Neutrino masses and mixing

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Contents

1 Introduction

The possibility of neutrinos having definite mass is a long-aged issue dating back to the first experimental observations in the fifties. The Standard Model (the theory which describes with an incredible accuracy particle physics) was not yet even formulated, nor were neutrino flavours different from the electron one ν_e yet discovered, when Bruno Pontecorvo postulated that neutrinos oscillate, that is the state vector of a neutrino is a superposition of state vectors of Majorana particles with different masses, in analogy with what was at that time known about the $K^0 - \bar{K}^0$ system. This hypothesis requires neutrinos to be of definite mass and that weak interaction do not conserve lepton charge, as well as strangeness.
In the sixties weak interactions were understood to fit well in a local non-abelian gauge theory unified to electromagnetism, where particles interact via massive vector bosons and the lepton masses arise from a weird yet astonishingly precise mechanism of spontaneous symmetry breaking. It took a little bit less than a decade to rule out the issue of taming the infinities cropping out the weak interactions loop diagrams and eventually demonstrate that this bewildering theory was indeed renormalizable. However, in this major theoretical model neutrinos, although present, are not given any mass and are thus represented by spinors of definite chirality: left-handed spinors. A few decades before, the young italian genius Ettore Majorana (mysteriously disappeared in 1938) was working over a very original theory of spinors in neat contrast with the by then established Dirac theory. The new mathematical objects the Sicilian scientist was proposing were suitable to represent particles with the property of being invariant under charge conjugation. Majorana spinors carry two degrees of freedom only, instead of the four carried by Dirac spinors. Since charge conjugation inverts the chirality, if neutrinos are Majorana particles, it is also possible to have three right-handed particles from the left-handed neutrinos of the Standard Model. In this way the electro-weak theory can be extended, including non-zero mass neutrinos.
The appearance of Grand Unified Theories (Pati and Salam 1973; Georgi and Glashow 1974) stimulated further the interest in neutrino mixing and oscillations, which both arise naturally in these models. However the ultimate confirmation of the massive nature of neutrinos came out the neutrino solar problem, in the 1980's. For twenty years the world scientific community had been puzzled by a discrepancy between the number of neutrinos produced by the nuclear reactions inside the sun, and those flowing through the earth. The measurements made by Ray Davies and John N. Bachall in the 1960's detected a flow of neutrinos that was in deficit of two thirds from the expected amount. Where had these two thirds of neutrinos gone? The particle physics community took it out on Davies and co., who were still convinced of having done a good job though. Someone then suggested that the Standard Solar Model, upon which the neutrino production inside the

sun was estimated, was wrong. This was not the case either, as it turned out ultimately in 1998, when the Super-Kamiokande collaboration in Japan revealed that neutrino oscillation do take place on the way from the sun to the earth, and the detectors used to collect neutrino signals were only sensible to the electron flavour, missing out the other two neutrino flavours: τ and μ. For not having stubbornly lied to the world for nearly fourty years, Ray Davies was eventually awarded the Nobel prize in 2002, together with Masatoshi Koshiba (who worked on Super-Kamiokande).

The Super-Kamiokande collaboration showed clearly that neutrinos oscillate between different generations, in particular the $\nu_\mu \leftrightarrow \nu_\tau$ oscillation was observed. The oscillations are evidence of mass differences among different flavour, but unfortunately don't give any information on the value of a single mass. The only thing we can deduce is a lower bound on the mass value: there is a mass eigenstate with a mass of at least $0.04\,eV$. Since the oscillations experiments results, that is in the last ten years, all the experimental efforts have been driven towards the goal of measuring this absolute mass value. In particular, great attention has been given to neutrinoless double-β decay experiments, whose theory had been developed in the 1980's. Currently many experiments are running or under construction to investigate further these processes which are very likely to reveal in short time the exact value of neutrino masses.

From a theoretical point of view the question is of course how to incorporate neutrino masses in the Standard Model, or how to modify the previous theory to allow neutrinos to be massive. Depending on neutrinos being Dirac or Majorana particles, different models, thoroughly analyzed in this work, are possible to describe the same physics. All these different frames were developed starting from the early 1970's, as hints of neutrinos having mass were already present. The general picture of the Standard Model is not modified, meaning that the neutrino mass term (both Dirac and Majorana) arises from the same symmetry breaking pattern that gives masses to quarks and charged leptons. A remarkable issue is the one related to the symmetries carried by different types of mass term. For example, would neutrinos be Dirac particles, then the total lepton charge should be conserved and processes like neutrinoless double-β decay forbidden. Conversely in a theory where neutrinos are Majorana the total lepton charge is not conserved.

All existing experimental data confirm the hypothesis that weak interactions do conserve lepton charge of all particles different from neutrinos. Therefore charged leptons are Dirac particles. Despite in principle both Dirac and Majorana schemes are possible for neutrinos, nowadays the most plausible scenario is the one with neutrinos being Majorana particles with definite mass and the lepton charge conservation violated at some energy scale higher than the electro-weak one. The smallness of neutrino masses compared to the quarks and charged leptons would then be naturally explained as inversely proportional to the very large energy scale at which the lepton charge con-

Particles	L_e	L_μ	L_τ
(e, ν_e)	1	0	0
(μ, ν_μ)	0	1	0
(τ, ν_τ)	0	0	1
Hadrons, $W^\pm$, Z^0, γ	0	0	0

servation is violated. The framework is that of an effective theory with the mass term arising upon symmetry breaking from a non-renormalizable operator killed by the first power of the energy scale. As it is always the case in high-energy physics, going to higher energies reveals physics that lower scales do not contemplate.

This work is aimed at giving a general picture of what is the current knowledge of neutrino masses and wants to be an introduction for those who are at their first touch with the topic. It is mainly a review of some key works in order to introduce the reader to this very challenging branch of particle physics.
The first chapter is a summary of the main features of the mass generation in the Standard Model, which is the background for the theory. In the second chapter different mass terms are analyzed and discussed in relation with experiments. The discussion merges then on the See-Saw mechanism, which is considered to be the best theoretical picture of our current knowledge about neutrino masses. The subsequent chapter on oscillations is inserted only for completeness and because oscillations crop out all over the place in the discussion, even though they are not the principal aim of this work. In particular, oscillations help introducing the discussion on experiments, that is the content of the last chapter.

2 The Standard Model: Quark mixing

In all our discussion we will assume that the interaction of neutrinos with other leptons and quarks is described by the $SU(2) \times U(1)_Y - symmetric$ Lagrangian of the Standard Model (SM). We will neglect the $SU(3)$ factor of the strong interactions, which is also present as a symmetry of the SM but is not related to our aims. In this theory the left-handed (LH) neutrinos and the corresponding LH charged leptons form $SU(2)$ doublets in the following fashion

$$\psi_{lL} = \begin{pmatrix} \nu_{lL} \\ l_{lL} \end{pmatrix}, \qquad l = e, \mu, \tau \tag{2.1}$$

while the right-handed (RH) components of the leptons are singlets with respect to this group. No RH neutrinos are present in the original formulation of the Standard Model and this is because neutrinos were still considered to be massless and so of definite chirality. As we see, the l-index runs over the three different generations of leptons. The LH quark fields are grouped in $SU(2)$ doublets of three generations (or families)

$$\begin{pmatrix} u_{L1} \\ d_{L1} \end{pmatrix}, \qquad \begin{pmatrix} c_{L2} \\ s_{L2} \end{pmatrix}, \qquad \begin{pmatrix} t_{L2} \\ b_{L2} \end{pmatrix} \tag{2.2}$$

The six RH quarks are $SU(2)$-singlets: $u_{R1}, d_{R1}; c_{R2}, s_{R2}; t_{R3}, b_{R3}$. For a more compact notation, we shall denote these three families by

$$L_{Lk} = \begin{pmatrix} u_{Lk} \\ d_{Lk} \end{pmatrix}, \qquad k = 1, 2, 3 \tag{2.3}$$

and the RH fields simply by u_{Rk}, d_{Rk}, with $k = 1, 2, 3$. As we have mentioned before, the Standard Model requires local $SU(2) \times U(1)_Y$ gauge invariance, which is obtained by substituting in the dynamical lagrangian-terms of each LH field the $SU(2) \times U(1)_Y$ covariant derivative, and for each RH field the $U(1)_Y$ covariant derivative. The $U(1)_Y$ factor is known as the group of weak-hypercharge. The theory then requires symmetry breaking

$$SU(2) \times U(1)_Y \to U(1)_{em} \tag{2.4}$$

in order for the leptons, the quarks and the intermediate gauge vector bosons to acquire masses, leaving the photon massless. The remaining unbroken $U(1)_{em}$ symmetry is the proper electromagnetic symmetry, which is generated by a *mixture* of the hypercharge generator, denoted Y, and an $SU(2)$ generator, which we may take to be $T_3 = 1/2\sigma_3$, that is

$$Q_{em} = Y + T_3. \tag{2.5}$$

The covariant derivatives which enter the Lagrangian are for the leptons

$$D_\mu \psi_{lL} = (\partial_\mu + i\frac{g_2}{2}\mathbf{W}_\mu + i\frac{g_1}{2}B_\mu)\psi_{lL} \tag{2.6}$$

$$D_\mu l_{lR} = (\partial_\mu + i\frac{g_1}{2}B_\mu)l_{lR}. \tag{2.7}$$

where $\mathbf{W}_\mu$ are the $SU(2)$ gauge vector bosons, B_μ is the $U(1)_Y$ gauge boson and g_1, g_2 are coupling constants. Analogously, for the quarks we have

$$D_\mu L_{Lk} = (\partial_\mu + i\frac{g_2}{2}\mathbf{W}_\mu - i\frac{g_1}{6}B_\mu)L_{Lk} \tag{2.8}$$

$$D_\mu u_{Rk} = (\partial_\mu - 2i\frac{g_1}{3}B_\mu)u_{Rk} \tag{2.9}$$

$$D_\mu d_{Rk} = (\partial_\mu - i\frac{g_1}{3}B_\mu)u_{Rk} \tag{2.10}$$

The dynamical Lagrangian for leptons and quarks then reads

$$\mathcal{L}_{dyn} = \mathcal{L}_{dyn}(lepton) + \mathcal{L}_{dyn}(quark) \tag{2.11}$$

where

$$\mathcal{L}_{dyn}(lepton) = i\psi^\dagger_{lL}\tilde{\sigma}^\mu D_\mu \psi_{lL} + il^\dagger_{lR}\sigma^\mu D_\mu l_{lR} \tag{2.12}$$

and

$$\begin{aligned}\mathcal{L}_{dyn}(quark) = {} & iL^\dagger_{Lk}\tilde{\sigma}^\mu[\partial_\mu + i(g_2/2)\mathbf{W}_\mu + (ig_1/6)B_\mu]L_{Lk} \\ & + iu^\dagger_{Rk}\sigma^\mu[\partial_\mu + (2ig_1/3)B_\mu]u_{Rk} \\ & + id^\dagger_{Rk}\sigma^\mu[\partial_\mu - (ig_1/3)B_\mu]d_{Rk}\end{aligned} \tag{2.13}$$

Sums over l and k indices are understood. So far all the fermions ar massless. They are given masses through the Higgs effect, by including Yukawa couplings in the Lagrangian. After symmetry breaking the choice of the vacuum for the Higgs field allows all the fermions to acquire masses. Is important to remark that no $SU(2) \times U(1)$-invariant mass terms are allowed in principle. Any attempt to construct mass terms would indeed involve something like $\bar{\chi}_R\psi_L$, for some spinors χ_R, ψ_L. Such a quantity fails to be gauge-invariant because of the spinor index-structure which does not permit a good contraction.

Let us briefly consider how the Higgs mechanism works. The scalar field enters the theory as an $SU(2)$ doublet

$$\Phi = \begin{pmatrix}\phi_A \\ \phi_B\end{pmatrix} \tag{2.14}$$

It interacts both with the fermions and the gauge bosons and is also present in the total Lagrangian of the SM in the Higgs potential

$$V(\Phi^\dagger\Phi) = \kappa(\Phi^\dagger\Phi)^2 - \mu^2(\Phi^\dagger\Phi) \tag{2.15}$$

If we take κ and μ^2 to be positive constants, this potential is clearly degenerate in its vacuum, and this spontaneously breaks the symmetry. Choose as a vacuum expectation value (vev) for the scalar field

$$\bar{\Phi} = \begin{pmatrix}0 \\ \frac{1}{\sqrt{2}}v\end{pmatrix}, \qquad v = \mu^2/\kappa \tag{2.16}$$

Expanding Φ about its vacuum expectation value (making use of the unitary gauge to avoid mixing terms with the Goldstone bosons) we have

$$\Phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}, \tag{2.17}$$

$H(x)$ is known as the Higgs field. The $SU(2) \times U(1)$ gauge invariant Yukawa couplings for both leptons and quarks are

$$\mathcal{L}_Y = -(f_{mn}\psi^{\dagger}_{mL} l_{nR}\Phi + h_{mn}L^{\dagger}_{mL} d_{nR}\Phi + k_{mn}L^{\dagger}_{mL} u_{nR}\tilde{\Phi}) + h.c. \tag{2.18}$$

with h_{mn}, f_{mn} and k_{mn} coupling matrices, with no further constraints. On symmetry breaking, this gives the mass term

$$\mathcal{L}_{mass} = -\frac{v}{\sqrt{2}}(f_{mn}l^{\dagger}_{mL} l_{nR} + h_{mn}d^{\dagger}_{mL} d_{nR} + k_{mn}u^{\dagger}_{mL} u_{nR}) + h.c. \tag{2.19}$$

Note that the mass term for the u-quarks has appeared because

$$\tilde{\Phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v + H(x)) \\ 0 \end{pmatrix} \tag{2.20}$$

We have of course neglected the Higgs-fermions interactions which as well arise in this process, as they don't bring about any mass term.
The mass term (2.19) involves mixture of the three generations of quarks, i.e. the form of (2.19) is not diagonal. However, any complex matrix can be put in diagonal form by making use of biunitary transformations. From now on disregard the lepton term and focus on the quarks only. The matrices h_{mn} and k_{mn} can be diagonalized as

$$h_{mn} = D^{\dagger}_{Lms} m^d_{st} D_{Rtn}, \qquad k_{mn} = U^{\dagger}_{Lms} m^u_{st} U_{Rtn} \tag{2.21}$$

with U_L, U_R, D_L, D_R independent unitary matrices and m^u, m^d diagonal matrices. After this substitution in (2.19), the theory turns out to be most directly described in terms of the true quark fields

$$\begin{aligned} d'_{Li} &= D_{Lij} d_{Lj}, \qquad & d'_{Ri} &= D_{Rij} d_{Rj}, \\ u'_{Li} &= U_{Lij} u_{Lj}, \qquad & u'_{Ri} &= U_{Rij} u_{Rj} \end{aligned} \tag{2.22}$$

The mass contribution to the lagrangian then becomes, dropping the primes on the new quark fields

$$\mathcal{L}_{mass} = -\sum_{i=1}^{3} [m^d_i (d^{\dagger}_{Li} d_{Ri} + d^{\dagger}_{Ri} d_{Li}) + m^u_i (u^{\dagger}_{Li} u_{Ri} + u^{\dagger}_{Ri} u_{Li})] \tag{2.23}$$

which is manifestly diagonal.
There are further issues about the substitution (2.22), as it affects the form of

the charged and neutral weak currents (fermion-gauge bosons interactions) and brings in the Cabibbo-Kobayashi-Maskawa matrix, but we will not investigate further this argument. Analogous to what we have seen for quarks is the story for charged leptons masses: the latter ones arise from couplings to the scalar field after symmetry breaking. We should remark at this stage that because the Higgs has been chosen to be a doublet and for the absence of RH neutrino fields, is not possible to generate masses for neutrinos in this theory and hence, in the original formulation of the Standard Model, neutrinos stay massless.

3 Neutrino mixing schemes

So far we have only briefly summarized the major features of the SM and the mixing scheme for quarks, which is a good starting point for the theoretical structure we are going to look at more closely for neutrinos. Let us start this section saying that there are several different schemes for neutrino mixing, whereas only one for quarks. This is because neutrinos are neutral particles, while quarks are charged. As we will briefly see, not carrying electric charge allows neutrinos to be interpreted both as Dirac and Majorana particles (quarks are strictly Dirac particles).

3.1 Dirac and Majorana particles

A Dirac particle is described by a four-component spinor

$$\psi_a = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix} \qquad \alpha, \dot{\beta} = 1, 2 \tag{3.1}$$

with λ_α and $\bar{\chi}^{\dot{\beta}}$ being two irreducible representations of $SL(2, \mathbb{C})$. The fact that ψ_a as a whole is not irreducible allows to project it in its irreducible components by making use of the matrix

$$\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{3.2}$$

In particular

$$\gamma_5\psi_L = \psi_L, \qquad \gamma_5\psi_R = -\psi_R \tag{3.3}$$

where

$$\psi_L = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \qquad \psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix} \tag{3.4}$$

are the LH and RH components of the Dirac spinor. Let us define the *Dirac conjugate* as the row object

$$\bar{\psi}_a = -i \begin{pmatrix} \chi^\alpha & \bar{\lambda}_{\dot{\beta}} \end{pmatrix} \tag{3.5}$$

We can further define the *Dirac charge conjugate* as

$$\psi_c = \mathcal{C}(\bar{\psi})^T \tag{3.6}$$

where $\mathcal{C}$ is the *charge conjugation matrix*, which has the following properties

$$\mathcal{C}\gamma_\alpha^T\mathcal{C}^{-1} = -\gamma_\alpha, \qquad \mathcal{C}^\dagger\mathcal{C} = 1, \qquad \mathcal{C}^T = -\mathcal{C} \tag{3.7}$$

However, neutral massive fermions can be described by simpler spinors carrying only two independent components instead of four, as it was originally proposed by Ettore Majorana in his major work in 1937. A Majorana spinor has the following form

$$\psi_a = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\beta}} \end{pmatrix} \tag{3.8}$$

If we now look at the charge conjugate, we easly find that, for a Majorana spinor

$$\psi_c = \psi, \tag{3.9}$$

that is charge conjugation leaves Majorana spinors invariant. This is the reason why, once they are given a definite mass, neutrinos can be seen as both Dirac and Majorana particles: being a Dirac particle doesn't require to be charged, but being charged requires to be Dirac.

Suppose now to have as many neutrino flavours as we want, say n. It is useful to separate LH and RH components in two n-component column vectors

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \\ . \\ . \\ . \end{pmatrix}, \qquad \nu_R = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ . \\ . \\ . \end{pmatrix}. \tag{3.10}$$

Say also there is an index l running over the n flavours, i.e. $l = e, \mu, \tau, \ldots$. The LH fields are those entering into the SM and representing originally massless neutrinos, while the RH do not enter the SM and are introduced in order to build a mass term.

For ν_L and ν_R let us define the charge conjugates

$$\nu_L^c \equiv \mathcal{C}(\bar{\nu}_L)^T, \qquad \nu_R^c \equiv \mathcal{C}(\bar{\nu}_R)^T. \tag{3.11}$$

It turns out that ν_L^c is a RH field, while ν_R^c is LH. Indeed, using the relation

$$\mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5^T, \tag{3.12}$$

we have

$$\frac{1}{2}(1-\gamma_5)\nu_L^c = \mathcal{C}[\bar{\nu}_L\frac{1}{2}(1-\gamma_5)]^T. \tag{3.13}$$

Moreover

$$\bar{\nu}_L\frac{1}{2}(1-\gamma_5) = \bar{\nu}_L, \tag{3.14}$$

hence we find

$$\frac{1}{2}(1-\gamma_5)\nu_L^c = \nu_L^c \tag{3.15}$$

and similarly

$$\frac{1}{2}(1+\gamma_5)\nu_R^c = \nu_R^c \tag{3.16}$$

Using the fields ν_L, ν_R, ν_L^c and ν_R^c, we can now start building up different mass terms.

3.2 Dirac mass term

Since the RH fields $\nu_{eR}, \nu_{\tau R}, \nu_{\mu R}$ do not exist in the SM, a mass term of the form

$$\mathcal{L}^D = - \sum_{\alpha,\beta=e,\mu,\tau,...} \bar{\nu}_{\alpha R} \, M^D_{\alpha\beta} \, \nu_{\beta L} + H.c., \tag{3.17}$$

known as a *Dirac mass term*, is in principle precluded. M^D is a $n \times n$ mass matrix, not diagonal, which mixes neutrino flavours. However it is a straightforward extension to add the three RH fields as singlets under the total SM gauge group, so that neutrinos become similar to the other massive fermion fields (i.e. quarks and charged leptons). In this way, the mass term (3.17) can be generated by the same Higgs mechanism which is responsible of the masse terms for the other fermions.
Now proceed in diagonalizing the matrix M^D, with the biunitary transformation scheme we mentioned before for quarks:

$$M^D = V m U^\dagger, \tag{3.18}$$

with U and V both unitary matrices and $m_{ik} = m_i \delta_{ik}$ ($m_i > 0$). Making use of (3.18), the mass term (3.17) becomes

$$\mathcal{L}^D = - \sum_{\alpha,\beta,i} \bar{\nu}_{\alpha R} \, V_{\alpha i} \, m_i \, (U^\dagger)_{i\beta} \, \nu_{\beta L} + H.c. = - \sum_{i=1}^{n} m_i \bar{\nu}_i \nu_i, \tag{3.19}$$

where

$$\nu_i = \nu_{iL} + \nu_{iR}, \qquad i = 1, \ldots, n \tag{3.20}$$

and

$$\begin{aligned} \nu_{iL} &= \sum_\beta (U^\dagger)_{i\beta} \, \nu_{\beta L} \\ \nu_{iR} &= \sum_\alpha (V^\dagger)_{i\alpha} \, \nu_{\alpha R} \end{aligned} \tag{3.21}$$

Inverting the last two relations we have

$$\nu_{\alpha L} = \sum_i U_{\alpha i} \, \nu_{iL}, \qquad \alpha = e, \mu, \tau, ... \tag{3.22}$$

The neutrino fields given in (3.20) are the n components of a massive multiplet

$$\nu^{'} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ . \\ . \\ . \\ \nu_n \end{pmatrix} \tag{3.23}$$

After this definition equation (3.22) may be further recast in the simpler form

$$\nu_L = U\,\nu_L^{'} \tag{3.24}$$

From (3.22) and (3.24) we see that the the LH neutrino fields which are present in the SM are linear combinations of LH neutrino fields having definite masses. Moreover (3.22) may be used to show that, with the mass term (3.17), the total Lagrangian is invariant under the global gauge transformations

$$\begin{aligned} &\nu_k \to e^{i\,\Lambda}\,\nu_k \\ &l \to e^{i\,\Lambda}\,l, \qquad l = e, \mu, \tau, \dots, \end{aligned} \tag{3.25}$$

with Λ a constant parameter independent of the flavour l. This invariance entails the conservation of the total lepton charge

$$L = \sum_{l=e,\mu,\tau,\dots} L_l \tag{3.26}$$

and assures as well that neutrinos with definite masses are Dirac particles. In fact charged leptons carry lepton charge one, thus in order for L to be conserved, the global gauge transofrmations (3.25) require massive neutrinos to carry a unit of lepton charge too (since Λ is the same for both leptons and neutrinos). In this way the lepton charge of ν_k is the opposite of the one of $\bar{\nu}_k$. On the other hand, for the SM fields $\nu_{\alpha L}$ is easy to see that (3.17) doesn't allow the individual lepton charges L_l to be conserved, unless the matrix M^d is diagonal (in that case global gauge transformations with l-dependent parameters are still symmetries of the Lagrangian); but the theory is nontheless invariant under the transformations

$$\begin{aligned} &\nu_{\alpha L} \to e^{i\Lambda}\,\nu_{\alpha L}, \qquad \nu_{\alpha R} \to e^{i\Lambda}\,\nu_{\alpha R} \\ &l \to e^{i\Lambda}\,l \end{aligned} \tag{3.27}$$

which imply the total lepton charge to be conserved.
The theory of Dirac massive neutrinos hence allows processes like

$$\mu^+ \to e^+ + \gamma, \qquad \mu^+ \to e^+ + e^- + e^+ \tag{3.28}$$

where the total lepton charge is conserved. Conversely, a process like neutrinoless double β-decay

$$(A, Z) \to (A, Z+2) + e^- + e^- \tag{3.29}$$

is forbidden.

3.3 Majorana mass term

By making use of the definitions we gave in section (3.1) we can construct different mass terms involving the fields ν_L^c, ν_R^c as well as ν_L, ν_R. In particular a left-handed Majorana mass term has the form

$$\mathcal{L}_L^M = -\frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau,...} \bar{\nu}_{\alpha L}^c \, M_{\alpha\beta L}^M \, \nu_{\beta L} + H.c. \tag{3.30}$$

Similarly a right-handed is given by

$$\mathcal{L}_R^M = -\frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau,...} \bar{\nu}_{\alpha R}^c \, M_{\alpha\beta R}^M \, \nu_{\beta R} + H.c. \tag{3.31}$$

Note that there are no global gauge transformations which leave (3.30) and (3.31) invariant, thus no lepton charge can be conserved in a theory with such mass terms. Hence there is no way of discerning between a neutrino and its own antiparticle, guaranteeing that the massive fields in this case represent true Majorana particles. Let us proceed to put the $n \times n$ mass matrix M^M in the standard diagonal form.
It is useful to take into account that M^M is symmetric; in fact, using (3.7) and the definitions of ν_L^c and ν_R^c we find

$$\bar{\nu}_L^c = -\nu_L^T C^{-1}, \qquad \bar{\nu}_R^c = -\nu_R^T C^{-1} \tag{3.32}$$

Now (3.30) becomes

$$\begin{aligned} \bar{\nu}_L^c M^M \nu_L &= (\bar{\nu}_L^c M^M \nu_L)^T = -(\nu_L^T C^{-1} M^M \nu_L)^T \\ &= \nu_L^T (C^{-1})^T (M^M)^T \nu_L = \bar{\nu}_L^c (M^M)^T \nu_L \end{aligned} \tag{3.33}$$

(a minus sign appears when we permute two fermionic fields) and hence

$$M = M^T \tag{3.34}$$

A symmetrical matrix can be diagonalized with a unitary transformation U in this way:

$$M = (U^\dagger)^T m \, U^\dagger \tag{3.35}$$

with m being a diagonal 3×3 matrix with positive eigenvalues. Inserting (3.35) into (3.30) we obtain

$$\mathcal{L}_L^M = -\frac{1}{2} \bar{n}_L^c \, m \, n_L - \frac{1}{2} \bar{n}_L \, m \, n_L^c, \tag{3.36}$$

having put

$$n_L = U^\dagger \, \nu_L, \qquad n_L^c = \mathcal{C} \, \bar{n}_L^T. \tag{3.37}$$

So the mass term can be recast in

$$\mathcal{L}^M = -\frac{1}{2}\sum_{k=1}^{n} m_k \bar{\varphi}_k \varphi_k \tag{3.38}$$

where

$$\varphi = n_L + n_L^c = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n \end{pmatrix}. \tag{3.39}$$

Again the LH neutrino fields ν_L are lineare combinations of massive neutrino fields φ_k. By inverting (3.37) and using (3.39) we have

$$\nu_{lL} = \sum_{k=1}^{n} U_{lk}\,\varphi_{kL} \tag{3.40}$$

From (3.39) we also have that the fields φ_k are Majorana fermions. In fact, after some easy steps (making use of the second relation in Eq. (3.37), we get

$$\varphi_k = \mathcal{C}\bar{\varphi}_k^T, \qquad k = 1, \ldots, n. \tag{3.41}$$

Being no lepton charge conserved in a theory with a Majorana mass term, processes like (3.29) are now allowed together with the other ones. This kind of mass term was first considered by Pontecorvo *et al* in 1969; subsequently many experiments (as we will see) have been addressed to study processes where lepton charge is not conserved which seem to go adrift from the fundamental pillars of the SM.

3.4 Dirac-Majorana mass term and See-Saw mechanism for mass generation

After having considered in detail the two different kind of mass term, the next step is to put them together to have

$$\mathcal{L}^{D-M} = -\frac{1}{2}\bar{\nu}_L^c\, M_L^M\, \nu_L - \frac{1}{2}\bar{\nu}_R\, M_R^M\, \nu_R^c - \bar{\nu}_R\, M^D\, \nu_L + H.c. \tag{3.42}$$

known as a $Dirac - Majorana$ mass term. Sums over the matrix indices are understood. Let us specify to the case of three neutrino flavours to make calculations easier in this case. The expression (3.42) can be put in a more compact form by defining the fields

$$\mathbf{n}_L = \begin{pmatrix} \nu_L^{'} \\ (\nu_R^{'})^c \end{pmatrix}, \qquad \nu_L^{'} = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}, \qquad \nu_R^{'} = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix}, \tag{3.43}$$

and hence writing

$$\mathcal{L}^{D-M} = -\frac{1}{2}\bar{\mathbf{n}}_L^c\, M\, \mathbf{n}_L + H.c. \tag{3.44}$$

Now M is a 6×6 matrix

$$M = \begin{pmatrix} M_L^M & (M^D)^T \\ M^D & M_R^M \end{pmatrix} \tag{3.45}$$

whose entries are 3×3 matrices given respectively by Dirac and Majorana mass matrices we saw before. The procedure to diagonalize the matrix (3.45) is the same of the one we showed for the Majorana mass term, so we won't repeat it here. The only difference to remark is that the sum in (3.38) now runs over 6 values (if we took into account n neutrino flavours then it would have been a sum over $2n$ values, being M a $2n \times 2n$ matrix). Hence we can diagonalize the Dirac-Majorana mass matrix and express flavour neutrino fields as linear combinations of fields with definite masses. Moreover, as we saw in the case of simple Majorana mass term, the massive neutrinos are Majorana particles, described by two degrees of freedom. Thus, as in the previous case, we can't distinguish between a neutrino and an antineutrino, because none of them carry a conserved lepton charge.

This theory is very powerful in describing the smallness of the neutrino masses and thence the difficulty in detecting them at the energy scale of the SM. In order to better understand this we need to look closer at the masses we get after diagonalizing the mass matrix.

As a first note, is crucial to stress that the mass block M^D in Eq. (3.45) is generated by the Higgs mechanism and then its entries must be proportional to the Higgs doublet vacuum expectation value $v_{sm} = 246\, GeV$, allowing at most an order $10^2\, GeV$. Conversely, the RH Majorana mass block M_R^M is invariant under the gauge symmetries of the SM and doesn't need to be generated by the Higgs mechanism after symmetry breaking, but can be present in the total lagrangian of the theory without spoiling its symmetries. This implies that the elements of M_R^M are in principle not bounded from above. However this Majorana mass term could be generated by the Higgs mechanism at a higher energy scale beyond the SM, as high as the grand-unification scale of Grand unified theories (GUT) of $\sim 10^{15}\, GeV$.

Let us see what we get after diagonalizing (3.45) in the case of just one flavour (the matrix being 2×2). The lagrangian (3.42) reduces to

$$\begin{aligned} \mathcal{L}^{D-M} &= -\frac{1}{2} m_L (\bar{\nu}_L)^c\, \nu_L - \frac{1}{2} m_R \bar{\nu}_R\, (\nu_R)^c + h.c. \\ &= -\frac{1}{2} (\bar{n}_L)^c\, M\, n_L + h.c. \end{aligned} \tag{3.46}$$

where

$$n_L \equiv \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}, \qquad M \equiv \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}. \tag{3.47}$$

Assume also that m_L, m_R and m_D are all real. To diagonalize, it turns out to be easier to write M in the following form

$$M = \frac{1}{2} Tr M + \mathbf{M} \tag{3.48}$$

with $\mathbf{M}$ defined as

$$\mathbf{M} = \begin{pmatrix} \frac{1}{2}(m_R - m_L) & m_D \\ m_D & \frac{1}{2}(m_R - m_L) \end{pmatrix}. \tag{3.49}$$

Being $\mathbf{M}$ symmetric is diagonalizable by an orthogonal transformation

$$\mathbf{M} = \mathcal{O}\, \mathbf{m}\, \mathcal{O}^T. \tag{3.50}$$

where the diagonal matrix $\mathbf{m}$ has entries

$$\mathbf{m}_{1,2} = \pm\frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2}. \tag{3.51}$$

which are the eigenvalues of $\mathbf{M}$. Now we can put this result together with the definition (3.48) to have for the eigenvalues of the matrix M

$$\mathbf{m}_{1,2}^{'} = \frac{1}{2}(m_R + m_L) \pm \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2}. \tag{3.52}$$

The following relation holds

$$M = \bar{\mathcal{O}}\, \mathbf{m}^{'}\, \bar{\mathcal{O}}^T \tag{3.53}$$

where $\bar{\mathcal{O}}$ is a different orthogonal matrix from the unbarred one. To be consistent with what we said in (3.35) let us write the eigenvalues of M in this way

$$m_i^{'} = \eta_i\, m_i, \qquad \eta_i = \pm 1, \qquad m_i = |m_i^{'}| \tag{3.54}$$

so that we can switch to a unitary transformation like in (3.35) by defining

$$U^\dagger = \sqrt{\eta}\, \bar{\mathcal{O}}^T \tag{3.55}$$

Now the mass term (3.46) has the form

$$\mathcal{L}^{D-M} = -\frac{1}{2}\sum_{i=1}^{2} m_i \bar{\nu}_i \nu_i \tag{3.56}$$

and the relation between flavour neutrinos and massive fields is

$$\begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} = U \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}, \qquad U = \bar{\mathcal{O}}\, (\sqrt{\eta})^*. \tag{3.57}$$

From (3.52) is easy to calculate the limit for $m_R \gg m_D$. Assume $m_L = 0$, so that the lepton number conservation is not violated by the LH Majorana mass term. The eigenvalues we get are

$$m_1 \simeq \frac{m_D^2}{m_R}, \qquad m_2 \simeq m_R, \tag{3.58}$$

that is, if the condition $m_R \gg m_D$ is enforced, we have one very heavy particle with mass m_2 and a light one with mass m_1. This result is still valid when we generalize to three flavours. Say we turn back to our 6×6 mass matrix (3.45) and put to zero the M_L^M 3×3 block. The mass matrix reduces to

$$M = \begin{pmatrix} 0 & (M^D)^T \\ M^D & M_R^M \end{pmatrix}. \tag{3.59}$$

If we assume, for what we explained before, that the entries of M_R^M are much bigger than those of M_D, then we can approximately diagonalize by blocks the total mass matrix. This procedure leads to a light 3×3 block

$$M_{light} \simeq M^D \, (M_R^M)^{-1} \, (M^D)^T \tag{3.60}$$

and a heavy 3×3 block $M_{heavy} \simeq M_R^M$. The six masses (three heavy and three light) are given by the eigenvalues of the two matrices M_{light} and M_{heavy}. Note that the structure of the masses is the same of that in the simpler case we saw before: the light mass is still quadratic in the Dirac mass and pulled down in magnitude by the inverse of the Majorana mass. This is what is known as the *See − Saw mechanism*. The smallness of the three light neutrino masses is explained as inversely proportional to the energy scale where the lepton number conservation is violated. The three heavy particles we get form this model are completely unrelated to the low-energy physics of the SM, but could show up at some higher scale. On the other hand the three light neutrinos are predicted to be Majorana particles, allowing processes where the lepton number conservation is violated.
It is very interesting and instructive to see how this model is related to an effective theory, where the effective Lagrangian of the theory is given by

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \frac{O_5}{\Lambda} + \frac{O_6}{\Lambda^2} + \ldots \tag{3.61}$$

Here $\mathcal{L}_{SM}$ is the original Lagrangian of the SM, while the other terms are non-renormalizable field operators which are not included in the original electroweak theory. These terms have energy (mass) dimensions grater than four and thus have to be rescaled by appropriate powers of the energy scale Λ. The first non-renormalizable term O_5 can be taken to be quadratic both in the Higgs and leptons doublets so that, on symmetry breaking, the Higgs mechanism produces neutrino mass terms like in (3.30), but now the entries of the mass matrix are of the order

$$M_L^M \sim \frac{v_{sm}}{\Lambda} \ll v_{sm}. \tag{3.62}$$

This explains while neutrino masses obtained from this model are extremely small compared to the quark's and lepton's masses, which are proportional to v_{sm}. Moreover, as we will investigate further, experiments on flavour oscillations give squared-mass differences, which can be used to roughly assess the validity of this model. There are mainly two different kinds of experiments: solar experiments and atmospheric experiments, with a hierarchy of squared-mass differences between the two. From atmospheric experiments we have

$$\Delta m^2_{atm} = 2.6 \pm 0.15 \times 10^{-3} eV^2 \tag{3.63}$$

while from solar experiments

$$\Delta m^2_{sol} = 7.92(1 \pm 0.09) \times 10^{-5} eV^2. \tag{3.64}$$

So if we roughly put $M_L^M \approx \sqrt{\Delta m^2_{atm}} \approx 0.05\, eV$ we get from (3.62) $\Lambda \sim 10^{15}\, GeV$ which is indeed the GUT energy scale. Hence for this model, if neutrinos are massive, the lepton number conservation is violated at GUT energies.

4 Flavour oscillations and the mixing matrix U

The mixing schemes we have discussed have as their principal consequence that of triggering *neutrino oscillations*. These are really important in determining experimentally the massive nature of neutrinos as they give differences of squared masses and the mixing angles upon which the matrix U depends. The key idea of oscillations is that, because flavour fields are linear combinations of massive fields, the state vector of any given flavour field after time-evolution turns into a linear combination of states of all types of neutrinos, meaning that any flavour neutrino is itself a superposition of other flavour neutrinos. Let us briefly see how this happens. We start with our mixing scheme

$$\nu_{\alpha L} = \sum_i U_{\alpha i}\, \nu_i \tag{4.1}$$

with the massive fields ν_i that can either be Majorana or Dirac. The state vector for the flavor neutrino is given by

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}\, |\nu_i\rangle, \tag{4.2}$$

where the ket $|\nu_i\rangle$ is the state vector of a massive neutrino with mass m_i and we consider the approximation where the masses are negligible compared to the corresponding momenta. The time-evolution evolves the state $|\nu_\alpha\rangle$ to

$$|\nu_\alpha\rangle_t = \sum_i U^*_{\alpha i}\, e^{-iE_i t}\, |\nu_i\rangle \tag{4.3}$$

The remarkable fact is that the *evolved* state is now itself a linear combination of all possible neutrino flavours. This is a consequence of (4.2) that, together with the unitarity of U yields

$$|\nu_i\rangle = \sum_\beta U_{\beta i}\, |\nu_\beta\rangle \tag{4.4}$$

and thus can be used to re-express the state evolved after time t as

$$|\nu_\alpha\rangle_t = \sum_\beta \mathcal{A}_{\nu_\beta;\nu_\alpha}(t)\, |\nu_\beta\rangle \tag{4.5}$$

$$\mathcal{A}_{\nu_\beta;\nu_\alpha}(t) = \sum_i U_{\beta i}\, e^{-iE_i t} U^*_{\alpha i} \tag{4.6}$$

The quantity $\mathcal{A}_{\nu_\beta;\nu_\alpha}(t)$ is the amplitude for the state transition $\nu_\alpha \to \nu_\beta$, which gives the following probability

$$\mathcal{P}_{\nu_\alpha \to \nu_\beta} = \left| \sum_i U_{\beta i}\, e^{-iE_i t} U^*_{\alpha i} \right|^2 . \tag{4.7}$$

For this probability to be non-zero, at least two neutrino masses must be different; in fact if the masses are all the same, for the unitarity of U the amplitude reduces to

$$\mathcal{A}_{\nu_\beta;\nu_\alpha}(t) = e^{-iEt}\sum_i U_{\beta i}U^*_{\alpha i} = e^{-iEt}\delta_{\alpha\beta}. \tag{4.8}$$

The same problem we have if the mixing matrix U is diagonal (i.e. no mixing). Eq. (4.7) can be expanded in

$$\mathcal{P}_{\nu_\alpha\to\nu_\beta} = \sum_{ij} U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}e^{-i\Delta m^2_{ij}D/2E}. \tag{4.9}$$

In the last expression $\Delta m^2_{ij} = m_i^2 - m_j^2$, $E_i = \sqrt{p^2+m_i^2} \simeq p + m_i^2/2p$ and $m_i \ll p$. D is the distance between the source of the neutrino beam and the detector and approximately equals the time t ($v \sim c = 1$). Thus we see that any experiment designed to detect neutrino oscillations cannot measure any individual mass, but only mass differences.
The probability depends both on $(n-1)$ mass squared differences and on the parameters entering in the mixing matrix U. This matrix depends on n^2 parameters, which are angles and phases, as any $n \times n$ unitary matrix can be constructed as a product of rotation matrices and unitary matrices made up of just phase factors. Not all these parameters are independent and it can be shown that the number of independent angles is $n(n-1)/2$ while the number of phases depends on whether the neutrinos are Dirac or Majorana: in the first case they are $(n-1)(n-2)/2$, in the second case $n(n-1)/2$. We will come back to this to see how crucial it is for the properties of the mixing matrix.

For completeness now briefly have a look at the oscillating behaviour of antineutrinos. The antineutrino state-vector is given by

$$|\bar\nu_\alpha\rangle = \sum_i U_{\alpha i}\,|\bar\nu_i\rangle, \qquad (\textit{Dirac case}) \tag{4.10}$$

$$|\bar\nu_\alpha\rangle = \sum_i U_{\alpha i}\,|\nu_i\rangle, \qquad (\textit{Majorana case}). \tag{4.11}$$

$\bar\nu_i$ (ν_i) is the state for an antineutrino (neutrino) with mass m_i. In both cases the amplitude for the transition $\bar\nu_\alpha \to \bar\nu_\beta$ is

$$\mathcal{A}_{\bar\nu_\beta;\bar\nu_\alpha}(t) = \sum_i U^*_{\beta i}\,e^{-iE_it}U_{\alpha i} \tag{4.12}$$

which yields the following probability

$$\mathcal{P}_{\bar\nu_\alpha\to\bar\nu_\beta} = \left|\sum_i U^*_{\beta i}\,e^{-iE_it}U_{\alpha i}\right|^2. \tag{4.13}$$

Comparing the amplitude for antineutrinos with the one for neutrinos we saw before, we see that the relation

$$\mathcal{A}_{\bar{\nu}_\beta;\bar{\nu}_\alpha}(t) = \mathcal{A}_{\nu_\alpha;\nu_\beta}(t) \tag{4.14}$$

holds, which entails for the probabilities

$$\mathcal{P}_{\nu_\alpha\to\nu_\beta} = \mathcal{P}_{\bar{\nu}_\beta\to\bar{\nu}_\alpha}. \tag{4.15}$$

Is easy to check that expressions (4.7) and (4.13) are invariant under transformations of the mixing matrix of the form

$$U_{\alpha i} \to U'_{\alpha i} = e^{-i\gamma_\alpha}\, U_{\alpha i}\, e^{i\delta_i} \tag{4.16}$$

with γ_α and δ_i real parameters. Hence the parameters of the mixing matrix U that appear in the probability cannot be absorbed by such transformations and this, as we said, reduces the number of phases upon which the mixing matrix depends to $(n-1)(n-2)/2$ in the Dirac case and to $n(n-1)/2$ in the Majorana case.
One can parametrize the mixing matrix U in a very similar fashion to that used for quark mixing (for reference see [4]). For a Dirac spinor this has the following form

$$U = \begin{pmatrix} c_{e2}s_{e3} & s_{e2}c_{e3} & s_{e3}\,e^{-i\delta} \\ -s_{e2}c_{\mu3} - c_{e2}s_{\mu3}s_{e3}\,e^{i\delta} & c_{e2}c_{\mu3} - s_{e2}s_{\mu3}s_{e3}\,e^{i\delta} & s_{\mu3}c_{e3} \\ s_{e2}c_{\mu3} - c_{e2}s_{\mu3}s_{e3}\,e^{i\delta} & -c_{e2}c_{\mu3} - s_{e2}s_{\mu3}s_{e3}\,e^{i\delta} & c_{\mu3}c_{e3} \end{pmatrix}. \tag{4.17}$$

Here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ and we identify the three angles $\theta_{e3}\,,\theta_{e2}$ and $\theta_{\mu3}$ and the phase δ as the parameters of U that can be measured experimentally.

5 The experimental side of the medal

Many experiments aimed to study oscillations have established, in the past twenty years, that neutrinos are indeed massive, albeit their masses are very small. Specifically, two different mass squared differences have been observed in solar and atmospheric experiments and then confirmed by other experiments made on earth. The experiment LSND measured a third mass difference which has not though be confirmed by subsequent experiments like KARMEN and MiniBooNE. Two mass squared differences need at least three different mass eigenstates identifiable as the three massive neutrino fields. Solar experiments have given a squared mass difference involving the masses m_1 and m_2 with $|m_1| > |m_2|$. On the other hand atmospheric experiments involve the third mass m_3. In total we have

$$\begin{aligned} \Delta m^2_{SOL} &= \Delta m^2_{21} = |m_2|^2 - |m_1|^2 \\ \Delta m^2_{ATM} &= \Delta m^2_{31} = |m_3|^2 - |m_1|^2. \end{aligned} \tag{5.1}$$

Δm^2_{ATM} can be either positive or negative depending on the hierarchy of the masses. We talk of normal hierarchy if $m_1 < m_2 < m_3$, while the inverted hierarchy is that where $m_3 < m_1 < m_2$. In the following table the experimental results obtained from solar and atmospheric experiments are shown. From these results we may see the hierarchy between the two

$\Delta m^2_{sun}(10^{-5}\ eV^2)$	$7.67^{+0.16}_{-0.17}$	$7.65^{+0.023}_{-0.020}$
$\Delta m^2_{atm}(10^{-3}\ eV^2)$	$2.39^{+0.11}_{-0.08}$	$2.40^{+0.012}_{-0.011}$
$\sin^2\theta_{12}$	$0.312^{+0.019}_{-0.018}$	$0.304^{+0.022}_{-0.016}$
$\sin^2\theta_{23}$	$0.466^{+0.073}_{-0.058}$	$0.50^{+0.07}_{-0.06}$
$\sin^2\theta_{13}$	0.016 ± 0.010	$0.010^{0.016}_{-0.011}$

Table 1: The results in the first column are taken from ref. [10], those in the second from ref.[11]

mass squared differences which we mentioned in previous chapters, that is the ratio between the two is

$$\Delta m^2_{ATM} \simeq 30\,\Delta m^2_{SOL}. \tag{5.2}$$

This experimental evidence matches with the mixing scheme where there are three neutrinos and two independent mass squared differences, with $\Delta m^2_{31} \simeq \Delta m^2_{32}$. From Table 1 we see that the the parameter θ_{13} is close to zero within the experimental error, while the other two angles are very large. If we now look back at the mixing matrix we saw at the end of the previous chapter, we see that substituting the values given in Table 1 within the errors, the matrix is well represented by the following one

$$U_{TB} = \begin{pmatrix} \sqrt{2}/3 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{5.3}$$

This matrix is know as $tri - bimaximal$ and to obtain it we made the following approximations

$$\theta_{13} = 0, \qquad \sin^2\theta_{12} = \frac{1}{3}, \qquad \sin^2\theta_{23} = \frac{1}{2}, \tag{5.4}$$

with the phase convention $\delta = 0$. Should not θ_{13} be zero, or at least to it compatible, there would be issues in CP violation and matter-antimatter asymmetry which have not yet been observed. In the last ten years the majority of data analysis have been made by taking the approximation $\theta_{13} = 0$, due to the fact that there has not been yet any evidence of finiteness for θ_{13}.

Whereas many experiments have hitherto confirmed that neutrinos do oscillate and thus have masses, what remains still unknown is the so-called absolute value of these masses. Oscillations, as we have already pointed out, can't give any information on what is the real value of a single neutrino mass and so to probe any exact value other kinds of experiments have been carried through: the most important are those on beta decay and neutrinoless double beta decay. But still, starting from our mass squared difference we get from oscillation experiments, we can give a first rough and rather naive approximation of what is the lower bound for the single mass. If we consider for example the data supplied by the MINOS experiment and published in 2006 (see ref.[15]), it was measured that $\Delta m^2_{23} = 0.0027\, eV^2$, consistent with what measured by the Super-Kamiokande neutrino detector in 1998. From this value, we can certainly say that at least one of the two masses involved in the squared difference has to be at least as big as the square root of Δm^2_{23}, that is $0.04\, eV$.
Once we know that a lower bound exists the even more interesting challenge is to investigate whether one can put also upper bounds to the mass value. This can be done through experiments on beta decay.

5.1 Beta decay

Measurement of the electron neutrino mass can be obtained by looking at the spectrum of nuclear beta decay. The quantity of interest is the Curie function, which is given by

$$K(T) = \left[(Q-T)\sqrt{(Q-T)^2 - m^2_{\nu_e}}\right]^{1/2} \tag{5.5}$$

in the case there is no neutrino mixing and by

$$K(T) = \left[(Q-T) \sum_{k=1}^{3} |U_{ek}|^2 \sqrt{(Q-T)^2 - m_{k\nu_e}^2} \right]^{1/2} \tag{5.6}$$

in the case of mixing. Here $Q = M_i - M_f - m_e$, with M_i and M_f the masses of the initial and final nuclei respectively. $T = E_e - m_e$ is the electron kinetic energy. Would m_{ν_e} be zero, then the Curie function would simply be a linear decreasing line, with the point $T = Q$ being the so-called end of the spectrum. Switching on the tiny neutrino mass, what we expect theoretically is that the end-point is shifted to the value $T = Q - m_{\nu_e}$ (in case of no mixing) and to $T = Q - m_{\nu_l}$ (in case of mixing), where ν_l is the lightest massive neutrino component of ν_e, which depends on the hierarchy scheme. Another feature expected analitically in this model with neutrino mixing is to have kinks in the Curie function corresponding to the energies $T_k = Q - m_k$; the sharpness of these kinks depends on the value of $|U_{ek}|^2$. Tritium has turned out to be the best candidate in these kind of experiments for its very small Q-value in beta decay. This is important as the relative number of events occurring below the end-point is inversely proportional to the cubic of Q; hence one can maximize this number by using small Q values.

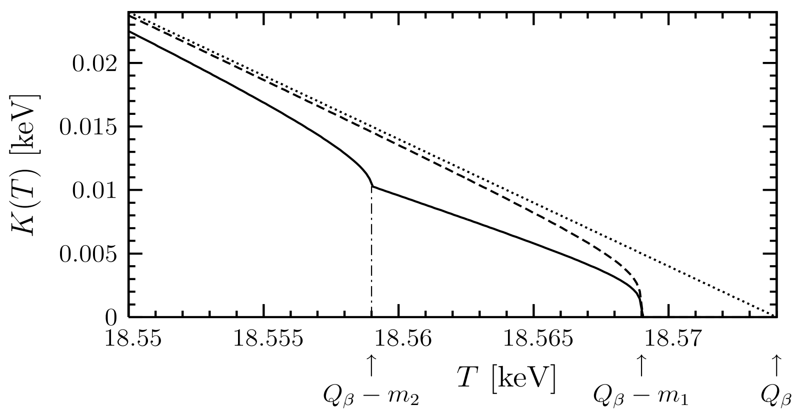

Figure 1: *The Curie function is represented by the dotted line for $m_{\nu_e} = 0$; by the dashed line in case there is no mixing and $m_{\nu_e} = 5\,eV$ and by the solid line for a mixing scheme like in (5.6) with $m_1 = 5\,eV$ and $m_2 = 15\,eV$.*

Hence the *important* region of the spectrum to look at is the one close to the end-point. With this technique experiments in Germany have been able, five years ago, to estimate the upper bound on the electron neutrino mass as $2.3\,eV$ [95% $C.L.$] (Mainz tritium experiment) and $2.5\,eV$ [95% $C.L.$] (Troitzk tritium experiment), but only in the case of one-generation Curie function given by (5.5). No kinks have yet been observed. However these results obtained in the framework of no mixing can be extended to the three neutrino mixing scheme by considering the following argument: since the impossibility of measuring any kinks in the Curie function is very likely due to the bad resolution of the instruments, one can assume that the following inequality holds

$$m_k \ll T - Q, \tag{5.7}$$

so that the expression (5.6) can be exanded (taking the square) as

$$\begin{aligned} K^2 &= (Q-T)^2 \sum_k |U_{ek}|^2 \sqrt{1 - \frac{m_k^2}{(Q-T)^2}} \\ &\simeq (Q-T)^2 \sum_k |U_{ek}|^2 \left[1 - \frac{1}{2}\frac{m_k^2}{(Q-T)^2}\right] \\ &= (Q-T)^2 \left[1 - \frac{1}{2}\frac{m_\beta^2}{(Q-T)^2}\right] \simeq (Q-T)^2 \sqrt{1 - \frac{m_\beta^2}{(Q-T)^2}} \\ &= (Q-T)\sqrt{(Q-T)^2 - m_\beta^2}, \end{aligned} \tag{5.8}$$

with m_β defined as

$$m_\beta^2 = \sum_k |U_{ek}|^2 m_k^2. \tag{5.9}$$

This approximation has rendered $K(T)$ in (5.6) depend on m_β the same way $K(T)$ in (5.5) depends on m_{ν_e}. Hence the upper limit both in the Mainz and Troitzk experiments, must be interpreted, in the case of three neutrino mixing, as a limit on this effective mass m_β.

The next important data taking is supposed to start in 2012 at KATRIN, Münster, where a $0.2\,eV$-sensible spectrometer is currently under construction. A very interesting analysis has been carried out recently where, starting from the data measured in oscillation experiments, the values of the single masses are plotted as functions of the lightest mass (chosen as unknown) in both the normal and inverted schemes. Figure (2) shows the behaviour of the masses and points out that after the lightest mass approaches approximately $2 \times 10^{-1}\,eV$ all the masses tend to be degenerate. The same behaviour can be studied plotting m_β as a function of the lightest mass in both schemes. Figure (3) helps understand which are the contributions of the single masses to m_β and also shows which regions will be better clarified by the forthcoming KATRIN experiment.

5.2 Neutrinoless Double Beta Decay

Neutrinoless Double Beta Decay (NDBD) is nowadays definitely the most exciting area of research to probe neutrino properties and capture the exact

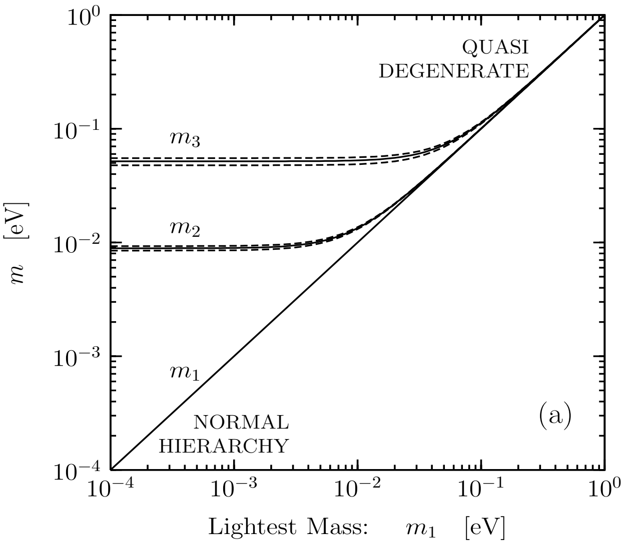

Figure 2: *Neutrino masses plotted as functions of the lightest mass in the normal hierarchy. The region where the KATRIN experiments is giving information is also shown.*

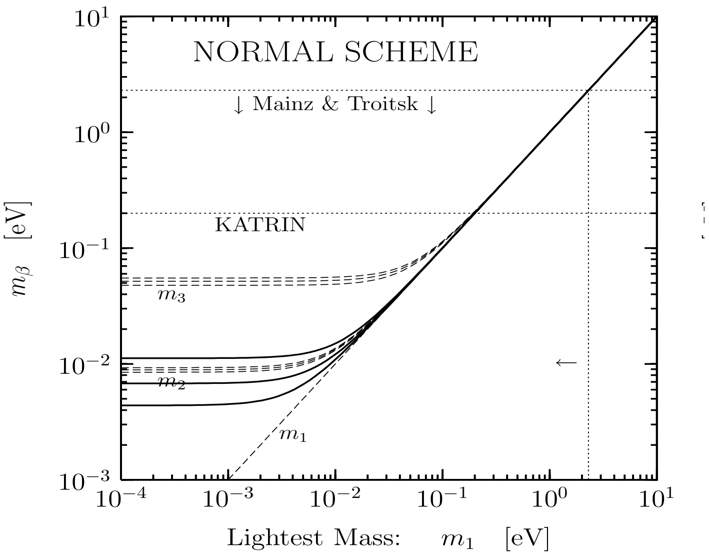

Figure 3: *Effective mass plotted as function of the lightest mass. Solid lines are the best fit value (middle one) and the 2σ ranges. Dashed lines show the individual masses with their 2σ ranges.*

values of neutrino masses. As we have already mentioned in previous chapters the decay, also referred to as $\beta\beta_{0\nu}$, is a transition where an initial nucleus decays to another nucleus with the spontaneous emission of two electrons:

$$(A, Z) \to (A, Z+2) + 2e^-. \tag{5.10}$$

A transition of this type clearly violates the lepton number conservation by two units and is possible only if neutrinos are Majorana particles. In the frame of the Standard Model this process is strictly forbidden so it's really new physics the one that might come out these experiments. The $\beta\beta_{0\nu}$ decay is a second order process in G_F which can be seen as mediated by a virtual massive light Majorana neutrino. Nonetheless, as it will be better explained, is important to stress that this is not the only possible picture, as the process could also be mediated by supersymmetric particles or by heavy neutrinos. The value of the mass appears in the decay rate of the nucleus, which is the inverse of the half-lifetime and is given by

$$[T^{0\nu}_{1/2}(X)]^{-1} = G^X_{0\nu}|\mathcal{M}^X_{0\nu}|^2 \frac{|m_\beta|^2}{m_e^2}. \tag{5.11}$$

In this expression $G^X_{0\nu}$ is the phase-space integral, which is exactly calculable, $\mathcal{M}^X_{0\nu}$ is the nuclear matrix element and m_β is the effective Majorana mass, linear combination of neutrino masses, as defined in Eq. (5.9). Note that it is also possible to have Majorana phases entering Eq. (5.11) as

$$m_\beta = \sum_k |U_{ek}|^2 m_k e^{i\alpha_k}. \tag{5.12}$$

To have CP invariance one requires $\alpha_i = 0, \pi$, but in principle these phases could cause cancellations such to make the effective mass smaller than any of m_k's. Indirectly, then, $\beta\beta_{0\nu}$ turns out to be a possible way to also measure the Majorana phases.

If one knows the matrix elements, once the $\beta\beta_{0\nu}$ decay is observed, it is possible to deduce the value of the mass m_β. A major problem though is represented by the matrix elements $\mathcal{M}^X_{0\nu}$, whose evaluation is not at all simple and actually different techniques have yielded conflicting results. The methods implemented for computing the nuclear matrix elements are those of many body-systems, the principal being the Quasi Particle Random Phase Approximation (QRPA), which unfortunately has turned out to be unrealistic in taking correlations into account. Currently great efforts are made in trying to modify this method in order to have more consistent results with the other principal technique, that is the Shell Model. A cautious review of both these two major technique is being carried on by many groups, even though to include all the missing correlations into the QRPA seems to be a pretty hard job, due to the many approximations one has to make.

The picture with light neutrinos being the exchanging particles in the process is not the only possible one though. Frameworks with heavy particles (or heavy neutrinos) or supersymmetric particles exchange that still violate the lepton charge conservation at high energy scales can reproduce the same electron spectrum as the one obtained with the exchange of the light Majorana neutrino. Should these other mechanism also happen with, by now unknown, non-negligible probability, then the possibility of extrapolating the light neutrino mass from $\beta\beta_{0\nu}$ would be in real danger. More specifically, if we choose the lepton charge violating energy scale to be of the order of $10^{12}\, eV$ and take $m_\beta \sim 0.1 - 0.5\, eV$, then the contributions to the decay amplitude of heavy particle and light particle exchange are of the same order, meaning that both pictures are possible. Very naively, the decay amplitudes in both case are given by

$$A_L \sim G_F^2 \frac{m_\beta}{k^2}, \qquad A_H \sim G_F^2 \frac{M_W^4}{\Lambda^5} \tag{5.13}$$

being Λ the heavy mass energy scale at which the lepton charge conservation is violated and k the typical light neutrino (virtual) momentum. The ratio between the two, with the chosen energy scale, is $\sim O(1)$. If we go to higher energy scales (like $3 \times 10^{12}\, eV$), then A_L becomes dominant.
So far there is no clear evidence of any $\beta\beta_{0\nu}$ observation, even though in 2002 there was a claim for an observation of neutrinoless double beta decay of ^{76}Ge with $T^{0\nu}_{1/2} = 1.9^{+1.00}_{-0.17} \times 10^{25}\, years$, which has been recently confirmed [9]. This claim has been long criticized but still there is no evidence for ruling it out completely, so new generation experiments are expected to give a substantial contribution to this issue. Hereafter we list the recent $\beta\beta_{0\nu}$ results, with the mass limits deduced by the authors from the half-life times. All limits are at 90% level of confidence, unless otherwise indicated. We see

Isotope	Half-life limit (y)	m_β (meV)
^{48}Ca	$> 1.4 \times 10^{22}$	$< 7200 - 44700$
^{76}Ge	$> 1.9 \times 10^{25}$	< 350 (H-M)
^{76}Ge	$> 1.6 \times 10^{25}$	$< 330 - 1350$ (IGEX)
^{82}Se	$> 2.7 \times 10^{22}$ (68%)	< 5000
^{100}Mo	$> 5.5 \times 10^{22}$	< 2100
^{116}Cd	$> 1.7 \times 10^{23}$	< 1700
^{128}Te	$> 7.7 \times 10^{24}$	$< 1100 - 1500$
^{130}Te	$> 5.5 \times 10^{23}$	$< 370 - 1700$
^{136}Xe	$> 4.4 \times 10^{23}$	$< 1800 - 5200$
^{150}Nd	$> 1.2 \times 10^{21}$	< 3000

that the most stringent bound is the one obtained in the Heidelberg-Moscow

^{76}Ge experiment, comparable with the IGEX experiment result, also on ^{76}Ge.

New generation experiments (CUORE and GERDA at the LNGS, Italy, or MAJORANA or Super-NEMO, this last one not started yet) have as their principal goal that of improving sensitivity in order to probe the inverted hierarchy (IH) of neutrino masses ($m \sim 10-50\, meV$). Such a goal is however far from the allowance of the current technology and so phased projects have been launched both in the USA and Europe. Also the claimed evidence for $\beta\beta_{0\nu}$ signal in the Heidelberg-Moscow experiment is one of the first goals to be achieved in short time.
Many experiments are being carried out for the search of neutrinoless double beta decay with different experimental techniques and isotopes. This is of fundamental importance for many reasons; first of all the observation of a signal may not be considered a real discovery without being confirmed by other experiments on different isotopes. Also the theoretical uncertainty on the nuclear matrix elements requires different experiments to get a reliable value for the effective mass. Finally, from experiments on different isotopes one can have deeper insights on whether the light neutrino scheme is indeed true or not: in fact the relative matrix elements for different nuclei depend on the exchange mechanism.
Huge strides have been made in the last ten years in developing advanced techniques for the investigation of $\beta\beta_{0\nu}$. It is very likely that this will be the field where experimental confirmations on the theoretical frame we have been discussing will soon be made. A key point, as usual, is how much governments will be keen on spending for very expensive experiments.

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