No-Hair Theorems

and introduction to Hairy Black Holes

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Abstract

Intensive research on Black holes properties has been carried out in the last 50 years. The most significant discoveries are the No hair Theorems. Some of these theorems are explained with a series of proofs. Bekenstein approach is studied as well as a generalised proof by Saa. A way to avoid these theorems is presented for scalar field hair and SU (N) Einstein Yang Mill theory with a negative cosmological constant and the MTZ model is introduced.

Introduction

We will start by giving some reminders on general notions used throughout this dissertation.

Reminders on general relativity

Einstein equations are described by $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, G the Newtonian constant and $T_{\mu\nu}$ the energy-momentum tensor. We remind the metric for Schwarzschild $ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$. The electrically charged black holes or the Reissner-Nordstrom black holes are characterised by the metric $ds^2 = -e^{2\alpha(r,t)} dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2$ [26]. This black hole has a non zero electromagnetic field which acts as a source of energy-momentum. The energy momentum tensor for electromagnetism is $T_{\mu\nu} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$ where $F_{\mu\nu}$ is the electromagnetic field strength tensor.

Killing Horizon

An asymptotically flat space-time is the one for which the future null infinity, the past null infinity and the spacelike infinity have the same structure as for Minkowski. The future event horizon is the surface beyond which timelike curves cannot escape infinity and an analogous definition holds for the past event horizon.

The most important feature of a black hole is its event horizon. An event horizon is a hypersurface separating the space-time points that are connected to other space-time points which are far away from the black hole (far enough to assume that the space-time is described by Minkowski metric) by a timelike path from the rest. The gradient $\partial_\mu f$ is normal to the hypersurfaceΣ. If the normal vector is null, the hypersurface is said to be null and the normal vector is also tangent to $\sum$. Null hypersurfaces are considered to be a collection of null geodesics $x^{\mu}(\lambda)$ called the generators of the hypersurface. The tangent vectors to these geodesics are called ξ^{μ} and they are proportional to the normal vectors. They also serve as normal vectors to the hypersurface.[26]

Now if a Killing vector field χ^{μ} is null along some null hypersurface $\sum$ we say that $\sum$ is a Killing horizon of χ^{μ}. The vector field χ^{μ} is normal to $\sum$ since a null surface cannot have to linearly independent null tangent vectors.

Event horizon and Killing horizon are closely related in space-times with time translation symmetry. Every event horizon $\sum$ in a stationary asymptotically flat space-time is a Killing horizon for some Killing vector field χ^{μ}[26].

Soliton

The soliton is an intrinsically non linear solution of the field equation with remarkable stability and particle like properties. This local travelling wave pulse with a coherent structure represented a revolution in the non linear science. The soliton is the result of a balance between two forces: one is linear and acts to disperse the pulse the other has the opposite effect it is non linear and acts to focus the pulse. Non linearity is essential for balancing the dispersion process.[9]

Note: throughout the dissertation the prime denotes the partial derivative with respect to the radial coordinate r and the convariant d'Alembertian is $\blacksquare = \nabla^{\mu}\nabla_{\mu}$

1. The uniqueness theorem

For the no hair theorems, the uniqueness theorem is essential. At the centre of the uniqueness problem lies the proof that the static electro-vacuum (electrovac) black hole space times (with no degenerate horizon) are described by the Reissner-Nordstrom (RN) metric, whereas the circular ones (i.e., the stationary and axisymmetric ones with integrable Killing fields) are given by the Kerr-Newman metric. A lot of work has been put towards classifying static black holes in vacuum. The works of Chase, Bekenstein, Hartle and Teitelboim show that stationary black hole solutions are hairless in a variety of theories where classical fields are coupled to Einstein gravity. The pioneering investigations in this field were attributed to Israel, Muller zum Hagen and Robinson. The alternative approach to the problem of the uniqueness of black hole solutions was proposed by Bunting and Masood-ul-Alam and then strengthened to the Einstein-Maxwell (EM) black holes. Heusler included the magnetically charged RN solution and static Einstein-σ-model case [28]. The condition of non-degeneracy of the event horizon was removed and it was shown that Schwarzschild black hole exhausted the family of all appropriately regular black hole space-times. It was revealed that RN solution comprised the family of regular black hole space-times under the restrictive condition that all degenerate components of black hole horizon carried a charge of the same sign.

In the late sixties, the main contributors to the proof of the uniqueness theorem for the asymptotically fat, stationary black hole solutions of the Einstein-Maxwell equations were Israel, Penrose and Wheeler. Using very rigorous proofs, they established that all stationary electro-vacuum (electrovac) black hole space-times are characterized by their mass, angular momentum and electric charge This result has a direct implication: all stationary black hole solutions can be described in terms of a small set of asymptotically measurable quantities. In the static case it was Israel who, in his pioneering work, was able to establish that both static vacuum and electrovac black hole space times are spherically symmetric. In particular he proved in one of his papers that one can consider the limiting external field as a gravitationally collapsing asymmetric body as static [10]. As a consequence a series of papers were published showing that the unique non-degenerate electrovac static black hole metrics are the Reissner-Nordstrom family [11].

It is only in 1989 these statements were disproved when several authors presented a counterexample within the framework of SU (2) Einstein-Yang-Mills (EYM) theory. The main argument was to say that although the new solution was static and had vanishing Yang-Mills charges, it was different from the Schwarzschild black hole and, therefore, not characterized by its total mass (one of the main reasons the discovery hasn't happened before was the belief that EYM equations admit no soliton solutions). Following this discovery a whole variety of new black hole configurations violating the generalized no-hair conjecture were found during the last few years. These include, for instance, black holes with Skyrme, dilaton or Yang-Mills-Higgs hair [8]. As a consequence of the diversity of new solutions, the different steps of the proof of the uniqueness theorem had to be reconsidered. In particular questions were asked as to whether there are steps in the uniqueness proof which are not sensitive to the details of the matter contents.

The classical uniqueness theorems were, established for space-times which are either circular or static. The circularity theorem by Kundt and Trumper and Carter[29] does not hold for the EYM system unless additional constraints are imposed and the the staticity theorem establishes the hypersurface orthogonality of the stationary Killing field for electrovac black hole space times with non-rotating horizons.

The uniqueness theorem for stationary and axisymmetric black holes is mainly based on the Ernst formulation of the Einstein -Maxwell equations. The key result consists in Carter's observation that the field equations can be reduced to a 2-dimensional boundary value problem. An identity due to Robinson then establishes that all vacuum solutions with the same boundary and regularity conditions are identical [31]. The uniqueness problem for the electrovac case remained open until Mazur and Bunting independently succeeded in deriving the desired divergence identities in a systematic way [29].

During the last years the discovery of new black hole solutions in theories with nonlinear matter fields led scientists to study topics related to the stationary problem of non-rotating black holes as well as the subject of the stationarity of these objects. Taking into consideration nonlinear matter models or general sigma models in the present research, the problems of black hole solutions of the late 1960s and 1970s are reconsidered. Historically the idea of a staticity theorem was put forward by Lichnerowicz for the simple case in which there was no black hole. He used the example of a stationary perfect fluid that was everywhere locally static i.e. its flow vector was aligned with the Killing vector [30]. The Killing vector itself would have the staticity property of being hypersurface orthogonal. Hawking extended the research by generalising the proof of staticity to the vacuum case. He considered black holes that were non-rotating i.e. the null generator of the horizon was aligned with the Killing vector. Following Hawking contribution, Carter considered an extension of this problem to the case of electromagnetic fields and obtained the desired result to some extent. Using the Arnowitt-Deser-Misner (ADM)formalism, Sudarsky and Wald considered an asymptotically flat solution to Einstein-Yang-Mills (EYM) equations with a Killing vector field which was timelike at infinity. Using the example of an asymptotically flat maximal slice with compact interior, they established that the solution is static when it had a vanishing Yang-Mills electric field on the static hypersurfaces. If an asymptotically flat solution possesses a black hole, then it is static when it has a vanishing electric field on the static hypersurface. They also presented a new derivation of the mass formula and proved that every stationary solution is an extremum of the ADM mass at fixed Yang-Mills electric charge. On the other hand, every stationary black hole solution is an extremum of the ADM mass at fixed electric charge, canonical angular momentum, and horizon area [27]. One should also mention the work of Sudarsky and Wald, in which they derived new integral mass formulas for stationary black holes in EYM theory. Using the notion of maximal hypersurfaces and combining the mass formulas, they obtained the proof that non-rotating Einstein-Maxwell (EM) black holes must be static and have a vanishing magnetic field on the static slices. Hawking's strong rigidity theorem which states that that the event horizon of a stationary black hole space-time is a Killing horizon represents the basis for the uniqueness theorem. The theorem establishes a connexion between the event horizon and the Killing horizon. The theorem requires that the matter fields obey well behaved hyperbolic field equations and that the stress-energy tensor satisfies the weak energy condition, the theorem asserts that the event horizon of a stationary black hole space-time is a Killing horizon. This also implies that either the null-generator Killing field of the horizon coincides with the stationary Killing field or space-time admits at least one axial Killing field. This theorem emphasizes that the event horizon of a stationary black hole had to be a Killing horizon; i.e., there had to exist a Killing field χ_μ in the spacetime which was normal to the horizon. If this field did not coincide with the stationary Killing field t_μ then it was shown that the space-time had to be axisymmetric as well as stationary. It follows that the black hole will be rotating; i.e., its angular velocity of the horizon Ω will be nonzero (Ω is defined by the relation $\chi_\mu = t_\mu + \Omega\phi_\mu$ where ϕ_μ is an axial Killing vector field) and the Killing vector field $\chi_\mu$will be spacelike in the vicinity of the horizon. The black hole will be enclosed by an ergoregion. On the other hand, if t_μ coincides with χ_μ (so that the black hole is non-rotating) and t_μ is globally timelike outside the black hole, then one can show that the spacetime is static. The standard black hole uniqueness theorem leaves an open question of the problem of the potential existence of additional stationary black hole solutions of EM equations with a bifurcate horizon which is neither static nor axisymmetric. The situation was recuperated by Wald. He showed that any non-rotating black hole in EM theory, the ergoregion of which was disjoint from the horizon, had to be static, even if the *t*m was not initially presupposed to be globally timelike outside the black hole. Chrusciel reconsidered the problem of *the strong rigidity theorem* and gave the corrected version of the theorem in which he excluded the previous assumption about maxi- mal analytic extensions which were not unique [32]. The uniqueness theorems for black holes are closely related to the problem of staticity. However, the uniqueness theorems are based on stronger assumptions than the *strong rigidity theorem*.

Namely, in the non-rotating case one requires staticity whereas in the rotating case the uniqueness theorem is established for circular space-times. The foundations of the uniqueness theorems were laid by Israel who established the uniqueness of the Schwarzschild metric and its Reissner-Nordstrom generalization as static asymptotically flat solutions of the Einstein and EM vacuum field equations. Then, Muller zum Hagen *et al.* in their works were able to weaken Israel's assumptions concerning the topology and regularity of the two-surface $V = -t_{\mu}t^{\mu} = constant$. Robinson generalized the theorem of Israel concerning the uniqueness of the Schwarzschild black hole. Finally, Bunting and Masood-ul-Alam excluded multiple black hole solutions, using the conformal transformation and the positive mass theorem. Lately, a generalization of the results to electro-vacuum space-times was achieved. The uniqueness results for rotating configurations, i.e., for stationary, axisymmetric black hole space-times, were obtained by Carter, completed by Hawking and Ellis and the next works of Carter and Robinson. They were related to the vacuum case. Robinson also gained a complicated identity which enabled him to expand Carter's results to electrovac spacetimes.

A quite different approach to the problem under consideration was presented by Bunting and Mazur. Bunting's approach was based on applying a general class of harmonic mappings between Riemannian manifolds while Mazur's was based on the observation that the Ernst equations describe a nonlinear s model on symmetric space [33]. A recent review which covers in detail various aspects of the uniqueness theorems for non-rotating and rotating black holes was provided by Heusler. Heusler and Straumann considered the stationary EYM and Einstein dilaton theories. They showed that the mass variation formula involves only global quantities and surface terms; their results hold for arbitrary gauge groups and any structure of the Higgs field multiplets. The same authors studied the staticity conjecture and circularity conditions for rotating black holes in EYM theories. It turned out that contrary to the Abelian case the staticity conjecture might not hold for non-Abelian gauge fields like the circularity theorem for these fields. Recently, it has been shown that in the non-Abelian case stationary black hole space-times with vanishing angular momentum need not to be static unless they have vanishing electric Yang-Mills charge[28]. Heusler demonstrated that any self-coupled, stationary scalar mapping (s model! from a domain of strictly outer communication, with a non-rotating horizon, has to be static. He also proved the no-hair conjecture for this model.

2.No hair theorems:

There are two kinds of no hair theorems in gravitational physics. The first one is the cosmic non hair theorem which leads to the conclusion that the inflation is a natural phenomenon and would validate this theory to explain the homogeneity and isotropy of the universe we observe today [12]. Here we are interested in the black hole no hair theorem.

Historically, a series of no hair theorems appeared when physicists began looking at the possible interaction of black holes with any kind of matter. Naturally the attention was turned to scalar fields, which makes the most realistic candidate. The no hair theorems excluded for a long time scalar fields, vector fields, massive vectors, spinors and Abelian Higgs hair from stationary black hole exterior. The turning point was the discovery of coloured back holes in Yang Mils theory and a series of solutions for hairy black holes have been found since then. Bekenstein was the first one to suggest a no hair theorem but it was quickly proved to be unstable so we suggested a new one which is the one we are interested in. The statement that black holes have no hair means that they can only be dressed by field that obey the Gauss law like the electromagnetic field. Conformal coupling to gravity permitted the discovery of extremal Reiner Nordstrom geometry because the scalar hair diverges at the horizon; this put the scalar fields under the no hair theorem.

The presence of a cosmological constant (positive or negative) the Kerr Newman solution to the Einstein Maxwell equations becomes the Kerr Newman de sitter solution whose space time is the asymptotically flat the sitter space. The presence of a cosmological constant changes the asymptotic behaviour and structure of space times. In our example all no hair theorems have been proves assuming that the space-time is asymptotically flat.

There are different approaches to prove the No Hair Theorems. Scaling arguments provide an efficient tool for proving nonexistence theorems in at space-time but they are restricted to highly symmetric situations, this arguments if considered in a more complex way lead to a first proof of the no-hair theorem for spherically symmetric scalar fields with arbitrary non-negative potentials. Another proof is based on a mass bound for spherically symmetric black holes and the circumstance that scalar fields (with harmonic action and non-negative potentials) violate the strong energy condition [8].

One of the most impressive solutions is the colored black hole solution of the Einstein-Yang-Mills (EYM) system. Although this solution was found to be unstable in the gravitational sector, non-Abelian hair is generic, and many other non-Abelian black holes were discovered after the colored black hole.[18]

3. **Bekenstein approach:**

In his 1995 paper [1] Bekenstein proves the no hair theorem for a black holes dressed with a multicomponent scalar field. This paper show significant modifications compared to his first works on no hair theorems. In fact Sudarsky showed that there are some exceptions to the no hair theorems as first formulated which lead Bekenstein to write a Novel no hair scalar hair for black holes.

We will follow step by step the proof.

Bekenstein takes the case of a static scalar field in a static black hole background

$$S_{\psi} = -\frac{1}{2}\int\left[\psi_{,\alpha}\psi_{,}^{\alpha} + V(\psi^2)\right](-g)^{1/2}dx^4 \quad (1)$$

Using the action, the field equation is obtained, multiplies by ψ and integrated over the black hole exterior at a given time. The result obtained is:

$$\int\left[g^{ab}\psi_{,a}\psi_{,b} + \psi^2 V'(\psi^2)\right](-g)^{1/2}dx = 0 \quad (2)$$

The metric g^{ab} is positive definite with the indices a and b being spatial coordinates

A theorem states that if $V(\psi^2)$ is non negative and vanishes only at some discrete values ψ_j then the field is constant outside the black hole and its value corresponds to one in the interval $[0, \psi_j]$. It is particularly the case for Klein Gordon field for which if we consider μ as the field's mass we have $V(\psi^2) = \mu^2$. Bechmann and Lechtenfeld objected to Bekhestein logic and claimed that an exponentially decaying scalar hair can be attached to a static spherical black hole (BL solution). However in the BL case the potential is not a closed expression and in some regions the potential is negative which makes it unphysical as it violates the condition $V(\psi^2) \geq 0$. It is necessary to highlight that the theorem fails for any field violating the condition $V(\psi^2) \geq 0$ for example in the case of the Higgs hair with a double well potential for which $V(\psi^2)$ is negative in some regions although some improvements have been made towards providing proofs of a couple of no-hair theorem for black holes in the Abelian Higgs model, in arbitrary dimension and for arbitrary horizon topology [13].

For his proof, Bekenstein considers a multiplet of scalar fields in to the following action:

$$S_{\psi,\chi,\ldots} = -\int \mathcal{E}(\mathcal{K},\mathcal{L},\mathcal{M},\ldots\psi,\chi,\ldots)(-g)^{1/2}\, d^4x \tag{3}$$

And from the first derivatives of ψ and χ, we can form

$$\mathcal{K} \equiv g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}$$

$$\mathcal{L} \equiv g^{\alpha\beta}\chi_{,\alpha}\chi_{,\beta}$$

$$\mathcal{M} \equiv g^{\alpha\beta}\chi_{,\alpha}\psi_{,\beta}$$

In nature there are no elementary fields therefore Bekenstein uses the most general form for a scalar field. He assumes also minimal coupling to gravity and that the energy density carried by the scalar field is non-negative. The energy momentum tensor corresponding to action in (3) is :

$$T_\alpha^\beta = -\mathcal{E}\delta_\alpha^\beta + 2(\partial\mathcal{E}/\partial\mathcal{K})\psi_{,\alpha}\psi_,^\beta + 2(\partial\mathcal{E}/\partial\,\mathcal{L})\chi_{,\alpha}\chi_,^\beta + (\partial\mathcal{E}/\partial\,\mathcal{M})\left(\chi_{,\alpha}\psi_,^\beta + \psi_{,\alpha}\chi_,^\beta\right) \tag{4}$$

The local energy density seen by an observer with four-velocity U^α is:

$$\rho = \mathcal{E} + 2\left[(\partial\mathcal{E}/\partial\mathcal{K})\left(\psi_{,\alpha}U^\alpha\right)^2 + (\partial\mathcal{E}/\partial\mathcal{L})\left(\chi_{,\alpha}U^\alpha\right)^2 + (\partial\mathcal{E}/\partial\mathcal{M})\chi_{,\alpha}U^\alpha\psi_{,\beta}U^\beta\right] \tag{5}$$

Where ρ is an energy density therefore it has to be positive or null. Like for a static black hole with scalar hair we suppose that the field has a time like killing vector. We can assume that $\rho = \mathcal{E}$ in Eq. (5) providing the observer moves along the Killing vector $\psi_{,a}U^\alpha = 0$. Therefore, for this specific field we have:

$$\mathcal{E} \geq 0 \tag{6}$$

and this proves the condition that the energy density ρ has to be positive or null. Going back to Eq. (3), if we take the case when $|v| \to 1$ one can see that the terms involving derivatives dominate $\mathcal{E}$. Combining this information with the condition (6) one can conclude that the dominant terms must be non negative (in our case v is a three velocity with which a second observer moves relative to Killing vector observer. For the case of a free falling frame of reference co-moving momentarily with the first observer we have $U^0 = 1/(1-v^2)^{1/2}$, and $U = 1/(1-v^2)^{1/2}$ where we remind that U^α is the four-velocity). $\psi_{,\alpha}U^\alpha$ and $\chi_{,\alpha}U^\alpha$ are positive provided the conditions below are satisfied:

$$\partial\mathcal{E}/\partial\mathcal{K} > 0, \partial\mathcal{E}/\partial\mathcal{L} > 0 \tag{7}$$

$$(\partial\mathcal{E}/\partial\mathcal{M})^2 \leq 4(\partial\mathcal{E}/\partial\mathcal{K})(\partial\mathcal{E}/\partial\mathcal{L}) \tag{8}$$

To proceed further, Bekenstein assumes (in the case of a spherically symmetric black hole) the existence of a self consistent asymptotically flat solution for the Einstein and the scalar field. In this case the metric outside the horizon can be written as:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + sin\theta^2 d\varphi^2) \tag{9}$$

The event horizon radius is at $r = r_h$ where $exp[\nu(r_h)] = 0$. This last equation has several solutions and the horizon always corresponds to the outer zero.

In this proof asymptotic flatness is assumed as well as the non triviality of the scalar field (this last assumption leads to the conclusion that ψ and χ depend on r). As a consequence we have $\nu(r) = \lambda(r) = O(r^{-1})$ as $\to \infty$.

Because of the coordinate invariance of the scalar's action, the energy momentum tensor obeys the conservation law:

$$T_{\mu\ ;\nu}^{\ \nu} = 0 \tag{10}$$

A well know result by Landau and Lifshitz shows that the r component of Eq. (10) can be written in the form (the prime here corresponds to the partial derivative with respect to r):

$$\left[(-g)^{1/2}T_r^{\ r}\right]' - (1/2)(-g)^{1/2}\left(\partial g_{\alpha\beta}/\partial r\right)T^{\alpha\beta} = 0 \tag{11}$$

T_μ^ν must be diagonal and $T_\theta^\theta = T_\varphi^\varphi$ because of the static and spherical symmetry of the solution we can write Eq (11) as:

$$\left(e^{\frac{\lambda+\nu}{2}}r^2T_r^{\ r}\right)' - (1/2)e^{\frac{\lambda+\nu}{2}}r^2\left[\nu' T_t^{\ t} + \lambda' T_r^{\ r} + 4T_\theta^{\ \theta}/r\right] = 0 \tag{12}$$

The terms containing the derivative of λ with respect to r cancel and we are left with the expression:

$$\left(e^{\nu/2}r^2T_r^{\ r}\right)' - (1/2)e^{\nu/2}r^2\left[\nu' T_t^{\ t} + 4T_\theta^{\ \theta}/r\right] = 0 \tag{13}$$

By using the symmetries condition and Eq. (4) we obtain the result: $T_r^{\ r} = T_t^{\ t} = -\mathcal{E}$. Using this in the right hand side of Eq. (13) and rearranging the derivatives, we obtain the following equation:

$$\left(e^{\nu/2} r^2 T_r^{\ r}\right)' = e^{\nu/2} r^2 \mathcal{E} \tag{14}$$

This is a very important equation and has a central role in Bekenstein proof.

The term at the horizon (boundary term) vanishes when we integrate Eq. (14) from $r = r_h$ to a generic r, because $e^{\nu} = 0$ and $T_r^{\ r}$ is finite at that boundary. After integration we obtain:

$$T_r^{\ r}(r) = -\frac{e^{-\nu/2}}{r^2} \int_{r_h}^{r} \left(e^{\nu/2} r^2\right)' \mathcal{E} dr \tag{15}$$

Sufficiently near the horizon $e^{\nu/2} r^2$ has to grow with r and this is because $e^{\nu} = 0$ is positive outside it the horizon and null on the horizon. Then from condition in (6) and Eq. (15) on can conclude that sufficiently near the horizon we have $T_r^{\ r} < 0$.

The differentiation of equation (14) is carried out to obtain

$$(T_r^{\ r})' = -e^{-\nu/2} r^{-2} \left(e^{\nu/2} r^2\right)' (\mathcal{E} + T_r^{\ r}) \tag{16}$$

And from Eq. (4) one can write:

$$\mathcal{E} + T_r^{\ r} = 2e^{-\nu}\left[(\partial\mathcal{E}/\partial\mathcal{K})\psi_{,r}^2 + (\partial\mathcal{E}/\partial\,\mathcal{L})\chi_{,r}^2 + (\partial\mathcal{E}/\partial\,\mathcal{M})\chi_{,r}\psi_{,r}\right] \tag{17}$$

We notice that $\mathcal{E} + T_r^{\ r} \geq 0$ everywhere because conditions (7) and (8) guarantee the positive definiteness of the quadratic form in Eq. (17). The previous conclusions about $e^{\nu/2} r^2$ and from Eq. (16) we can conclude that like $T_r^{\ r}$, sufficiently near the horizon the derivative of $T_r^{\ r}$ with respect to r is negative.

If we take Eq. (16) and apply the condition $e^{\nu/2} \to 1$ asymptotically, we have that $T_r^{\ r} < 0$

To guarantee asymptotic flatness of the solution $\mathcal{E}$ must decrease at least as r^{-3} when $r \to \infty$.

We can then conclude that and $|T_r^{\ r}|$ decreases asymptotically as $r^{-2}$and that the integral in Eq. (15) converges. Asymptotically we also have that $(T_r^{\ r})' < 0$ so we can deduce that as $r \to \infty$ $T_r^{\ r}$ must be positive and decreasing with increasing$r$. Remember that we found near the horizon $T_r^{\ r} < 0$ and $(T_r^{\ r})' < 0$. Taking everything into consideration, Bekenstein states that in some interval $[r_a, r_b]$ we have $(T_r^{\ r})' > 0$ and that $T_r^{\ r}$ changes sign at some r_c with

$r_a < r_c < r_b$ being positive in $[r_c, r_b]$. Bekenstein shows then that this conclusion is gravitationally unstable. To do so he goes back to the Einstein equations:

$$e^{-\lambda}(r^{-2} - r^{-1}\lambda') - r^{-2} = 8\pi G T_t{}^t = -8\pi G T \mathcal{E} \quad (18)$$

$$e^{-\lambda}(r^{-1}\nu' + r^{-2}) - r^{-2} = 8\pi G T_r{}^r \quad (19)$$

To solve Eq. (18) we use the following equation

$$e^{-\lambda} = 1 - 8\pi G r^{-1} \int_{r_h}^{r} \mathcal{E} r^2 dr - 2GMr^{-1} \quad (20)$$

In Eq. (20) M is a constant of integration which can basically be considered to be the bare mass of the black hole, G the gravitational constant and $\lambda = \lambda(r)$. We require that $e^{\lambda} \to \infty$ for $r \to r_h$ so that $2GM = r_h$. Moreover we have $\lambda = O(r^{-1})$ because of the condition on asymptotic flatness which requires that$\mathcal{E} = O(r^{-3})$. Having e^{ν} would be incompatible with a regular black hole solution because this would mean a change in the metric signature. This taken into consideration and using Eq. (20) Bekenstein comes to the conclusion that $e^{\lambda} \geq 1$ throughout the black hole exterior.

The next step is to write Eq (19) in the following form

$$e^{-\nu/2} r^{-2} \left(r^2 e^{\nu/2}\right)' = [4\pi G T_r{}^r + (1/2r)] e^{\lambda} + 3/2r \quad (21)$$

Using the inequality $e^{\lambda}/2 + 3/2r > 2$ one can conclude another inequality which is:

$$e^{-\nu/2} r^{-2} \left(r^2 e^{\nu/2}\right)' = [4\pi G T_r{}^r + (1/2r)] e^{\lambda} + 3/2r > 4\pi r\, G T_r{}^r$$

Remember we found that in $[r_c, r_b]$ $T_r{}^r > 0$ so we have the implication that $e^{-\nu/2} r^{-2} \left(r^2 e^{\nu/2}\right)' > 0$ in this interval. Moreover using Eq. (16) we have that $(T_r{}^r)' < 0$ thorough our interval. The next step is essential in the proof because with it, Bekenstein establishes the No hair theorem for static spherically symmetric black holes. Earlier we have determined that $(T_r{}^r)' > 0$ throughout the bigger interval$[r_a, r_b]$. Therefore there is a contradiction in our inequalities and solving these contradiction means accepting that the scalar field componentsψ, χ, ... are constant thorough the black hole exterior. These constants must have values such that all components of $T_\mu{}^\nu$ vanish identically i.e. values such that:

$$\mathcal{E}(0,0,0, \ldots \psi, \chi, \ldots) = 0 \quad (22)$$

In his argument Bekenstein uses the trivial solution for the scalar equation as a boundary condition and in order to obtain a trivial solution for the scalar equation in the free empty space, such values for the scalar field components satisfying Eq. (22) must exist. The important conclusion that can be made is that the black hole solution must be Schwarchild. The black hole would have been Reissner-Nordstrom black hole in the case it was electrically and/or magnetically charged and the scalar fields uncoupled to electromagnetism.

This theorem is very important and lead to several applications. One of them is using a analogous argument for the Higgs field with an action similar to the one in Eq. (1). To exclude Higgs hair, we suppose the potential $V(|\psi|^2)$ has several wells and we assume the presence of a global minimum which is $V = 0$. The energy density of the field is positive definite and we can choose ψ_0 to be one of the values of ψ for which $V = 0$. ψ_0 can serve as a boundary condition for an asymptotically flat solution which requires that the energy density vanishes as $r \rightarrow \infty$ .but according to the theorem throughout the black holes exterior, $\psi = \psi_0$ which is sufficient to rule out Higgs hair.

5.<u>Saa approach</u>

In his paper published in 2008 [7] Saa presents a generalisation of Bekenstein method. He adopts the conventions previously used in [14] to presents a theorem that excludes finite scalar hairs of any asymptotically flat static and spherically symmetric black hole solution. However in his paper he doesn't consider situations where the divergence of the scalar field is not related to the singularity and where a scalar hair is. The divergence of scalar fields play an important part in the existence of hairs and this point was further investigated in Zannias paper [16] considered to exist. To do so he chooses the action

$$S[g,\phi] = \int d^4x(-g)^{1/2}\{f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\} \tag{23}$$

Where $f(\phi)$ and $h(\phi)$ are positive

In the literature the most common non minimally coupling for the scalar field is $f(\phi) = 1 - \xi\phi^2$ and $h(\phi) = 1$. The Bekenstein method allows Saa to construct the exact solution from the solutions of the minimally coupled case. The case $\xi = \frac{1}{6}$ corresponds to the conformal coupling case and there is also a method to generate solution for arbitrary ξ which is explored in [15]

He proceeds as follows; he provides a covariant method to provide solutions for our action.

We start from the minimally coupled case with the action:

$$\bar{S}[\bar{g},\bar{\phi}\,] = \int d^4x(-g)^{1/2}\{\bar{R} - \bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi}\} \tag{24}$$

We have two set of Euler Lagrange equations:

For the first action in Eq. (1) we obtain:

$$f(\phi)R_{\mu\nu} - h(\phi)\partial_\mu\phi\partial_\nu\phi - D_\mu D_\nu f(\phi) - \frac{1}{2}g_{\mu\nu}\blacksquare f(\phi) = 0 \tag{25a}$$

$$2h(\phi)\blacksquare\phi + h^{'}(\phi)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + f^{'}(\phi) \tag{25b}$$

And for the action in Eq. (2) we have:

$$\bar{R}_{\mu\nu} - \partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = 0 \tag{26a}$$

$$\bar{\blacksquare}\bar{\phi} = 0 \tag{26b}$$

Saa considers the conformal transformation $g_{\mu\nu} = \Omega^2\bar{g}_{\mu\nu}$ in order to obtain the relation between Eq. (25) and Eq. (26). The choice for the conformal transformation allows the curvature scalar to transform as $R\left(\Omega^2\bar{g}_{\mu\nu}\right) = \Omega^{-2}\bar{R} - 6\Omega^{-3}\overline{\blacksquare\Omega}$

He chooses deliberately

$$f(\phi) = \Omega^{-2} \tag{27}$$

Using the conformal transformation and Eq. (1) we get:

$$S\left[\Omega^2\bar{g},\phi\right] = \int d^4\,x(-g)^{1/2}\left\{\bar{R} - \left(\frac{3}{2}\left(\frac{d}{d\phi}\ln f(\phi)\right)^2 + \frac{h(\phi)}{f(\phi)}\right)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} \tag{28}$$

The next step is to define a new function $\bar{\phi}(\phi)$ as

$$\bar{\phi}(\phi) = \int_a^\phi d\xi\left(\frac{3}{2}\left(\frac{d}{d\xi}\ln f(\xi)\right)^2 + \frac{h(\xi)}{f(\xi)}\right) \tag{29}$$

The result obtained is $S\left[\Omega^2\bar{g},\bar{\phi}(\phi)\right] = \bar{S}[\bar{g},\bar{\phi}\,]$ (here we are using an arbitrary a which is determined by boundary conditions).

Because of the assumption made earlier on the positiveness of f and h leads to the consequence that the right handed side of Eq. (29) is a monotonically increasing function of ϕ. Therefore the Eq. (27) and (29) represent a covariant transformation because it is independent of any symmetry assumption and this transformation that maps (a one-to one map) a solution $(g_{\mu\nu}, \phi)$ of both equations in (25) into a solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ of both equations in (26). If $\bar{g}_{\mu\nu}$ admits a killing vector ξ for which $\pounds_{\xi}\bar{\phi} = 0$ then ξ is also a killing vector for$g_{\mu\nu}$ and this is true because the transformation used also preserves symmetries. This fact leads us to a very important conclusion for the rest of the proof: if we know all solutions$(\bar{g}_{\mu\nu}, \bar{\phi})$ with a given symmetry we know all $(g_{\mu\nu}, \phi)$ with the same symmetry.

For the set of equations in (26) some properties of the asymptotically flat static and spherically symmetric solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ were investigated in the paper [17] and the solutions are well known and given by two parameters (λ, r_0) family of solutions presented here:

$$\bar{\phi} = \left(2(1-\lambda^2)\right)^{1/2} \ln \mathcal{R} \tag{30a}$$

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\mathcal{R}^{2\lambda} dt^2 + \left(1 - \frac{r_0^2}{r^2}\right) \mathcal{R}^{-2\lambda} \left(dr^2 + r^2 d\Omega^2\right) \tag{30b}$$

We have $\mathcal{R} = \frac{r - r_0}{r + r_0}$ and we can choose the parameter λ to be positive and smaller than 1.

It is interesting to notice that by using the transformation $r' = r(1 + r_0/r)^2$ and for $\lambda = 1$, the solution is the exterior vacuum scharwchild solution with the horizon at $r_0^{'} = 4r_0$. Due to the fact that that the surface $r = r^0$ is not a horizon, if we take the case $0 \leq \lambda < 1$ our set of equations in (30) does not represent a black hole. If we calculate the scalar curvature, we find that in this specific case it represent a naked singularity.

This shows that the proof is in accordance with Bekenstein no hair theorem because the only black hole solution for the set of equation in (30) corresponds to the case where $\lambda = 1$ which is true for the usual Schwarzschild solution (when $\phi = 0$). The used conformal transformation does not allow $f(\phi) \to \infty$ for any $r \neq r_0$.

The important result of Saa approach is that using the transformation in Eq. (26) and Eq. (29) any flat static and spherically symmetric solution of Eq. (25) can be obtainded from Eq.(30).

This leads to the theorem:

The only asymptotically flat static and spherically symmetric exterior solution of the system governed by the action $S[g,\phi] = \int d^4x(-g)^{1/2}\{f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\}$, with the field ϕ finite everywhere is the Shwarchild solution.

6. Solution on black holes in 4D

It has been proven that solution of hairy black holes exists. Most of them are unstable. In this section we will discuss the different types of hairy black holes and the stability of solution.

We will depart from action which encompasses the characteristics of the system and derive Einstein and scalar equations from which we will deduce whether the system is stable or not.

Dressing a black hole with non minimally coupled scalar field hair:

This section is mainly based on paper in [4]. This paper investigates the possibility of dressing a black hole with a classical non-minimally coupled scalar field in 4 dimensions. The model includes a cosmological constant. The action describing the system is:

$$S = \int d^4(-g)^{1/2}[1/2(R-2\Lambda) - 1/2(\nabla\phi)^2 - 1/2(\xi R\phi^2) - V(\phi)] \quad (31)$$

In this paper the simplest case for the self interacting scalar potential is considered which is $V(\phi) = 0$. In Eq. (1) Λ is the cosmological constant and R is the Ricci scalar curvature. For the coupling constant ξ, we have $\xi = 0$ in the case of a minimally coupled scalar field and in the case of conformal coupling the most common form is used, i.e. $\xi = 1/6$.

If we take the variation of our action we obtain the Einstein equations and the scalar field equation

$$[1-\xi\phi^2]G_{\mu\nu} + g_{\mu\nu}\Lambda = (1-2\xi)\nabla_\mu\phi\nabla_\nu\phi + (2\xi - 1/2)g_{\mu\nu}(\nabla\phi)^2 - 2\xi\phi\nabla_\mu\nabla_\nu\phi + 2\xi g_{\mu\nu}\phi\nabla^\rho\nabla_\rho\phi \quad (32)$$

$$\nabla_\mu\nabla^\mu\phi = \xi R\phi \quad (33)$$

We want to eliminate higher order derivatives of the metric from Eq (3). To do that we can use the scalar field equation (33) to substitute for $\nabla_\mu\nabla^\mu\phi$ in the expression of the Ricci scalar. On way of obtaining the Ricci scalar is by taking the trace for (32) and we obtain:

$$R = \frac{(1-6\xi)(\nabla\phi)^2 + 4\Lambda}{1-\xi(1-6\xi)\phi^2} \quad (34)$$

The next step is to assume that the scalar field ϕ depends on on the radial coordinate, then we consider a static spherically symmetric black hole geometry with the metric:

$$ds^2 = -\left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}\right) exp\big(2\delta(r)\big)dt^2 + \left(1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (35)$$

We can obtain the Einstein equations for this new system:

$$\frac{2}{r^2}(1-\xi\phi^2)m' = \xi\Lambda\phi^2 + \left(\frac{1}{2} - 2\xi\right)N\phi'^2 - \xi\phi\phi' N' - 2\xi N\phi\phi'' - \frac{4N}{r}\xi\phi\phi' \quad (36a)$$

$$\frac{2}{r}(1-\xi\phi^2)\delta' = (1-2\xi)\phi'^2 - 2\xi\phi\phi'' + 2\xi\phi\phi'\delta' \quad (36b)$$

There is also a scalar field equation for the system:

$$N\phi'' + \left(N\delta' + N' + \frac{2N}{r}\right)\phi' - \xi R\phi = 0 \quad (37)$$

Where $N(r)$ is defined as:

$$N(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3} \quad (38)$$

We are interested in black hole solutions with regular non extremal event horizon at $r = r_h$ and we always assume that asymptotically the geometry approaches anti-de Sitter (AdS) space (negative cosmological constant). We want to able to numerically integrate Eq. (36a), Eq. (36b) and Eq. (37). To do so we eliminate the Ricci scalar curvature from Eq. (37) using Eq. (34) and we eliminateϕ'' from both the Einstein equations (36a) and (36b).

These assumptions allows the scalar field to take the following form

$$\phi = \phi_\infty + O(r^{-k}) \quad (39)$$

Here $k > 0$. We can use the scalar field equation (37) to find an expression for k. By solving the polynomial $k^2 - 3k + 12\xi = 0$ we obtain:

$$k = \frac{3}{2}\left[1 \pm (1 - 16\xi/3)^{1/2}\right] \quad (40)$$

In Eq. (39) ϕ_∞ is a constant however, in order to maintain the consistency between the Einstein equations (36a) and (36b) and Eq. (34) we need to have the condition $\phi_\infty = 0$.

Because the potential is null in our system k doesn't depend on the cosmological constant. Let us analyse the results found so far and what the different values of k correspond to. The geometry has to asymptotically approach AdS space in a manner compatible with Ricci scalar curvature in Eq. (34) so we can already rule out the case where ϕ diverges as $r \to \infty$. We need the scalar field to converge to zero at infinity therefore for all$\xi > 0$, k has a positive real part. The case of a non-minimally coupled scalar field in AdS with a non zero scalar potential is studied in [18]. Regarding the stability of the solution the paper came to the conclusion that the scalar field ϕ must oscillate around zero with decreasing amplitude as r tends to infinity. This happens specifically when k is no longer real but has a non zero imaginary part which corresponds to the case $\xi > 3/16$. In the case where $\xi < 0$ we have one root of k which is negative and one root which is positive, this means that the scalar field diverges at infinity. In the case $\xi < 3/16$ the scalar field ϕ monotonically decays to zero from its value on the event horizon.

We want to know how δ' and m' behave. For m', after substituting the expression for ϕ given in Eq. (39) into Einstein equation (36a) we obtain the approximation $m' \sim O(r^{-2k+2})$. We use the second Einstein equation (36b) to obtain the approximation $\delta' \sim O(r^{-2k-1})$ therefore $\delta \rightarrow \delta_\infty + O(r^{-2k-1})$ with δ_∞ being a constant. It is useful to add that near $r = r_h$ we have $\delta(r) = \delta(r_h) + O(r - r_h)$. Using [19] which studies a similar situation we have that when $\xi \geq 3/16$ the constant k is complex and its real part is exactly equal to $3/2$ this leads to the conclusion that $m \sim O(ln(r))$ as $r \rightarrow \infty$. Moreover k is real when we take the positive root in Eq.(40) we have $k > 3/2$. In this case $m \rightarrow M + O(r^{-2k+3})$ as $r \rightarrow \infty$ and $m(r)$ converges to the constant M at infinity.

It is well known that stable and non trivial solutions exists when $\xi = 0$ or 1/6. Moreover the no hair theorem in our system has been proven everywhere in space except when $\Lambda < 0 \; and \; \xi > 0$. It is therefore useful to study the existence and stability of hairy black holes in this specific case but we will only find the solution for which our conformal transformation is valid, namely:

$$\bar{g}_{\mu\nu} = \Omega g_{\mu\nu} \tag{41}$$

To find these solutions we use a specific method which is presented in [20] for conformally coupled scalar fields which consists in integrating numerically our minimally coupled field equations in Eq. (36a) and Eq. (36b) and the solutions are transformed back to the non-minimally coupled system.

To illustrate the results we can show two typical solutions (numerically found)

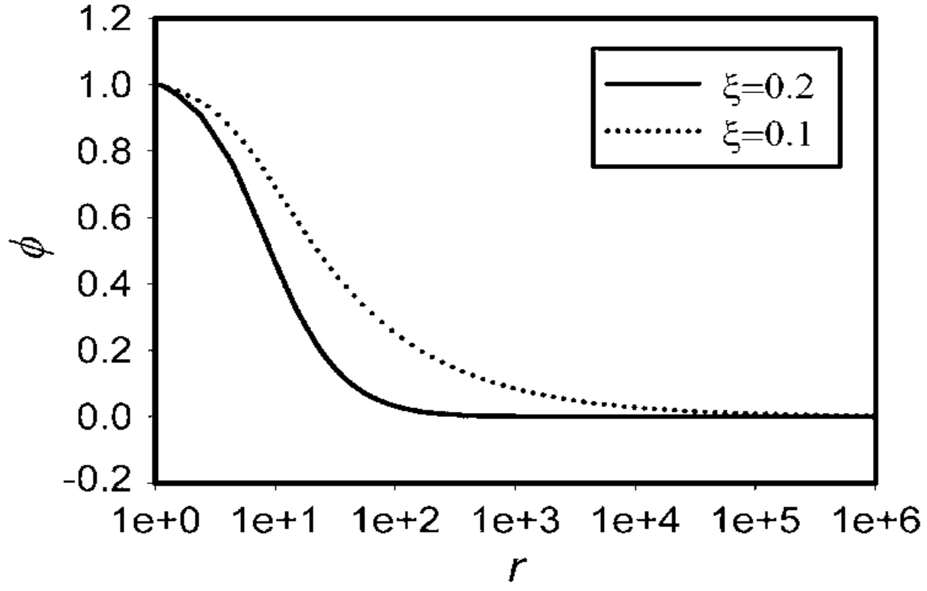

Figure.1: Examples of typical hairy black-hole solutions with a non-minimally coupled scalar field, when ξ = 0.1 (dotted) and ξ = 0.2 (solid). For these solutions, the event horizon radius is taken to be $r = r_h = 1.0541$ for the ξ = 0.1 solution and $r = r_h = 1.118$ for the ξ = 0.2 solution, the cosmological constant $\Lambda = -0.1$ and the value of the scalar field at the event horizon$\phi(r_h) = 1$. Solutions for other values of the parameters Λ , r_h, ξ and $\phi(r_h)$ behave similarly.

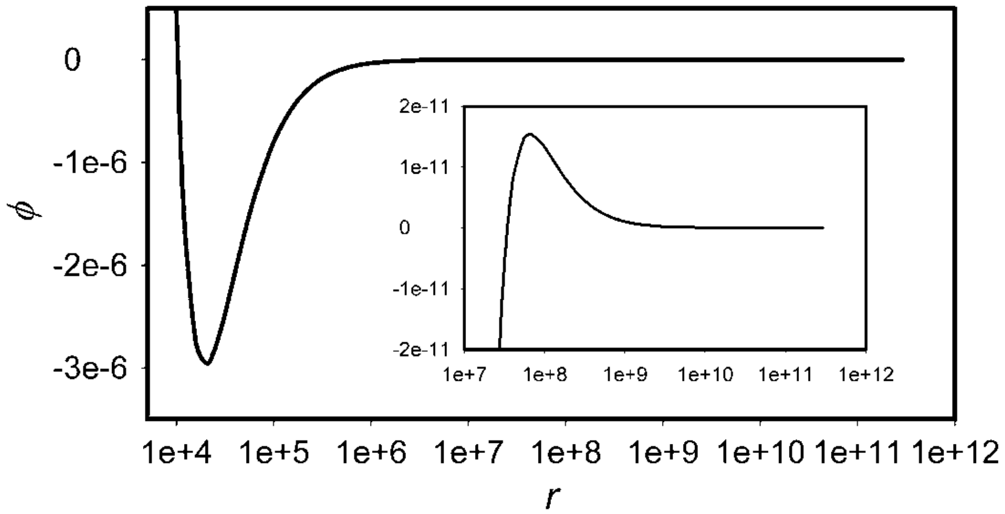

Figure.2: Example of a typical hairy black-hole solution with a non-minimally coupled scalar field, when ξ = 0.2 > 3/16. The values of the other parameters are as in Figure.1. The first oscillation in ϕ about zero can be seen in the main graph, while the inset shows the second oscillation.

The paper in [21] establishes that many of the black hole solutions in literature are unstable when subjected to a small perturbation. Let us discuss the stability of our system.

A linear spherically symmetric perturbation of the metric and scalar field is considered. The perturbation for the scalar field is $\delta\phi$ and following the method used in [19] we can obtain a perturbation equation for ψ :

$$\psi = r(1-\xi\phi^2)^{-1/2}[1-\xi(1-6\xi)\phi^2]^{1/2}\delta\phi$$

The perturbation equation for the system represents a perturbation which is periodic in time therefore it has the standard Schrödinger form:

$$\sigma^2\psi = -\frac{\partial^2\psi}{\partial r_*^2} + \mathcal{U}\psi \tag{42}$$

Here r_* is the tortoise coordinate in our asymptotically AdS space. We can choose a specific value for the constant of integration such that the tortoise coordinate lies in the interval.$]-\infty, 0]$ and it is related to the radial coordinate by $\frac{dr_*}{dr} = \frac{e^{-\delta}}{N}$.

The perturbation potential that we call $\mathcal{U}$ is given by:

$$\mathcal{U} = \left(Ne^{2\delta}/r^2\right)[1 - N\mathcal{A}^{-2}\mathcal{B}^2 - \Lambda r^2 - \Lambda r^2\mathcal{A} - \Lambda\xi r^2\phi^2\mathcal{A}^{-1}(2-\xi\phi^2) + 8\Lambda\xi r^3\phi\phi'\mathcal{A}^{-1}\mathcal{B}^{-1} + 4\Lambda\xi r^2\mathcal{C}^{-2} + 16\Lambda\xi^2 r^2\phi^2\mathcal{A}^{-1}\mathcal{C}^{-1} + \Lambda r^4\phi'^2\mathcal{A}^{-1}\mathcal{B}^{-2}\mathcal{C} - r^2\phi'^2\mathcal{B}^{-2}\mathcal{C}] \tag{43}$$

It vanishes at the event horizon $r = r_h$ and at infinity it follows the approximation $\mathcal{U} \sim \frac{r^2 2\Lambda^2}{9}(1-6\xi)$. For the case $\xi = 1/6$ the potential remains bounded at infinity. For the case $\xi < 1/6$ this means that the potential diverges to positive infinity like $+r^2$ as $r \to \infty$ and diverges to negative infinity like $-r^2$ $\xi > 1/6$. Now to check the stability we do it numerically for those solutions with $\xi < 1/6$ for which the perturbation potential $\mathcal{U}$ is positive everywhere outside the vent horizon, we can conclude that the perturbation equation has no bound state solutions and the black hole is linearly stable.

Now for $\xi > 1/6$ it is a slightly more complicated case because the potential diverges to $-\infty$ as $r_* \to 0$. The method used is to define a new variable y such that $y = -r_*$ and y is positive or null and if $r \to \infty$ we have $y \to 0$. The idea is to examine the zero mode solutions of Eq. (42) which are time independent solutions of our perturbation equation. The new perturbation equation in terms of our new variable y can be written as:

$$\sigma^2\psi = -\frac{\partial^2\psi}{\partial y^2} + \mathcal{U}\psi \tag{44}$$

We can show the graph corresponding to the potential in figure 3:

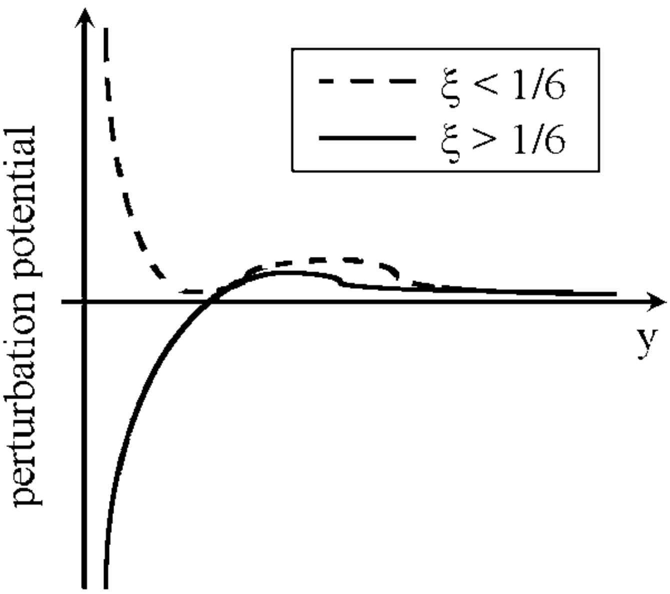

Figure.3: Sketch of the perturbation potential U as a function of $y = -r*$ for the black-hole solutions shown in Figure.1

Now in [22] an important result shows that the number of bound states of the Eq. (44) is equal to the number of zeros of the zero mode $f(y)$ such that $f(0) = 0$. If we define the zero mode as g and impose on it suitable initial conditions at the event horizon we can find zero modes $g = g(r)$ of the equivalent perturbation equation are found numerically using:

$$-N^2e^{2\delta}g'' - Ne^{\delta}\left(Ne^{\delta}\right)'g' + \mathcal{U}g = 0 \tag{45}$$

as $r \to \infty$ we examine the form of the solution of (45) it is a well known result that g behaves like $r^{-}\ell$ where:

$$\ell = \frac{1}{2}\left[1 \pm (9 - 48\xi)^{1/2}\right] \tag{46}$$

We are interested in the case $\xi > 1/6$ because the real part of ℓ is positive and as $r \to \infty$ $g(r)$ tends to zero.

For example we can take the black hole solutions in figure 1 and we can plot the the corresponding zero mode function $g(r)$, because these are numerical solution we will describe them like they are in the paper.

Here is the figure:

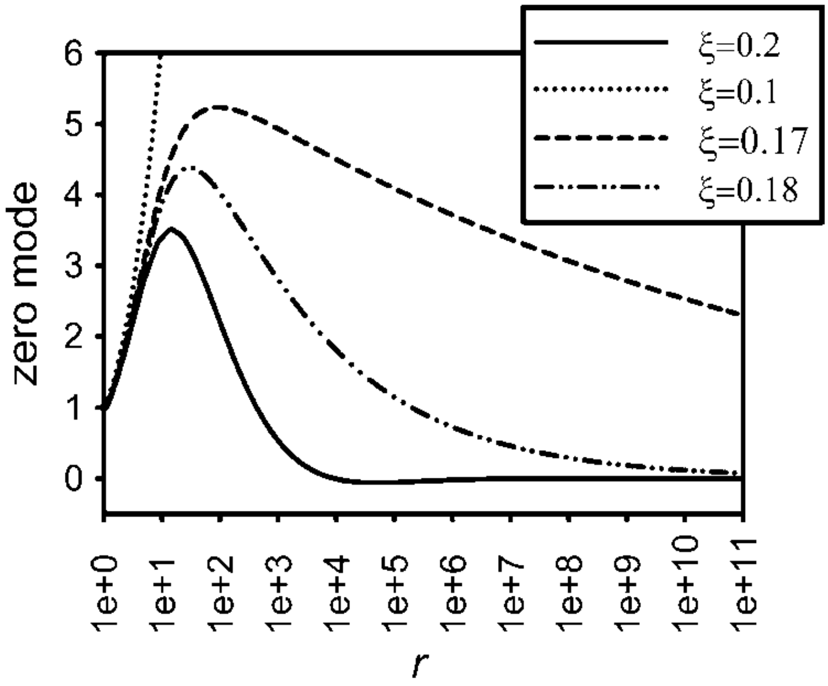

Figure.4: The zero mode solutions of the perturbation equation (45) for the equilibrium black-hole solutions plotted in Figure.1, and the corresponding solutions when ξ = 0.17 and 0.18. The cosmological constant is $\Lambda = -0.1$.

The zero mode functions have no zeros and simply increase away from their value at the event horizon, For $\xi > 1/6$ it depsnds on whether $\xi \leq 3/16$ or $\xi > 3/16$. If $1/6 < \xi \leq 3/16$ then from $\ell = \frac{1}{2}\left[1 \pm (9 - 48\xi)^{1/2}\right]$ the constant ℓ is real and the zero mode function g monotonically decrease to zero as $r \rightarrow \infty$. However for $\xi > 1/6$ the zero mode functions have at least one zero and may oscillate many times with decreasing amplitudes as $r \rightarrow \infty$..This mean that for $\xi > 3/16$ there is at least ne bound state solution of the perturbation equation With $\sigma^2 < 0$ and the black holes are unstable.

Hairy black hole solution of Einstein yang mills theory with a negative cosmological constant.

Over the last years much has been learned about the classical interaction of Yang–Mills fields with the gravitational field of Einstein's general relativity. Most investigations have concentrated on Yang–Mills fields with the gauge group SU (2) starting with Bartnik and Mckinnon's discovery of globally regular and asymptotically flat numerical solutions. Their global existence was analytically proved and many further properties like stability of these particle-like or Soliton solutions and the corresponding black hole solutions were investigated numerically as well as analytically. Moreover, many different matter fields can be minimally coupled to the gravitational and Yang–Mills fields, and corresponding spherically symmetric solutions have been, mostly numerically, but sometimes also analytically studied.

The proof presented here and based on [4] is a generalisation of black hole solutions of the SU (2) Einstein –Yang-Mill equations in four dimensional asymptotically flat space-time found in [3]. The difference between the two paper is that [3] considers topological black holes too whereas paper in [4] considers only spherically symmetric black holes. In this proof Λ is the cosmological constant, R is the Ricci scalar and the metric signature is $(-,+,+,+)$. We are interested in static spherically symmetric Soliton and black hole solutions of the field equations which are derived by varying the action. This proof is more complex than the ones given before.

Here we will present the model and the boundary conditions, the solutions are found numerically and are too complicated to investigate.

For the SU (N) EYM theory the action negative cosmological constant is:

$$S = \frac{1}{2}\int d^4x(-g)^{1/2}\left[R - 2\Lambda - TrF_{\mu\nu}\, F^{\mu\nu}\right] \tag{47}$$

And the subsequent field equations are:

$$T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\,, \qquad D_\mu F_\nu{}^\mu = \nabla_\mu F_\nu{}^\mu + \left[\mathrm{A}_\mu, F_\nu{}^\mu\right] = 0 \tag{48}$$

The YM stress energy tensor is defined by:

$$T_{\mu\nu} = TrF_{\mu\lambda}\, F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}TrF_{\lambda\sigma}\, F^{\lambda\sigma} \tag{49}$$

The metric is written in standard Schwarzschild like coordinate and corresponds to:

$$ds^2 = -\mu S^2 dt^2 + \mu^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{50}$$

Like we have seen before, the metric functions μ and S depend on the radial coordinate only i.e. $\mu = \mu(r)$ and $S = S(r)$. The metric function $\mu(r)$ can be written as:

$$\mu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3} \tag{51}$$

According to [23] the most spherically symmetric ansatz for the SU(N) gauge potential is:

$$A = \mathcal{A}dt + \mathcal{B}dr + \frac{1}{2}(C - C^H)d\theta - \frac{i}{2}[(C + C^H)sin\theta + Dcos\theta]d\phi \tag{52}$$

Using a choice of Gauge specified in [23] and because we know that we are only interested in purely magnetic solutions so we can set $\mathcal{A} \equiv 0$ and $\mathcal{B} \equiv 0$ from the beginning but we define these matrices nonetheless. Matrices $\mathcal{A}$ and $\mathcal{B}$ depend only on the radial coordinate r and are purely imaginary traceless diagonal (N×N) matices. In Eq. (52) we defined C^H is the hermitian conjugate to C and $\mathcal{D}$ is a constant (N×N) matice defined by

$$D = diag(N - 1, N - 3, \ldots, -N + 3, -N + 1) \tag{53}$$

For $j = 1, \ldots, N - 1$ the matrix C is nilpotent and defined as:

$$C_{j,j+1} = \omega_j(r) e^{i\gamma_j(r)} \tag{54}$$

Where ω_j are gauge field functions.

For any $j = 1, \ldots, N - 1$ if $\omega_j(r)$ are non zero one of the YM equations becomes

$$\gamma_j = 0 \tag{55}$$

This is proven in [23] and we will just assume this result here.

A direct consequence is a new expression for the ansatz in Eq. (52) becomes:

$$A = \frac{1}{2}(C - C^H)d\theta - \frac{i}{2}[(C + C^H)sin\theta + Dcos\theta]d\phi \tag{56}$$

This anzats for the Yang Mills potential is not the only possible one in SU (N) EYM. All spherically symmetric SU(N) gauge potentials can be found in [26].

The gauge field is described by $N - 1$ functions $\omega_j(r)$ because the only non-zero entries of the matrix C are $C_{j,j+1} = \omega_j(r)$. The ansatz in Eq. (56) is particularly convenient because we have exactly $N-1$ YM equations for the $N-1$ gauge field functions ω_j. $\forall j=1,\ldots,N-1$. For our YM fields equation we have:

$$r^2\mu\omega''_j + \left(2m - 2r^3p_\theta - \frac{2\Lambda r^3}{3}\right)\omega_j' + W_j\omega_j = 0 \tag{57}$$

With $p_\theta = \frac{1}{4r^4}\sum_{j=1}^{N}\left[\left(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j\right)^2\right]$ (58)

And $W_j = 1 - \omega_j^2 + \frac{1}{2}\left(\omega_{j-1}^2 + \omega_{j+1}^2\right)$ (59)

In this case, the Einstein equations take the form

$m' = \mu G + r^2 p_\theta$ and $\frac{S'}{S} = \frac{2G}{r}$ (60)

S and m are unknown functions and we have $G = \sum_{j=1}^{N-1}\omega_j'^2$. As a result we have $N+1$ ordinary differential equations for $N+1$unknown functions,$m(r)$, $S(r)$, $\omega_j(r)$. For each j independently it is useful to note that the field equations (57) and (60)are invariant under transformation

$$\omega_j(r) \to -\omega_j(r) \tag{61}$$

and under the substitution

$$j \to N-j \tag{62}$$

The boundary conditions are very important in this proof. Because the cosmological constant is negative, there is no cosmological horizon. However the field equations are singular at the origin $r=0$ at the event horizon $r=r_h$ and at infinity when $r\to\infty$. Let us briefly present the boundary conditions.

<u>At the origin</u>

The boundary conditions at the origin are most complicated of the three singular points.

We proceed by assuming that $m(r)$, $S(r)$, $\omega_j(r)$ have regular Taylor series expansions (expansions about the singular point about $r=0$).

We present the Taylor expansions here:

$$m(r) = m_0 + m_1 r + m_2 r^2 + O(r^3) \tag{63a}$$

$$S(r) = S_0 + S_1 r + S_2 r^2 + O(r^3) \quad (63b)$$

$$\omega_j(r) = \omega_{j,0} + \omega_{j,1} r + \omega_{j,3} r^2 + O(r^3) \quad (63c)$$

The metric and the curvature are regular at the origin because we impose the condition $S_0 \neq 0$ (since the metric only involves derivatives of S, S_0 is otherwise arbitrary) and m_i, S_i, $\omega_{j,i}$ are constants.

As a result, the constants and the metric functions are defined as

$$m_0 = m_1 = m_2 = S_1 = \omega_{j,1} = 0 \quad (64)$$

and $\omega_{j,0} = \pm[j(N-j)]^{1/2}$ (65)

Because of the invariance of the metric under transformation (61) we can take the positive root square in Eq. (65). To determine the values of the remaining constants we substitute the Taylor expansion into the field equations (57) and (60).

At the event horizon:

At $r = r_h$ where the metric function $\mu(r)$ has a single zero we assume that here is a regular non extremal event horizon for black hole solutions and this condition fixes the value of $m(r_h)$:

$$2m(r_h) = r_h - \frac{\Lambda r_h^3}{3} \quad (66)$$

Like for the singular point at the origin we use taylor expansions of the field variables. We assume that the variables $m(r)$, $\omega_{j(r)}$, $S(r)$ have regular Taylor expansions about $r = r_h$

$$m(r) = m(r_h) + m'(r_h)(r - r_h) + O(r - r_h)^2 \quad (67a)$$

$$S(r) = S(r_h) + S'(r_h)(r - r_h) + S_2 r^2 + O(r - r_h) \quad (67b)$$

$$\omega_j(r) = \omega_j(r_h) + \omega_j{}'(r_h)(r - r_h) + O(r - r_h)^2 \quad (67c)$$

We set $\mu(r_h) = 0$ in (57) and this fixes the derivatives ofthe gauge field functions $\omega_j(r)$ at the horizon $r = r_h$:

$$\omega_j{}'(r) = \frac{W_j(r_h)\omega_j(r_h)}{2m(r_h) - 2r_h^3 p^\theta(r_h) - (2\Lambda r_h^3/3)} \quad (68)$$

For a fixed cosmological constant the Taylor expansions in (67a), (67b), (67c) are determined by $N+1$ quantities $\omega_j(r_h)$, $S(r_h)$, r_h. Without loss of generality we can consider$\omega_j(r_h) > 0$.

At the event horizon we need to have a condition which weakly constrains the values of the gauge fields functions $\omega_j(r_h)$. Moreover if want the event horizon to be non-extremal we have to have the condition: $2m(r_h) = 2r_h^2 p^\theta(r_h) < 1 - \Lambda r_h^2$

At infinity

As $r \to \infty$ we put a condition on the metric in Eq (50): it has to approaches AdS space-time the field variables $m(r)$, $\omega_{j(r)}$, $S(r)$converge to constant values.

In a similar way as before we assume that the field variable have Taylor expansion as $r \to \infty$:

$$m(r) = M + O(r^{-1}) \quad (69a)$$

$$S(r) = 1 + O(r^{-1}) \quad (69b)$$

$$\omega_j(r) = \omega_{j,\infty} + O(r^{-1}) \quad (69c)$$

Regarding the values of $\omega_{j,\infty}$ if the cosmological constant is negative there no constraints on the values of $\omega_{j,\infty}$ and the AdS black holes will be magnetically charged. But we can impose the condition that the space-time is asymptotically flat and with a null cosmological constant and constrain the values to be

$$\omega_{j,\infty} = \pm[j(N-j)]^{1/2} \quad (70)$$

According to [4] this condition means that these solutions have no global magnetic charge so at infinity they are indistinguishable from Schwarzschild black holes. At the singular point which is infinity the fact that the boundary conditions less restrictive when the cosmological constant is negative leads to the expectation of many more solution in that specific case.

MTZ model

The MTZ black hole, named after Cristian Martinez, Ricardo Troncoso and Jorge Zanelli, is a black hole solution for (3+1)-dimensional gravity with a conformally coupled self-interacting scalar field The model includes a positive cosmological constant, the space-time is asymptotically locally AdS and the event horizon is a surface of constant negative curvature.

The advantage of this model is that it reproduces the local propagation properties ok Klein Gordon field equations on Minkowski space-time better than minimally coupled fields. Moreover this model allows non trivial black hole solutions.

We will here present the MTZ model and sketched the derivation of two solutions based on [6]

In this model the action is:

$$S = \frac{1}{2} = \int d^4x(-g)^2 \left[R - 2\Lambda - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2 - 2\alpha\phi^4 - \frac{1}{8\pi}F^{\mu\nu}F_{\mu\nu}\right] \quad (71)$$

We have the presence of a scalar field ϕ conformally coupled to gravity with and α is a coupling constant, a electromagnetic field $F_{\mu\nu}$ and a quatric self-interaction potential. Euler LaGrange Equations are obtained by varying the action with respect to the scalar field the Maxwell potential and the metric respectively:

$$\blacksquare\phi - \frac{1}{6}R\phi - 4\alpha\phi^3 = 0 \quad (72a)$$

$$\nabla^\mu F_{\mu\nu} = 0 \quad (72b)$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}^{\phi} + T_{\mu\nu}^{EM} \quad (72c)$$

And the stress energy tensors are:

$$T_{\mu\nu}^{\phi} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}\left[g_{\mu\nu}\blacksquare - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right]\phi^2 - \alpha g_{\mu\nu}\phi^4 \quad (72d)$$

$$T_{\mu\nu}^{EM} = \frac{1}{4\pi}\left(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \quad (72e)$$

We deliberately chose a non-minimally coupled scalar field with a quadric self interaction so that (72a) and (72b) are invariant under conformal transformations of the form $\phi \to \Omega^{-1}\phi$ a $F_{\mu\nu} \to F_{\mu\nu}$ and $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and equations and the stress energy tensors transform as$T_{\mu\nu}^{\phi} \to \Omega^{-2}T_{\mu\nu}^{\phi}$, $T_{\mu\nu}^{EM} \to \Omega^{-2}T_{\mu\nu}^{EM}$ under the same transformation.

Identically to $T^{EM} \coloneqq T_{\mu\nu}^{EM}g^{\mu\nu}$ the trace of $T_{\mu\nu}^{\phi}$ vanishes on shell

$$T^{\phi} \coloneqq T^{\phi}_{\mu\nu} g^{\mu\nu} = \phi \left[\blacksquare\phi - \frac{R}{6}\phi - 4\alpha\phi^3 \right] \tag{73}$$

Using the system f equations (72a), (72c), (72d) and (72e) and taking the trace of equation (72c) we obtain an important relation between the cosmological constant and Ricci scalar:

$$R = 4\Lambda \tag{74}$$

In the case where $\phi \equiv \phi_0$ with $\phi_0 \neq 0$ ($\phi_0$being a constant) we obtain a new system for the field equations in (72):

$$\left(1 - \frac{\phi_0^2}{6}\right) G_{\mu\nu} + (\Lambda + \alpha\phi_0^4) g_{\mu\nu} = T^{EM}_{\mu\nu} \tag{75a}$$

$$R + 24\alpha\phi_0^2 = 0 \tag{75b}$$

$$\nabla^{\mu} F_{\mu\nu} = 0 \tag{75c}$$

These are Einstein equations with an effective Newton constant which given by: $G_{eff} = \left(1 + \frac{\Lambda}{36\alpha}\right)^{-1} G$ and the case where$\left(1 + \frac{\Lambda}{36\alpha}\right) < 0$ is unphysical because it corresponds to repulsive gravitational forces. This case is further investigated in [25].

Again using (75b) we and taking the trace of (75a) we obtain:

$$\phi_0^2 = -\frac{\Lambda}{6\alpha} \tag{76}$$

We notice that using Eq. (76) the Eq. (75b) gives Eq. (74) again and that (75a) takes a simpler form:

$$\left(1 + \frac{\Lambda}{36\alpha}\right) \left[G_{\mu\nu} + \Lambda g_{\mu\nu}\right] = T^{EM}_{\mu\nu} \tag{77}$$

In this proof, because it seems to admit a wider range of solutions, we are particularly interested in the case where the coupling constant is α tuned with the cosmological constant as:

$$\alpha = -\frac{\Lambda}{36} \tag{78}$$

So a distinction can be made between these kind of theories which can be called special theories where $\alpha = -\frac{\Lambda}{36}$ and the generic theories where $\alpha \neq -\frac{\Lambda}{36}$.

The field equations for special theories become

$$T_{\mu\nu}^{EM} = 0 \tag{79a}$$

$$R - 4\Lambda = 0 \tag{79b}$$

$$\nabla^{\mu} F_{\mu\nu} = 0 \tag{79c}$$

In the case $\phi_0^2 = 6$. There is an important remark to be made on $T_{\mu\nu}^{EM} = 0$ because it implies that $F_{\mu\nu} = 0$. However it doesn't mean that the gravitational field is unconstrained.

In this paper only static spherically solutions for solutions for the case $F_{\mu\nu} = 0$ are explored and the scalar field depends on the radial coordinate r only ($\phi = \phi(r)$). The space is defined by:

$$ds^2 = -N_2(r)dt^2 + N_1 dr^2 + r^2 d\Omega^2 \tag{80}$$

Instead of (72a) we will use a simpler field

$$\blacksquare\phi - \frac{2}{3}\Lambda\phi - 4\alpha\phi^3 = 0 \tag{81}$$

In [25] the exact solution was found to be

$$ds^2 = -N(r)dt^2 + N^{-1}dr^2 + r^2 d\Omega^2 \tag{82}$$

There are two sets of solutions MTZ1 and MTZ2.

The first set of solutions with a constant scalar field$\phi_0^2 = -\frac{\Lambda}{6\alpha}$ is MTZ1.

By imposing the condition $G_{\mu\nu} = -\Lambda g_{\mu\nu}$ on Eq. (82) we can obtain $N(r)$ for generic theories. For special theories the only constraint on Eq. 982) is Eq. (74)

The conditions in special theories are less restrictive than the ones for generic the theories therefore we will have a wider set of solutions for special theories.

In fact for for generic theories we have the following set of equations:

$$N(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 \text{ and } \phi(r) = \left(\frac{-\Lambda}{6\alpha}\right)^{1/2} \tag{83}$$

For special theories we have two integration constants instead of one:

$$N(r) = 1 - \frac{2M}{r} + \frac{Q}{r^2} - \frac{\Lambda}{3} r^2 \text{ and } \phi(r) = (6)^{1/2} \quad (84)$$

In this paper, the singularity at $r = 0$ is hiden behind the event horizon because only static black hole solutions with a sensible stress energy momentum tensor are of interest and we want $T_{\mu\nu}^{\phi}$ to satisfy the appropriate energy conditions for that. It should be added that these solution are extremely unstable under perturbation of the metric.

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