 10-dimensional SYM to a point, and is a non-perturbative formulation of Type IIB superstrings. This model is the prime candidate for the emergent noncommutative gravity, discussed later.

The Non-abelian Born-Infield (NBI) model differs slightly from IKKT, by the dynamical degrees of freedom. In the IKKT, the size of the matrices were considered to be variable, whereas in the NBI model, the size is set and only the components of the matrix can fluctuate. This modification was introduced to calculate interaction between D-branes.

M(atrix) theory, provides a formulation of M-theory. If this correct, the compactification of this model should lead to a non-perturbative matrix formulation of type IIA superstrings, since type IIA is obtained from M-theory when compactifed on a circle. The idea is to compactify the 9th dimension, instead of the 11th in M(atrix) theory, to get type IIA. Then, we see that SYM should provide a light-front description of type IIA superstring theory. Because we are now interpreting dimension 9 as the dimension of M-theory which is compactified, the fundamental objects which carry the momentum p are no longer D_0-branes but D_1-branes (or D-strings) with longitudinal momentum $\frac{N}{R}$. Dijkgraaf, Verlinde and Verlinde first argued that 2 dimensional SYM in the large N limit should correspond to light-front IIA superstring theory.

In order to make this paper as self-contained as possible, we start with a review of the bosonic string, superstrings, dualities etc..., so the reader doesn't need to have studied string theory before.

The third chapter presents the different matrix models developed: the BFFS, IKKT and NBI models. After having constructed the actions, we analyse the solutions and we discuss their symmetries. We show how they are related to each other via compactification on a circle, on a torus and on a non-commutative torus. Finally, we talk about the non-perturbative description of type IIA superstrings, developed from the BFSS model by Dijkgraaf, Verlinde and Verlinde.

The fourth chapter, is on noncommutative geometry, which is the geometry of M(atrix) theory. Since this is a rather complicated domain of pure mathematics, we present only some points relevant for our purpose, like the derivation of noncommutative Yang-Mills theory from ordinary Yang-Mills, how they are related by the Seiberg-Witten map and how matrix models can be obtained from them.

As we said before, one of the only things we know about M-theory is its low energy limit 11-dimensional Supergravity. Thus, we present in the fifth chapter, how to construct the Lagrangian of this theory, and we derive the equations of motion for the graviton, the gravitino and the 3-form potential.

Finally, as a conclusion, we talk about recent developments and the applications of the M(atrix) model to cosmology with the presentation of a noncommutative emergent gravity.

An appendix on differential forms can be found at the end, since we use this formalism to simplify the expression of some the equations in supergravity, which are derived in more detail right after.

2 String theories, dualities and D-branes

This section is made for the good understanding of the M(atrix) model. We cover all the fundamental ideas, from classical string theory to D-branes. Finally, an alternative to M(atrix) theory, the AdS/CFT correspondance, the only other non-pertubative aspect of M-theory, is briefly presented.

2.1 The bosonic string

To write the bosonic string action, we use the analogy of the point particle. To do that, we consider p-branes, which are the fundamental p-dimensional objects. A particle would then be a 0-brane, a string a 1-brane and so forth.

2.1.1 The point particle action

The action of a point particle in a D-dimensionnal space time is simply given by:

$$S = -m\int ds = -m\int \sqrt{-\eta_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}}d\tau \tag{1}$$

where the dot represents the derivative respect to τ. The signature of the metric is $(-,+,....,+)$, the indices run between 0 and $D-1$ since we are in D dimensions. A particle sweeps out a line in space time, which is called the particle worldline, as shown in:

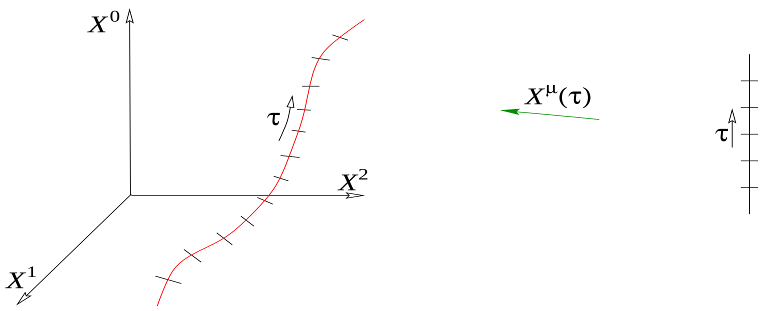

Figure 1: *A particle worldline. The function $X^{\mu}(\tau)$ embeds the worldline in spacetime.*

As we said before, a particle is a 0-brane, so by analogy, we can construct a generalised action for p-brane in a $p+1$-dimensional space time:

$$S = -T_p\int d\mu_p \tag{2}$$

T_p is called the brane tension and it's equal to $\frac{1}{2\pi\alpha'}$ and α' is related to the string length l_s by $\alpha' = \frac{l_s^2}{2}$. $d\mu_p$ is the $(p+1)$-dimensional volume element, which is just a generalization of the point particle 1-dimensional ds, given by:

$$d\mu_p = \sqrt{-h}d^{p+1}\sigma \tag{3}$$

where $h = det(h_{\alpha\beta})$, $h_{\alpha\beta} = \gamma_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ and $\alpha, \beta = 0, ..., p$.

2.1.2 The Nambu-Goto and Polyakov action

Since a string is a 1-brane, it sweeps out a 2-dimensional surface in Minkowski spacetime ($\gamma_{\mu\nu} = \eta_{\mu\nu}$), just like a particle sweeps out a line. This surface is called the world sheet of the string and is parametrised by 2 coordinates $\sigma^0 = \tau$ and $\sigma^1 = \sigma$, as it's shown in:

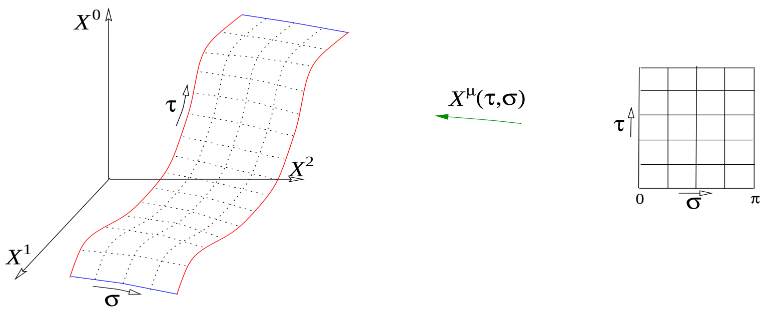

Figure 2: *The string worldsheet. The function $X^\mu(\tau,\sigma)$ embeds the worldsheet in spacetime.*

Using $h_{\alpha\beta} = \gamma_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, we can write $h_{\alpha\beta}$ as:

$$\mathbf{h}_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}X' \\ \dot{X}X' & X' \end{pmatrix}$$

By taking the determinant of it, we can write the action for a string from the p-brane action:

$$S = -T\int d\sigma d\tau \sqrt{(\dot{X}X')^2 - \dot{X}^2X'^2} \tag{4}$$

X' is the derivative respect to σ and $\dot{X}$ is the derivative respect to τ. This action is called the Nambu-Goto action. The problem with this action is that the square root makes the quantization hard. To get get rid of this problem, we use the Polyakov action, which is classically equivalent to the Nambu-Goto action and much easier to quantize. For this, we introduce the worlsheet metric $g_{\alpha\beta}(\sigma,\tau)$. The action then becomes:

$$S = -T\int d\tau d\sigma \sqrt{-g} g^{\alpha\beta} h_{\alpha\beta} = -T\int d\tau d\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{5}$$

We see that the square root isn't here anymore and a new field apeared, $g^{\alpha\beta}$. One should be careful with the notation, $g^{\alpha\beta}$ is the worldsheet metric, whereas $\eta_{\mu\nu}$ is the spacetime metric.

The Polyakov action is invariant under:

• The Poincaré group:

$$X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu \tag{6}$$

• The Weyl transformation:

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = e^\phi g_{\alpha\beta} \tag{7}$$

• A reparametrization of the coordinates $\sigma^\alpha \rightarrow \sigma'^\alpha(\sigma^\alpha)$. This symmetry is a diffeomorphism (an isomorphism for smooth manifolds. It is an invertible function that maps one differentiable manifold to another, such that both the function and its inverse are smooth).

2.1.3 Equations of motion

We are going to solve the equation of motion for the Polyakov action, and show that it is indeed equivalent to the Nambu-Goto action.

First, let's find the equation of motion for X^μ. Using the Euler-Lagrange equation, and noticing that the action depends only on the derivative of X^μ, we have:

$$\partial_\alpha\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}\Big) = 0$$

The equation of motion for X^μ is then:

$$\partial_\alpha(\sqrt{-g} g^{\alpha\beta} \partial_\beta X^\mu) = 0 \tag{8}$$

When we vary the action respect to the induced metric $g^{\alpha\beta}$, we have:

$$\delta S = -T\int d\tau d\sigma \Big[\delta(\sqrt{-g}) g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \sqrt{-g}\delta g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu\Big]\eta_{\mu\nu} \tag{9}$$

Using the identity $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$, we find the equation of motion for $g^{\alpha\beta}$:

$$\partial_\alpha X \partial_\beta X = \frac{1}{2} g_{\alpha\beta} g^{\rho\sigma} \partial_\rho X \partial_\sigma X \tag{10}$$

We can see that if we plug this result back in the Polyakov action, we recover the Nambu-Goto action. This equation gives:

$$T_{\alpha\beta} \equiv \partial_\alpha X \partial_\beta X - \frac{1}{2} g_{\alpha\beta} g^{\rho\sigma} \partial_\rho X \partial_\sigma X = 0 \tag{11}$$

which is the energy-momentum tensor on the (1+1)-dimensional worldsheet. The condition $T_{\alpha\beta} = 0$ is called the Virasoro constraint. It is a conserve current associated to the translation symmetry of the action.

2.1.4 Boundary conditions and solutions

In order to have well defined solutions, we need to precise the boundary conditions. We set the induced metric to be flat (due to a conformal invariance of the action). Since there two types of strings, closed and open, we have different boundary conditions. We want the action to be invariant under the shifts $X^\mu \to X^\mu + \delta X^\mu$. Then, when we vary the action, we have an additional boundary term:

$$\delta S = T \int d\tau d\sigma (\eta^{\alpha\beta} \partial_\alpha \partial_\beta X^\mu) \delta X^\mu - T \int d\tau \left[X'_\mu \delta X^\mu \right]_{\sigma=0}^{\sigma=2\pi} \tag{12}$$

We want this term to vanish, so we have:

<u>Closed strings</u>: They sweeps out a cylinder in spacetime, so the boundary condition should be:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{13}$$

<u>Open strings</u>: Here, there are two choices of boundary conditions;

• *Neumann boundary conditions*: $X'^\mu(\tau,\sigma)\big|_{\sigma=0,2\pi} = 0$. It means that the string can end anywhere in spacetime.

• *Dirichlet boundary conditions*: $\dot{X}^\mu(\tau,\sigma)\big|_{\sigma=0,2\pi} = 0$. Integrating this condition over τ, sets the spacetime location on where the string ends. Therefore, this is equivalent to fixing the endpoints of the string, and we have $\delta X^\mu(\tau,\sigma)\big|_{\sigma=0,2\pi} = 0$. We will see later that the string ends on a D_p-brane, which is a hypersurface with p space-like dimensions, and one time-like dimension.

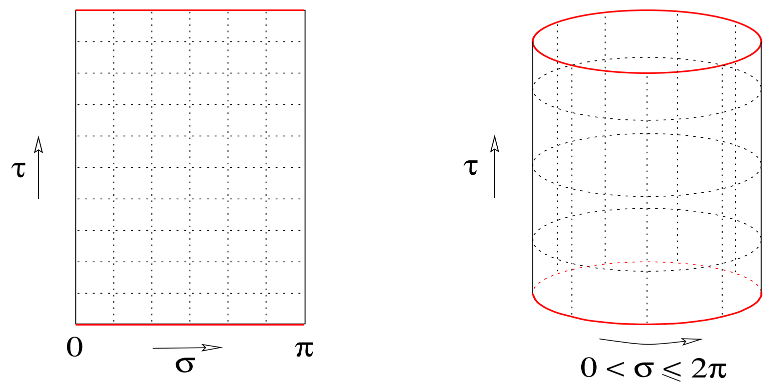

Figure 3: *The worlsheet of an open string (left) and of a closed string (right).*

The solution of the equation of motion $\Box X^\mu = \partial^\alpha \partial_\alpha X^\mu = \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} = 0$, is, in the most general case:

$$X^\mu(\tau, \sigma) = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma) \tag{14}$$

where X_R^μ is a wave moving towards more positive σ and X_L^μ is a wave moving towards more negative σ. For open strings, the left-moving and right-moving waves are related to each other by the boundary conditions at the end points. The closed string has no endpoints, so we work with the periodic conditions defined above. Then to describe properly closed string we need to compactify the worldsheet coordinate σ.

To solve the equations of motions, we introduce the light-cone coordinates $\xi^\pm = \tau \pm \sigma$ and $\partial_\pm = \frac{\partial}{\partial \xi^\pm}$. The equation of motion becomes:

$$\partial_+ \partial_- X^\mu = 0 \tag{15}$$

and the general solution then is:

$$X^\mu(\tau, \sigma) = X_R^\mu(\xi^-) + X_L^\mu(\xi^+) \tag{16}$$

When $\sigma \longrightarrow \sigma + 2\pi$, ξ_+ and ξ_- increase and decrease respectively by 2π. The periodicity condition of the closed string gives:

$$X_R^\mu(\xi^-) + X_L^\mu(\xi^+) = X_R^\mu(\xi^- - 2\pi) + X_L^\mu(\xi^+ + 2\pi) \tag{17}$$

By simply putting the ξ^+ on the right-hand side and the ξ^- on the left-hand side, we have:

$$X_R^\mu(\xi^-) - X_R^\mu(\xi^- - 2\pi) = +X_L^\mu(\xi^+ + 2\pi) - X_L^\mu(\xi^+) \tag{18}$$

The modification is quite simple but we can now see that both waves are dependent of each other. Since the ξ's are independent , it means that both sides of the equation (which are the derivatives of X respect to $\xi_\pm$) must vanish. So, they are periodic functions with period 2π, and we can write the mode expansions:

$$\begin{cases} \frac{dX_L^\mu(\xi^+)}{d\xi^+} = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-\imath n\xi_+} \\ \frac{dX_R^\mu(\xi^-)}{d\xi^-} = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-\imath n\xi_-} \end{cases}$$

Where $\tilde{\alpha}_n^\mu$ and α_n^μ represent the oscillatory modes of the string satisfying:

$$(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \quad , \qquad (\alpha_n^\mu)^* = \alpha_{-n}^\mu$$

When one integrates these equations:

$$\begin{cases} X_R^\mu(\xi^-) = \frac{1}{2}x_0^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu\xi^- + \imath\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-\imath n\xi^-} \\ X_L^\mu(\xi^+) = \frac{1}{2}x_0^\mu + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu\xi^+ + \imath\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-\imath n\xi^+} \end{cases} \tag{19}$$

The constraint given by (18 yields to:

$$2\pi\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu = 2\pi\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu$$

and therefore:

$$\tilde{\alpha}_0^\mu = \alpha_0^\mu \tag{20}$$

The final solution for a closed string is:

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + \imath\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-\imath n\tau}}{n}(\tilde{\alpha}_n^\mu e^{-\imath n\sigma} + \alpha_n^\mu e^{\imath n\sigma}) \tag{21}$$

With the definition of $P^\mu_\alpha(\tau,\sigma) = \frac{\delta S}{\delta \dot{X}_\mu} = T\partial_\alpha X^\mu$, we can relate the momentum P^μ_0 to α^μ_0 with:

$$P^\mu_0(\tau,\sigma) = T\dot{X}^\mu = \int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'}\dot{X}^\mu(\tau,\sigma) = \int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'}(\sqrt{2\alpha'}\alpha^\mu_0 + ...) = \sqrt{\frac{2}{\alpha'}}\alpha^\mu_0 \tag{22}$$

where the dots are the terms that vanish when we integrate. We have the relation

$$\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}P^\mu_0 \tag{23}$$

The solution for the open string is obtained from the closed string solution by imposing an extra condition $X^\mu(\tau,\sigma) = X^\mu(\tau,-\sigma)$. In terms of $\xi^\pm$, we have:

$$\begin{cases} X^\mu_R(\xi^-) = \frac{1}{2}x^\mu_0 - \frac{1}{2}x'^\mu_0 + \sqrt{\frac{\alpha'}{2}}\alpha^\mu_0\xi^- + \imath\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\imath n\xi^-} \\ X^\mu_L(\xi^+) = \frac{1}{2}x^\mu_0 + \frac{1}{2}x'^\mu_0 + \sqrt{\frac{\alpha'}{2}}\alpha^\mu_0\xi^+ + \imath\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\imath n\xi^+} \end{cases} \tag{24}$$

where x'^μ_0 is an arbitrary integration constant. We find:

$$X^\mu(\tau,\sigma) = x^\mu_0 + \sqrt{2\alpha'}\alpha^\mu_0\tau + \imath\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\imath n\tau}\cos(n\sigma) \tag{25}$$

with

$$\alpha^\mu_0 = \sqrt{2\alpha'}P^\mu_0 \tag{26}$$

In the light-cone coordinates, the Virasoro constraints (11) gives for the closed string:

$$T_{+-} = T_{-+} = 0$$

$$T_{++}(\xi^+) = \frac{1}{2}(\partial_+ X^\mu_L)^2 = \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\widetilde{\alpha}_{n-m}\cdot\widetilde{\alpha}_m e^{2\imath n\xi^+} = \sum_{n=-\infty}^{\infty}\widetilde{L}_n e^{2\imath n\xi^+} = 0$$

$$T_{--}(\xi^-) = \frac{1}{2}(\partial_- X^\mu_R)^2 = \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_{n-m}\cdot\alpha_m e^{2\imath n\xi^-} = \sum_{n=-\infty}^{\infty}L_n e^{2\imath n\xi^-} = 0 \tag{27}$$

For the open string, we only have the constraints with untilded quantities.

2.1.5 Quantization and the string spectrum

The pattern we follow to quantize the bosonic string is similar to the one used ion quantum field theory, for the quantization of the scalar field. With (22), we calculate the Poisson bracket:

$$\left\{P^\mu(\sigma,\tau),P^\nu(\sigma',\tau)\right\}=\left\{X^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\right\}=0 \tag{28}$$

$$\left\{P^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\right\}=\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{29}$$

In terms of $\tilde{\alpha}_n^\mu$ and α_n^μ, we have:

$$\left\{\tilde{\alpha}_m^\mu,\tilde{\alpha}_n^\nu\right\}=\left\{\alpha_m^\mu,\alpha_n^\nu\right\}=\imath m\eta^{\mu\nu}\delta_{m+n,0} \tag{30}$$

$$\left\{\alpha_m^\mu,\tilde{\alpha}_n^\nu\right\}=0 \tag{31}$$

Now that we have the Poisson brackets, all we need to quantized the theory is to replace them by commutators as $\{\cdots\}\rightarrow\imath\,[\cdots]$. By defining the annihilation operator $a_m^\mu=\frac{1}{\sqrt{m}}\alpha_m^\mu$ and the creation operator $a_m^{\dagger\mu}=\frac{1}{\sqrt{m}}\alpha_{-m}^\mu$ like in classical quantum mechanics, we have:

$$\left[a_m^\mu,a_n^{\dagger\nu}\right]=\left[\tilde{a}_m^\mu,\tilde{a}_n^{\dagger\nu}\right]=\eta^{\mu\nu}\delta_{m,n} \tag{32}$$

$$\left[X_\alpha^\mu,P_\beta^\nu\right]=\imath\eta^{\mu\nu}\delta_{\alpha\beta} \tag{33}$$

Since this is an ensemble of harmonic oscillators, the Fock space is built by applying the creation operator on the ground state satisfying:

$$\langle 0|0\rangle=1$$

$$a_m^\mu|0\rangle=0$$

$$|n\rangle=\frac{((a_m^\mu)^\dagger)^n}{\sqrt{n!}}|0\rangle$$

and

$$a_m^\mu(a_m^\mu)^\dagger=n|n\rangle$$

$$\langle m|n\rangle = \delta_{nm}$$

There is however a problem in the construction when we look at $\left[a_m^0, a_m^{\dagger 0}\right] = -1$. This creates states with negative norms, which make the theory inconsistent since we would have a non-unitary theory (with negative probabilities). The Visaroso constraints get ride of these states in the string spectrum. One can also specify the momentum k^μ which is the eigenvalue of P_0^μ, carried by the state:

$$|\phi\rangle = a_{n_1}^{\dagger\nu_1} a_{n_2}^{\dagger\nu_2} ... a_{n_n}^{\dagger\nu_n} |k;0\rangle$$

We now express the Hamiltonian in terms of L and $\widetilde{L}$. It is given by:

$$H = \int_0^\pi d\sigma(\dot{X}_\mu P_0^\mu - \mathcal{L}) = \frac{T}{2}\int_0^\pi d\sigma(\dot{X}^2 + X'^2) \tag{34}$$

When we insert the mode expansions we find for the closed string:

$$H = \sum_{n=-\infty}^{\infty} (\alpha_{-n}\cdot\alpha_n + \widetilde{\alpha}_{-n}\cdot\widetilde{\alpha}_n) = \widetilde{L}_0 + L_0 \tag{35}$$

and for the open string:

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty} (\alpha_{-n}\cdot\alpha_n) = L_0 \tag{36}$$

The Visaroso operators L_m in quantum theory are defined by their normal-ordered expression (we place all lowering operators to the right):

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} : \alpha_{m-n}\cdot\alpha_n :$$

Because of (30), only $L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n$ is affected by the normal ordering. The commutator of two oscillators is a (positive) constant. Therefore, the general form of a quantum version of L_0 will defer from the normal ordered one by a constant a: $L_0 \longrightarrow L_0 - a$. We now define the physical states $|phys\rangle$ of the full Hilbert space, the states which obey the quantum version of the Visaroso constraint, and we have:

$$(L_0 - a)|phys\rangle = 0 \tag{37}$$

$$L_n|phys\rangle = 0 \tag{38}$$

The Lorentz invariance implies that the massive modes form a representation of the Lorentz group $SO(D-1)$ and the massless modes form a representation of the Lorentz group $SO(D-2)$.

• The open string spectrum

The mass of the open string is given by $m^2 = -P_0^2$. With (37), (26), and the expression of L_0, we get:

$$m^2 = -P_0^2 = -\frac{1}{2\alpha'}\alpha_0^2 = \frac{1}{\alpha'}\Big(\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n - a\Big) = \frac{1}{\alpha'}(N-a) \tag{39}$$

where $N \in \mathbb{N}$ is called the level number.

Ground state: $N = 0$. This state is realized when all the oscillators are in vaccum and it's given by $|k;0\rangle$. The mass of this state is:

$$m^2 = -\frac{a}{\alpha'} \tag{40}$$

This state describes a tachyon (a particle travelling faster than the speed of light), since the mass is imaginary. This is where we see that the bosonic string is not a consistent theory since the vaccum is unstable.

First excited state: $N = 1$. The only way to get the first excited state is to apply α^{μ}_{-1} on the vacuum:

$$|k\rangle = \alpha^{\mu}_{-1}|k;0\rangle$$

The mass is given by

$$m^2 = \frac{1}{\alpha'}(1-a) \tag{41}$$

The index μ takes its values in the $(D-2)$ transverse coordinates (see 3.6). Thus, the state $\alpha^{\mu}_{-1}|k;0\rangle$ belongs to a vector representation of $SO(24)$. Since Lorentz invariance requires the mass to be equal to zero, we find $a = 1$.

Second excited state: $N = 2$. We can obtain this state by either:

$$|k_1\rangle = \alpha^{\mu}_{-2}|k;0\rangle$$

which has 24 states, or

$$|k_2\rangle = \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|k;0\rangle$$

which has $\frac{24\cdot 25}{2} = 300$ states. The total number of states is the number of states of a traceless second-rank tensor of $SO(25)$, which correspond to a $spin = 2$ particle. Since $a = 1$, any state with $N \geqslant 2$ have positive mass.

• The closed string spectrum

The spectrum of the right-moving modes of closed string is the same as the open string and the closed string states are tensor products of left-movers and right-movers. For the left-moving modes, we have the extra condition:

$$(L_0 - a)|phys\rangle = (\tilde{L}_0 - a)|phys\rangle = 0 \tag{42}$$

Adding and subtracting them, gives the quantum constraints:

$$(L_0 + \tilde{L}_0 - 2)|phys\rangle = 0$$

$$(L_0 - \tilde{L}_0)|phys\rangle = 0 \tag{43}$$

where we set $a = 1$. This lead us to the mass relation:

$$m^2 = \frac{4}{\alpha'}(N - 1)$$

with

$$N = \widetilde{N} \tag{44}$$

Ground state: $N = 0$. The ground state $|k; 0\rangle$ has a mass:

$$m^2 = -\frac{4}{\alpha'}$$

which is again a tachyon.

First excited state: $N = 1$. The state which is massless, is:

$$|k\rangle = \alpha^{\mu}_{-1}|k; 0\rangle \otimes \tilde{\alpha}^{\nu}_{-1}|k; 0\rangle \tag{45}$$

It has $24^2 = 576$ states. This tensor contain a symmetric and traceless part that transforms under $SO(24)$. It is a massless $spin = 2$ particle: the graviton.

There is two big problems with the boconic string;

Firstly, both open and closed string spectrum contain a tachyon, which violate causality and unitarity. Secondly none of the spectrum contain fermion, which is a bit of a problem since fermions are very important in physics.

Then, one need a new symmetry to make fermions appear from bosons. This is where supersymmetry arise, and therefore we define supersymmetric strings: superstrings.

2.2 Superstrings

Supersymmetry (SUSY), although it hasn't been observed yet, is a very powerful symmetry. Indeed, it makes string theory consistent since the spectrum of the strings contain fermions, and is free of tachyon. SUSY, which has been first discovered in the context of string theory and then adapted to four-dimensional particles, was introduced in two different equivalent ways[1]:

• The Ramond-Neveu-Schwarz (RNS) formalism, which is the original approach and which uses two-dimensional world-sheet supersymmetry. In this case, the world-sheet action takes the form:

$$S = -\frac{T}{2}\int d^2\xi(\partial_a X^\mu \partial^a X_\mu - \imath\overline{\psi}^\mu \rho^a \partial_a \psi_\mu)$$

This is the approach we will use afterwards to calculate the mode expansions and the spectrum of superstrings.

• The Green-Schwarz (GS) formalism, which uses a map that embeds the string world sheet into superspace instead of just spacetime in the bosonic string. Therefore, the advantage is to make the spacetime supersymmetry obvious since the superspace is supersymmetric by definition, and it doesn't require the GSO projection, needed in the RNS superstring to get ride of the tachyons. However, the quantization of the theory is much harder and hasn't been totally fulfilled yet.

2.2.1 The type of superstrings

There are five different types of superstring theories, all living in 10 dimensions. Type I and both heterotic theories have $\mathcal{N}=1$ SUSY whereas types IIA and IIB have $\mathcal{N}=2$ SUSY.

[1]It might be possible that the two approaches are actually not exactly equivalent

• Type I: This is the first superstrings theory. It is only one which contains both closed and open strings, but they aren't oriented (they have the same chirality). The symmetry gauge group is $SO(32)$.

• Type IIA: For closed strings, there are two ways to choose the chiralities of the left and right moving modes. If we choose the chiralities to be of opposite signs, then we have type IIA. It contains D_p-branes with p even. it is the strong limit coupling of Type IIA

• Type IIB: This is obtained by chosing the same chirality for the modes. It contains D_p-branes with p odd.

• Heterotic $E_8 \times E_8$ and $SO(32)$: The heterotic strings theories are the most promising in discribing the physical world. The original thing here is that it uses the formalism of both 26-dimensional bosonic string for the left-moving modes, and 10-dimensional superstrings for the right moving modes. The gauge groups naturally appear when we compactify the extra dimensions. The only possible tori which have the required properties for the theory to be consistent must have the Lie algebra $E_8 \times E_8$ or $SO(32)$.

2.2.2 Dualities

• T-duality: This duality, which is a perturbative duality, relates two different theories that were thought to be unrelated, by saying that the geometry of the extra dimensions are physically equivalent. Every theories that are related by this duality should actually be considered as only one. It relates R to $\tilde{R} = \frac{l_s^2}{R}$. For open strings, it interchanges the usual Neumann boundary conditions with Dirichlet boundary conditions. The types of superstring related by the T-duality are:

$$T : IIA \leftrightarrow IIB$$

$$T : E_8 \times E_8 \leftrightarrow SO(32)$$

• S-duality: Also called Strong/weak duality, it is a duality that relates the string coupling constant g_s to $\tilde{g}_s = \frac{1}{g_s}$. This is therefore a non-perturbative duality and allows to get non-perturbative results from a perturbative analysis. The different theories related by S-duality are:

$$S : IIA \leftrightarrow E_8 \times E_8$$

$$S : I \leftrightarrow SO(32)$$

$$S : IIB \leftrightarrow IIB$$

We show in the next chapter, how T-duality arises from closed and open strings, and how in the case of open strings, it implies the existence of D-branes.

2.3 RNS superstring

The study of superstring theory is not much different from the bosonic string, and we are going to follow the exact same pattern.

I recall the action of the RNS superstring given earlier:

$$S = -\frac{T}{2} \int d^2\xi (\partial_a X^\mu \partial^a X_\mu - \imath \overline{\psi}^\mu \rho^a \partial_a \psi_\mu) \tag{46}$$

where $\rho^a (a = 0, 1)$ are 2-dimensional Dirac matrices and satisfy the algebra $\{\rho^a, \rho^b\} = 2\eta^{ab}$. We have included Majorana spinors which belong to the representation of the Lorentz group $SO(1, 9)$ (since we are in 10 dimensions). This action is invariant under translations and supersymmetry transformations:

$$\delta_\epsilon X^\mu = \overline{\epsilon} \psi^\mu$$

$$\delta_\epsilon \psi^\mu = \rho^a \partial_a X^\mu \epsilon \tag{47}$$

where ϵ is the parameter of the supersymmetry group transformations and is an infinitesimal Majorana spinor.

The bosonic fields X^μ still possess the commutation relation (29), and the spinors statisfy the anti-commutation relation $\{\psi^\mu, \psi^\nu\} = 0$. The spinor ψ is a Majorana spinor and has two components

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}$$

We have the condition $\psi^* = \psi$ in order to keep the action real. The fermionic part of the action written in terms of the spinor components is:

$$\overline{\psi}^{\mu}\rho^{a}\partial_{a}\psi_{\mu}=\psi_{-}^{\mu}\partial_{+}\psi_{-}^{\mu}+\psi_{+}^{\mu}\partial_{-}\psi_{+}^{\mu} \tag{48}$$

The equation of motion are in fact the massless Dirac equations:

$$\partial_{+}\psi_{-}^{\mu}=\partial_{-}\psi_{+}^{\mu}=0 \tag{49}$$

where ψ_{-}^{μ} is the the right-mover and ψ_{+}^{μ} is the the left-mover and satisfy the anticommutation relation:

$$\{\psi_{\pm}^{\mu}(\sigma,\tau),\psi_{\pm}^{\nu}(\sigma',\tau)\}=\pi\eta^{\mu\nu}\delta_{(+,-)}\delta(\sigma-\sigma') \tag{50}$$

Like the bosonic string, we have the Virasoro constraints. We still have the current associated to the translation invariance, but since we have a new (super)symmetry, we have an additional current , associated to the invariance of the action under supersymmetry transformations (47). Their expressions are:

$$\begin{gathered}T_{ab}=\partial_{a}X^{\mu}\partial_{b}X^{\mu}+\frac{1}{4}\overline{\psi}^{\mu}\rho_{a}\partial_{b}\psi_{\mu}+\frac{1}{4}\overline{\psi}^{\mu}\rho_{b}\partial_{a}\psi_{\mu}-(trace)\\ J_{a}=\frac{1}{2}\rho^{b}\rho_{a}\psi_{\mu}\partial_{b}X^{\mu}\end{gathered} \tag{51}$$

When we use the light cone coordinates, we find T_{+-} and T_{-+} identically equal to zero and the non-zero components of both currents are:

$$\begin{cases}T_{++}=\partial_{+}X\partial_{+}X+\frac{i}{2}\psi_{+}\partial_{+}\psi_{+}=0\\ T_{--}=\partial_{-}X\partial_{-}X+\frac{i}{2}\psi_{-}\partial_{-}\psi_{-}=0\end{cases} \tag{52}$$

$$\begin{cases}J_{+}=\psi_{+}^{\mu}\partial_{+}X_{\mu}=0\\ J_{-}=\psi_{-}^{\mu}\partial_{-}X_{\mu}=0\end{cases} \tag{53}$$

2.3.1 Boundary conditions and solutions

Lets now find the expression of the mode expansion of the strings. When we vary the action (46) respect to ψ_- and ψ_+, we find that it vanishes if they respect the Dirac equation given above and we have the condition:

$$\delta S = \int d^2\sigma \big[\psi_+\delta\psi_+ - \psi_-\delta\psi_-\big]_{\sigma=2\pi} - \big[\psi_+\delta\psi_+ - \psi_-\delta\psi_-\big]_{\sigma=0} = 0 \tag{54}$$

Open strings. There are two possibilities, corresponding to the two ends of the strings:

$$\psi_+^\mu(\sigma,\tau) = \pm\psi_-^\mu(\sigma+2\pi,\tau) \tag{55}$$

The sign is a matter of convention, so we can set the sign at $\sigma = 0$ to be $+$. At the end of the string, there are still two possible choices:

- *Ramond (R) sector*: $\psi_+^\mu|_{\sigma=2\pi} = \psi_-^\mu|_{\sigma=2\pi}$. This gives the fermions.
- *Neveu-Schwarz (RS) sector*: $\psi_+^\mu|_{\sigma=2\pi} = -\psi_-^\mu|_{\sigma=2\pi}$. This gives the bosons.

Given the two different types of boundary conditions, we then have two different ways to expand the spinor fields in Fourier series:

$$R : \begin{cases} \psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}} b_r^\mu e^{-\imath r(\tau-\sigma)} \\ \psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}} b_r^\mu e^{-\imath r(\tau+\sigma)} \end{cases} \tag{56}$$

$$NS : \begin{cases} \psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-\imath r(\tau-\sigma)} \\ \psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-\imath r(\tau+\sigma)} \end{cases} \tag{57}$$

Closed strings. The boundary conditions, like the open string, give two fermionic modes; left-moving sector ψ_+^μ and right-moving sector ψ_-^μ. There are two possibilities for periodic conditions to make the boundary term (54) vanish:

$$\psi_\pm^\mu(\sigma,\tau) = \pm\psi_\pm^\mu(\sigma+2\pi,\tau) \tag{58}$$

A positive sign gives periodic boundary condtions whereas a negative sign gives antiperiodic boundary conditions. We can decide to give the left or right mover either periodic (R) or antiperiodic (RS) conditions. Thus we have for the right-mover of the closed string:

$$or \begin{cases} \psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}} b_r^\mu e^{-2\imath r(\tau-\sigma)} & (R) \\ \psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2\imath r(\tau-\sigma)} & (NS) \end{cases} \tag{59}$$

and for the left-mover:

$$or \begin{cases} \psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}} \tilde{b}_r^\mu e^{-2\imath r(\tau+\sigma)} & (R) \\ \psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2\imath r(\tau+\sigma)} & (NS) \end{cases} \tag{60}$$

Since we can pair together any of the sectors, we have four different combinations for the closed string, and their correspondant particle states:

R-R: Bosonic

NS-NS: Bosonic

R-NS: Fermionic

NS-R : Fermionic

2.3.2 Quantization and the superstring spectrum

Now we need to do the canonical quantization of the superstring since we have only classical supersymmetric strings. The procedure is similar to the bosonic string quantization. The oscillatory modes in the expansion obey the same comutation relation (32) as the bosonic string. Similarly, for the fermions the anticommutation relation (50) becomes in terms of the oscillatory modes b_r^μ:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{(r+s,0)} \tag{61}$$

One should be careful with the (same) notation we've been using and not be confused with the Poisson brackets for classical bosonic fields, and anticommutators for both classical and quantized spinor fields.

One can also notice that, like in the bosonic string, we have negative norm states because of the spacetime metric ($\eta^{00} = -1$).

We now calculate the expressions of the constraints (52) and (53) in terms of the oscillatory mode a_m^μ and b_r^μ:

$$\begin{cases} T_{++} = \displaystyle\sum_{n=-\infty}^{\infty} \Big[\frac{1}{2}\sum_{m=-\infty}^{\infty} \widetilde{\alpha}_{n-m}\cdot\widetilde{\alpha}_m + \frac{1}{4}\sum_r (2r-n)\widetilde{b}_{n-r}\cdot\widetilde{b}_r\Big] e^{-2\imath n\xi^-} = \sum_{n=-\infty}^{\infty} \widetilde{L}_n e^{-2\imath n\xi^-} \\ T_{--} = \displaystyle\sum_{n=-\infty}^{\infty} \Big[\frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}\cdot\alpha_m + \frac{1}{4}\sum_r (2r-n) b_{n-r}\cdot b_r\Big] e^{-2\imath n\xi^-} = \sum_{n=-\infty}^{\infty} L_n e^{-2\imath n\xi^-} \end{cases} \tag{62}$$

$$\begin{cases} J_{+} = \displaystyle\sum_r \Big[\sum_{m=-\infty}^{\infty} \widetilde{\alpha}_m\cdot\widetilde{b}_{r-m}\Big] e^{-2\imath r\xi^-} = \sum_r \widetilde{G}_r e^{-2\imath r\xi^-} \\ J_{-} = \displaystyle\sum_r \Big[\sum_{m=-\infty}^{\infty} \alpha_m\cdot b_{r-m}\Big] e^{-2\imath r\xi^-} = \sum_r G_r e^{-2\imath r\xi^-} \end{cases} \tag{63}$$

This is used to determine the string mass like in the bosonic string case. The physical states $|phys\rangle$ satisfy the conditions:

$$(L_0 - a)|phys\rangle = 0$$

$$L_n|phys\rangle = 0$$

$$G_r|phys\rangle = 0 \tag{64}$$

where a is equal to 0 in the R sector and $\frac{1}{2}$ in the NS sector. The first one gives the mass:

$$m^2 = \frac{1}{\alpha'}\Big[\Big(\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{r>0} r b_{-r}\cdot b_r\Big) - a\Big] = \frac{1}{\alpha'}(N-a) \tag{65}$$

To construct the spectrum, we have to consider the R and NS sectors independently. The ground state $|0\rangle$ must be annihilated by the annihilation operator in both sectors:

$$\alpha_m^\mu|0\rangle = b_r^\mu|0\rangle = 0 \qquad for\ r, m > 0 \tag{66}$$

Open string spectrum in the NS sector

Ground state: $N = 0$. The ground state has a mass:

$$m^2 = -\frac{1}{2\alpha'} \tag{67}$$

which is a tachyon. It is a unique ground state which corresponds to a state of spin 0. The excited states are obtained by acting raising operators and are also bosons.

First excited state: $N = \frac{1}{2}$. The reason why this state is $N = \frac{1}{2}$ and not 1 is because to construct the first excited state we need to apply the raising operator with the smallest value r or n. Then the good operator is $b^\mu_{-1/2}$ and the state is massless. It is a vector of $SO(8)$.

Open string spectrum in the R sector

With (61) we get $\{b^\mu_0, b^\nu_0\} = \eta^{\mu\nu}$, which is similar to the 10-dimensional Dirac algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$, up to factor 2. Hence, We define $b^\mu_0 = \frac{1}{\sqrt{2}}\Gamma^\mu$ and conclude that all the states that these operators act on are 10-dimensional spinors and then are fermions.

Ground state: $N = 0$. To analyze the ground state, we have to take into account the supercurrent constraint J_- (63) and the last condition in (64). The zero mode part of it ($r = 0$) obviously obeys:

$$G_0|phys\rangle = 0$$

with $G_0 = \alpha_0 \cdot b_0 + \sum_{n\neq 0} \alpha_{-n} \cdot b_n$. Since we have identified b^μ with Γ^μ and α^μ_0 with p^μ_0, and since the physical fermionic ground state $|0_f\rangle$ is defined by (66), we find that it satisfies the 10-dimensional massless Dirac equation:

$$\alpha_0 \cdot b_0|0_f\rangle = 0 \tag{68}$$

Then, the fermionic ground state is a massless Dirac spinor in 10 dimensions. The ground state in the R sector is a 32-component spinor since b_0 is a 32×32 matrix. In ten dimensions, one can impose both Majorana (spinor equal to his complex conjugate) and Weyl condition on spinors. Then, there are two different ground states which have two possible chiralities and then is degenerate. This condition gives rise of two different theories as we will see afterwards.

First excited state: $N = 1$. Here since the value of r and n are the same, the first excited state is obtained by applying either α_{-1} or b_{-1} on the ground state. The state has a mass $m^2 = \frac{1}{\alpha'}$.

2.3.3 The GSO projection

We said at the start of this section that supersymmetry get rid of the problems encountered in the bosonic string. This is not quite done yet because although we do have fermions and bosons, the ground state of the NS sector is a tachyon. Besides, one can see that the spectrum is not spacetime supersymmetric since there is no fermion with the same mass as the tachyon. An operation which consists of projecting the spectrum in a particular allow us to remove the tachyon from the spectrum. This operation is called the $GSO\ projection$ noted P_{GSO} was introduced by Gliozzi, Schrek and Olive. The physical states $|phys\rangle$ are replaced by $P_{GSO}|phys\rangle$.

<u>NS sector</u>

In this sector, the projector is defined with:

$$P_{GSO} = \frac{1}{2}\Big(1 - (-1)^F\Big) \tag{69}$$

where $F = \displaystyle\sum_{r=1/2}^{\infty} b_{-r} \cdot b_r$ is called fermion number operator. This is different from zero only if F is odd. Then, the projector keep only states with an odd number of b's and remove those with an even number. The ground state which has no an even number F is eliminated, and the first excited state becomes the ground state with $m^2 = 0$.

<u>R sector</u>

The expression of the projector is the same as before, but the definition of F changes:

$$P_{GSO} = \frac{1}{2}\Big(1 - (-1)^F\Big) \tag{70}$$

where $(-1)^F = \pm\Gamma^{11} \cdot (-1)^{\sum_{r\geqslant 1} b_{-r}\cdot b_r}$ is called Klein operator and $\Gamma^{11} = \Gamma^0\Gamma^1...\Gamma^9$, $(\Gamma^{11})^2 = 1$ and $\{\Gamma^\mu, \Gamma^{11}\} = 0$. The spinors that satisfy:

$$\Gamma^{11}\psi = \pm\psi \tag{71}$$

are said to have positive or negative chirality and the chirality operator is $P_\pm = 1 \pm \Gamma^{11}$.

The "new" ground state (NS) boson $b^\mu_{-1/2}|0\rangle$ is massless and has $D - 2 = 8$ polarizations (corresponding to the number of transverse dimensions in the DLCQ see 3.6). The ground state (R) fermion

$|0_f\rangle$ is a massless Majorana-Weyl spinor and has $\frac{1}{4}2^{D/2} = 8$ polarisations (a fermion in D dimensions has $2^{D/2}$ components and the Majorana and Weyl conditions each divide the number of components by 2). Hence the number of fermions and bosons are the same, as required by supersymmetry, have the same mass and then form a supermultiplet.

Closed string spectrum

As mentioned before, there are 4 possible sectors: $NS-NS$, $NS-R$, $R-NS$ and $R-R$. We can choose the right and left moving R sectors ground sates to have the same or opposite chirality. This correspond to 2 different theories:

• Type IIB: We define the type IIB theory with the left and right moving sector ground states to have the same chirality, chosen to be positive. Therefore the two R sectors have the same parity. Let us denoted them by $|0_f\rangle_R^+$. In this case, the ground state (massless) in the type IIB closed string spectrum are given in the $R-R$ sector, by the tensor product of the ground state in the R sector with itself:

$$|0_f\rangle_R^+ \otimes |0_f\rangle_R^+ \tag{72}$$

In the $NS-NS$ sector, the ground state is the tensor product of the "new" ground state $\widetilde{b}^i_{-1/2}|0\rangle_{NS}$ (after the GSO projection got rid of the tachyon) with itself:

$$\widetilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS} \tag{73}$$

In the $NS-R$ sector, the ground state is the tensor product of the "new" ground state in the NS sector and the ground state in the R sector:

$$\widetilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |0_f\rangle_R^+ \tag{74}$$

In the $R-NS$ sector, the ground state is the tensor product of the ground state in the R sector and the "new" ground state in the NS sector:

$$|0_f\rangle_R^+ \otimes b^i_{-1/2}|0\rangle_{NS} \tag{75}$$

All these states are massless.

• Type IIA: The left and right moving R sector ground states are chosen to have opposite chiralities. The massless states in the spectrum are in the $R-R$ sector the tensor product of the two ground state both in the R sector, but with opposite chirality:

$$|0_f\rangle_R^- \otimes |0_f\rangle_R^+ \tag{76}$$

In the $NS-NS$ sector, the ground state is the same as in type IIB:

$$\widetilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS} \tag{77}$$

In the $NS-R$ sector, the ground state is the tensor product of the "new" ground state in the NS sector and the ground state in the R sector with positive chirality.

$$\widetilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |0_f\rangle_R^+ \tag{78}$$

In the $R-NS$ sector, the ground state is the tensor product of the ground state in the R sector with negative chirality, and the "new" ground state in the NS sector:

$$|0_f\rangle_R^- \otimes b^i_{-1/2}|0\rangle_{NS} \tag{79}$$

The massless spectrum of each Type II closed string contain 2 Majorana-Weyl gravitinos and therefore they form $\mathcal{N}=2$ multiplets. There are 64 states in each of the four massless sectors:

• **NS-NS**: This sector is the same for both type IIA and IIB. The spectrum contains a scalar called dilaton (one state), an antisymmetric 2-form gauge field ($n = \frac{8\times7}{2} = 28$ states) and a symmetric traceless rank-2 tensor, the graviton ($n = \frac{10\times7}{2} = 35$ states).

• **NS-R** and **R-NS**: Each of these sectors contain a spin $3/2$ gravitino ($n = \frac{8\times7\times6}{6} = 56$ states) and a spin $1/2$ fermion called dilatino (8 states). In IIB, the 2 gravitinos have the same chirality, whereas in IIA they have opposite chirality.

•**R-R**: These states are bosons, obtained by tensoring a pair of Majorana-Weyl spinors. In the IIA case, the 2 Majorana-Weyl spinors have opposite chirality and one obtains a 1-form (vector) gauge field (8 states) and a 3-form gauge field (56 states). In the IIB case the 2 Majorana-Weyl spinors have the same chirality and one obtains a 0-form (scalar) gauge field (1 state), a 2-form gauge field (28 states) and a 4-form gauge field with a self-dual field strength (35 states). Because of the self duality of the field strength, the number of states is divided by 2.

Everything is summarize in the following:

	NS-NS	R-R
IIA	$g_{\mu\nu}, \phi, B_{\mu\nu}$	$A^{(1)}_{\mu}, A^{(3)}_{\mu\nu\rho}$
IIB	$g_{\mu\nu}, \phi, B_{\mu\nu}$	$A^{(0)}, A^{(2)}_{\mu\nu}, A^{(4)}_{\mu\nu\rho\sigma}$

2.4 M-theory

2.4.1 Relations to superstrings theories and supergravity

The following drawing shows pretty much everything we know about M-theory which is an 11-dimensional theory.

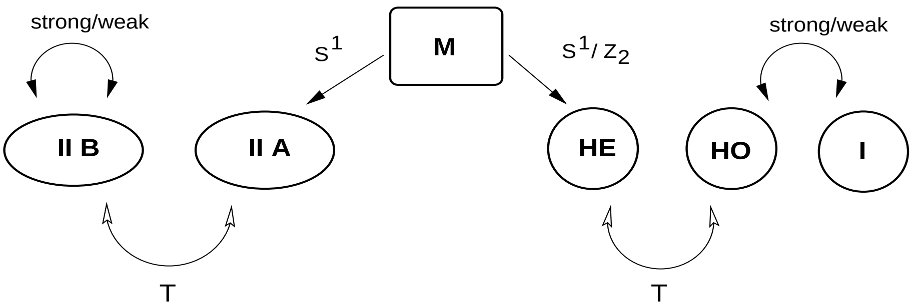

Figure 4: *Relation between M-theory and Superstrings theories.*

◦ M-theory with a longitudinal coordinate x^{11} compactified on a circle S^1 gives the 10-dimensional Type IIA string theory. We also say that it is the strong coupling limit of Type IIA and Heterotic $E_8 \times E_8$

◦11-dimensional supergravity is the low energy limit of M-theory.

◦ M-theory compactified on a torus is dual to Type IIB compactified on a circle.

◦ In the non-compactified limit, it doesn't contain strings, but a three-form gauge field A_3 and M-branes. Such fields can couple to the M-branes, electrically to a $M2$-brane (2-dimensional super-membrane), and magnetically to a $M5$-brane. From its relationship with supergravity, it must also contain the graviton (bosonic field with 44 components), the gravitino (fermionic field with 128 components) and the 3-form potential (bosonic field with 84 components).

2.4.2 D-Branes

We just show here how T-duality imply the existence of D-branes. In the next chapter, we'll present some aspects of D-brane dynamics.

T-duality for closed strings Before talking about T-duality, we need to introduce the notion of winding number. This notion appears when we compactify the parameter σ of the string on a circle to get a cylinder.

The periodic condition then becomes:

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) + 2\pi m R \tag{80}$$

where R is the radius of the cylinder on which the string is compactified on, and m is the winding number which correspond to the number of time we wind the string around. Since the cylinder is oriented, m can be negative. One can define the winding number $w = \frac{mR}{\alpha'}$ and get:

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) + 2\pi \alpha' w \tag{81}$$

The expansion (19) still holds, but α_0 is not equal to $\widetilde{\alpha}_0$ anymore. (81) gives the new condition:

$$2\pi\sqrt{\frac{\alpha'}{2}}\widetilde{\alpha}_0 = 2\pi\sqrt{\frac{\alpha'}{2}}\alpha_0 + 2\pi\alpha' w \quad \Longrightarrow \quad \widetilde{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'}w \tag{82}$$

For non-compact closed string we find the momentum:

$$p = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma(\dot{X}^\mu_L + \dot{X}^\mu_R) = \frac{1}{\sqrt{2\alpha'}}(\alpha_0 + \tilde{\alpha}_0) \tag{83}$$

The full coordinate $X(\tau, \sigma)$ is:

$$X(\tau, \sigma) = x_0 + \alpha' p \tau + \alpha' w \sigma + \imath\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-\imath n \tau}}{n}(\widetilde{\alpha}_n e^{-\imath n\sigma} + \alpha_n e^{\imath n \sigma}) \tag{84}$$

Solving (82) and (83) simultaneously, we get:

$$\begin{cases} \alpha_0 = \sqrt{\frac{\alpha'}{2}}(p-w) \\ \widetilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}(p+w) \end{cases} \tag{85}$$

The mass of a given string state is given by $(\alpha_0)^2 + (\widetilde{\alpha}_0)^2$. Let us consider now the operator $e^{-\imath ap}$ which translates states along the direction x by a distance a. If one decides to compactify x^9, then x^9 lives on a circle of radius R, the translation operator that translate by $2\pi R$ has no effect on the states. Thus, $e^{-\imath 2\pi Rp}$ is a unit operator, and then the states have momentum along x that is quantized and take values:

$$p = \frac{n}{R}, \qquad n \in \mathbb{Z} \tag{86}$$

Using this result, (85) can be written as:

$$\begin{cases} \alpha_0 = \sqrt{\frac{\alpha'}{2}}(\frac{n}{R} - \frac{mR}{\alpha'}) \\ \widetilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}(\frac{n}{R} + \frac{mR}{\alpha'}) \end{cases} \tag{87}$$

The T-duality is now obvious; (87) is invariant under the simultaneous exchanges:

$$n \leftrightarrow m \quad , \quad R \leftrightarrow R' = \frac{\alpha'}{R} \tag{88}$$

T-duality for open strings Here again, the expansion (25) still holds, and the momentum p is quantized in the same way as the closed string $p = \frac{n}{R}$. With (24), we define $\widetilde{X}^\mu(\tau,\sigma)$ as:

$$\widetilde{X}^\mu(\tau,\sigma) = X^\mu_L(\tau,\sigma) - X^\mu_R(\tau,\sigma)$$

and find:

$$\widetilde{X}^\mu(\tau,\sigma) = x'^\mu_0 + \sqrt{2\alpha'}\alpha^\mu_0\sigma + \sqrt{2\alpha'}\alpha^\mu_0 \sum_{n\neq 0} \frac{\alpha^\mu_n}{n} e^{-\imath n\tau}\sin(n\sigma) \tag{89}$$

For (25), we had the boundary condition $\partial_\sigma X^\mu(\tau,\sigma)\big|_{\sigma=0,2\pi} = 0$, which is Neumann. For (89), the boundary condition is no longer Neumann, but Dirichlet since we have $\partial_\tau \tilde{X}^\mu(\tau,\sigma)\big|_{\sigma=0,2\pi} = 0$. In other words, a Neumann boundary condition for X is equivalent to a Dirichlet boundary condition for $\widetilde{X}$.

When we compare:

$$\widetilde{X}^{\mu}(\tau, 2\pi) - \widetilde{X}^{\mu}(\tau, 0) = \sqrt{2\alpha'}\alpha_0^{\mu}\pi = 2\pi\alpha' p = 2\pi\alpha' \frac{n}{R} \tag{90}$$

and

$$X^{\mu}(\tau, 2\pi) - X^{\mu}(\tau, 0) = 2\pi R n \tag{91}$$

we see that $\widetilde{X}$ and X, are equivalent under $R \leftrightarrow R' = \frac{\alpha'}{R}$. Therefore, we can conclude that the T-duality swaps the boundary conditions of a string. We have compactified only one dimension. So the string ends are free to move in any of [(p+1)-1]-dimensions which are not T-dualized. They constitute a p-dimensional hypersurface called D_p-brane.

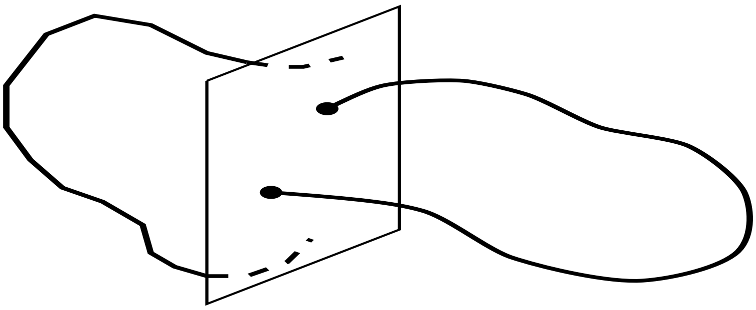

Figure 5: *Open strings ending on a D_p-brane*

2.5 A non-perturbative formulations of M-theory: the AdS/CFT correspondance

We briefly present some aspects of the AdS/CFT duality. Before talking about the correspondance, we introduce the notion of Conformal Field Theory (CFT) and Anti-de Sitter (AdS) space.

2.5.1 Conformal field theory

A conformal field theory is a field theory which is invariant under conformal transformations. A conformal transformation is used to transform an infinity space into a compact space. One can see it as a stereographic projection:

The physical manifold $\widetilde{S}$ is essentially the stereographic projection of the compact manifold S. It conserves the metric up to a scale. We also call it conformal compactification. The metrics associated

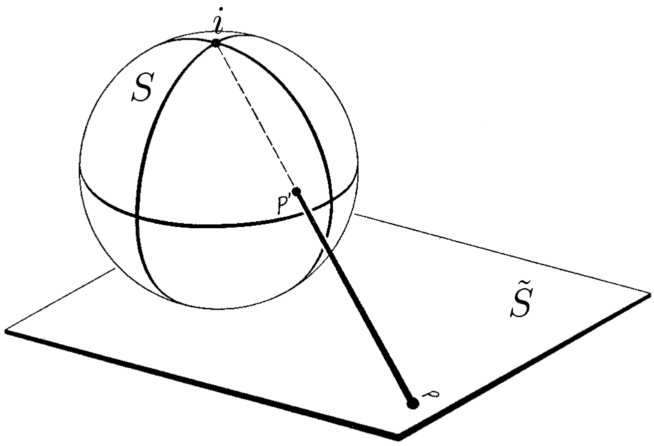

Figure 6: *The stereographic projection of* $\mathbb{S}^3$ *in* $\mathbb{R}^3$

to each manifolds are related by a conformal factor $\omega(x)$ which to be determinated by the equation $\bar{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the metric corresponding to the flat space and g, the metric of the compactified space. One should not make the confusion between a conformal compactification and the Weyl transformation, where the factor ω does not depend on the coordinates.

As an example, we calculate the conformal factor of the transformation $\mathbb{R}^3 \to \mathbb{S}^3$.

The standard metric of $\mathbb{S}^3$ is:

$$h = d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\varphi^2 \tag{92}$$

The metric of $\mathbb{R}^3$ is $\bar{h} = dx^2 + dy^2 + dz^2$ in Cartesian coordinates. In spherical coordinates it's:

$$\bar{h} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{93}$$

With $0 \leqslant \psi \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$

We can see that to statisfied the conformal equation, r must be a function of ψ. Hence: $r = r(\psi)$ and $dr = r' d\psi$

$$\omega^{-2}(r'^2 d\psi^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) = d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\varphi^2 \tag{94}$$

$$\begin{cases} \omega^{-2} r'^2 = 1 \\ \omega^{-2} r^2 = \sin^2\psi \end{cases} \tag{95}$$

By solving this equation we get the value of $r = k \tan \frac{\psi}{2}$, with $k \in \mathbb{R}$. Thanks to the system, we can have ω:

$$\omega^{-2} = \frac{\sin^2 \psi}{r^2}$$

$$\omega = \frac{r}{\sin \psi} = \frac{k \tan \frac{\psi}{2}}{\sin \psi} = \frac{k}{1 + \cos \psi} \tag{96}$$

An example of conformally invariant field theory is Yang-Mills theory. It is also invariant in its quantum version if we have the conditions of $\mathcal{N} = 4$ in 4-dimensions.

2.5.2 Anti-de Sitter space

An AdS space is a maximally symmetric (same number of symmetries as ordinary Euclidean space) spacetime geometry, with negative scalar curvature and isometry $SO(2, p)$. It is the Lorentzian analogue of p-dimensional hyperbolic space (Riemannian manifold with constant sectional curvature -1). On the $(p + 3)$-dimensional space with the metric:

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} X_i^2 \tag{97}$$

the $(p + 2)$-dimensional AdS_{p+2} can be represented by the hyperboloid:

$$R^2 = X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 \tag{98}$$

With the coordinate transformation:

$$X_0 = R \cosh \rho \cos \tau$$

$$X_i = R \sinh \rho \Omega_i$$

$$X_{p+2} = R \cosh \rho \sin \tau \tag{99}$$

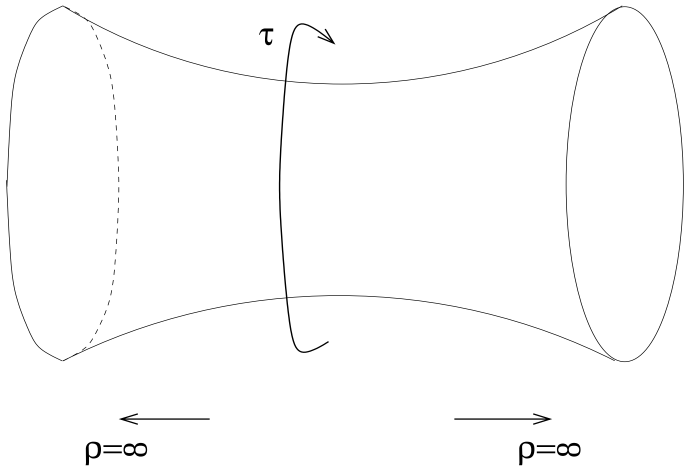

Figure 7: *AdS_{p+2} space as a hyperboloid in $\mathbb{R}^{2,p+1}$, with closed timelike curves along the τ direction.*

where $\rho \geqslant 0$ and $0 \leqslant \tau \leqslant 2\pi$, the metric on AdS_{p+2} is:

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2) \tag{100}$$

We see that when $\rho = 0$. A serie expansion leads to a metric with the topology of $S^1 \times \mathbb{R}^{p+1}$:

$$ds^2 = R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2)$$

We introduce another coordinate θ: $\tan\theta = \sinh\rho$, with $0 \leqslant \theta \leqslant \frac{\pi}{2}$, and transform the metric to get:

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2) \tag{101}$$

We see that the metric is conformally related to the Einstein static universe. Therefore, AdS_{p+2} can be conformally mapped into one-half (since $\theta \leqslant \frac{\pi}{2}$) of the whole Einstein static universe. We need to set the boundary condition at $\theta = \frac{\pi}{2}$ to make the Cauchy problem (initial data value problem) well posed. In general, if a spacetime can be conformally compactified into a region which has the same boundary structure as one-half Einstein static universe (has a metric of the form like (101)), the spacetime is called asymptotically AdS.

2.5.3 The correspondence

The AdS/CFT (also called Maldacena) correspondance [36] [60] states that there is a complete equivalence between conformally invariant quantum field theories and superstrings theory in a special spacetime geometry. A collection of a large number N of coincident p-branes produces a spacetime geometry with a horizon (like a black hole horizon). Near the horizon, this geometry can be approximated by a product of an anti-de Sitter space and a compact manifold (like a sphere). The main example of this correspondence is obtained by considering N coincident D_3-branes in the type IIB superstring theory. Then, we have the equivalence between:

• $\mathcal{N} = 4$ SYM theory in 4-dimensions, with gauge group $SU(N)$ and coupling constant g_{YM}, which is a gauge theory known to be conformally invariant in $3 + 1$ dimensions.

• The type IIB superstring theory in 10 dimensions, on $AdS_5 \times S^5$, where both AdS_5 and S^5 have the same radius and where the string coupling is $g_s = g_{YM}^2$.

One of the feature of the duality is the identification of the isometry group of AdS to the conformal symmetry group of the flat space.

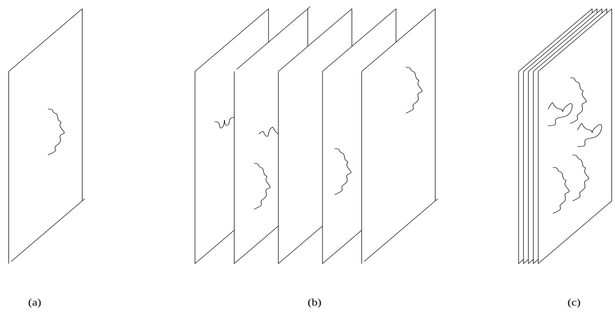

Figure 8: *From left: Single D-brane; well separated D-branes; coincident D-branes*

The AdS/CFT conjecture is that type IIB, in the $AdS_5 \times S^5$ background is dual to $\mathcal{N} = 4$, $D = 3 + 1$ SYM with gauge group $SU(N)$. I present here how the conjecture arises:

The brane action is defined on the $3 + 1$ dimensional brane worldvolume of Type IIB superstring theory, and it contains the $\mathcal{N} = 4$ SYM Lagrangian, which is known to be conformally invariant. For a D_3-brane, the action is:

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2)$$

$$f = 1 + \frac{R^4}{r^4} \tag{102}$$

with $R^4 = 4\pi g_s \alpha'^2 N$.

In the near horizon region where $r \ll R$, f can be aproximate by $f = \frac{R^4}{r^4}$, and the metric becomes:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \tag{103}$$

Which is exactly the geometry of $AdS_5 \times S^5$.

We have just presented here the general idea of AdS/CFT and how the correspondence arose in the first place. It is a very wide subject and it exists much more correspondences in addition to the one presented here. Although it has been proved to be very useful, is not the only non-perturbative formulation of M-theory. We now turn to the main subject; the M(atrix) models, an alternative to AdS/CFT. Although they don't have the same properties, we will see at the end of the chapter what are the connections between them.

3 M(atrix) models

We said earlier that M-theory doesn't contain strings but M2- and M5-branes, which make the theory very complicated. However, one great conjecture states that M-theory reduces to a simple matrix model, which is a supersymmetric quantum theory with matrix degrees of freedom. In the case of the BFFS model, these degrees of freedom turn out to be D_0-branes. We review the different models BFSS, IKKT, and NBI, how they are related to each other, how noncommutative geometry arises from these relations and what are the connections to AdS/CFT.

3.1 The BFSS model

This model which has been the first to be developed by Banks, Fischler, Shenker and Susskind [5], is based on the idea that M-theory can be described in the infinite momentum frame (IMF) by a theory where the only dynamical degree of freedom are D_0-branes. The key idea is to interpret the 9 space dimensions (the X^i fields) of the D_0-brane matrix model as the transverse dimensions of an eleven dimensional theory in the IMF. We start by introducing the IMF and the D-brane effective action. Then, the appearance of D_0-branes as dynamical degree of freedom is discussed.

3.1.1 The Infinite Momentum Frame (IMF)

This frame was introduced by Weinberg, to simplify problems in perturbative quantum field theory. It is a frame which is highly boosted in the momentum P direction until it becomes much larger than any other momenta in the problem.

In M-theory, we separate the components of the eleven dimensional coordinates in three parts:

$$x^{\mu} = (t, x^{i}, x^{11}) \tag{104}$$

where $i = 1, ..., 9$. This coordinates are called the transverse coordinates and sometimes written as $x^{\perp}$. We compactify the longitudinal coordinate x^{11} on a circle:

$$x^{11} = x^{11} + 2\pi R \tag{105}$$

Since in the IMF we boost along longitudinal momentum p^{11}, the great advantage is that only positive p^{11} matter whereas the zero or negative ones do not appear. However, a boost is not a symmetry of Lorentz invariant theories which have been compactified in the direction of the boost, so the Lorentz invariance in M(atrix) theory is no longer explicit, if still present. We have the same condition (86) as before, due to the compactification:

$$p^{11} = \frac{N}{R} \tag{106}$$

with N an integer stricly positive. In order to recover the 11-dimensional M-theory we need to uncompactify, with the conditions:

$$R \to \infty \qquad and \qquad \frac{N}{R} \to \infty \tag{107}$$

The other great advantage of working in the IMF is that it has a transverse Galilean symmetry, which leads to a nonrelativistic form of the equations. For example, the Galilean transformation for transverse momenta is

$$p^{i} \to p^{i} + p^{11} v^{i} \tag{108}$$

and the energy of a massless particle boosted in the longitudinal direction x^{11} is $E = \frac{p_{\perp}^{2}}{2p^{11}}$. We see that the longitudinal momentum p^{11} plays the role of the mass in the IMF Galilean theory.

3.1.2 M-theory, Type IIA and the conjecture

By definition, 10-dimensional Type IIA is equivalent to 11-dimensional M-theory with a dimension compactified on a circle with the radius of compactification $R \to 0$. Given this relationship, one can relates objects in both theories to each other. The correspondances include the following:

1. The string coupling constant is related to the radius of compactification by:

$$R = g^{2/3} l_p = g l_s \tag{109}$$

2. The photon of IIA in the R-R sector is the photon called Kaluza-Klein photon $g_{\mu 11}$ which arises from the compactification in eleven dimensional supergravity.

3. The only objects in the theory which carry R-R photon charge are the D_0-branes. They are point particles (in 10D) which carry longitudinal momentum $p_{11}^{D_0} = \frac{1}{R}$.

Consequently, D_0-branes are good candidates to be the dynamical degrees of freedom (parton) of M-theory in the IMF. Since the dynamics of D-branes is governed the reduction of SYM theory to $p+1$ dimensions (or by the Dirac-Born-Infield action in a purely bosonic theory), a collection of N D_0-branes is described by 10-dimensional $U(N)$ SYM reduced to $0+1$ dimensions, i.e. by $N \times N$ hermitian matrix quantum mechanics. The conjecture follows:

M-theory in the infinite momentum frame (IMF) is exactly equivalent to the $N \to \infty$ limit of D_0-branes supersymmetric matrix quantum mechanics, described by the 10-dimensional $U(N)$ SYM $\mathcal{N} = 1$, reduced to 0+1 dimension.

3.1.3 D-Brane action from dimensional reduction of $10D$ Super Yang-Mills

It was shown by Leigh that the equation of motion for a D-brane in a purely bosonic theory are precisely those of the Dirac-Born-Infield action. The electrodynamics on a fluctuating D_p-brane, in an arbitrary background, is described by the action:

$$S_{DBI} = -\frac{T_p}{g_s} \int d^{p+1}\xi \sqrt{-det(g_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} \tag{110}$$

where $g_{\alpha\beta}$ and $B_{\alpha\beta}$ are the pull-backs of the spacetime supergravity fields to the D_p-brane world-volume. If we make some assumptions, this action can be simplified. We consider $B_{\alpha\beta} = 0$, the spacetime background flat so that $g_{\mu\nu} = \eta_{\mu\nu}$ and the geometry of the brane flat as well. The pull-back of the metric on the D_p-brane is now:

$$g_{\alpha\beta} \simeq \eta_{\alpha\beta} + \partial_\alpha X^a \partial_\beta X_a \tag{111}$$

and the DBI action (110) becomes:

$$S_{DBI} = -\frac{T_p(2\pi\alpha')^2}{4g_s} \int d^{p+1}\xi \Big[F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{(2\pi\alpha')^2}\partial_\alpha X^a \partial_\beta X_a - \frac{T_p}{g_s} + o(F^4) \Big] \tag{112}$$

where $(\alpha, \beta) = 0, ..., p$ and $a = (p+1), ..., (D-1)$.

This is the action for a $U(1)$ gauge theory in $(p+1)$ dimensions with $(9-p)$ scalar fields X^a. It is actually the action that would result from the dimensional reduction to $(p+1)$ dimensions of abelian Yang-Mills theory in 10 dimensions with action:

$$S_{YM} = -\frac{1}{4g_{YM}^2} \int d^{10}x F_{\mu\nu}F^{\mu\nu} \tag{113}$$

if we identify the coupling constants:

$$g_{YM}^2 = \frac{g_s}{T_p(2\pi\alpha')^2} \tag{114}$$

This lead us a generalization to a non-abelian supersymmetric case:

The low energy dynamics of N parallel coincident D_p-branes in flat space is described in static gauge by the dimensional reduction to $(p+1)$-dimensions of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge group $U(N)$ in ten dimensions, with action:

$$S_{SYM} = \frac{1}{4g_{YM}^2} \int d^{10}x \ Tr\Big[- F_{\mu\nu}F^{\mu\nu} + 2\imath \ \overline{\psi}\gamma^\mu D_\mu \psi \Big] \tag{115}$$

where the covariant derivative is $D_\mu\psi = \partial_\mu\psi - \imath[A_\mu, \psi]$, ψ is a Majorana-Weyl spinor of $SO(1,9)$ in 10 dimensions, and the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \imath[A_\mu, A_\nu]$. This action is invariant under SUSY transformations:

$$\delta_\epsilon A_\mu = \frac{\imath}{2}\bar{\epsilon}\Gamma_\mu \psi \tag{116}$$

$$\delta_\epsilon \psi = -\frac{1}{2} F_{\mu\nu}[\Gamma^\mu, \Gamma^\nu]\epsilon \tag{117}$$

Now we can construct a SYM theory in $p+1$ dimensions, which is the D_p-brane action, by dimensional reduction of the 10-dimensional SYM. This is done by assuming that all the fields A_μ are independent of coordinates $p+1, ..., 9$. Then, the $10D$ field A_μ decomposes into $(p+1)$-dimensional $U(N)$ gauge field A_m with $m = 0, ..., p$, and and $9-p$ scalar fields X^i, with transverse indices $i = p+1, ..., 9$. If we consider the bosonic part of (115), we get that the strength field tensor decomposes as:

$$F_{\mu\nu}^2 = F_{mn}^2 + F_{mj}^2 + F_{ij}^2 \tag{118}$$

On the brane, there is no dependance on the X^i, so the derivatives in the i direction vanish:

$$F_{mn} = \partial_m A_n - \partial_n A_m + \imath [A_m, A_n] \tag{119}$$

$$F_{mj} = \partial_m X_j + \imath [A_m, X_j] \equiv D_m X_j \tag{120}$$

$$F_{ij} = \imath [X_i, X_j] \tag{121}$$

We get:

$$S_{D_p} = -\frac{T_p g_s (2\pi\alpha')^2}{4} \int d^{p+1}\xi \;\; Tr\Big[F_{mn}F^{mn} + 2D_m X^i D^m X_i + [X_i, X_j]^2\Big] + fermions \tag{122}$$

3.1.4 D_0-Brane mechanics

From the previous subsection, we know that the dynamics of N D_0-branes in the low energy limit in flat 10-dimensional spacetime is the dimensional reduction of $\mathcal{N} = 1$ SYM in 10 dimensions to $0+1$ dimension. The 10 dimensional gauge field A_μ splits into 9 transverse scalars X^m, and one dimensional gauge field A_0. We get a supersymmetric matrix quantum mechanics for X^i's and θ's in the adjoint representation of $U(N)$ with the Lagrangian:

$$\mathcal{L}_{D_0} = \frac{1}{2g_s\sqrt{\alpha'}} Tr\Big[D_0 X^i D_0 X^i + 2\theta^\top D_0\theta - \frac{1}{2}[X^i, X^j]^2 - 2\theta^\top \gamma_i [\theta, X^i]\Big] \tag{123}$$

where we have used the following:

$\psi = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$, is a Majorana-Weyl spinor and θ is a real 16 components spinor. The gamma matrices Γ^μ are real symmetric 16×16 of $SO(1,9)$ given by:

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}$$

From (119), (120) and (121), we had:

$$F_{ij} = \imath[X_i, X_j] \qquad F_{0j} = \partial_0 X_j + \imath[A_0, X_j] \equiv D_0 X_j \tag{124}$$

$$D_j\theta = \imath[X_i, \theta] \qquad D_0\theta = \partial_0\theta + \imath[A_0, \theta] \tag{125}$$

Changing the units to those where $l_s \equiv \sqrt{\alpha'} = 1$ and $l_p = 1$, and introducing:

$$Y^i = \frac{X^i}{g_s^{1/3}} \tag{126}$$

which is more convenient for the 11-dimensional interpretation, (123) becomes:

$$\mathcal{L}_{D_0} = Tr\left(\frac{1}{2R_{11}} D_t Y^i D_t Y^i - \frac{1}{4} R_{11}[Y^i, Y^j]^2 - \theta^\top D_t\theta - R_{11}\theta^\top \gamma_i[\theta, Y^i]\right) \tag{127}$$

where $D_t = \partial_t + \imath A_0$. One can simplify (123) by choosing the gauge $A_0 = 0$. The Hamiltonian, written in 11 dimensions, associated to (128) is:

$$\mathcal{H}_{D_0} = R_{11}\, Tr\left[\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}[Y^i, Y^j]^2 + \theta^\top\gamma_i[\theta, Y^i]\right] \tag{128}$$

where Π is the canonical conjugate to Y, and half of the θ's are canonical conjugate momenta of the other half.

3.1.5 M(atrix) theory objects: supergravitons and membranes

• **Supergravitons** The simplest states of the Hamiltonian is when the matrices Y^i are diagonal with only one nonvanishing component and all θ's equal to zero. Then we get:

$$E = \frac{R}{2} p_i^2 = \frac{p_i^2}{2p_{11}} \tag{129}$$

where p_i is the energy eigenvalue of Π_i. This corresponds to a single D_0-brane. Each of these states are accompanied by the fermionic superpartners and they form a representation of the algebra of 16 θ's with $2^{16/2} = 2^8 = 256$ components. This is exactly the number of states of the supergraviton in $11D$ supergravity arising from the graviton, the 3-form and the gravitino (256=44+84+128).

A more general eigenstate has a form of the diagonal $N \times N$ matrix:

$$Y^i = \begin{pmatrix} Y^i_{(1)} & & \\ & \ddots & \\ & & Y^i_{(N)} \end{pmatrix} \tag{130}$$

where the diagonal matrix elements are the coordinates of the D_0-branes. It describes a state of N supergravitons, where the matrices Y^k are $N_k \times N_k$ matrices and the longitudinal momentum of the kth graviton is $p = N_k/R$.

• **Membranes** Since M-theory is the strong coupling of type IIA, it must have membranes in its spectrum. We use two different ways to show of membranes are obtained from the M(atrix) model action (128).

First, we see how we can get the supermembrane action from the D_0-brane action. It was Townsend who first pointed out the connection between these two and said that a membrane should be considered as a collection of D_0-branes. To make this connection, we use the following:

We begin with a pair of unitary operator U, V with the relations:

$$UV = e^{\frac{2\imath\pi}{N}} VU$$

$$U^N = 1, \qquad V^N = 1 \tag{131}$$

U and V may be written as exponential of canonical variables p and q:

$$U = e^{\imath p}, \qquad V = e^{\imath q} \tag{132}$$

satisfying the commutation relation:

$$[q, p] = \frac{2\imath\pi}{N} \tag{133}$$

They can be represented on a N dimensional Hilbert space, where they form a basis such that any matrix Z can be written as:

$$Z = \sum_{n,m=1}^{N} Z_{nm} U^m V^m \tag{134}$$

One can interpret these coordinates in terms of the quantum mechanics of particles, with coordinates p q, on a torus. Therefore, due to the commutation relation of p and q, the space is sometimes called ”noncommuting torus”. In the limit of large N, the noncommuting torus behaves like a phase space, and we have the correspondance between the two spaces:

$*$ The trace of an operator is replaced by N times the integral over the torus:

$$TrA \rightarrow N \int dpdqA(p, q) \tag{135}$$

$*$ The commutator of two operators is replaced by $1/N$ times the Poisson brackets:

$$[A, B] \rightarrow \frac{1}{N}\{A, B\} = \frac{1}{N}(\partial_q A \partial_p B - \partial_q B \partial_p A) \tag{136}$$

If one promotes Y^i and θ of (128) as operators depending on p and q and operates the changes, one gets the Lagrangian:

$$\mathcal{L} = \frac{p_{11}}{2} \int dpdq(\dot{Y}^i(p, q))^2 - \frac{1}{p_{11}} \int dpdq(\partial_q Y^i \partial_p Y^j - \partial_q Y^j \partial_p Y^i)^2 + fermions \tag{137}$$

and the Hamiltonian:

$$H = \frac{1}{2p_{11}} \int dpdq \Pi_i^2(p, q) + \frac{1}{p_{11}} \int dpdq(\partial_q Y^i \partial_p Y^j - \partial_q Y^j \partial_p Y^i)^2 + fermions \tag{138}$$

which exactly the Hamiltonian for the $11D$ supermembrane in the IMF [58].

Another way to see the emergence of membrane is from the classical equations of motion:

$$[Y^\mu,[Y^\mu,Y^\nu]]=0 \qquad , \qquad [X^\mu,\gamma_\mu\theta_\alpha]=0 \tag{139}$$

An infinite membrane stretched out in the $8,9$ plane is given by:

$$Y^8=R_8\sqrt{N}p \qquad , \qquad Y^9=R_9\sqrt{N}q \tag{140}$$

and all other Y's and θ's are equal to zero. p and q are infinite dimensional matrices ($N\to\infty$) satisfying (133), and R_8 and R_9 are the compactification radii. Since the commutator of Y^8 and Y^9 is equal to a complex number, the equations (139) are satisfied.

One can compute the tension of the brane in both M(atrix) and M-theory, to check whether they match, which could be a first step in the proof of the conjecture. The calculation done in [5], shows that the tensions actually agree.

3.1.6 The symmetries

The Hamiltonian (128) has a Galilean symmetry, which one can see by defining the center of mass of the system by:

$$Y(c.m.)=\frac{1}{N}TrY \tag{141}$$

A translation is defined by $Y\to Y+c\mathbb{I}$, where c is a constant (adding a multiple of the identity to Y).This has no effect on the commutator because the identity $\mathbb{I}$ commutes with all Y, and has no effect on the equations of motion. Similarly, the Hamiltonian has a rotation invariance. The center of mass momentum is defined by:

$$P(c.m.)=Tr\Pi=\frac{N}{R}\dot{Y}(c.m.) \tag{142}$$

With the expression of $p_{11} = \frac{N}{R}$, the momentum reads[2]:

$$P(c.m.) = p_{11}\dot{Y}(c.m.) \tag{143}$$

A Galilean boost is defined by $\dot{Y} \to \dot{Y} + c\mathbb{I}$, where c is a constant (adding a multiple of the identity to $\dot{Y}$). This once again has no effect on the equation of motion. Hence, the whole Hamiltonian has a full Galilean invariance.

The Lorentz invariance is broken because a boost is not a symmetry of the IMF. In M(atrix) theory, it has not been proved yet, and there are actually very little evidence of it. However, in [6] the model has been used to describe the properties of the Schwarzschild black holes in $7+1$ dimensions, by describing it as a Boltzmann gas made of D_0-branes. Compactified on T^3 and with the assumption $N \sim S$, it properly describes the energy-entropy relation and the Hawking temperature. Their results actually rely on the Lorentz invariance of Matrix model, which is investigated further in details in [23]. They consider the Hawking radiation in Matrix model for the case $N \gg S$, and get the correct evaporation rate of the black hole. Their result about Hawking radiation is independent of the boost parameter, and thus, gives support of the Lorentz invariance of the Matrix model.

Also, in [3], they present a formulation of a matrix model which manifestly possesses the general coordinate invariance when they identify the large N matrices with differential operators. In order to build a matrix model which has the local Lorentz invariance, they investigate how the $SO(1,9)$ Lorentz symmetry and the $U(N)$ gauge symmetry are mixed together. They find that the bosonic part of the model reproduces the Einstein gravity in the classical low-energy limit. Finally, they give a proposal to build a matrix model which has $\mathcal{N} = 2$ SUSY and reduces to the type IIB supergravity in the classical low-energy limit.

3.2 The IKKT model

We saw that BFSS provides a nonperturbative description of M-theory. Another model was introdced in [26] to describe type IIB. We show how this model has been constructed.

[2]Note that the expression of P(c.m.) is of the form $p = mv$, since p_{11} plays the role of the mass. The Galilean form of this relation is due to the IMF formulation.

3.2.1 The action

As we said before, a consequence of the opposite chiralities in the theories is the difference of D-branes we find. In type IIA we have D_p-branes with p even and in type IIB we have D_p-branes with p odd. The analogue of the D_0-brane is the D-instanton ($p = -1$). Since the Lagrangian in BFSS is expressed in terms of D_0-branes, one can expect to be able to formulate the fundamental Lagrangian of IKKT in terms of D-instanton i.e by 10-dimensional SYM reduced to a point.

The starting point is obviously the Nambu-Goto form of the Green-Schwartz action of type IIB superstring theory:

$$S_{GS} = -T \int d^2\sigma \Big[\sqrt{-(\epsilon^{ab}\partial_a \widetilde{X}^\mu \partial_b \widetilde{X}^\nu)^2} + 2\imath\epsilon^{ab}\partial_a \widetilde{X}^\mu \overline{\Psi}\gamma_\mu \partial_b \Psi\Big] \tag{144}$$

ψ is a Majorana spinor with 16 components and $\mu = 0, ..., 9$. This action can be written [41] in the so called Schild form:

$$S_{Schild} = \int d^2\sigma \Big[\alpha\Big(\frac{1}{4\sqrt{g}}\{\widetilde{X}^\mu, \widetilde{X}^\nu\}^2 - \frac{\imath}{2}\overline{\Psi}\gamma^\mu\{\widetilde{X}_\mu, \Psi\}\Big) + \beta\sqrt{g}\Big] \tag{145}$$

where $\{\widetilde{X}^\mu, \widetilde{X}^\nu\}$ are the poisson brackets defined by:

$$\{\widetilde{X}^\mu, \widetilde{X}^\nu\} = \epsilon^{ab}\partial_a \widetilde{X}^\mu \partial_b \widetilde{X}^\nu \tag{146}$$

One can show that this action is classically equivalent to the Green-Schwartz action by calculating the equation of motion for $\sqrt{g}$. By solving the Euler-Lagrange equation we have:

$$-\frac{\alpha}{4(\sqrt{g})^2}(\epsilon^{ab}\partial_a \widetilde{X}^\mu \partial_b \widetilde{X}^\nu)^2 + \beta = 0$$

and by isolating $\sqrt{g}$

$$\sqrt{g} = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\sqrt{(\epsilon^{ab}\partial_a \widetilde{X}^\mu \partial_b \widetilde{X}^\nu)^2} \tag{147}$$

we recover the GS action if we plug this in the Schild action. The IKKT model is obtained from the Schild action (145) by replacing the bosonic $\widetilde{X}_\mu(\sigma^0, \sigma^1)$ and fermionic $\Psi_\alpha(\sigma^0, \sigma^1)$ fields by hermitian N dimensional matrices. We denote X^{ab}_μ the bosonic matrices and ψ^{ab}_α the fermionic matrices. In the limit where N is large, we have the correspondance:

$$\imath\{.,.\} \Rightarrow [.,.] \tag{148}$$

and

$$\int d^2\sigma\sqrt{g} \Rightarrow Tr \tag{149}$$

Hence, the action of the IKKT model is:

$$S = \alpha\, Tr\Big[-\frac{1}{4}\,[X^\mu, X^\nu]^2 - \frac{1}{2}\overline{\psi}^\alpha \gamma^\mu_{\alpha\beta}\,\big[X_\mu, \psi^\beta\big]\Big] + \beta N \tag{150}$$

where the dynamical variable is N (the size of the matrices), and is the analogue of $\sqrt{g}$ in the Schild action (145). It means that the size of the matrix is not fixed. The bosonic part of this action can be obtained from a matrix model developed in the 80's by Eguchi and Kawai[3] [21]. The IKKT model can also be obtained by reducing SYM in 10 dimensions to a point.

3.2.2 Symmetries

Since IKKT model is constructed from 10 dimensional SYM, it has many inherent symmetries coming from SYM theory, and is invariant under:

○ Shifts:

$$X_\mu \to X_\mu + a_\mu \mathbb{I} \tag{151}$$

where a_μ is a c-number

○ $SO(10)$ transformations (rotations):

$$X_\mu \to \Lambda_\mu{}^\nu X_\nu \tag{152}$$

where Λ is a generator of the group $SO(10)$. We will see later how this property leads us to the BFSS model.

○ $SU(N)$ gauge symmetry:

$$X_\mu \to U^{-1} X_\mu U \tag{153}$$

[3]The action of the Eguchi-Kawai model is $S = -\frac{1}{4}\sum_{\mu,\nu} Tr(U_\mu U_\nu U_\mu^{-1} U_\nu^{-1} - \mathbb{I})$. By setting $U_\mu = exp(aX_\mu)$ and taking the limit $a \to 0$, one obtains the bosonic part of IKKT.

where U is a generator of the gauge group $SU(N)$.

∘ Supersymmetry transformations:

$$\delta_{Q_1} X_\mu = \bar{\epsilon}\gamma_\mu \psi \qquad , \qquad \delta_{Q_2} X_\mu = 0 \tag{154}$$

$$\delta_{Q_1} \psi = [X_\mu, X_\nu]\gamma^{\mu\nu}\epsilon \qquad , \qquad \delta_{Q_2}\psi = \eta \tag{155}$$

where ϵ and η are the parameters of the SUSY transformations.

3.2.3 Classical solutions

The equation of motion for the Schild action when $\Psi = 0$ are:

$$\{\widetilde{X}^\mu, \{\widetilde{X}_\mu, \widetilde{X}_\nu\}\} = 0 \qquad , \qquad \{\widetilde{X}^\mu, \gamma_\mu \Psi_\alpha\} = 0 \tag{156}$$

The equation of motion for the IKKT model can be obtained either by operating the previous changes, or by solving the Euler-Lagrange equation. In both case we find:

$$[X^\mu, [X_\mu, X_\nu]] = 0 \qquad , \qquad [X^\mu, \gamma_\mu \psi_\alpha] = 0 \tag{157}$$

To solve the equation of motion, one can see that (157) are similar to (139) for the BFSS model. Then, they have solutions of the same form (140), associated to static D-strings along the 1st axis.

$$X_\mu = \left(\frac{T}{2\pi}q, \frac{L}{2\pi}p, 0, ..., 0\right) \qquad , \qquad \psi_\alpha = 0 \tag{158}$$

where p and q are $N \times N$ matrices statisfying the commutation relation (133). This solution is for one string. The case of two parallel static D-strings separated by a distance b along the second axis is obtained by considering X_μ's as matrices with two diagonal blocks:

$$\mathbf{X_0} = \begin{pmatrix} \frac{T}{2\pi}q & 0 \\ 0 & \frac{T}{2\pi}q \end{pmatrix}, \qquad \mathbf{X_1} = \begin{pmatrix} \frac{L}{2\pi}p & 0 \\ 0 & \frac{L}{2\pi}p \end{pmatrix} \qquad \mathbf{X_2} = \begin{pmatrix} \frac{b}{2} & 0 \\ 0 & -\frac{b}{2} \end{pmatrix}$$

The generalization for one D_p-brane with $p > 1$ is:

$$X_\mu = \Big(\frac{T}{2\pi}q_1, \frac{L}{2\pi}p_1, ..., \frac{T}{2\pi}q_{\frac{p+1}{2}}, \frac{L}{2\pi}p_{\frac{p+1}{2}}, 0, ..., 0\Big) \qquad , \qquad \psi_\alpha = 0 \tag{159}$$

where there are $\frac{p+1}{2}$ pairs of operator p, q. The solution for multi-brane can be obtained similarly as for two static D-strings.

3.3 The NBI model

This model gives a description of Type IIB superstrings, just like the IKKT model. The necessity of introducing another model describing the same things comes from the calculation of the interaction between D_p-branes using solution (159). The results reproduce those from superstring calculations only at large distances. The modification of the IKKT model [22] is the introduction of a new dynamical variable replacing N: an hermitian matrix Y^{ab} with positive eigenvalues, which is the analogue of $\sqrt{g}$ is the Schild action. The integration over this new variable Y^{ab} yields to the non-abelian Born-Infield action (NBI) which reproduces the Nambu-Goto version of the Green-Schwarz action of IIB (144).

3.3.1 The action

In the IKKT model, the size of Y^{ab} is set to be N and only the element of the matrix fluctuate. From the Schild action (145):

$$S^{cl} = -\alpha \;\, Tr\Big[\frac{1}{4}Y^{-1}[X_\mu, X_\nu]^2 + \frac{1}{2}\;\, \overline{\psi}\gamma^\mu[X_\mu, \psi]\Big] + \beta\, Tr\, Y \tag{160}$$

The equation of motion for Y^{ab}:

$$\frac{\alpha}{4}\Big(Y^{-1}[X_\mu, X_\nu]^2 Y^{-1}\Big)_{ab} + \beta\,\delta_{ab} = 0 \tag{161}$$

yields to the solution of (160):

$$Y = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\sqrt{-[X_\mu, X_\nu]^2} \tag{162}$$

We see that (162) is very similar to (147). This is how we see that Y is the equivalent to $\sqrt{g}$. Putting

(162) into (160) gives the non-abelian Born-Infield (NBI) action:

$$S_{NBI}^{cl} = \sqrt{\alpha\beta}\ Tr\Big[\sqrt{-[X_\mu, X_\nu]^2} - \frac{\alpha}{2}\ \overline{\psi}\gamma^\mu[X_\mu, \psi]\Big] \tag{163}$$

The NBI matrix model action is defined by the action:

$$S_{NBI} = -\alpha\ Tr\Big[\frac{1}{4}Y^{-1}[X_\mu, X_\nu]^2 + \frac{1}{2}\ \overline{\psi}\gamma^\mu[X_\mu, \psi]\Big] + V(Y) \tag{164}$$

where Y is a hermitian $N \times N$ matrix with positive eigenvalues, and the potential V is:

$$V(Y) = \beta\, Tr\, Y + (N - \frac{1}{2})\, Tr\ \ln(Y)$$

The action is invariant under SUSY transformations:

$$\delta_\epsilon \psi = \frac{\imath}{4}\big\{Y^{-1}, [X_\mu, X_\nu]\big\}\gamma^{\mu\nu}\epsilon$$

$$\delta_\epsilon X_\mu = \imath\overline{\epsilon}\gamma_\mu\psi \tag{165}$$

One can prove that this model reproduces the Nambu-Goto version of the Green-Schwarz action of type IIB superstrings (144) [35, 22].

3.3.2 D-brane solutions

The equations of motion of X_μ and ψ_α of the action (163) are:

$$\Big[X^\mu, \big\{Y^{-1}, [X_\mu, X_\nu]\big\}\Big] = 0 \quad , \qquad [X_\mu, \gamma^\mu\psi_\alpha] = 0 \tag{166}$$

The solutions of the NBI matrix model are of the same form (159) as the IKKT model.

3.4 Relation between the models

The relation between the IKKT and the NBI models has been discussed in the previous subsection is quite straightforward. We now turn to the relation to BFSS and IKKT[4]. Since BFSS describesM-theory, and IKKT describes type IIB, one might ask whether the two models are related, just as the two superstring theories are. The relation actually is that when we compactify on a circle the Euclidean version of IKKT model, it gives the BFFS model at finite temperature. The compactification of matrix models has been studied in the first place by Connes, Douglas and Schwarz [15].

3.4.1 Compactification on a circle

When we compactify on a circle in the X^i direction, one should have the gauge equivalence:

$$UX^iU^{-1} = X^i + 2\pi R_i \mathbb{I} \qquad ; \qquad UX^jU^{-1} = X^j \qquad ; \qquad U\psi^\alpha U^{-1} = \psi^\alpha \tag{167}$$

where R_i is the radius of compactification and U is the unitary matrix of the gauge group transformation. These equation can't be satisfied unless X^i and ψ^α aren't finite matrices but operators in an infinite-dimensional Hilbert space $\mathcal{H}$. The solutions are:

$$X^i = A^i(\sigma) + 2\imath\pi R_i \frac{\partial}{\partial\sigma} \qquad ; \qquad X^j = A^j(\sigma) \qquad ; \qquad \psi^\alpha = \Psi^\alpha(\sigma) \tag{168}$$

$$(Uf)(\sigma) = e^{\imath\sigma} f(\sigma) \tag{169}$$

where $0 \leqslant \sigma \leqslant 2\pi$ is the coordinate compactified on S^1 and A^i's and Ψ's are hermitian operators in $\mathcal{H}$.

The reasons why we consider an Euclidean version of the model are:

$*$ The BFSS model is obtained by a reduction of the 10-dimensional SYM to $0+1$ dimension. This breaks the Lorentz invariance $SO(1,9)$, and the theory is only invariant under the little group $SO(9)$, which corresponds to spatial rotations.

$*$ The IKKT model is obtained by a reduction of the 10-dimensional SYM to $0+0$ dimension (a point), and the theory is invariant under $SO(1,9)$.

[4]A numerical approach, using the Monte Carlo simulation, has been used in [31] to find a relation between the two models, by identifying them to two other models which are equivalent: EK (Eguchi-Kawai) and cQEK (continuum quenched EK) models [25].

Taking the Euclidean version of IKKT allow us to use the $\mathbb{R}^{10}$ metric and thus, have a $SO(10)$ symmetry group. This way, one can compactify in any direction X^i (with i=0, ..., 9) and end up the same symmetry group as BFSS, $SO(9)$. From the IKKT action (150), one can insert the solutions (168) and (169) with the time X^0 compactified on S^1:

$$S = C\int d\sigma Tr\Big[2\sum_{i=1}^{9}(\nabla_0 A^i)^2 + \sum_{i,j=1}^{9}[A^i, A^j]^2 + 2\Psi^\alpha\sigma^0_{\alpha\beta}\nabla_0\Psi^\beta + 2\sum_{i=1}^{9}\Psi^\alpha\sigma^i_{\alpha\beta}[A^i, \Psi^\beta]\Big] \tag{170}$$

C is a constant, and $(\nabla_0)f(\sigma) = \imath R_0\frac{\partial}{\partial\sigma} + [A^0, f](\sigma)$. This is an action of a matrix quantum mechanics with compact Euclidean time direction. One can see that it's very similar to the BFSS action. In fact, it is equivalent to BFSS quantum mechanics at finite temperature.

This can be easily generalized to the compactifiation of more than one dimension on a torus. This is how noncommutative geometry arises.

3.4.2 Compactification on a torus T^2

The torus appears when we compactify more than one dimension. In a general case, the correspondence between a torus and spheres is $T^d \simeq (S^1)^d$. If one compactifies X^1 and X^2, the new set of equations follows from (176):

$$U_1X^1U_1^{-1} = X^1 + 2\pi R_1\mathbb{I} \quad ; \quad U_1X^iU_1^{-1} = X^i \quad ; \quad U_1\psi^\alpha U_1^{-1} = \psi^\alpha \tag{171}$$

$$U_2X^2U_2^{-1} = X^2 + 2\pi R_2\mathbb{I} \quad ; \quad U_2X^iU_2^{-1} = X^i \quad ; \quad U_2\psi^\alpha U_2^{-1} = \psi^\alpha \tag{172}$$

where R_i's are the radii of compactification. From these equations, it follows that $U_1U_2U_1^{-1}U_2^{-1}$ commutes with X^i's and ψ^α's. Then, it can be written as a scalar operator:

$$U_1U_2 = \lambda U_2U_1 \tag{173}$$

where $\lambda = e^{2\imath\pi\theta}$ is a complex constant. If $\lambda = 1$, U_1 and U_2 commute, and then we have about a commutative torus. The solutions of (171) and (172) follow from (169):

$$X^1 = A^1(\sigma^1, \sigma^2) + 2\imath\pi R_1 \frac{\partial}{\partial\sigma^1} \quad ; \quad X^2 = A^2(\sigma^1, \sigma^2) + 2\imath\pi R_2 \frac{\partial}{\partial\sigma^2} \quad ; \quad X^i = A^i(\sigma^1, \sigma^2) \tag{174}$$

$$\psi^\alpha = \Psi^\alpha(\sigma^1, \sigma^2) \quad ; \quad (U_k f)(\sigma^1, \sigma^2) = e^{\imath\sigma^k} f(\sigma^1, \sigma^2) \tag{175}$$

with $i \neq 1, 2$, $k = 1, 2$. The coordinates compactified on the torus T^2 take values between 0 and 2π.

3.4.3 Compactification on a noncommutative torus T^d_θ

If one wants to compactify d dimensions in a D-dimensional Hilbert space $\mathcal{H}$, we will get a torus T^d with the equations:

$$U_j X^k U_j^{-1} = X^k + 2\pi R_k \mathbb{I}\delta_j^k \quad ; \quad U_j X^k U_l^{-1} = X^l \quad ; \quad U_j \psi^\alpha U_j^{-1} = \psi^\alpha \tag{176}$$

where $j, k = 1, ..., d$ and $l = D - d, ..., D$. The solutions are a generalization of (174) and (175), and the following relation still holds:

$$U_j U_k = e^{2\imath\pi\theta^{jk}} U_k U_j \tag{177}$$

Like in the previous section, the parameter in the relation can be set to be equal to 1, so we get a commutative torus. However, if it is different from 1, the torus will be noncommutative, caracterized by the parameter θ, which is a constant $d \times d$ antisymmetric matrix. One can restrict the action of either BFSS or IKKT to be solution of (176). This leads to SYM on a noncommutative torus [34].

3.5 Relation to AdS/CFT

In AdS/CFT we derived field theories from string theories by considering their large N limit. It has been shown in [36] that they contain in their Hilbert space excitations describing supergravity, and conjectured that the field theories are dual to the full quantum string theory on various spacetime. This duality can be used to give a definition of M-theory on flat spacetime as the large N limit of the field theories. Since the field theories can be defined non-perturbatively, this definition of M-theory is non-perturbative. The most obvious difference with M(atrix) theory is with the signification of N. In AdS/CFT, the N is related to the curvature and the size of the space where the theory is defined, whereas, in M(atrix) theory, it is interpreted as the momentum along a compact direction. However, in both cases, in the large N limit, we get flat and non-compact spaces. One of the crucial difference is that in AdS/CFT, it is much more clear how to recover supergravity in the large N limit. In M(atrix) theory, it hasn't been established yet whether the model is consistent with 11-dimensional supergravity. Since the Yang-Mills matrix model is defined in only 10 dimensions, it is not evident that it is the appropriate theory to describe 11-dimensional supergravity. However, 10-dimensional IIA supergravity is the dimensional reduction of 11-dimensional supergravity, so one can investigate if we can get IIA supergravity from Matrix string theory (see 3.7), which is itself obtained from M(atrix) theory. See [9] and [43] for an extension of AdS/CFT correspondence to the Matrix model of D-particles in the large N limit (generalized AdS/CFT correspondence). This could lead to a map between the two theories and enable us to take advantage of both approaches by using new tools from one description to be used in the other.

3.6 A better BFSS model

The BFSS model presented at the beginning of this chapter is supposed to described all the physics contained in M-theory. However, there are some restrictions in it. First, it has to be formulated in the IMF frame. Also, the dimensions of the matrices have to be taken to infinity. Another formulation proposed by Susskind [50] allows to get rid of the infinite value of N, if, instead of the IMF, one works in the DLCQ (Discrete Light-Cone Quantization) framework. The constraints on SYM implied by the dualities of M-theory, which were supposed to be true only for large N are in fact also true for N finite as long as we work in the context of DLCQ. The IMF and DLCQ are considered to be similar when $N \to \infty$, but when N is finite, they are different. In the DLCQ framework, the coordinate which is compactified is not the space-like coordinate x^{11}, but the light-like coordinate $x^- = \frac{1}{\sqrt{2}}(t - x^{11})$. Then, the quantized momentum is $p^- = \frac{N}{R}$. The new conjecture is that M-theory in the DLCQ is exactly described by $U(N)$ SYM, with N finite. To check the conjecture, one can work on perturbative or non-perturbatives evidences [11].

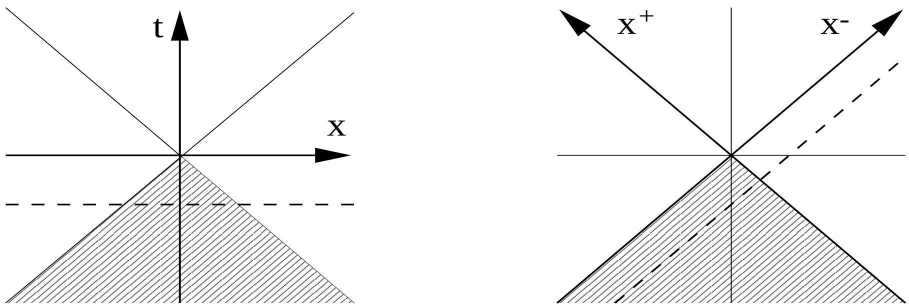

Figure 9: *Change of variables to the light cone frame for the position and time coordinates.*

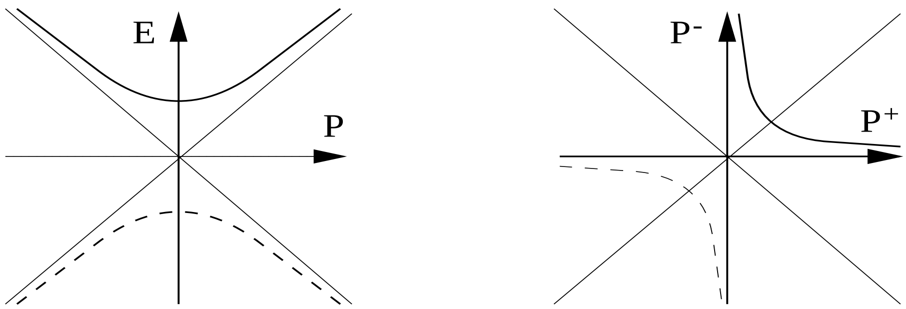

Figure 10: *Change of variables to the light cone frame for the momentum and energy coordinates.*

3.7 M(atrix) string theory

An interesting feature of M(atrix) theory is that with a few modifications, it can be used to give a nonperturbative definition of string theory. If we consider M(atrix) theory compactified in dimension 9 on a circle, we have a SYM theory in $1+1$ dimensions. In the BFSS model, this corresponds to M-theory compactified on T^2.

From the Hamiltonian for the BFSS model, we are going to derive the Hamiltonian for the matrix string theory. We start from (128), which I recall for convenience:

$$\mathcal{H}_{D_0} = R_{11}\, Tr\, \left[\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}[Y^i, Y^j]^2 + \theta^\top \gamma_i[\theta, Y^i]\right] \tag{178}$$

where $i = 1, ..., 9$. Type IIA is obtained from M-theory via the compactification of the 11th dimension (conventionally) on a circle S^1. Since the BFSS model describes M-theory, we need to compactify one transverse dimension of the BFSS Hamiltonian to be able to find a M(atrix) model for Type IIA. We compactify the 9th dimension on a circle of radius R_9. We apply a T-duality transformation along the S^1 directions, so we can identify Y^9 with the covariant derivative $R_9 D_\sigma$, where σ is the compactified coordinate running from 0 to 2π. The conjugate momentum is identified with the electric field $E = R_9\Pi_9$. This leads to the new Hamiltonian:

$$\mathcal{H} = \frac{R_{11}}{2\pi}\int\frac{d\sigma}{R_9}\,Tr\,\left[\frac{1}{2}\Pi_i\Pi_i + \frac{R_9^2}{4}(DY^i)^2 + R_9\theta^\top D\theta + \frac{1}{2R_9^2}E^2 + \frac{1}{4}[Y^i,Y^j]^2 + \theta^\top\gamma_i[\theta,Y^i]\right] \quad (179)$$

where $i = 1, ..., 8$. If we rescale the coordiantes as $Y^i \to R_9^{-1/2}Y^i$, we get:

$$\mathcal{H} = \frac{R_{11}}{2\pi}\int d\sigma\,Tr\,\left[\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}(DY^i)^2 + \theta^\top D\theta + \frac{1}{R_9^3}(E^2 + [Y^i,Y^j]^2) + \frac{1}{R_9^{3/2}}\theta^\top\gamma_i[\theta,Y^i]\right] \quad (180)$$

Once again, Type IIA is obtained from M-theory from compactifying the 11th dimension, which relates the string coupling constant g_s to the radius R_{11} by $g_s = (R_{11}/l_p)^{3/2}$. Since in the M(atrix) model, the 11th dimension is already compactified, we needed to compactify the 9th one. So, to arrive to a matrix string point of view, we need to interchange the role of the 9th and 11th direction, by defining the string scale $l_s = \sqrt{\alpha'}$ and string coupling constant g_s in terms of R_9 and the 11-dimensional Planck length l_p by:

$$R_9 = g_s l_s, \qquad l_p = g_s^{1/3} l_s \quad (181)$$

or $g_s = (R_9/l_p)^{3/2}$. From this we obtain the final result in string units where $l_s = 1$:

$$\mathcal{H} = \frac{R_{11}}{2\pi}\int d\sigma\,Tr\,\left[\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}(DY^i)^2 + \theta^\top D\theta + \frac{1}{g_s^2}(E^2 + [Y^i,Y^j]^2) + \frac{1}{g_s}\theta^\top\gamma_i[\theta,Y^i]\right] \quad (182)$$

The 8 scalar fields Y^i's and the 8 fermionic fields θ are $N\times N$ hermitian matrices. The fields transform under the representation of the symmetry group $SO(8)$ of transversal rotations. This Hamiltonian is of the form of the Green-Schwarz light-front string Hamiltonian of Type IIA, except that the fields are represented by non-commuting matrices. The eigenvalues of matrix coordinate Y^i are the coordinates of the fundamental Type IIA string, since in the original BFSS model, they represented the coordinates of the D_0-branes. T-duality transformation along the S^1 directions, turned the D_0-brane, from the BFSS model, into Type IIA D-strings.

4 Noncommutative geometry

The idea of noncommutative geometry is the replacement of the commutative algebra of function on a manifold by a noncommutative deformation of it. To make this construction, we start from a definition given to some geometric notion using algebra of functions with commutative geometry, and we replace these notions by noncommutative algebra. This type of geometry was introduced by von Neumann as ”pointless geometry” because in a quantum phase space, points are replaced by cells of size $\hbar$. The points of a quantized spacetime become fuzzy and are replaced with cells whose size is set by the noncommutative length scale θ. A string is replaced by a certain finite number of elementary volumes of ”fuzz”, each of which can contain one quantum mode. After defining the new noncommutative operations and constructing noncommutative Yang-Mills, we show how to get M(atrix) models from a noncommutative geometric approach.

4.1 Formalism

Let's consider two fields ϕ and ψ. A noncommutative field theory can be seen as a deformation of a classical quantum field theory by using the star product instead of the point product:

$$\phi(x)\cdot\psi(x) \Longrightarrow \phi(x)\star\psi(x) = e^{\frac{\imath}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^i}}\phi(x+\xi)\psi(x+\zeta)|_{\xi=\zeta=0} = \phi(x)\cdot\psi(x)+\frac{\imath}{2}\theta^{ij}\partial_i\phi\partial_j\psi+o(\theta^2) \tag{183}$$

This product is associative but obviously not commutative. Geometrically, the star product can be seen as generating a deformation of the ordinary canonical transformations, induced by θ. If we set θ to zero, we recover normal geometry. The commutation relation are:

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = \imath\theta^{\mu\nu} \tag{184}$$

$$[x^\mu, p^\nu]_\star = \imath\hbar\delta^{\mu\nu} \tag{185}$$

$$[p^\mu, p^\nu]_\star = 0 \tag{186}$$

Where $\theta^{\mu\nu}$ is a real antisymmetric matrix. In general, given any ordinary field theory, one obtains a noncommutative field theory by replacing all the dot products by star products

4.2 Yang-Mills, Noncommutative Yang-Mills and the appearance of matrices

Yang-Mills is the name given to non-abelian theories that have been constructed from abelian electromagnetic Maxwell's theory. The Maxwell equations are:

$$\partial_\mu F^{\mu\nu} = -j^\nu$$

$$\partial_\mu (*F^{\mu\nu}) = 0 \tag{187}$$

which are obtained from the action:

$$S = \int d^4x \Big(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Big) \tag{188}$$

with the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory has the symmetry group $U(1)$ which is abelian. In electrodynamics, we have the scalar fields ϕ action:

$$S = \int d^4x \Big(\partial_\mu \phi^* \partial^\mu \phi - V(\phi^*, \phi) \Big) \tag{189}$$

Which is also invariant under $U(1)$ and transforms as:

$$\phi \to \phi' = e^{-\imath\alpha} \phi \tag{190}$$

$$\phi^* \to \phi'^* = e^{\imath\alpha} \phi^* \tag{191}$$

Where α is the parameter of the group. This action is actually invariant only in the case of a global symmetry, where α doesn't depend on the coordiates x^μ. To make the action invariant under local transformation, one needs to introduce covariant derivatives:

$$\partial_\mu \phi \to D_\mu \phi = \partial_\mu \phi + \imath e A_\mu \phi \tag{192}$$

$$\partial_\mu \phi^* \to D_\mu \phi^* = \partial_\mu \phi^* - \imath e A_\mu \phi^* \tag{193}$$

where there is a coupling between the magnetic potential A_μ and the scalar fields ϕ. e is the coupling constant, and A_μ has to transforms as:

$$A_\mu \rightarrow A'_\mu = \frac{1}{e}\partial_\mu \alpha(x) \tag{194}$$

One can obtain the field strength $F_{\mu\nu}$ by taking the commutator of the covariant derivatives:

$$[D_\mu, D_\nu] = \imath e F_{\mu\nu} \tag{195}$$

We now turn to non-abelian groups, with gauge group $SU(N)$, where the fields transform under the transformation rules:

$$\phi_a \rightarrow \phi'_a = U_a^b \phi_b$$

$$(\phi^a)^* \rightarrow (\phi'^a)^* = (\phi^b)^* (U_b^a)^\dagger \tag{196}$$

where U is a element of the group. For a local symmetry invariance, the generalisation of the abelian transformations are:

$$\partial_\mu \phi_a \rightarrow D_\mu \phi_a = \partial_\mu \phi_a - \imath g A_\mu^k (T^k)_a^b \phi_b \tag{197}$$

$$A_\mu^k \rightarrow (A_\mu^k)' = \left[U(A_\mu^k T^k - \frac{\imath}{g} U^{-1} \partial_\mu U) U^{-1} \right]_a^b \tag{198}$$

where g is called Yang-Mills coupling and T^i's are the generators of the group, forming the algebra of the group:

$$\left[T^i, T^j\right] = \imath f^{ijk} T^k \tag{199}$$

The action is then invariant under the transformations (196). Since the symmetry is local, U is:

$$U_a^b = \left[exp(\imath \epsilon^k(x) T^k)\right]_a^b \tag{200}$$

where $\epsilon^i(x)$ are the group parameters. The Yang-Mills field strength is again obtained from the commutator of the covariant derivatives:

$$[D_\mu, D_\nu] = -\imath g \Big(\partial_\mu A_\nu^i - \partial_\nu A_\mu^i - \imath g \left[A_\mu^i, A_\nu^j\right] \Big) (T^i)_b^a \tag{201}$$

where we have:

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \imath g \left[A^i_\mu, A^j_\nu\right] \tag{202}$$

$F^i_{\mu\nu}$ which transforms as $\delta_{\epsilon^i} F^i_{\mu\nu} = -\imath \left[F^i_{\mu\nu}, \epsilon^i\right]$ under the gauge transformation:

$$\delta_{\epsilon^i} A^i_\mu = \partial_\mu \epsilon^i - \imath \left[A^i_\mu, \epsilon^i\right] \tag{203}$$

allows us to write the generalisation of the Maxwell action for a non-abelian gauge group:

$$S = \int d^4x \Big(-\frac{1}{4} F^{i\mu\nu} F^i_{\mu\nu} \Big) \tag{204}$$

and the Maxwell becomes:

$$D_\mu F^{i\mu\nu} = -J^\mu$$

$$D_\mu (*F^{i\mu\nu}) = 0 \tag{205}$$

Now that we have constructed the Yang-Mills action for a non-abelian gauge group, we just need to replace the dot product by the star product to obtain the expression of noncommutative Yang-Mills:

$$S_{NYM} = \int Tr \Big(-\frac{1}{4} \widehat{F}^{\mu\nu} \widehat{F}_{\mu\nu} \Big) \tag{206}$$

where the field strength is given by:

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - \imath [\widehat{A}_\mu, \widehat{A}_\nu]_\star = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - \imath [\widehat{A}_\mu, \widehat{A}_\nu] + o(\theta, (\partial \widehat{A})^2) \tag{207}$$

and its variation $\widehat{\delta}_{\widehat{\epsilon}^i} \widehat{F}^i_{\mu\nu} = -\imath \left[\widehat{F}^i_{\mu\nu}, \widehat{\epsilon}^i\right]$. We can expand $o(\theta, (\partial \widehat{A})^2)$ to the first order in θ:

$$o(\theta, (\partial \widehat{A})^2) = \frac{1}{2} \theta^{\rho\sigma} (\partial_\rho \widehat{A}_\mu \partial_\sigma \widehat{A}_\nu - \partial_\rho \widehat{A}_\nu \partial_\sigma \widehat{A}_\mu) + o(\theta^2)$$

This action is invariant under the transformation:

$$\widehat{A}_\mu \longrightarrow U \star \widehat{A}_\mu \star U^{-1} + \imath U \star \partial_\mu U^{-1} \tag{208}$$

with

$$U(x^\mu) = e^{\imath\theta^{-1}_{\mu\nu}a^\mu x^\nu} \tag{209}$$

and $U \star U^{-1} = U^{-1} \star U = \mathbb{I}$.

4.2.1 The Seiberg-Witten map

Now that we have a noncommutative generalization of gauge theories, we can work out the relationship with string theory. It turns out that noncommutative gauge theories arising from open strings theory imply that open string theory can always be thought of giving rise to ordinary gauge theory. One can see here a contradiction but in 1999, Seiberg and Witten [44] proposed a map that relates ordinary Yang-Mills vector potential A_μ with parameter ϵ and gauge transformation (203), to noncommutative Yang-Mills vector potential $\widehat{A}_\mu(A_\mu)$ with parameter $\widehat{\epsilon}^i(A_\mu, \epsilon^i)$ and gauge transformation:

$$\widehat{\delta}_{\widehat{\epsilon}^i}\widehat{A}_\mu = \widehat{\partial}_\mu\widehat{\epsilon}^i + \imath\widehat{A}_\mu \star \widehat{\epsilon}^i - \imath\widehat{\epsilon}^i \star \widehat{A}_\mu \tag{210}$$

such that we have:

$$\widehat{A}_\mu(A_\mu) + \widehat{\delta}\widehat{A}_\mu(A_\mu) = \widehat{A}_\mu(A + \delta A) \tag{211}$$

We write $\widehat{A}_\mu(A_\mu) = A_\mu + A'_\mu(A_\mu)$ and $\widehat{\epsilon}^i(A_\mu, \epsilon^i) = \epsilon^i + \epsilon'^i(\epsilon^i, A_\mu)$, with A' and ϵ' function of A and ϵ of order θ. When we expand (211) in power of θ using (183), we find:

$$A'_\mu(A_\mu+\delta_{\epsilon^i}A_\mu)-A'_\mu(A_\mu)-\partial_\mu\epsilon^i-\imath\left[\epsilon', A_\mu\right]-\imath\left[\epsilon^i, A'_\mu\right] = -\frac{1}{2}\theta^{\rho\sigma}(\partial_\rho\epsilon^i\partial_\sigma A_\mu+\partial_\sigma A_\mu\partial_\rho\epsilon^i)+o(\theta^2) \tag{212}$$

The solution of this equation to the first order in $\theta = \delta\theta$ is:

$$\begin{cases} \widehat{A}_\mu(A_\mu) - A_\mu = -\frac{1}{4}\delta\theta^{\rho\sigma}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\} \\ \widehat{\epsilon}^i(A_\mu, \epsilon^i) = \frac{1}{4}\delta\theta^{\rho\sigma}\{\partial_\rho\epsilon^i, A_\sigma\} \end{cases} \tag{213}$$

where $\{....\}$ are anticommutators. From this we get the first order relation between field strength:

$$\widehat{F}_{\mu\nu} - F_{\mu\nu} = \frac{1}{4}\delta\theta^{\rho\sigma}\Big(2\{F_{\mu\rho}, F_{\nu\sigma}\} - \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\}\Big) \tag{214}$$

These equations are called the Seiberg-Witten equations and are differential equation determining the map to all orders in θ.

For a rank one gauge field with constant F we have:

$$\delta\widehat{F} = -\widehat{F}\delta\theta\widehat{F} \tag{215}$$

which has the solution:

$$\widehat{F} = (1 + F\theta)^{-1}F \tag{216}$$

Deformation of gravity can be induced from a noncommutative gauge theory with position-dependent noncommutativity $\theta^{\mu\nu}(x^\mu)$ using this map.

4.2.2 The appearance of matrices from NYM

It was noticed that gravity is contained in the dynamics of noncommutative gauge theory through the observation that spacetime translations of noncommutative gauge fields are equivalent to gauge transformations. From (208), (209), and the identity $e^{\imath k\cdot x} \star \varphi(x^\mu) \star e^{-\imath k\cdot x} = \varphi(x^\mu - \theta^{\mu\nu}k_\nu)$, we find:

$$A_\mu(x^\mu) \longrightarrow A_\mu(x^\mu + a^\mu) - \theta^{-1}_{\mu\nu}a^\nu \tag{217}$$

Since a^ν is a constant shift, the noncommutativity of the field strength (207) disappear. Hence, the translation symmetry is a gauge symmetry and noncommutative gauge theories provide toy models of general relativity (simplified set of equations that can be used to understand a mechanism that is also useful in the non-simplified theory).

In order to identify the gravity gauge theory, we need to formulate the noncommutative gauge theory in an independent spacetime coordinates background. Then we introduce covariant coordinates:

$$X_\mu = \theta^{-1}_{\mu\nu}x^\nu + A_\mu \tag{218}$$

that we use to rewrite (207) as:

$$F_{\mu\nu} = -\imath\,[X_\mu, X_\nu]_\star + \theta^{-1}_{\mu\nu} \tag{219}$$

We can use this to rewrite the NYM action (206) as:

$$S = -\frac{1}{4} Tr\Big(-\imath \, [X_\mu, X_\nu] + \theta^{-1}_{\mu\nu}\Big)^2 \tag{220}$$

This action is expressed in terms of operators X_μ which are no longer regarded as position coordinates since we are not in a spacetime background. It is totally spacetime independent and X_μ are abstract objects of an infinite-dimensional (since we have no restriction on μ) matrix algebra. It is therefore called a Matrix model. We can get the equations of motions:

$$[X_\mu, [X_\mu, X_\nu]] = 0 \tag{221}$$

with the vacuum solution ($A_\mu = 0$) satisfying:

$$[X_\mu, X_\nu] = -\imath\theta^{-1}_{\mu\nu} \tag{222}$$

One can notice that the equation of motion (221) is similar to the equation of motion (157) for the IKKT matrix model. If one sets $\theta = 0$, we have from (220) a matrix model for a commutative YM theory:

$$S = \frac{1}{4} Tr\Big([X_\mu, X_\nu] \, [X_\mu, X_\nu] \Big) \tag{223}$$

This model will be used later on in the construction of the Emergent noncommutative gravity. One can also construct the BFSS and IKKT model independently of string theory, from a noncommutative approach [46].

5 Low energy limit of M-theory: 11D supergravity

This theory was built in the late 70's by Cremmer, Julia and Schrerk [17], as an attempt for a grand unified theory. The first superstring revolution in the mid 80's saw this theory abandoned for superstring theory. But, during the second superstring revolution in mid 90's, it was discovered that the strong coupling limit is 11-dimensional supergravity. We present how the Lagrangian was originally constructed , as well as the derivation of the equations of motion. To avoid any too complicated calculation, the most important results are given here, and the details are explained in appendix C.

5.1 The construction of the Lagrangian

Since supergravity is the unification of general relativity and supersymmetry, it must contains the graviton, which is a $spin = 2$ boson, and its superpartner, the gravitino, which is a $spin = \frac{3}{2}$ fermion. Thus the natural starting point for the action is the Einstein-Hilbert action and the Rarita-Schwinger action (which is the equivalent of the Dirac equation but for $s = \frac{3}{2}$ particles):

$$S_1 = \int dx^D \sqrt{g}\Big[\frac{1}{4}R + \frac{1}{2}\overline{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho\Big] \tag{224}$$

The $\Gamma^{\mu\nu\rho}$ are the 32 dimensional Pauli matrices and satisfy the 11 dimensional Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta_{\mu\nu}\mathbb{I}_{32}$$

$$\Gamma^{\mu_1 \cdots \mu_n} = \Gamma^{[\mu_1}\Gamma^{\mu_2}...\Gamma^{\mu_n]}$$

The covariant derivative is given by:

$$D_\nu(\omega)\psi_\mu = \partial_\nu\psi_\mu - \frac{1}{4}\omega_{\nu ab}\,\Gamma^{ab}\psi_\mu$$

This action is invariant under the SUSY transformations:

$$\delta_Q\psi_\mu = D_\mu(\omega)\epsilon \tag{225}$$

$$\delta_Q e^a_\mu = \overline{\epsilon}\Gamma^a\psi_\mu \quad \Longrightarrow \quad \delta_Q g_{\mu\nu} = \delta_Q(\eta_{ab}e^a_\mu e^b_\nu) = 2\overline{\epsilon}\Gamma_\mu\psi_\nu$$

where we have used the tetrad basis $e^a = e^a_\mu dx^\mu$. When we vary the action we get:

$$\delta_Q S = \int dx^D \sqrt{g}\Big[\frac{1}{2}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\overline{\epsilon}\Gamma^{(\mu}\psi^{\nu)} - \frac{1}{2}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\overline{\epsilon}\Gamma^{(\mu}\psi^{\nu)}\Big]\delta_Q g^{\mu\nu} \tag{226}$$

which is identically equal to zero. We have used the symmetry of the Riemmann tensor ($R_{\mu[\nu\alpha\beta]=0}$). Then, we have proved that the variation of the action (224) is equal to zero. However, a third order fermionic term has been neglected. We need to add an additional term to cancel them. The good one is the kinetic term for the three form potential $C_{\mu\nu\rho}$:

$$S_2 = \int dx^D \sqrt{g}\Big[-\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big] \tag{227}$$

where $G_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]}$. By rewriting this as:

$$-\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} = -\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G_{\alpha\beta\gamma\delta}g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}$$

we see that, according to the variation law (225), we are going to get a new contribution $-\frac{1}{24}\bar{\epsilon}\Gamma^{\mu}\psi^{\nu}(G)^2_{\mu\nu}$. There is two indices of G contracted by the variation of $\sqrt{g}$ (see (C.2.1)). We must balance this additional term by adding something involving G with two indices contracted with an unknown quantity X: $\overline{\psi}_{\mu}(XG)^{\mu\rho}\psi_{\rho}$. We then should modify the SUSY transformation of ψ_{μ} as:

$$\delta_Q\psi_{\mu} = D_{\mu}(\omega)\epsilon + (ZG)_{\mu}\epsilon \equiv \hat{D}_{\mu}(\omega)\epsilon \tag{228}$$

Where Z is the unknown quantity that contracts 3 indices of G. The action now looks like:

$$S_3 = \int dx^D\sqrt{g}\Big[\frac{1}{4}R + \frac{1}{2}\overline{\psi}_{\mu}\Gamma^{\mu\nu\rho}D_{\nu}(\omega)\psi_{\rho} - \frac{1}{4\cdot 48}\overline{\psi}_{\mu}(XG)^{\mu\rho}\psi_{\rho} - \frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big] \tag{229}$$

The method to find the expression of $(XG)^{\mu\rho}$ is to write down all the possible terms consistent with the tensor structure with two free indices $\mu\nu$. The Γ matrices and $g^{\mu\nu}$ are the objects used for the construction since they can raise and lower indices. We consider $G^{\alpha\beta\gamma\delta}$ with its all four indices. First, we contract all of them. We need two free, so we have two possible terms: $g^{\mu\nu}\Gamma_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta}$ or $\Gamma^{\mu\nu}{}_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta}$. If we contract two of them, we simply have $\Gamma_{\alpha\beta}G^{\mu\nu\alpha\beta}$. If we contract 3, we have $\Gamma^{\mu}{}_{\alpha\beta\gamma}G^{\nu\alpha\beta\gamma}$ which can be decomposed into symmetric and antisymmetric parts in $\mu\nu$. After reorganizing this we have:

$$(XG)^{\mu\nu} = \underbrace{a\,\Gamma^{(\mu}{}_{\alpha\beta\gamma}G^{\nu)\alpha\beta\gamma} + b\,g^{\mu\nu}\Gamma_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta}}_{symmetric} + \underbrace{c\,\Gamma^{\mu\nu}{}_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} + d\,\Gamma_{\alpha\beta}G^{\mu\nu\alpha\beta} + e\,\Gamma^{[\mu}{}_{\alpha\beta\gamma}G^{\nu]\alpha\beta\gamma}}_{antisymmetric} \tag{230}$$

where a, b, c, d, e are real constants. It turns out that the only remaining constants are $c = -\frac{1}{8\cdot 4!}$ and

$d = -\frac{12}{8\cdot 4!}$, and therefore we have the expression of $(XG)^{\mu\nu}$:

$$(XG)^{\mu\nu} = -\frac{1}{8\cdot 4!}\Gamma^{\mu\nu}{}_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} - \frac{12}{8\cdot 4!}\Gamma_{\alpha\beta}G^{\mu\nu\alpha\beta} \tag{231}$$

The same analysis for $(ZG)^{\mu}$ gives:

$$(ZG)^{\mu} = -\frac{1}{2\cdot 144}\Gamma^{\alpha\beta\gamma\delta}{}_{\mu}G_{\alpha\beta\gamma\delta} + \frac{8}{2\cdot 144}\Gamma^{\alpha\beta\gamma\delta}\delta^{\alpha}_{\mu}G^{\alpha\beta\gamma\delta} \tag{232}$$

The variation (228) is now:

$$\delta_Q\psi_\mu = D_\mu\epsilon - \frac{1}{2\cdot 144}\Big(\Gamma^{\alpha\beta\gamma\delta}{}_{\mu} - 8\Gamma^{\beta\gamma\delta}\delta^{\alpha}_{\mu}\Big)\epsilon G_{\alpha\beta\gamma\delta} \tag{233}$$

We now perform the variation of (229) to see wether everything cancels out and if we have a consistent action. We consider only the new terms $\psi_\mu(XG)^{\mu\rho}\psi_\rho$ and $G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}$:

$$\delta_Q S_3 = \int dx^D\Big[\underbrace{\delta_Q\Big(-\frac{\sqrt{g}}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big)}_{I} + \underbrace{\delta_Q\Big(\frac{\sqrt{g}}{4\cdot 48}\overline{\psi}_\mu(XG)^{\mu\rho}\psi_\rho\Big)}_{II}\Big] \tag{234}$$

$$\delta_Q(S_3)_I = \delta_Q\Big(-\frac{\sqrt{g}}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big) = -\frac{1}{4\cdot 48}G^2\ \delta_Q(\sqrt{g}) + \sqrt{g}\ \delta_Q\Big(-\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G_{\alpha\beta\gamma\delta}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}\Big) \tag{235}$$

with (246) and (225) we find:

$$\delta_Q(S_3)_I = -\frac{1}{24}\Big((G^2)_{\mu\nu} - \frac{1}{8}G^2g_{\mu\nu}\Big)\overline{\epsilon}\ \Gamma^{(\mu}\psi^{\nu)} \tag{236}$$

We must calculate the variation of $\bar{\psi}$ in order to be able to find the variation of $(S_3)_{II}$. With:

$$\delta_Q\bar{\psi}_\mu = (D_\mu\epsilon)^\top\Gamma^0 + \frac{1}{2\cdot 144}\bar{\epsilon}\Big(\Gamma^{\alpha\beta\gamma\delta}{}_{\mu} + 8\Gamma^{\beta\gamma\delta}\delta^{\alpha}_{\mu}\Big)G_{\alpha\beta\gamma\delta} \tag{237}$$

we find:

$$\delta(S_3)_{II} = -\frac{2}{3\cdot 32\cdot (12)^3}\Big[\bar{\epsilon}\Big(\Gamma^{\rho\sigma\eta\tau}{}_{\mu} + 8\Gamma^{\sigma\eta\tau}\delta^{\rho}_{\mu}\Big)\Gamma^{\mu\xi\nu}\Big(\Gamma^{\rho\sigma\eta\tau}{}_{\xi} - 8\Gamma^{\sigma\eta\tau}\delta^{\rho}_{\xi}\Big)\psi_\nu\Big]G_{\rho\sigma\eta\tau}G_{\alpha\beta\gamma\delta} \tag{238}$$

After some calculation using identities of the Clifford algebra of the gamma matrices (see [37]), we find that the variation of the total action S_3 (229) does not vanish, and we have the left term:

$$\delta_Q S_3 = \int d^D x \Big[\frac{9}{4\cdot(12)^4}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho} G_{\alpha_1\ldots\alpha_4} G_{\beta_1\ldots\beta_4}\,\bar\epsilon\,\Gamma_{[\mu\nu}\psi_{\rho]}\Big] \tag{239}$$

Once again, we need to balance this undesired term by adding something new. Based on the form of (301), one makes the ansatz of the compensating term, called the Chern-Simons term:

$$S_{CS} = \int d^D x \Big[\frac{1}{4\cdot(12)^4}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho} G_{\alpha_1\ldots\alpha_4} G_{\beta_1\ldots\beta_4}\, C_{\mu\nu\rho}\Big] \tag{240}$$

with a super-transformation of the potential satisfying:

$$\delta_Q C_{\mu\nu\rho} = a\,\bar\epsilon\,\Gamma_{[\mu\nu}\psi_{\rho]} \tag{241}$$

with $a \in \mathbb{R}$. When we perform the variation of the CS term, one finds:

$$\delta_Q S_{CS} = \int d^D x \Big[\frac{3}{4\cdot(12)^4}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho} G_{\alpha_1\ldots\alpha_4} G_{\beta_1\ldots\beta_4}\, a\,\bar\epsilon\,\Gamma_{[\mu\nu}\psi_{\rho]}\Big] \tag{242}$$

which is identical to (301). If one sets $a = 3$, the action $S = S_3 + S_{CS}$ vanishes, and the total Lagrangian of 11-dimensional Supergravity finally reads:

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{4}\,eR + \frac{1}{2}\,e\bar\psi_\mu\Gamma^{\mu\nu\rho}D_\nu\left(\frac{\omega+\hat\omega}{2}\right)\psi_\rho - \frac{1}{4\cdot 48}\,eG_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} \\ & -\frac{1}{4\cdot 48}\,e\left(\bar\psi_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\,\bar\psi^\alpha\Gamma^{\gamma\delta}\psi^\beta\right)\left(\frac{G_{\alpha\beta\gamma\delta}+G_{\alpha\beta\gamma\delta}}{2}\right) \\ & +\frac{1}{4\cdot 144^2}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}\,. \end{aligned} \tag{243}$$

where we added the fermionic terms and used the notations:

$\omega_{\mu ab} = \omega^{(0)}_{\mu ab} + \frac{1}{4}\left[\bar\psi_\alpha\Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta - 2\left(\bar\psi_\mu\Gamma_b\psi_a - \bar\psi_\mu\Gamma_a\psi_b + \bar\psi_b\Gamma_\mu\psi_a\right)\right]$

$\hat G_{\mu\nu\rho\sigma} = G_{\mu\nu\rho\sigma} + 6\bar\psi_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]}$ $\qquad \hat\omega_{\mu ab} = \omega_{\mu ab} - \frac{1}{4}\bar\psi_\alpha\Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta$

5.2 Equation of motion

5.2.1 Graviton $g_{\mu\nu}$

We now derive the equation of motion for the graviton. The action to be considered here is the Einstein-Hilbert action coupled to the field strength $G_{\mu\nu\rho\sigma}$. Therefore, we should expect to find the Einstein equation plus an additional term:

$$S = \int d^D x \sqrt{g} R - \frac{1}{48} \int d^D x \sqrt{g} e G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} = S + S' \tag{244}$$

• Let us consider first the variation of the Einstein-Hilbert action:

$$\delta S = \int d^D x \Big[\delta(\sqrt{g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{g} \delta(g^{\mu\nu}) R_{\mu\nu} + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} \Big] \tag{245}$$

Using the identity:

$$\delta(\sqrt{g}) = \frac{1}{2} \frac{1}{\sqrt{g}} \delta g = \frac{1}{2} \frac{1}{\sqrt{g}} g g^{jk} \delta g_{jk} = \frac{1}{2} \sqrt{g} g^{jk} \delta g_{jk} = -\frac{1}{2} \sqrt{g} g^{jk} \delta g_{jk} \tag{246}$$

We find:

$$\delta S = \int d^D x \sqrt{g} \Big[-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} \Big] \delta g^{\mu\nu} + \int d^D x \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} \tag{247}$$

We see that the first term is the Einstein tensor. Therefore, we want the second term to vanish. Using Stoke's theorem, one can show that the term $\delta R_{\mu\nu}$ does not contribute (see appendix C).

• Using the variation $\delta\sqrt{g}$, given above, we write the variation of the second part of the action:

$$\delta S' = -\frac{e}{48} \int d^D x \Big[-\frac{1}{2} \sqrt{g} g_{\alpha\beta} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} + 4\sqrt{g} G_{\alpha\nu\rho\sigma} G_{\beta}^{\nu\rho\sigma} \Big] \delta g^{\alpha\beta} \tag{248}$$

The equation of motion for the whole action now reads:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{e}{48} \Big[-\frac{1}{2} g_{\alpha\beta} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} + 4\sqrt{g} G_{\alpha\nu\rho\sigma} G_{\beta}^{\nu\rho\sigma} \Big] \tag{249}$$

By contracting this with $g^{\beta\alpha}$, we get:

$$R = \frac{e}{144} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} \tag{250}$$

5.2.2 3-form potential $C_{\mu\nu\rho}$

The part of the supergravity action involving the 3-form potential $C_{\mu\nu\rho}$ is:

$$\mathcal{L} = \frac{1}{48} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} + \frac{1}{144^2} \epsilon^{\alpha_1 \dots \alpha_4 \beta_1 \dots \beta_4 \gamma_1 \gamma_2 \gamma_3} G_{\alpha_1 \dots \alpha_4} G_{\beta_1 \dots \beta_4} G_{\gamma_1 \gamma_2 \gamma_3} \tag{251}$$

To find the equation of motion, one can solve the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial C_{ijk}} - \partial_\xi \left[\frac{\partial \mathcal{L}}{\partial(\partial_\xi C_{ijk})} \right] = 0$. Using the following identities:

$$\frac{\partial G_{\mu\nu\rho\sigma}}{\partial(\partial_\xi C_{ijk})} = \delta^{\xi ijk}_{\mu\nu\rho\sigma} \tag{252}$$

and

$$\delta^{\alpha_1 \dots \alpha_4}_{\beta_1 \dots \beta_4} G^{\beta_1 \dots \beta_4} = 4! G^{\alpha_1 \dots \alpha_4} \tag{253}$$

we have the equation of motion:

$$\partial_\xi G^{\xi ijk} + \frac{18}{144^2} \epsilon^{\alpha_1 \dots \alpha_4 \beta_1 \dots \beta_4 ijk} G_{\alpha_1 \dots \alpha_4} G_{\beta_1 \dots \beta_4} = 0 \tag{254}$$

This equation of motion can be expressed in a simpler form using the language of forms (totally anti-symmetric tensors). I recall some of the definitions and properties of this language in the appendix B. The first term $\partial_\xi G^{ijk}$ must be of the form $*d(*G)$ since we want to rewrite the equation in term of three form (number of free indices). Indeed, G is a 4-form. Since we are in 11 dimensions, $*G$ is a $11 - 4 = 7$-form. $(d * G)$ is a 8-form and so $(*d * G)$ is a 3-form. Starting from the coordinates generalization $G = \frac{1}{4!} G_{\mu_1 \mu_2 \mu_3 \mu_4} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_4}$, and using $\epsilon_{\nu_1 \dots \nu_{11}} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \nu_4 \dots \nu_{11}} = -8! \delta^{\alpha_1 \alpha_2 \alpha_3}_{\nu_1 \nu_2 \nu_3}$, we end up with:

$$*d(*G)_{\nu_1 \nu_2 \nu_3} = \partial^\xi G_{\xi \nu_1 \nu_2 \nu_3} \tag{255}$$

Doing the same analysis, we find that the second term should be written as $*(G \wedge G)$:

$$\left[* (G \wedge G) \right]_{\nu_1 \nu_2 \nu_3} = \frac{1}{4!} \epsilon_{\nu_1 \nu_2 \nu_3 \alpha_1 \dots \alpha_4 \beta_1 \dots \beta_4} G^{\alpha_1 \dots \alpha_4} G^{\beta_1 \dots \beta_4} \tag{256}$$

Using (255) and (256), we can rewrite the equation of motion as follow:

$$d(*G) + \frac{1}{2} G \wedge G = 0 \tag{257}$$

5.2.3 Gravitino ψ_μ

The part of the Lagrangian involving the gravitino is:

$$\mathcal{L} = \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu\Big(\frac{\omega+\hat{\omega}}{2}\Big)\psi_\rho - \frac{1}{4\cdot 48}\Big(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\gamma\delta}\psi^\beta\Big)\Big(\hat{G}_{\alpha\beta\gamma\delta} - 3\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\Big) \quad (258)$$

When we solve the Euler-Lagrange equation for $\bar{\psi}$ and ψ, we find the equation of motion:

$$\begin{aligned} 0 = \frac{1}{2}\Big[\Gamma^{\xi\nu\rho} D_\nu(\hat{\omega}) - \frac{1}{96}\Big(\Gamma^{\xi\rho}_{\alpha\beta\gamma\delta} + 12\delta^\xi_\alpha \Gamma_{\gamma\delta}\delta^\rho_\beta\Big)\hat{G}^{\alpha\beta\gamma\delta}\Big]\psi_\rho \\ -\frac{1}{64}\Gamma^{\xi\nu\rho}\Gamma^{ab}\psi_\rho \bar{\psi}_\alpha \Gamma^{\alpha\beta}_{\nu ab}\psi_\beta + \frac{1}{64}\Gamma^{\xi\beta}_{\nu ab}\psi_\beta \bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho \\ +\frac{1}{64}\Big(\Gamma^{\xi\nu}_{\alpha\beta\gamma\delta}\psi_\nu \bar{\psi}^{[\alpha}\Gamma^{\beta\gamma}\psi^{\delta]} - \delta^\xi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu\Big) \end{aligned} \quad (259)$$

We find that the last four terms vanish using the Cremmer-Julia-Scherk Fierz identity:

$$\begin{aligned} \frac{1}{8}\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu \bar{\psi}_\alpha\,\Gamma_{\beta\gamma} \;-\; \frac{1}{8}\Gamma_{\beta\gamma}\,\psi_\nu \bar{\psi}_\alpha\,\Gamma^{\mu\nu\alpha\beta\gamma\delta} \\ -\frac{1}{4}\Gamma^{\mu\nu\alpha\beta\delta}\psi_\nu \bar{\psi}_\alpha \Gamma_\beta \;+\; \frac{1}{4}\Gamma_\beta \psi_\nu \bar{\psi}_\alpha \Gamma^{\mu\nu\alpha\beta\delta} \\ -2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\psi_\nu \bar{\psi}_\alpha \Gamma_\beta \;-\; 2\,\Gamma_\beta \psi_\nu \bar{\psi}_\alpha g^{\beta[\alpha}\Gamma^{\delta\mu\nu]} \\ +2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\,\bar{\psi}_\alpha \Gamma_\beta \psi_\nu \;=\; 0 \end{aligned} \quad (260)$$

and using the identity:

$$3\Big(\Gamma^{\mu\rho}_{\alpha\beta\gamma\delta} + 12\delta^\mu_\alpha \Gamma_{\gamma\delta}\delta^\rho_\beta\Big)\psi_\rho \hat{G}^{\alpha\beta\gamma\delta} = \Gamma^{\mu\nu\rho}\Big(\Gamma_\nu^{\alpha\beta\gamma\delta} - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_\nu\Big)\psi_\rho \hat{G}_{\alpha\beta\gamma\delta} \quad (261)$$

the equation of motion can be written is a simple form:

$$\Gamma^{\mu\nu\rho}\hat{D}_\nu \psi_\rho = 0 \quad (262)$$

where the covariant derivative is:

$$\hat{D}_\nu \psi_\rho = D_\nu(\hat{\omega})\psi_\rho - \frac{1}{2\cdot 144}\Big(\Gamma_\nu^{\alpha\beta\gamma\delta} - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_\nu\Big)\psi_\rho \hat{G}_{\alpha\beta\gamma\delta}$$

6 Discussion

The relationship between matrix theory and noncommutative geometry is not very clear and is still an active area of research. There is a lot of aspects of both theories that we couldn't cover due to a lack of time. In [46], they show how to get BFSS and IKKT directly from noncommutative geometry. The solution of 11-dimenional gravity supergravity, can be found in [20] for the $M2$-brane and in [24] for the $M5$-brane. In [37], they show how to get Type IIA superstrings from 11-dimensional supergravity, and they calculate intersecting $M5$-$M5$ branes solutions. A very little progress has been made recently in "pure" models of M-theory or string theory in general. However, their applications to other fields has been investigated especially in cosmology for the study of black holes for example [6] [23]. In [40] they propose an alternative model of inflation based on a recent formulation in terms of coherent states of noncommutative quantum field theory. A very recent model has been proposed, based on the IKKT model [33] [47] [48] [49] called emergent noncommutative gravity. They show that the Yang-Mills matrix model action for noncommutative $U(N)$ gauge theory (223), describres $SU(N)$ gauge theory coupled to gravity. Those kind of models have noncommutative branes as solutions, which, when embedded in $\mathbb{R}^{10}$, give rise to a dynamical effective metric, governing the dynamics of the fields on the brane. The resulting geometry is therefore dynamical governed by the matrix model and its effective action which contains the Einstein-Hilbert term. One can say that gravity emerges from noncommutative gauge theory. Having an effective metric simplifies the quantization, since, the metric is not the fundamental degree of freedom. What is quantized is the matrix model action rather than the Einstein-Hilbert action. As we saw in this paper, different models can be obtained from NYM. But there is a prime candidate as a model for an emergent noncommutative gravity. Indeed, the theory is expected to be finite leading to the identification of the Planck scale, and therefore provide a well-defined quantum theory of fundamental interactions including gravity. This is possible in the case of maximally supersymmetry, which is the IKKT model in 10 dimensions. The strong point of this model is that it solves the cosmological constant problem since the results they obtain are in good agreement with observation, which hasn't been the case so far. On the other hand, an analog of the Schwarzschild solution is yet to be found.

A Useful identities

• $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$ • $\frac{\partial G_{\mu\nu\rho\sigma}}{\partial(\partial_\xi C_{ijk})} = \delta^{\xi ijk}_{\mu\nu\rho\sigma}$ • $\delta^{\alpha_1\ldots\alpha_4}_{\beta_1\ldots\beta_4}G^{\beta_1\ldots\beta_4} = 4!G^{\alpha_1\ldots\alpha_4}$

• $\epsilon_{\nu_1\ldots\nu_{11}}\epsilon^{\alpha_1\alpha_2\alpha_3\nu_4\ldots\nu_{11}} = -8!\delta^{\alpha_1\alpha_2\alpha_3}_{\nu_1\nu_2\nu_3}$ • $D_{[\mu}D_{\nu]}\epsilon = \frac{1}{8}R^{\alpha\beta}_{\mu\nu}\Gamma_{\alpha\beta}\epsilon$ • $\overline{\Gamma^j\epsilon} = (-1)^{\frac{(j+1)j}{2}}\bar{\epsilon}\Gamma^j$

• $3\Big(\Gamma^{\mu\rho}_{\alpha\beta\gamma\delta} + 12\delta^\mu_\alpha\Gamma_{\gamma\delta}\delta^\rho_\beta\Big)\psi_\rho\hat{G}^{\alpha\beta\gamma\delta} = \Gamma^{\mu\nu\rho}\Big(\Gamma^{\alpha\beta\gamma\delta}_\nu - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_\nu\Big)\psi_\rho\hat{G}_{\alpha\beta\gamma\delta}$

• $\frac{1}{8}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu\bar{\psi}_\alpha\,\Gamma_{\beta\gamma} \;-\; \frac{1}{8}\,\Gamma_{\beta\gamma}\,\psi_\nu\bar{\psi}_\alpha\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}$
$-\;\frac{1}{4}\,\Gamma^{\mu\nu\alpha\beta\delta}\psi_\nu\bar{\psi}_\alpha\Gamma_\beta \;+\; \frac{1}{4}\,\Gamma_\beta\psi_\nu\bar{\psi}_\alpha\Gamma^{\mu\nu\alpha\beta\delta}$
$-\;2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\psi_\nu\bar{\psi}_\alpha\Gamma_\beta \;-\; 2\,\Gamma_\beta\psi_\nu\bar{\psi}_\alpha g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}$
$+\;2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\,\bar{\psi}_\alpha\Gamma_\beta\psi_\nu \;=\; 0$ (Cremmer-Julia-Scherk Fierz identity)

• $\Gamma^{a_j\ldots a_1}\Gamma_{b_1\ldots b_k} = \sum_{l=0}^{\min(j,k)} l! \begin{pmatrix} j \\ l \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \delta^{[a_1}_{[b_1}\cdots\delta^{a_l}_{b_l}\,\Gamma^{a_j\ldots a_{l+1}]}{}_{b_{l+1}\ldots b_k]}$

• $\overline{\psi}_\mu\Gamma^{(j)}\epsilon = -(-1)^{\frac{j(j+1)}{2}}\bar{\epsilon}\Gamma^{(j)}\psi_\mu$ • $e^{\imath k\cdot x}\star\varphi(x^\mu)\star e^{-\imath k\cdot x} = \varphi(x^\mu - \theta^{\mu\nu}k_\nu)$

B The language of Differential Forms

B.1 Wedge product

Given a p-form and a q-form, we can construct a $(p+q)$-form using the wedge product $A\wedge B$ by taking the antisymmetrized tensor product:

$$(A\wedge B)_{\mu_1\ldots\mu_{(p+q)}} = \frac{(p+q)!}{p!q!}A_{[\mu_1\ldots\mu_p}B_{\mu_{p+1}\ldots\mu_{p+q}]} \tag{263}$$

For example, if we have two 1-forms we have

$$A_\mu\wedge B_\nu = A_\mu B_\nu - A_\nu B_\mu = 2A_{[\mu}B_{\nu]} \tag{264}$$

We also have the following properties:

$$A\wedge B = (-1)^{p+q}B\wedge A$$

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

$$\omega_{\mu_1 \dots \mu_p} \wedge \omega_{\mu_1 \dots \mu_p} = 0 \tag{265}$$

if p is odd. The coordinates generalisation of a p-form is:

$$\omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{266}$$

B.2 Exterior derivative

The exterior derivative d allows us to differentiate a p-form to obtain a $(p+1)$-form as follow:

$$(dA)_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} \tag{267}$$

It satisfies the Leibniz rule:

$$d(\omega \wedge \xi) = (d\omega) \wedge \xi + (-1)^p \omega \wedge (d\xi) \tag{268}$$

where ξ is a p-form, and the property

$$d^2 A = d(dA) = 0 \tag{269}$$

for any form.

B.3 Hodge duality

We define the hodge star operator on a D-dimensional manifold as a map from p-forms to $(D-p)$-forms:

$$(*A)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!}\epsilon^{\nu_1 \dots \nu_p}_{\mu_1 \dots \nu_{D-p}} A_{\nu_1 \dots \nu_p} \tag{270}$$

$$(**A) = (-1)^{s+p(D-p)} A \tag{271}$$

For example, the hodge dual of the field strength $F_{\mu\nu}$ is:

$$*F_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}_{\mu\nu} F_{\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \tag{272}$$

C Supergravity calculations

C.1 The construction of the Lagrangian

Einstein-Hilbert and Rarita-Schwinger action:

$$S_1 = \int dx^D \sqrt{g}\Big[\frac{1}{4}R + \frac{1}{2}\overline{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho\Big] \tag{273}$$

The $\Gamma^{\mu\nu\rho}$ are the 32 dimensional Pauli matrices and satisfy the 11 dimensional Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta_{\mu\nu}\mathbb{I}_{32}$$

$$\Gamma^{\mu\nu} = \Gamma^{[\mu}\Gamma^{\nu]}$$

The gamma matrices can be expressed with the Pauli matrices τ_i:

$$\Gamma^0 = -\imath\tau_2 \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \tau_3 \quad , \quad \Gamma^1 = \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_2 \otimes \tau_1$$

$$\Gamma^2 = -\tau_2 \otimes \mathbb{I} \otimes \tau_1 \otimes \tau_2 \otimes \tau_1 \quad , \quad \Gamma^3 = -\tau_2 \otimes \mathbb{I} \otimes \tau_3 \otimes \tau_2 \otimes \tau_1$$

$$\Gamma^4 = -\tau_2 \otimes \tau_1 \otimes \tau_2 \otimes \mathbb{I} \otimes \tau_1 \quad , \quad \Gamma^5 = -\tau_2 \otimes \tau_3 \otimes \tau_2 \otimes \mathbb{I} \otimes \tau_1$$

$$\Gamma^6 = -\tau_2 \otimes \tau_2 \otimes \mathbb{I} \otimes \tau_1 \otimes \tau_1 \quad , \quad \Gamma^7 = -\tau_2 \otimes \tau_2 \otimes \mathbb{I} \otimes \tau_3 \otimes \tau_1$$

$$\Gamma^8 = -\tau_2 \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \tau_2 \quad , \quad \Gamma^9 = \tau_1 \otimes \mathbb{I}_{16}$$

$$\Gamma^{10} = \tau_3 \otimes \mathbb{I}_{16}$$

The covariant derivative is given by:

$$D_\nu(\omega)\psi_\mu = \partial_\nu\psi_\mu - \frac{1}{4}\omega_{\nu ab}\,\Gamma^{ab}\psi_\mu$$

This action is invariant under the SUSY transformations:

$$\delta_Q\psi_\mu = D_\mu(\omega)\epsilon \tag{274}$$

$$\delta_Q e^a_\mu = \bar{\epsilon}\Gamma^a\psi_\mu \quad \Longrightarrow \quad \delta_Q g_{\mu\nu} = \delta_Q(\eta_{ab}e^a_\mu e^b_\nu) = 2\bar{\epsilon}\Gamma_\mu\psi_\nu$$

When we vary the action we get:

$$\delta_Q S_1 = \int dx^D\sqrt{g}\Big[\frac{1}{4}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta_Q g^{\mu\nu} + \frac{1}{2}\delta_Q\overline{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho - \frac{1}{2}\overline{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)\delta_Q\psi_\rho$$

$$+\frac{3}{2}\overline{\psi}_{[\mu}\delta_Q g^{\mu\alpha}\Gamma^{\nu\rho}{}_\alpha D_\nu(\omega)\psi_\rho\Big] \tag{275}$$

The derivation of the first part of the variation (Einstein equation) is derived in the section (C.2.1). The last term involves fermionic fields to the third order, which are gonna be cancelled by terms yet to be added to the action. We can neglect them for the time being. Let's calculate the second term, rewritten with (274) as $\frac{1}{2}\overline{D_\mu\epsilon}\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho$. By integrating by parts:

$$\int dx^D\sqrt{g}\Big[\frac{1}{2}\overline{D_\mu\epsilon}\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho\Big] = \frac{1}{2}\bar{\epsilon}\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho + \int dx^D\sqrt{g}\Big[-\frac{1}{2}\bar{\epsilon}\Gamma^{\mu\nu\rho}D_\mu(\omega)D_\nu(\omega)\psi_\rho\Big] \tag{276}$$

On the right-hand side, only the second term remains. Using the identity:

$$D_{[\mu}D_{\nu]}\epsilon = \frac{1}{8}R_{\mu\nu}{}^{\alpha\beta}\Gamma_{\alpha\beta}\epsilon$$

and the fact that (276) is antisymmetric in $[\mu\nu]$, (276) becomes:

$$\int dx^D\sqrt{g}\Big[\frac{1}{2}\overline{D_\mu\epsilon}\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho\Big] = \int dx^D\sqrt{g}\Big[-\frac{1}{16}R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\psi_\rho\Big] \tag{277}$$

We can expand (277) using the Clifford algebra identity:

$$\Gamma^{a_j\ldots a_1}\Gamma_{b_1\ldots b_k} \;=\; \sum_{l=0}^{\min(j,k)} l! \begin{pmatrix} j \\ l \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \delta^{[a_1}_{[b_1}\cdots\delta^{a_l}_{b_l}\,\Gamma^{a_j\ldots a_{l+1}]}{}_{b_{l+1}\ldots b_k]}\,. \tag{278}$$

With this, we have:

$$R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\psi_\rho = R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Big[\Gamma_{\alpha\beta}{}^{\mu\nu\rho} + 6\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]} + 6\delta^{[\rho}_{[\alpha}\delta^\nu_{\beta]}\Gamma^{\mu]}\Big]\psi_\rho$$

$$R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\psi_\rho = R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Big[\Gamma_{\alpha\beta}{}^{\mu\nu\rho} + 6\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\Big]\psi_\rho + \bar{\epsilon}\Big[4R_\mu{}^\rho\Gamma^\mu - 2R\Gamma^\rho\Big]\psi_\rho$$

$$R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\psi_\rho = R_{\mu\nu}{}^{\alpha\beta}\bar{\epsilon}\Big[\Gamma_{\alpha\beta}{}^{\mu\nu\rho} + 6\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}{}_{\beta]}\Big]\psi_\rho + 4\Big[R^{\mu\rho} - \frac{1}{2}Rg^{\mu\rho}\Big]\bar{\epsilon}\Gamma_\mu\psi_\rho$$

The second term finally reads:

$$\int dx^D\sqrt{g}\Big[\frac{1}{2}\overline{D_\mu\epsilon}\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho\Big] = \int dx^D\sqrt{g}\Big[-\frac{1}{16}R^{\alpha\beta}_{\mu\nu}\bar{\epsilon}\Big[\Gamma^{\mu\nu\rho}_{\alpha\beta}+6\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}_{\beta]}\Big]\psi_\rho-\frac{1}{4}\Big[R^{\mu\rho}-\frac{1}{2}Rg^{\mu\rho}\Big]\bar{\epsilon}\Gamma_\mu\psi_\rho\Big] \tag{279}$$

Doing the same calculation with the third term $-\frac{1}{2}\overline{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)\delta_Q\psi_\rho$, we have a similar result:

$$\int dx^D\sqrt{g}\Big[-\frac{1}{2}\overline{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)D_\rho(\omega)\epsilon\Big] = \int dx^D\sqrt{g}\Big[-\frac{1}{16}R^{\alpha\beta}_{\nu\rho}\bar{\psi}_\mu\Big[\Gamma^{\mu\nu\rho}_{\alpha\beta}+6\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}_{\beta]}\Big]\epsilon-\frac{1}{4}\Big[R^{\rho\mu}-\frac{1}{2}Rg^{\rho\mu}\Big]\bar{\psi}_\mu\Gamma_\rho\epsilon\Big] \tag{280}$$

We need to switch the position of the ϵ and ψ to compare them. Then with the identity:

$$\overline{\psi}_\mu\Gamma^{(j)}\epsilon = -(-1)^{\frac{j(j+1)}{2}}\bar{\epsilon}\Gamma^{(j)}\psi_\mu \tag{281}$$

and by adding them, we find:

$$\begin{aligned}\int dx^D\sqrt{g}\Big[\frac{1}{2}\overline{D_\mu\epsilon}\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho - \frac{1}{2}\overline{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)D_\rho(\omega)\epsilon\Big] = \int dx^D\sqrt{g}\Big[&\frac{1}{16}(-1-1)R_{\mu\nu\alpha\beta}\bar{\epsilon}\Gamma^{\mu\nu\alpha\beta}\psi_\rho \\ &-\frac{6}{16}(1-1)R^{\alpha\beta}_{\mu\nu}\bar{\epsilon}\delta^{[\rho}_{[\alpha}\Gamma^{\mu\nu]}_{\beta]}\psi_\rho \\ &-\frac{1}{4}(-1-1)\Big[R^{\mu\rho}-\frac{1}{2}Rg^{\mu\rho}\Big]\bar{\epsilon}\Gamma_\mu\psi_\rho\Big]\end{aligned}$$

The second line obviously vanishes as well as the first one due to the symmetry of the Riemmann tensor ($R_{\mu[\nu\alpha\beta]=0}$). Putting altogether in the variation (275), we get:

$$\delta_Q S = \int dx^D\sqrt{g}\Big[\frac{1}{2}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\bar{\epsilon}\Gamma^{(\mu}\psi^{\nu)} - \frac{1}{2}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\bar{\epsilon}\Gamma^{(\mu}\psi^{\nu)}\Big]\delta_Q g^{\mu\nu} \tag{282}$$

The term to add in order to cancel out third order fermionic terms is the kinetic term for the three form potential $C_{\mu\nu\rho}$:

$$S_2 = \int dx^D\sqrt{g}\Big[-\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big] \tag{283}$$

where $G_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]}$. By rewriting this as:

$$-\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} = -\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G_{\alpha\beta\gamma\delta}g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}$$

we get a new contribution $-\frac{1}{24}\bar{\epsilon}\Gamma^{\mu}\psi^{\nu}(G)^2_{\mu\nu}$, that we must balance with G with two indices contracted with an unknown quantity X: $\overline{\psi}_{\mu}(XG)^{\mu\rho}\psi_{\rho}$. We then should modify the SUSY transformation of ψ_μ as:

$$\delta_Q\psi_\mu = D_\mu(\omega)\epsilon + (ZG)_\mu\epsilon \equiv \hat{D}_\mu(\omega)\epsilon \tag{284}$$

The action now looks like:

$$S_3 = \int dx^D\sqrt{g}\Big[\frac{1}{4}R + \frac{1}{2}\overline{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho - \frac{1}{4\cdot 48}\overline{\psi}_\mu(XG)^{\mu\rho}\psi_\rho - \frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big] \tag{285}$$

$(XG)^{\mu\nu}$ is given by:

$$(XG)^{\mu\nu} = -\frac{1}{8\cdot 4!}\Gamma^{\mu\nu}{}_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} - \frac{12}{8\cdot 4!}\Gamma_{\alpha\beta}G^{\mu\nu\alpha\beta} \tag{286}$$

and $(ZG)^\mu$ is:

$$(ZG)^\mu = -\frac{1}{2\cdot 144}\Gamma^{\alpha\beta\gamma\delta}{}_{\mu}G_{\alpha\beta\gamma\delta} + \frac{8}{2\cdot 144}\Gamma^{\alpha\beta\gamma\delta}\delta^{\alpha}_{\mu}G^{\alpha\beta\gamma\delta} \tag{287}$$

The variation (284) is now:

$$\delta_Q\psi_\mu = D_\mu\epsilon - \frac{1}{2\cdot 144}\Big(\Gamma^{\alpha\beta\gamma\delta}{}_{\mu} - 8\Gamma^{\beta\gamma\delta}\delta^{\alpha}_{\mu}\Big)\epsilon G_{\alpha\beta\gamma\delta} \tag{288}$$

The variation of the new terms $\psi_\mu(XG)^{\mu\rho}\psi_\rho$ and $G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}$ reads:

$$\delta S_3 = \int dx^D\Big[\underbrace{\delta_Q\Big(-\frac{\sqrt{g}}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big)}_{I} + \underbrace{\delta_Q\Big(\frac{\sqrt{g}}{4\cdot 48}\overline{\psi}_\mu(XG)^{\mu\rho}\psi_\rho\Big)}_{II}\Big] \tag{289}$$

$$\delta(S_3)_I = \delta_Q\Big(-\frac{\sqrt{g}}{4\cdot 48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\Big) = -\frac{1}{4\cdot 48}G^2\,\delta_Q(\sqrt{g}) + \sqrt{g}\,\delta_Q\Big(-\frac{1}{4\cdot 48}G_{\mu\nu\rho\sigma}G_{\alpha\beta\gamma\delta}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}\Big) \tag{290}$$

with (307) and (274) we find:

$$\delta(S_3)_I = -\frac{1}{4\cdot 48}G^2\Big(-\frac{1}{2}\sqrt{g}\,g_{\mu\nu}2\,\overline{\epsilon}\,\Gamma^{(\mu}\psi^{\nu)}\Big) - \frac{4}{4\cdot 48}(G^2)_{\mu\nu}2\,\overline{\epsilon}\,\Gamma^{(\mu}\psi^{\nu)}$$

$$\delta(S_3)_I = -\frac{1}{24}\Big((G^2)_{\mu\nu} - \frac{1}{8}G^2 g_{\mu\nu}\Big)\overline{\epsilon}\,\Gamma^{(\mu}\psi^{\nu)} \tag{291}$$

The variation of $(S_3)_{II}$ is:

$$\delta(S_3)_{II} = -\frac{1}{4\cdot 48}\delta_Q\Big(\overline{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\overline{\psi}^\alpha\Gamma^{\gamma\delta}\psi^\beta\Big)G_{\alpha\beta\gamma\delta} \tag{292}$$

We lower the indices of the ψ and $\overline{\psi}$ with the metric g and we expand the variation:

$$\delta(S_3)_{II} = -\frac{1}{4\cdot 48}\Big[\delta_Q\overline{\psi}_\mu\Big(\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12g^{\mu[\alpha}\Gamma^{\gamma\delta}\psi^{\beta]}\Big) - \Big(\overline{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta} + 12\bar{\psi}^{[\alpha}\Gamma^{\gamma\delta}g^{\beta]\nu}\Big)\delta_Q\psi_\nu\Big]G_{\alpha\beta\gamma\delta} \tag{293}$$

It is more complicated because we have to determine the variation of $\bar{\psi}$, using the variation (284):

$$\delta_Q\bar{\psi}_\mu = \delta_Q\Big(\psi_\mu^\top\Gamma^0\Big) = \Big(\delta_Q\psi_\mu\Big)^\top\Gamma^0 = \Big(D_\mu\epsilon - \frac{1}{2\cdot 144}[\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\Gamma^{\beta\gamma\delta}\delta_\mu^\alpha]G_{\alpha\beta\gamma\delta}\epsilon\Big)^\top\Gamma^0 \tag{294}$$

$$= (D_\mu\epsilon)^\top\Gamma^0 - \frac{1}{2\cdot 144}\Big(\overline{\Gamma^{\alpha\beta\gamma\delta}\epsilon} - 8\overline{\Gamma^{\beta\gamma\delta}\delta_\mu^\alpha}\epsilon\delta_\mu^\alpha\Big)G_{\alpha\beta\gamma\delta} \tag{295}$$

$\delta_Q\bar{\psi}$ is determined using the identity:

$$\overline{\Gamma^j\epsilon} = (-1)^{\frac{(j+1)j}{2}}\bar{\epsilon}\Gamma^j \tag{296}$$

where j is a number of indices, and is equal to:

$$\delta_Q\bar{\psi}_\mu = (D_\mu\epsilon)^\top\Gamma^0 + \frac{1}{2\cdot 144}\bar{\epsilon}\Big(\Gamma^{\alpha\beta\gamma\delta}{}_\mu + 8\Gamma^{\beta\gamma\delta}\delta_\mu^\alpha\Big)G_{\alpha\beta\gamma\delta} \tag{297}$$

Using (288) and (297), (293) is now:

$$\begin{aligned}\delta(S_3)_{II} = -\frac{1}{32\cdot(12)^3}\Big[&\bar{\epsilon}\Big(\Gamma^{\rho\sigma\eta\tau}{}_\mu + 8\Gamma^{\sigma\eta\tau}\delta_\mu^\rho\Big)\Big(\Gamma^{\mu\nu\alpha\beta\gamma\delta} + 12g^{\mu[\alpha}\Gamma^{\gamma\delta}\psi^{\beta]\nu}\Big)\psi_\nu\\ &-\overline{\psi}_\nu\Big(\Gamma^{\mu\nu\alpha\beta\gamma\delta} - 12g^{\nu[\alpha}\Gamma^{\gamma\delta}g^{\beta]\mu}\Big)\Big(\Gamma^{\rho\sigma\eta\tau}{}_\mu - 8\Gamma^{\sigma\eta\tau}\delta_\mu^\rho\Big)\epsilon\Big]G_{\rho\sigma\eta\tau}G_{\alpha\beta\gamma\delta}\end{aligned} \tag{298}$$

Using the identity:

$$(\Gamma_{\alpha\beta\gamma\delta\nu} + 8\,\Gamma_{\beta\gamma\delta}\eta_{\nu\alpha})\Gamma^{\mu\nu\rho}\,G^{\alpha\beta\gamma\delta} = 3\,(\Gamma^{\mu\rho}{}_{\alpha\beta\gamma\delta} + 12\,\delta^{\mu}_{\alpha}\,\Gamma_{\gamma\delta}\,\delta^{\rho}_{\beta})G^{\alpha\beta\gamma\delta} \tag{299}$$

(298) becomes:

$$\begin{aligned}\delta(S_3)_{II} = -\frac{1}{3\cdot 32\cdot(12)^3}\Big[\bar{\epsilon}\Big(\Gamma^{\rho\sigma\eta\tau}{}_{\mu} + 8\Gamma^{\sigma\eta\tau}\delta^{\rho}_{\mu}\Big)\Gamma^{\mu\xi\nu}\Big(\Gamma^{\rho\sigma\eta\tau}{}_{\xi} - 8\Gamma^{\sigma\eta\tau}\delta^{\rho}_{\xi}\Big)\psi_{\nu} \\ +\overline{\psi}_{\nu}\Big(\Gamma^{\rho\sigma\eta\tau}{}_{\xi} + 8\Gamma^{\sigma\eta\tau}\delta^{\rho}_{\xi}\Big)\Gamma^{\nu\xi\mu}\Big(\Gamma^{\rho\sigma\eta\tau}{}_{\mu} - 8\Gamma^{\sigma\eta\tau}\delta^{\rho}_{\mu}\Big)\epsilon\Big]G_{\rho\sigma\eta\tau}G_{\alpha\beta\gamma\delta}\end{aligned} \tag{300}$$

The variation of the total action S_3 (229) does not vanish and the remaining term is:

$$\delta_Q S_3 = \int d^D x\Big[\frac{9}{4\cdot(12)^4}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}\,\bar{\epsilon}\,\Gamma_{[\mu\nu}\psi_{\rho]}\Big] \tag{301}$$

To balance it, one makes the ansatz of the compensating term, called the Chern-Simons term:

$$S_{CS} = \int d^D x\Big[\frac{1}{4\cdot(12)^4}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}\,C_{\mu\nu\rho}\Big] \tag{302}$$

with the super-transformation of the potential satisfying:

$$\delta_Q C_{\mu\nu\rho} = 3\,\bar{\epsilon}\,\Gamma_{[\mu\nu}\psi_{\rho]} \tag{303}$$

The total action $S = S_3 + S_{CS}$ vanishes, and the final Lagrangian of 11-dimensional Supergravity reads:

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\,eR + \frac{1}{2}\,e\bar{\psi}_{\mu}\Gamma^{\mu\nu\rho}D_{\nu}\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_{\rho} - \frac{1}{4\cdot 48}\,eG_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} \\ &\quad -\frac{1}{4\cdot 48}\,e\left(\bar{\psi}_{\mu}\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_{\nu} + 12\,\bar{\psi}^{\alpha}\Gamma^{\gamma\delta}\psi^{\beta}\right)\left(\frac{G_{\alpha\beta\gamma\delta} + G_{\alpha\beta\gamma\delta}}{2}\right) \\ &\quad +\frac{1}{4\cdot 144^2}\,\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}\,.\end{aligned} \tag{304}$$

with the following notations:

$$
\begin{aligned}
D_\nu(\omega)\psi_\mu &= \partial_\nu\psi_\mu - \frac{1}{4}\,\omega_{\nu ab}\,\Gamma^{ab}\,\psi_\mu \\
G_{\mu\nu\rho\sigma} &= 4\,\partial_{[\mu}\,C_{\nu\rho\sigma]} \\
\hat{G}_{\mu\nu\rho\sigma} &= G_{\mu\nu\rho\sigma} + 6\,\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]} \\
\omega_{\mu ab} &= \omega^{(0)}_{\mu ab} + \frac{1}{4}\left[\bar{\psi}_\alpha\Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta - 2\left(\bar{\psi}_\mu\Gamma_b\psi_a - \bar{\psi}_\mu\Gamma_a\psi_b + \bar{\psi}_b\Gamma_\mu\psi_a\right)\right] \\
\hat{\omega}_{\mu ab} &= \omega_{\mu ab} - \frac{1}{4}\,\bar{\psi}_\alpha\,\Gamma_{\mu ab}{}^{\alpha\beta}\,\psi_\beta
\end{aligned}
$$

C.2 Equation of motion

C.2.1 Graviton $g_{\mu\nu}$

We vary the Einstein-Hilbert action coupled to the field strength $G_{\mu\nu\rho\sigma}$:

$$S = \int d^Dx\sqrt{g}R - \frac{1}{48}\int d^Dx\sqrt{g}eG_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} = S + S' \qquad (305)$$

• Let's consider first the variation of the Einstein-Hilbert action:

$$\delta S = \int d^Dx\Big[\delta(\sqrt{g})g^{\mu\nu}R_{\mu\nu} + \sqrt{g}\delta(g^{\mu\nu})R_{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}\Big] \qquad (306)$$

Using the identity:

$$\delta(\sqrt{g}) = \frac{1}{2}\frac{1}{\sqrt{g}}\delta g = \frac{1}{2}\frac{1}{\sqrt{g}}gg^{jk}\delta g_{jk} = \frac{1}{2}\sqrt{g}g^{jk}\delta g_{jk} = -\frac{1}{2}\sqrt{g}g^{jk}\delta g_{jk} \qquad (307)$$

We find:

$$\delta S = \int d^Dx\sqrt{g}\Big[-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\Big]\delta g^{\mu\nu} + \int d^Dx\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} \qquad (308)$$

The second term must vanish. To prove this we calculate the variation of the Ricci tensor given by:

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\nu\mu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\lambda\mu} \qquad (309)$$

$\delta R_{\mu\nu}$ becomes:

$$\delta R_{\mu\nu} = \partial_\lambda \delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\nu\mu} + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\nu\mu} - \delta\Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\lambda\mu} - \Gamma^\lambda_{\nu\rho}\delta\Gamma^\rho_{\lambda\mu} \tag{310}$$

The variation $\delta\Gamma^\mu_{\nu\rho}$ is the difference of two connections, and therefore is itself a tensor. We can thus take its covariant derivative:

$$\nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\rho_{\nu\sigma} \tag{311}$$

Then, the variation of the Riemann tensor is:

$$\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}) \tag{312}$$

and the contribution of this term in the action is written as:

$$\delta S = \int d^D x \sqrt{g} \nabla_\sigma \Big[g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu}) \Big] \tag{313}$$

But this is an integral with respect to the natural volulme element of the covariant divergence of a vector. So, by Stoke's theorem, this is equal to a boundary contribution at infinity, which is equal to zero since we make the variation of the action vanish at infinity. Then, the term from $\delta R_{\mu\nu}$ does not contribute.

• Using the variation $\delta\sqrt{g}$, given above, we write the variation of the second part of the action:

$$\delta S' = -\frac{e}{48}\int d^D x \Big[\delta\sqrt{g} G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + \sqrt{g}\delta(G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}) \Big]$$

$$= -\frac{e}{48}\int d^D x \Big[-\frac{1}{2}\sqrt{g} g_{\alpha\beta} G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + 4\sqrt{g} G_{\alpha\nu\rho\sigma}G_\beta^{\nu\rho\sigma} \Big] \delta g^{\alpha\beta} \tag{314}$$

The equation of motion for the whole action now reads:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{e}{48}\Big[-\frac{1}{2} g_{\alpha\beta} G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma} + 4\sqrt{g} G_{\alpha\nu\rho\sigma}G_\beta^{\nu\rho\sigma} \Big] \tag{315}$$

By contracting this with $g^{\beta\alpha}$, we obtain:

$$R = \frac{e}{144} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} \tag{316}$$

C.2.2 3-form potential $C_{\mu\nu\rho}$

The part of the supergravity action involving the 3-form potential $C_{\mu\nu\rho}$ is:

$$\mathcal{L} = \frac{1}{48} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} + \frac{1}{144^2} \epsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \gamma_1 \gamma_2 \gamma_3} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} G_{\gamma_1 \gamma_2 \gamma_3} \tag{317}$$

To find the equation of motion, one can solve the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial C_{ijk}} - \partial_\xi \left[\frac{\partial \mathcal{L}}{\partial(\partial_\xi C_{ijk})} \right] = 0$. Using the following identities:

$$\frac{\partial G_{\mu\nu\rho\sigma}}{\partial(\partial_\xi C_{ijk})} = \delta^{\xi ijk}_{\mu\nu\rho\sigma} \tag{318}$$

and

$$\delta^{\alpha_1 \ldots \alpha_4}_{\beta_1 \ldots \beta_4} G^{\beta_1 \ldots \beta_4} = 4! G^{\alpha_1 \ldots \alpha_4} \tag{319}$$

we have the equation of motion:

$$\begin{aligned}
0 &= \frac{1}{144^2} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \gamma_1 \gamma_2 \gamma_3} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \delta^{ijk}_{\gamma_1 \gamma_2 \gamma_3} \\
&\quad - \frac{\partial}{\partial x^\xi} \Bigg[-\frac{1}{48} \Big\{ \delta^{\xi ijk}_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} + g^{\mu\tau_1} g^{\nu\tau_2} g^{\rho\tau_3} g^{\sigma\tau_4} G_{\mu\nu\rho\sigma} \delta^{\xi ijk}_{\tau_1 \ldots \tau_4} \Big\} \\
&\qquad\qquad + \frac{2}{144^2} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \gamma_1 \gamma_2 \gamma_3} \delta^{\xi ijk}_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} C_{\gamma_1 \gamma_2 \gamma_3} \Bigg] \\
0 &= \partial_\xi G^{\xi ijk} + \frac{3!}{144^2} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 ijk} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \\
&\quad - \frac{2}{144^2} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 \gamma_1 \gamma_2 \gamma_3} \delta^{\xi ijk}_{\alpha_1 \ldots \alpha_4} \underbrace{G_{\beta_1 \ldots \beta_4} \partial_\xi C_{\gamma_1 \gamma_2 \gamma_3}}_{\text{since } dG = 0} \\
0 &= \partial_\xi G^{\xi ijk} + \frac{3!}{144^2} \varepsilon^{\alpha_1 \ldots \alpha_4 \beta_1 \ldots \beta_4 ijk} G_{\alpha_1 \ldots \alpha_4} G_{\beta_1 \ldots \beta_4} \\
&\quad + \frac{2 \cdot 4!}{144^2} \varepsilon^{\beta_1 \ldots \beta_4 \xi \gamma_1 \gamma_2 \gamma_3 ijk} G_{\beta_1 \ldots \beta_4} \underbrace{\partial_\xi C_{\gamma_1 \gamma_2 \gamma_3}}_{\partial_{[\xi} C_{\gamma_1 \gamma_2 \gamma_3]}}
\end{aligned} \tag{320}$$

and then:

$$\partial_\xi G^{\xi ijk} + \frac{18}{144^2}\epsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4 ijk}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4} = 0 \tag{321}$$

The first term $\partial_\xi G^{ijk}$ must be of the form $*d(*G)$:

$$G = \frac{1}{4!}G_{\mu_1\mu_2\mu_3\mu_4}dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_4}$$

$$(*G)_{\mu_5\ldots\mu_{11}} = \frac{1}{4!}\epsilon_{\mu_5\ldots\mu_{11}\mu_1\ldots\mu_4}G^{\mu_1\ldots\mu_4}$$

$$d(*G)_{\nu_1\ldots\nu_8} = 8\partial_{[\nu_1}\frac{1}{4!}\epsilon_{\nu_2\ldots\nu_8]\alpha_1\ldots\alpha_4}G^{\alpha_1\ldots\alpha_4}$$

$$*d(*G)_{\nu_1\nu_2\nu_3} = \frac{1}{8!}\epsilon_{\nu_1\ldots\nu_{11}}8\partial^{[\nu_4}\frac{1}{4!}\epsilon^{\nu_5\ldots\nu_{11}]\alpha_1\ldots\alpha_4}G_{\alpha_1\ldots\alpha_4}$$

$$*d(*G)_{\nu_1\nu_2\nu_3} = \frac{1}{7!4!}\epsilon_{\nu_1\ldots\nu_{11}}\epsilon^{\alpha_1\ldots\alpha_4[\nu_5\ldots\nu_{11}}\partial^{\nu_4]}G_{\alpha_1\ldots\alpha_4}$$

$$*d(*G)_{\nu_1\nu_2\nu_3} = \frac{1}{7!3!}\epsilon_{\nu_1\ldots\nu_{11}}\epsilon^{\alpha_1\alpha_2\alpha_3\nu_4\ldots\nu_{11}}\frac{1}{8}\partial^{\nu_4}G_{\alpha_1\ldots\alpha_4}$$

$$*d(*G)_{\nu_1\nu_2\nu_3} = \partial^\xi G_{\xi\nu_1\nu_2\nu_3} \tag{322}$$

since $\epsilon_{\nu_1\ldots\nu_{11}}\epsilon^{\alpha_1\alpha_2\alpha_3\nu_4\ldots\nu_{11}} = -8!\delta^{\alpha_1\alpha_2\alpha_3}_{\nu_1\nu_2\nu_3}$.

The second term should be written as $*(G \wedge G)$:

$$G \wedge G = \frac{1}{(4!)^2}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_4} \wedge dx^{\beta_1} \wedge \ldots \wedge dx^{\beta_4}$$

$$\big[*(G \wedge G)\big]_{\nu_1\nu_2\nu_3} = \frac{1}{4!}\epsilon_{\nu_1\nu_2\nu_3\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4}G^{\alpha_1\ldots\alpha_4}G^{\beta_1\ldots\beta_4} \tag{323}$$

Using all this, we can rewrite the equation of motion as follow:

$$*d(*G) + (4!)^2(\frac{18}{144^2}) * (G \wedge G) = 0$$

$$d(*G) + \frac{1}{2}G \wedge G = 0 \tag{324}$$

C.2.3 Gravitino ψ_μ

The part of the Lagrangian involving the gravitino is:

$$\mathcal{L} = \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu\Big(\frac{\omega+\hat{\omega}}{2}\Big)\psi_\rho - \frac{1}{4\cdot 48}\Big(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\gamma\delta}\psi^\beta\Big)\Big(\hat{G}_{\alpha\beta\gamma\delta} - 3\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\Big) \quad (325)$$

We are going to solve the Euler-Lagrange equation for $\bar{\psi}$, which reduces to

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_\xi} = 0$$

. Let split the Lagrangian as:

$$\begin{cases} \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \\ \mathcal{L}_1 = \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu\Big(\frac{\omega+\hat{\omega}}{2}\Big)\psi_\rho \\ \mathcal{L}_2 = -\frac{1}{4\cdot 48}\Big(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\gamma\delta}\psi^\beta\Big)\Big(\hat{G}_{\alpha\beta\gamma\delta} - 3\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\Big) \end{cases} \quad (326)$$

We first consider $\mathcal{L}_2$. With $\frac{\partial \bar{\psi}^\mu}{\partial \psi_\xi} = \delta^\xi_\alpha$ and the notations given earlier, we directly get:

$$\begin{aligned} \frac{\partial \mathcal{L}_2}{\partial \bar{\psi}_\xi} = -\frac{1}{4\cdot 48}\Big(\Gamma^{\xi\nu}{}_{\alpha\beta\gamma\delta}\psi_\nu + 12\delta^\xi_\alpha \Gamma_{\gamma\delta}\psi_\beta\Big)\Big(\hat{G}^{\alpha\beta\gamma\delta} - 3\bar{\psi}^{[\alpha}\Gamma^{\beta\gamma}\psi^{\delta]}\Big) \\ -\frac{3}{4\cdot 48}\delta^\xi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\Big(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\gamma\delta}\psi^\beta\Big) \end{aligned} \quad (327)$$

For $\mathcal{L}_1$, we first need to expand it, to make all the $\overline{\psi}$'s appear (one can also use the chain rule):

$$D_\nu\Big(\frac{\omega+\hat{\omega}}{2}\Big)\psi_\rho = \partial_\nu\psi_\rho - \frac{1}{4}\Big[\omega^{(0)}_{\nu ab} + \frac{1}{4}\Big(\overline{\psi}_\alpha \Gamma^{\alpha\beta}_{\nu ab}\psi_\beta - 2(\overline{\psi}_\nu \Gamma_b \psi_a - \overline{\psi}_\nu \Gamma_a \psi_b + \overline{\psi}_b \Gamma_\nu \psi_a)\Big)\Big]\Gamma^{ab}\psi_\rho + \frac{1}{32}\overline{\psi}_\alpha \Gamma^{\alpha\beta}_{\nu ab}\psi_\beta \Gamma^{ab}\psi_\rho$$

When we multiply by $\frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}$, we get:

$$\mathcal{L}_1 = \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \frac{1}{8}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Big[\omega^{(0)}_{\nu ab} - \frac{1}{2}(\overline{\psi}_\nu \Gamma_b \psi_a - \overline{\psi}_\nu \Gamma_a \psi_b + \overline{\psi}_b \Gamma_\nu \psi_a) + \frac{1}{8}\overline{\psi}_\alpha \Gamma^{\alpha\beta}_{\nu ab}\psi_\beta\Big]\Gamma^{ab}\psi_\rho \quad (328)$$

By taking the derivative of this, we find:

$$
\begin{aligned}
\frac{\partial \mathcal{L}_1}{\partial \overline{\psi}_\xi} = \frac{1}{2}\Gamma^{\xi\nu\rho}\partial_\nu\psi_\rho - \frac{1}{8}\Gamma^{\xi\nu\rho}\Big[\omega^{(0)}_{\nu ab} - \frac{1}{2}(\overline{\psi}_\nu\Gamma_b\psi_a - \overline{\psi}_\nu\Gamma_a\psi_b + \overline{\psi}_b\Gamma_\nu\psi_a) + \frac{1}{8}\overline{\psi}_\alpha\Gamma^{\alpha\beta}_{\nu ab}\psi_\beta\Big]\Gamma^{ab}\psi_\rho \\
-\frac{1}{64}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma^{\xi\beta}_{\nu ab}\psi_\beta\Gamma^{ab}\psi_\rho \\
-\frac{1}{8}(\bar{\psi}_\mu\Gamma^{\mu\nu\rho})\Big[-\frac{1}{2}\partial_{\bar{\xi}}(\overline{\psi}_\nu\Gamma_b\psi_a - \overline{\psi}_\nu\Gamma_a\psi_b + \overline{\psi}_b\Gamma_\nu\psi_a)\Gamma^{ab}\psi_\rho\Big]
\end{aligned}
\tag{329}
$$

When we calculate the equation of motion for ψ, we find that, combine to the equation of motion for $\bar{\psi}$, the last part of (329) vanishes, and the rest can be written in a condensed form:

$$
\frac{\partial \mathcal{L}}{\partial \overline{\psi}_\xi} = \frac{1}{2}\Gamma^{\xi\nu\rho}\Big[D_\nu(\hat{\omega}) - \frac{1}{32}\overline{\psi}_\alpha\Gamma^{\alpha\beta}_{\nu ab}\psi_\beta\Gamma^{ab}\Big]\psi_\rho + \frac{1}{64}\Gamma^{\xi\beta}_{\nu ab}\psi_\beta\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho \tag{330}
$$

Putting everything together

$$
\begin{aligned}
0 = \frac{1}{2}\Gamma^{\xi\nu\rho}\Big[D_\nu(\hat{\omega}) - \frac{1}{32}\overline{\psi}_\alpha\Gamma^{\alpha\beta}_{\nu ab}\psi_\beta\Gamma^{ab}\Big]\psi_\rho + \frac{1}{64}\Gamma^{\xi\beta}_{\nu ab}\psi_\beta\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho \\
-\frac{1}{4\cdot 48}\Big(\Gamma^{\xi\nu}_{\alpha\beta\gamma\delta}\psi_\nu + 12\delta^\xi_\alpha\Gamma_{\gamma\delta}\psi_\beta\Big)\Big(\hat{G}^{\alpha\beta\gamma\delta} - 3\bar{\psi}^{[\alpha}\Gamma^{\beta\gamma}\psi^{\delta]}\Big) \\
-\frac{3}{4\cdot 48}\delta^\xi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\Big(\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha\Gamma^{\gamma\delta}\psi^\beta\Big)
\end{aligned}
$$

and by re-ordering the terms, we have the equation of motion:

$$
\begin{aligned}
0 = \frac{1}{2}\Big[\Gamma^{\xi\nu\rho}D_\nu(\hat{\omega}) - \frac{1}{96}\Big(\Gamma^{\xi\rho}_{\alpha\beta\gamma\delta} + 12\delta^\xi_\alpha\Gamma_{\gamma\delta}\delta^\rho_\beta\Big)\hat{G}^{\alpha\beta\gamma\delta}\Big]\psi_\rho \\
-\frac{1}{64}\Gamma^{\xi\nu\rho}\Gamma^{ab}\psi_\rho\bar{\psi}_\alpha\Gamma^{\alpha\beta}_{\nu ab}\psi_\beta + \frac{1}{64}\Gamma^{\xi\beta}_{\nu ab}\psi_\beta\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho \\
+\frac{1}{64}\Big(\Gamma^{\xi\nu}_{\alpha\beta\gamma\delta}\psi_\nu\bar{\psi}^{[\alpha}\Gamma^{\beta\gamma}\psi^{\delta]} - \delta^\xi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu\Big)
\end{aligned}
\tag{331}
$$

The last four terms vanish using the Cremmer-Julia-Scherk Fierz identity:

$$
\begin{aligned}
\frac{1}{8}\,\Gamma^{\mu\nu\alpha\beta\gamma\delta}\,\psi_\nu\bar{\psi}_\alpha\,\Gamma_{\beta\gamma} \;-\; \frac{1}{8}\,\Gamma_{\beta\gamma}\,\psi_\nu\bar{\psi}_\alpha\,\Gamma^{\mu\nu\alpha\beta\gamma\delta} \\
-\,\frac{1}{4}\,\Gamma^{\mu\nu\alpha\beta\delta}\psi_\nu\bar{\psi}_\alpha\Gamma_\beta \;+\; \frac{1}{4}\,\Gamma_\beta\psi_\nu\bar{\psi}_\alpha\Gamma^{\mu\nu\alpha\beta\delta} \\
-\,2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\psi_\nu\bar{\psi}_\alpha\Gamma_\beta \;-\; 2\,\Gamma_\beta\psi_\nu\bar{\psi}_\alpha g^{\beta[\alpha}\Gamma^{\delta\mu\nu]} \\
+\,2\,g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\,\bar{\psi}_\alpha\Gamma_\beta\psi_\nu \;=\; 0
\end{aligned}
\tag{332}
$$

and with the identity:

$$3\Big(\Gamma^{\mu\rho}_{\alpha\beta\gamma\delta} + 12\delta^{\mu}_{\alpha}\Gamma_{\gamma\delta}\delta^{\rho}_{\beta}\Big)\psi_\rho \hat{G}^{\alpha\beta\gamma\delta} = \Gamma^{\mu\nu\rho}\Big(\Gamma^{\alpha\beta\gamma\delta}_{\nu} - 8\Gamma^{\beta\gamma\delta}\delta^{\alpha}_{\nu}\Big)\psi_\rho \hat{G}_{\alpha\beta\gamma\delta} \tag{333}$$

the equation of motion can be written is a simple form:

$$\Gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho = 0 \tag{334}$$

where we have used:

$$\hat{D}_\nu\psi_\rho = D_\nu(\hat{\omega})\psi_\rho - \frac{1}{2\cdot 144}\Big(\Gamma^{\alpha\beta\gamma\delta}_{\nu} - 8\Gamma^{\beta\gamma\delta}\delta^{\alpha}_{\nu}\Big)\psi_\rho \hat{G}_{\alpha\beta\gamma\delta}$$

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