GENERALIZED GEOMETRY AND THREE-FORM SUPERGRAVITY

MICHAEL KUMARESAN

Submitted in partial fulfilment of the requirements for the degree of Master of Science of Imperial College London

Date: September 23^{rd}, 2011.

Contents

Introduction

Advances in theoretical physics have often relied on the elegance and technical rigor of mathematical formalism. "The development of Newton's theory of mechanics and the simultaneous development of the techniques of calculus constitute a classic example of this phenomenon," writes Eguchi [17]. And the list goes on. Maxwell's theory of electromagnetism rests on fundamental ideas in the theory of vector calculus; Einstein's formulation of gravitation relies heavily on the theory of differential geometry; and Yang and Mills' ideas on gauge theory are pivoted by the theory of fiber bundles. In all cases, the mathematical formalism is crucial to understanding the physical implications of the science, and often gives many insights into the way in which the physics works. Michael Reed sums this up in the best manner in his text, *Modern Mathematical Physics*. He notes,

> "It is a common fallacy to suppose that mathematics is important for physics only because it is a useful tool for making computations. Actually, mathematics plays a more subtle role which in the long run is more important. When a successful mathematical model is created for a physical phenomenon, that is, a model which can be used for accurate computations and predictions, the mathematical structure of the model itself provides a new way of thinking about the phenomenon. Put slightly differently, when a model is successful, it is natural to think of the physical quantities in terms of the mathematical objects which represent them and to interpret similar or secondary phenomena in terms of the same model. Because of this, an investigation of the internal mathematical structure of the model can alter and enlarge our understanding of the physical phenomenon [16]."

Reed's point is well-uttered: the abstractions of mathematics prove to be such an effective instrument in deciphering the order and the beauty of nature, and should remain central in our study of physical phenomena.

Not only has the development of physics been marked by its ability to be captured in elegant mathematical frameworks, it has also shown a remarkable "unifying" trend. Consider the simplest example of Maxwell's equations for electromagnetism. In units where $c = 1$, they are given by:

$$\nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

These are beautiful equations, and our first clue to thinking about them in a unified way is simply to observe that the electric and magnetic fields in the equations are intrinsically tied to each other. One affects the other. It simply does not make sense to think of electricity and magnetism as disparate phenomena. Rather, to remain consistent we must think of them as a single, unified object - the electromagnetic field. But this is not really the heart of the issue. If one examines more closely, one finds that Maxwell's equations reveal a crucial underlying symmetry:

Lorentz covariance. We may write Maxwell's equations using the modern language of differential forms to make this more manifest (please see Appendix for details).

One cannot begin to overstate the importance of symmetry in theoretical physics. First, symmetry is fundamentally pleasing and aesthetically satisfying. Second, in many a situation, it can help to simplify a problem considerably. But thirdly and most importantly, it lies at the heart of conservation laws in physics. The deep relationship between symmetry and conservation laws is stated in one of the most important theorems in theoretical physics, Noether's theorem. The best way to reveal the symmetry properties of a theory is through an action formulation. Noether's theorem states that if the action is invariant under some group of transformations, then there exists one or more conserved quantities. In particle physics, this is crucial, as it allows us to label particles by their conserved quantum numbers. Returning to the problem at hand, in constructing an action for Maxwell's theory, one introduces a vector potential. This allows us to apply Poincaré's Lemma and locally solve Maxwell's equations. There is, however, a redundancy in the vector potential formulation expressed by another local symmetry: gauge symmetry. Local gauge symmetry has numerous important consequences - it is responsible for local charge conservation (via Noether's theorem); but more importantly, it is absolutely central to understanding the dynamical nature of the theory. In other words, it allows us to talk about the dynamics and interactions of the fundamental particles and forces that govern Maxwell's theory. It is in fact the Lorentz and gauge symmetry of the Maxwell field equations that endow the theory with its unified character.

But the unified nature of physical phenomena is not restricted to Maxwell's theory of electromagnetism. Salam and Weinberg applied the gauge principle in their model of the electroweak interactions to put electromagnetism and the weak force into a unified framework. And with the advancement of quantization methods, our classical theories for the fundamental forces have now been quantized successfully. Quantum Electrodynamics represents the quantized version of classical Maxwell theory and has been unified with the weak force to give a consistent quantum theory of the electroweak interactions. Quantum Chromodynamics describes the successful quantization procedure of the strong force. The combination of the electroweak theory and QCD yields what is arguably the greatest triumph in 20th century theoretical physics: the Standard Model of particle physics. In fact, this "unifying" theme is not only limited to particle physics. Unification appears to pervade throughout the entire field of physics. Einstein's theory of relativity shows a "merging" of space and time into a spacetime continuum; and Hawking's work shows deep connections between the fields of black holes and classical statistical mechanics.

While the Standard Model has provided some outstanding predictions in the realm of high energy physics and has helped to unify three of the four fundamental forces in nature, it is also fraught with problems. Perhaps the biggest flaw is that it fails to provide a consistent theory of quantum gravity; that is, it cannot incorporate the fundamental force most akin to our everyday experience: gravity. This is because of issues of non-renormalizibility in the construction of a quantum field theory of gravity. It is no secret that the biggest problem physicists would like to solve is the incompatibility of quantum mechanics and general relativity. Any theory that seeks to describe physics at the fundamental level must do so, and the Standard Model falls short in this regard. Additionally it fails to explain other fundamental problems such as the confinement of quarks and the cosmological constant problem.

At this point, it is worth making a short digression and making a few more comments on symmetry that will help us to move forward. The example above shows that symmetry is a fundamental tool in the theoretical physicist's toolbox. We noted that the unified nature of the theory of electromagnetism arises from the Lorentz and gauge symmetries of the theory. We can broadly classify symmetries into two categories. Lorentz transformations are an example of a spacetime symmetry (as are Poincaré symmetries and general coordinate transformations), while gauge transformations are an example of an internal symmetry (as is the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$). Another vital property of symmetries is that they may be "hidden." That is, a symmetry may exist in nature but may be spontaneously broken. When a theory is symmetric with respect to a symmetry group, then spontaneous symmetry breaking occurs if one element of the group is required to be distinct. This concept was applied by Salam and Weinberg to unify electromagnetism and the weak force. They conjectured that the electroweak field's $SU(2) \times U(1)$ symmetry group was broken to the Maxwell $U(1)$ gauge group. This is why we observe electromagnetic symmetry and not the hidden electroweak symmetry. This suggests an obvious way to proceed when we attempt to attack the problems present within the Standard Model: find more symmetries! Symmetries have been so successful in providing unifying frameworks for physical theories that it seems obvious to try to identify more symmetries in nature. Of course, there are two ways to approach this. First, we may look for more internal symmetries. That is, we may attempt to find a Lie group under which the theory is symmetric and hope that this gives new predictions that may resolve our problems. Or, we may look for more spacetime symmetries. Extra dimensions is an obvious candidate here; but there also exists another spacetime symmetry in nature - supersymmetry. This is a symmetry between bosons and fermions (the two types of particles found in the Standard Model) and provides several answers to some of the problems present within the Standard Model. In particular, it plays a vital role in the theory of supergravity - which attempts to unify gravity, supersymmetry and electromagnetism into a single framework.

Since its conception, supergravity has played a key role in the development of high energy physics [18, 19]. In particular, it emerges as the gauge theory of supersymmetry; that is, if we are to promote supersymmetry to a local symmetry, gravity naturally arises [20]. Consider the commutator of two supersymmetry transformations with anticommuting parameters ϵ_1 and ϵ_2 given by

$$[\delta(\epsilon_1), \delta(\epsilon_2)] \sim \bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_\mu$$

where P_μ is the generator of space-time translations. The equation above shows that two successive supersymmetry transformations result in a space-time translation. Now suppose that the parameters ϵ_1 and ϵ_2 are a function of the spacetime coordinate. Then, two consecutive supersymmetry transformations will yield a local translation parameter given by $\xi^\mu(x) = (\bar{\epsilon}_1 \gamma^\mu \epsilon_2)(x)$; however, we know that local translations are the infinitesimal form of general coordinate transformations. Consequently, any theory that is endowed with local supersymmetry invariance must also be invariant under general coordinate transformations. Thus, the metric will appear as a dynamical field, rendering any locally supersymmetric theory into a theory of gravity. Associated with any local gauge symmetry is a gauge field A_μ transforming as $\delta_\epsilon A_\mu = \partial_\mu \epsilon + \ldots$, where ϵ is the parameter of the gauge transformation. In the case of local supersymmetry, the gauge transformation parameter is a spinorial object. The corresponding field (the Rarita-Schwinger field), $\Psi^\alpha_\mu(x)$, represents a particle of helicity $\frac{3}{2}$ known as the gravitino (the superpartner of the graviton).

The reason why supergravity is appealing as a physical theory is that supersymmetry imposes strict constraints on the dynamics and field content, giving rise to rich and interesting mathematical structure [21]. Fields that typically arise in a gravity supermultiplet, apart from the graviton and gravitinos, are p-form gauge fields, which are generalizations of the electromagnetic gauge field and other matter fields such as scalar and spinor fields. Moreoever, supersymmetry also has the nice property of alleviating the divergent ultraviolet behavior of quantum field theories [22]. This was primarily why supergravity was originally conceived as a fundamental theory capable of eliminating the non-renormalizable divergences that arose in the construction of quantum field theories of gravity. Finally, the particle content and symmetries of supergravity models gave the theory a viable framework for the unification of all the fundamental forces [23, 24]. The consensus today, however, is that although local supersymmetry improves the high energy behavior of quantum gravity, it is an effective rather than fundamental theory of nature.

Supergravity theories provide a wonderful platform for tackling phenomenological issues in particle physics and cosmology [24, 25]; but they also play a prominent role in the context of string theory. The massless sector of the spectrum of superstring theories is described by supergravity [26] and so by studying the behavior of classical supergravity solutions, one retrieves valuable information about the low energy dynamics of superstring theories. In addition, many results established in supergravity, such as dualities connecting different coupling regimes of various supergravity theories, can be generalized to the superstring level [27]. Among the various supergravity theories, 11-dimensional supergravity occupies an exalted position. Eleven is the maximal space-time dimension in which a supergravity theory can be constructed and possess no particle with helicity greater than two [28]. The field content of $d = 11$ supergravity is fairly simple. It consists of the graviton field, a Majorana gravitino field and a 3-form gauge field. The supergravity theory in eleven dimensions was originally constructed [29] in order to obtain supergravity theories in lower dimensions via Kaluza-Klein dimensional reduction [30]. These methods have fallen out of favor as they do not produce realistic models in four dimensions. The current perspective on $d = 11$ supergravity is that it is the low energy effective field theory of M-theory. M-theory currently purports to be our best guess at grand unification of the fundamental forces and a consistent theory of quantum gravity. It goes without saying the importance of studying supergravity and gaining new insights into the phenomenology contained therein.

So why the long-winded discussion on mathematics, unification, the Standard Model and supergravity? In the above discussion, I have attempted to draw attention to two key points. Firstly, I have highlighted the fact that one of the main themes in the history of physics has been unification. "Time and time again diverse phenomena have been understood in terms of a small number of underlying principles and building blocks" [31]. I have argued that despite the shortcomings of the Standard Model in unifying fundamental physics, supergravity has shown potential to fix them, and is worth every bit of our attention. Moreover, I have also argued that with a suitable mathematical framework, new physical insights can be gained. This begs the question, is there a neat mathematical formalism that will give us an elegant, unified and geometrical picture of supergravity? Does there exist a geometrical framework in which we can account for the extra degrees of freedom beyond the metric which are present in supergravity? The answer turns out to be yes. Generalized complex geometry [33, 34], developed fairly recently by Hitchin and Gualtieri, "is the study of structures on a generalized tangent space $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$" [37]. It provides a beautiful framework for studying and understanding theories of supergravity. The formalism has already been applied to Type IIB Supergravity [35], M-Theory and Superpotentials [36], and most recently to Type II Supergravity Theories [37]. Our goal will be to consider a simpler four dimensional model of supergravity to which we will apply the mathematical formalism of generalized complex geometry. (It turns out that this model in fact has plenty of similarities to the eleven dimensional model of supergravity described above). In studying the simple model in this new framework, we will attempt to understand and draw some new insights into both supergravity and string theory and comment on the usefulness of generalized geometry in understanding supergravity theories. The task at hand is quite a large one, and given the time frame of this project, we unfortunately cannot supply all the details. Thus, in this paper, we present the initial construction of the mathematics for the four dimensional theory of supergravity that we will consider, and give some remarks concerning ideas worth developing beyond this dissertation.

The layout of the paper is as follows. We begin by providing an overview of Hitchin's generalized geometry in Part 1. We follow this with a discussion of the four dimensional model of supergravity that we will study in Part 2. We present the initial mathematical construction of the linear, differential and metric structure of our extended generalized geometry in Part 3. And we provide an account of the ideas worth developing beyond this project in Part 4. Finally, since this paper draws analogies heavily from the study of differential geometry, we have also included several appendices (Parts 5-8) to bring the reader up to speed in the basics of differentiable manifolds, Riemannian geometry and fibre bundle theory. We have also included a section on gauge theories. These are by no means comprehensive and many of the proofs of key results have been omitted. The intention in including the appendices is to give a broad overview of important results; to provide a foundation and some motivation for understanding some of the constructions present in generalized geometry; and to give the reader an appreciation for the vital role that geometry plays in the development and understanding of physical theories. We follow texts [1]-[11] quite closely, particularly Nakahara.

At this point, I would like to express my sincere thanks to several people who have been of great help to me during the months in which this research was undertaken. I would like to express my gratitude to Dr. Daniel Waldram of the Theoretical Physics Group at Imperial College London - for his wonderful patience and ability to break down complex ideas and enunciate them in the clearest of fashions. Thank you for your encouragement and support; for insisting on going through difficult calculations; for meticulous attention to detail; and for opening my eyes to the research process. I could not have asked for a better advisor! I would also like to thank Jaime Gonzalez de Chaves Otaola, Giovanni Guccione and Jason Lewis for many a helpful discussion.

This dissertation is dedicated to Dr. Partha Debroy and Dr. Robert Uglesich of the Department of Physics at The Cooper Union for the Advancement of Science and Art. Thank you for your constant support, encouragement and for inspiring me to pursue research in physics. Finally, I would like to dedicate this project to my parents - for their unrelenting love and support.

> "How marvelous are Your works, Oh Lord, and how profound Your thoughts! Wonderful is Your workmanship, and that my soul knows very well..."

Part 1. Generalized Geometry

Generalized complex geometry is the area of mathematics research concerned with geometric structures that unify the seemingly disparate fields of complex and symplectic geometry. The generalized structures that we will discuss were initially introduced by N. Hitchin [33]; and have subsequently been developed by Gualtieri [34]. We follow Gualtieri's work and Baraglia's [41] work fairly closely. Generalized geometry has plenty of advantages. Namely, it provides a fruitful platform for investigating geometries of nonlinear sigma models [42]; its natural $O(d,d)$ metric provides a framework within which we can discuss T-duality and other symmetries of string theory; and finally, with the development of generalized Calabi-Yau manifolds, it sheds new light on certain aspects of string theory. But what we are particularly concerned with is the structure it provides for us to understand supergravity. With this in mind, we will proceed by describing the basic properties of generalized geometry in this chapter. We begin with the linear algebra of the generalized tangent space and then describe the differential structure of such spaces. Finally, we outline several specific tools that will be required to express supergravity in the language of generalized geometry.

Generalized Tangent Spaces

Let $\mathcal{M}$ be an m-dimensional manifold. Consider the direct sum of the tangent and cotangent spaces, denoted $E \simeq T(\mathcal{M}) \oplus T^*(\mathcal{M})$. We refer to this space as the generalized tangent space. Elements of the generalized tangent space are pairs $(V,\lambda) = V + \lambda$ with $V \in T(\mathcal{M})$ and $\lambda \in T^*(\mathcal{M})$. Since V and λ have components given by V_i and λ_i for $i = 1, \ldots, m$, we can treat $(V,\lambda) = V + \lambda$ as a generalized vector with components given by:

$$W^I = \begin{pmatrix} V_i \\ \lambda_i \end{pmatrix}$$

The space E is naturally endowed with a symmetric inner product given by:

$$\langle V + \lambda, W + \omega \rangle = \frac{1}{2}\left(\lambda\left(W\right) + \omega\left(V\right)\right)$$

We can recast this in matrix form:

$$\langle V + \lambda, W + \omega \rangle = \frac{1}{2}\begin{pmatrix} V & \lambda \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} W \\ \omega \end{pmatrix}$$

Making this more explicit using components, we find:

$$\langle V + \lambda, W + \omega \rangle = \frac{1}{2}\begin{pmatrix} V_i & \lambda_i \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1}_{m\times m} \\ \mathbf{1}_{m\times m} & 0 \end{pmatrix}_{ij}\begin{pmatrix} W_j \\ \omega_j \end{pmatrix}$$

This defines a flat metric:

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{1}_{m\times m} \\ \mathbf{1}_{m\times m} & 0 \end{pmatrix}$$

η is a symmetric real matrix, and can be diagonalized by a similarity transformation. η has signature (m,m) and it can be shown that η is invariant under the orthogonal group $O\left(T(\mathcal{M}) \oplus T^*(\mathcal{M})\right) \simeq O(m,m)$. Its Lie algebra satisfies

$$\mathfrak{so}\left(E\right) = \left\{Q : Q^T M + MQ = 0\right\}$$

This suggests a natural special orthogonal subgroup action on the space. That is, we can specify a canonical orientation on $E \simeq T(\mathcal{M}) \oplus T^*(\mathcal{M})$ as follows. Note that we may decompose the highest exterior power in the following way:

$$\wedge^{2m}\left(T(\mathcal{M}) \oplus T^*(\mathcal{M})\right) = \wedge^m T(\mathcal{M}) \oplus \wedge^m T^*(\mathcal{M})$$

There is a natural pairing between $\wedge^k T(\mathcal{M})$ and $\wedge^k T^*(\mathcal{M})$ given by $(v^*, u) = det\left(v_i^*(u_j)\right)$ for $v^* = v_1^* \wedge \cdots \wedge v_k^* \in \wedge^k T^*(\mathcal{M})$ and $u = u_1 \wedge \cdots \wedge u_k \in \wedge^k T(\mathcal{M})$. Thus, we can write $\wedge^{2m} T(\mathcal{M}) \oplus T^*(\mathcal{M}) \simeq \mathbb{R}$, allowing us to identify a canonical orientation on $E \simeq T(\mathcal{M}) \oplus T^*(\mathcal{M})$ by specifying some real number. To preserve both this orientation and the inner product, we require $SO\left(T(\mathcal{M}) \oplus T^*(\mathcal{M})\right) \simeq SO(m, m)$. Thus, there is a natural special orthogonal group action on our space E.

Differential Structure

A key piece of structure on our geometry is a bilinear, skew-symmetric bracket on E known as the Courant bracket.

Definition. The Courant bracket is the bilinear form $[\,,\,]$ on E given by $[u, v]_C = [X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y))$ where $u = X + \xi$ and $v = Y + \eta$.
We can write this in terms of the interior derivative as:

$$[u, v]_C = [X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)$$

for $u = X + \xi, v = Y + \eta \in E$. Recall that the Lie bracket acts on sections of the tangent bundle on a manifold. Thus, the Courant bracket generalizes the action of the Lie bracket to sections of the generalized tangent bundle. It is interesting to note that while the Courant bracket is skew-symmetric, it does not satisfy the Jacobi identity. To see the extent of the failure, see [41].

Symmetries

For a smooth manifold $\mathcal{M}$, the symmetries of the Lie bracket on the tangent bundle $\pi : T\mathcal{M} \longrightarrow \mathcal{M}$ are given by the bundle automorphism (F, f), where $F : T\mathcal{M} \longrightarrow T\mathcal{M}$ and $f : \mathcal{M} \longrightarrow \mathcal{M}$ are diffeomorphisms. We require that F preserves the bracket. That is,

$$F([X, Y]) = [F(X), F(Y)] \;\forall\; X, Y \in T\mathcal{M}$$

$F = f_*$ is the push forward of the tangent space, and is defined at the point p as

$$f_* : T_p(\mathcal{M}) \longrightarrow T_{f(p)}(\mathcal{M})$$

Since F is a diffeomorphism, so is f_*. Thus the Lie bracket is invariant under diffeomorphism symmetry. Analogously, we can define a generalized bundle automorphism (F, f) for $F : T\mathcal{M} \oplus T\mathcal{M}^* \longrightarrow T\mathcal{M} \oplus T\mathcal{M}^*$ and $f : \mathcal{M} \longrightarrow \mathcal{M}$. We require that F preserves the Courant bracket:

$$F\left([X + \xi, Y + \eta]_C\right) = [F(X + \xi), F(Y + \eta)]_C \;\forall\; X + \xi, Y + \eta \in T\mathcal{M} \oplus T\mathcal{M}^*$$

Clearly $F = f_* \oplus f^*$ (where f_* is the push forward of the tangent space and f^* is the pull back of the cotangent space) defines an automorphism on $T\mathcal{M} \oplus T\mathcal{M}^*$. Thus we have diffeomorphism invariance on our geometry.

The Courant bracket also possess a non-trivial automorphism defined by forms which is not present in the Lie bracket. Let $B \in \wedge^2 T^*(\mathcal{M})$ be a closed two-form.

Let $X \in T(\mathcal{M})$ be a vector field. Define the map $X \longrightarrow i_X B$ from $T(\mathcal{M}) \longrightarrow T^*(\mathcal{M})$. Then B acts as an endomorphism of E by $B(X+\xi) = BX = i_X B$. We may exponentiate B to find:

$$e^B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

This allows us to define an orthogonal bundle mapping. The exponential $e^B \in SO(E)$ then acts as

$$e^B : X + \xi \longrightarrow X + \xi + i_X B$$

This transformation is known as a B-transformation. It can be thought of as a shear transformation, shearing in the $T^*(\mathcal{M})$ direction [34].

Proposition. *Let $u = X + \xi$ and $v = Y + \eta$ be sections of $T\mathcal{M} \oplus T^*\mathcal{M}$ and B be a 2-form. Then $\left[e^B u, e^B v\right] = e^B [u,v] - i_X i_Y dB$.*

Proof. $\left[e^B u, e^B v\right] = [X + \xi + i_X B, Y + \eta + i_Y B] = [u,v] + [X, i_Y B] + [i_X B, Y] = [u,v] + \mathcal{L}_X i_Y B - \frac{1}{2} d i_X i_Y B - \mathcal{L}_Y i_X B - \frac{1}{2} d i_Y i_X B = [u,v] + (\mathcal{L}_X i_Y - \mathcal{L}_Y i_X + d i_Y i_X) B = [u,v] + (\mathcal{L}_X i_Y - i_Y d i_X) B = [u,v] + (\mathcal{L}_X i_Y - i_Y d i_X) B - i_Y i_X dB + i_Y i_X dB = [u,v] + (\mathcal{L}_X i_Y - i_Y \mathcal{L}) B + i_Y i_X dB = [u,v] + i_{[X,Y]} B - i_X i_Y dB = e^B [u,v] - i_X i_Y dB$ □

Since B is closed, the Courant bracket is invariant under the action of a closed 2-form. It can be shown that diffeomorphisms and B-transformations make up the totality of automorphisms of the Courant bracket [34]. The automorphism group consists of the semi-direct product group $Diff(\mathcal{M}) \ltimes Z^2(\mathcal{M})$.

Let us summarize what we have described above. We are interested in maps that preserve the structure of the generalized tangent bundle. Given a vector space V, one can form the vector space $E \simeq V \oplus V^*$ with the natural bilinear form with signature (d,d). The group of transformations of E preserving the bilinear form is $O(E) \simeq O(d,d)$. The Lie algebra $\mathfrak{so}(E)$ of $O(E)$ [and $SO(E)$] consists of matrices which are skew adjoint with respect to the bilinear form. Finally, diffeomorphism and 2-form gauge invariance arise naturally in the construction.

Generalized Metrics

We now wish to introduce some additional structure to the generalized tangent bundle which will help to define more geometry on our space. In regular differential geometry, we endowed our tangent space with a Riemannian metric. Our goal now will be to introduce a generalization of the Riemannian metric on $E \simeq T(\mathcal{M}) \oplus T^*(\mathcal{M})$ which unifies the 2-form B and the conventional metric g into a single object. From a generalized metric, one may construct generalizations of the Levi-Civita connection, the Hodge star, the inner product for forms as well as the Laplacian operator. Since, we will not be needing all this, we refer the interested reader to [41].

We may motivate the construction of a generalized Riemannian metric by noting that for a Riemannian manifold $(\mathcal{M}, g)$, the metric is required to be positive definite. We thus make the following definition.

Definition. Let E be a generalized tangent bundle on $\mathcal{M}$. A generalized metric V is a positive definite subbundle of rank $n = dim\mathcal{M}$, that is, the restriction of the form $(\ ,\ )$ to V is positive definite.

We can construct the generalized metric by considering a reduction of the $O(d,d)$ structure of E to a $O(d) \times O(d)$ structure. We split E into two d-dimensional subbundles given by $E = C_+ \oplus C_-$. We require that η splits into a positive definite metric on C_+ and a negative metric on C_-, and thus the $O(d) \times O(d)$ subgroup structure is required to preserve each individual metric.

The splitting $E = C_+ \oplus C_-$ defines a generalized metric by

$$G = \eta\mid_{C_+} - \eta\mid_{C_-}$$

G is a symmetric automorphism of E since $G = G^*$ and $G : E \longrightarrow E$ defines the positive and negative eigenspaces $C_\pm$. We may write G in terms of the usual Riemannian metric g as

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

We may write this in a more general form, incorporating the B field:

$$G = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

Generalized Tetrads

With a generalized metric, we are now equipped to define a generalized non-coordinate basis. These tetrads will transform in representations of $O(d) \times O(d)$ rather than the usual $SO(3,1)$ Lorentz group. Consider a basis of one forms $E_A \in E^*$ with $A = 1, \ldots, 2d$. We require that G and η be expressed in the following form:

$$\eta = E^T \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{-1} \end{pmatrix} E$$

$$G = E^T \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} E$$

For two sets of tetrads, $e^a_\pm$ and inverse $\hat{e}_{\pm a}$, satisfying the usual relations given by

$$g_{mn} = e^a_{\pm m} e^b_{\pm n} \delta_{ab}$$

$$g^{mn} = \hat{e}^m_{\pm a} \hat{e}^n_{\pm b} \delta^{ab}$$

we find explicitly that

$$E = \frac{1}{\sqrt{2}} \begin{pmatrix} e_+ - \hat{e}^T_+ B & \hat{e}^T_+ \\ -(e_- + \hat{e}^T_- B & \hat{e}^T_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}^T_+(g - B) & \hat{e}^T_+ \\ -\hat{e}^T_-(g + B & \hat{e}^T_- \end{pmatrix}$$

The first d tetrads give a basis for C_+ and the second d tetrads give a basis for C_-.

Generalized Lie Derivatives

Since the Lie derivative is invaluable in differential geometry, we now turn our attention to defining a generalized Lie derivative. Recall that the Lie derivative acting on $(2,0)$-tensor fields satisfies

$$(\mathcal{L}_V T)(X,Y) = \mathcal{L}_V(T(X,Y)) - T(\mathcal{L}_V(X,Y)) - T(X, \mathcal{L}_V Y)$$

Also recall that for two vector fields X and Y, we define the Lie derivative with respect to X by

$$\mathcal{L}_X Y = d i_X Y + i_X dY$$

Analogously, we wish to define our generalized Lie derivative so that it satisfies the two equations above [43]. Thus we want our generalized Lie derivative to satisfy

$$(\mathbb{L}_v G)(x,y) = (\mathbb{L}_v(G(x,y)) - G(\mathbb{L}_v x, y) - G(x, \mathbb{L}_v y)$$

Thus, for two generalized vectors $V = X + \xi$ and $U = Y + \eta$ we defined the generalized Lie derivative by

$$\mathbb{L}_V U = \mathcal{L}_X Y + \mathcal{L}_X \eta - i_Y d\xi$$

This is known as the Dorfman derivative. We may also write this in terms of a bracket operation known as the Dorfman bracket:

$$[X+\xi, Y+\eta]_D = [X,Y] + \mathcal{L}_X \eta - i_Y d\xi$$

This naturally reduces to the usual Lie derivative if the B fields vanish.

We note that for ordinary vectors, the Lie derivative satisfies $\mathcal{L}_V = [v,w]$. However, for generalized vectors this does not hold; i.e., $\mathbb{L}_V U \neq [V,U]_C$ (where $[\ ,\ ]_C$ is the Courant bracket).

Proposition. *Two generalized vectors $V = v + \lambda$ and $W = w + \mu$ satisfy $[V,W]_C = \frac{1}{2}(\mathbb{L}_V W - \mathbb{L}_W V)$.*

Proof. $[V,W]_C = [v,w] + (\mathcal{L}_v\mu - \mathcal{L}_w\lambda - \frac{1}{2}d(i_v\mu - i_w\lambda) = [v,w] + \mathcal{L}_v\mu - \mathcal{L}_w\lambda - \frac{1}{2}\mathcal{L}_v\mu + \frac{1}{2}i_v d\mu + \frac{1}{2}\mathcal{L}_w\lambda - \frac{1}{2}i_w d\lambda = \frac{1}{2}[v,w] - \frac{1}{2}[w,v] + \frac{1}{2}\mathcal{L}_v\mu - \frac{1}{2}\mathcal{L}_w\lambda + \frac{1}{2}i_v d\mu - \frac{1}{2}i_w d\lambda = \left(\frac{1}{2}[v,w] + \frac{1}{2}\mathcal{L}_v\mu - \frac{1}{2}i_w d\lambda\right) - \left(-\frac{1}{2}[w,v] - \frac{1}{2}\mathcal{L}_w\lambda + \frac{1}{2}i_v d\mu\right) = \frac{1}{2}(\mathbb{L}_V W - \mathbb{L}_W V)$ □

For completeness, we end the section by stating the definition of a generalized Killing vector field.

Definition. A generalized Killing vector field V is a vector field satisfying the relation $\mathbb{L}_V G = 0$.

At this point, it is worth closing the section by making an important remark. The machinery we have outlined above gives an elegant description of geometries that consist of both a metric and a 2-form field. Such geometries arise in string theory and supergravity; in particular, they arise in the NSNS region of type II supergravity. The appealing feature of what we have described is the natural $SO(d,d)$ group action on the manifold. But in type II string theory, $SO(d,d)$ is a part of a larger group of symmetries - the "U-duality" group. Thus, the mathematics we have described above can be extended to the exceptional U-duality group, $E_{d(d)}$. This provides a nice language to describe M-theory [39] as well as other supergravity theories [36]. An obvious example of such an "extension" is the mathematics that we will consider in Part 3; that is, we consider $E \simeq T(\mathcal{M}) \oplus \wedge^2(T^*\mathcal{M})$ instead of $E \simeq T(\mathcal{M}) \oplus T^*(\mathcal{M})$ described above.

Part 2. Three-Form Supergravity

Rather than treating the usual eleven-dimensional supergravity action, we will consider a dualized version of it, where "one of the scalar auxiliary fields is replaced by the four-form field strength of the three-form potential" [38]. The theory can best be described as a "toy model" [38] of the theory of a membrane in eleven dimensions, with the "three-form supergravity sharing some of the structure of eleven-dimensional supergravity" [38]. Ovrut and Waldram point out that a clear advantage of this formulation of four-dimensional supergravity is that it provides a nice off-shell description of eleven-dimensional supergravity with all necessary auxiliary fields included [38]. In particular, "it is also strictly this three-form version of $N = 1$ supergravity that appears in the reduction of higher-dimensional supergravities containing form potentials of degree three or higher" [38].

We begin by considering the original $N = 1$, $d = 4$ supergravity bosonic lagrangian $\mathcal{L}$ given by:

$$\frac{\kappa^2}{e}\mathcal{L} = \frac{1}{2}R + \frac{1}{3}b^2 - \frac{1}{3}M_1^2 - \frac{1}{3}M_2^2$$

M_1 and M_2 are auxiliary scalar fields, b_μ is an auxiliary vector field, κ is a constant factor, and R is the Ricci scalar. Note also that e denotes the determinant of the non-coordinate basis e_μ^a satisfying $g_{\mu\nu} = \eta_{ab}\, e_\mu^a e_\nu^b$ on the manifold. The equations of motion for the auxiliary fields can be found by applying the Euler-Lagrange equations of classical mechanics:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial M_1} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu M_1)}\right) &= 0 \;\Rightarrow M_1 = 0\\ \frac{\partial\mathcal{L}}{\partial M_2} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu M_2)}\right) &= 0 \;\Rightarrow M_2 = 0\\ \frac{\partial\mathcal{L}}{\partial b_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu b_\nu)}\right) &= 0 \;\Rightarrow b^\nu = 0\end{aligned}$$

We conclude that all auxiliary fields should vanish. Now, we dualize M_2 by setting $M_2 = \frac{1}{4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2\mu_3\mu_4}$, where $F_{\mu_1\ldots\mu_4} = 4\partial_{\mu_1}A_{\mu_2\mu_3\mu_4}$. We may choose $F_{\mu_1\ldots\mu_4}$ in such a way since locally, a four-form on a four dimensional manifold is always closed. The constant 4 is a normalization factor, and A is determined up to a gauge transformation; that is, $A \longrightarrow A + B,\ B\epsilon Z^4\,(\mathcal{M})$.

Substituting back into the lagrangian, we obtain

$$\begin{aligned}\frac{\kappa^2}{e}\mathcal{L} &= \frac{1}{2}R + \frac{1}{3}b^2 - \frac{1}{3}M_1^2 - \frac{1}{3}\left(\frac{1}{4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2\mu_3\mu_4}\right)^2\\ &= \frac{1}{2}R + \frac{1}{3}b^2 - \frac{1}{3}M_1^2 - \frac{1}{3}\left(\frac{1}{16}\cdot 4!\cdot F^2\right)\\ &= \frac{1}{2}R + \frac{1}{3}b^2 - \frac{1}{3}M_1^2 - 2\cdot detg\cdot F^2\end{aligned}$$

where we have used $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\mu_1\mu_2\mu_3\mu_4} = 4!\cdot det(g)$. We can now find a field equation for the three-form field A:

$$\frac{\partial\mathcal{L}}{\partial A_{\mu_1\mu_2\mu_3}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_{\mu_1\mu_2\mu_3})}\right) = 0$$

$$\Rightarrow -4\partial_{\mu_1}\left(\sqrt{|g|}\partial^{\mu_1}A^{\mu_2\mu_3\mu_4}\right) = 0$$

$$\Rightarrow \partial_{\mu_1}\left(\sqrt{|g|}F^{\mu_1\mu_2\mu_3\mu_4}\right) = 0$$

This last equation can be written as follows: $\partial_{\mu_1}\sqrt{|g|}F^{\mu_1\mu_2\mu_3\mu_4} = 0 \Longrightarrow \nabla_{\mu_1}F^{\mu_1\mu_2\mu_3\mu_4} = 0$. We show this below. Consider the following three identities:

(1) $\frac{\delta g}{g} = g^{\alpha\beta}\delta g_{\alpha\beta}$

(2) $\Gamma^{\mu}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\right)$

(3) $\nabla_\mu T^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^\mu\right)$

We prove each part separately:

(1) $\widetilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$. Multiplying both sides by $g^{\mu\alpha}$ gives: $g^{\mu\alpha}\widetilde{g}_{\mu\nu} = g^{\mu\alpha}g_{\mu\nu} + g^{\mu\alpha}\delta g_{\mu\nu} = \delta^\alpha_\nu + g^{\mu\alpha}\delta g_{\mu\nu}$. Taking the determinant of both sides yields: $det(g^{\mu\alpha}\widetilde{g}_{\mu\nu}) = det(\delta^\alpha_\nu + g^{\mu\alpha}\delta g_{\mu\nu})$. Using the identity $det(1+\delta M) \simeq 1 + Tr(\delta M)$, one obtains: $\frac{\widetilde{g}}{g} = \frac{g+\delta g}{g} = 1 + Tr(g^{\mu\alpha}\delta g_{\mu\nu}) = 1 + g^{\mu\nu}\delta g_{\mu\nu} \longrightarrow \frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu}$.

(2) Identity 1 implies $\frac{\partial_\alpha g}{g} = g^{\mu\nu}\partial_\alpha g_{\mu\nu}$. The Christoffel Symbols are given by $\Gamma^\mu_{\alpha\nu} = \frac{1}{2}g^{\mu\beta}(\partial_\nu g_{\beta\alpha} + \partial_\alpha g_{\beta\nu} - \partial_\beta g_{\alpha\nu})$. $\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\beta}\partial_v g_{\beta\mu}$, so $\frac{\partial_\alpha g}{g} = g^{\mu\nu}\partial_\alpha g_{\mu\nu} = 2\Gamma^\mu_{\mu\nu} = \frac{2}{\sqrt{-g}}\cdot\frac{-\partial_\alpha g}{2\sqrt{-g}} = \frac{2}{\sqrt{-g}}\partial_\alpha(\sqrt{-g})$. $\Rightarrow \Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\right)$.

(3) $\nabla_\mu T^\mu = \partial_\mu T^\mu + \Gamma^\mu_{\mu\alpha}T^\alpha = \partial_\mu T^\mu + \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\right)T^\alpha = \frac{1}{\sqrt{-g}}(\sqrt{-g}\partial_\mu T^\mu + \partial_\alpha\left(\sqrt{-g}\right)T^\alpha) = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^\mu\right)$. $\square$

Generalizing identity (3), we may write $\partial_{\mu_1}\sqrt{|g|}F^{\mu_1\mu_2\mu_3\mu_4} = 0 \Longrightarrow \nabla_{\mu_1}F^{\mu_1\mu_2\mu_3\mu_4} = 0$. We have found that the field strength is covariantly constant. $\sqrt{|g|}F^{\mu_1\mu_2\mu_3\mu_4}$ is totally antisymmetric in its indices, and hence is proportional to the epsilon symbol $\epsilon^{\mu_1\mu_2\mu_3\mu_4}$. We take $\epsilon^{0123} = 1$. We can write:

$$\sqrt{|g|}F^{\mu_1\mu_2\mu_3\mu_4} = \lambda\epsilon^{\mu_1\mu_2\mu_3\mu_4} \Longrightarrow F^{\mu_1\mu_2\mu_3\mu_4} = \lambda\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$$

where $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$ is the epsilon tensor and λ is a proportionality constant. Substituting back into the auxiliary field gives:

$$M_2 = \frac{1}{4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\left(\lambda\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\right) = -6\lambda$$

We observe that the effect of dualizing the auxiliary field M_2 by introducing F gives a non-vanishing expectation value.

Now, rewriting M_2 in terms of $A_{\mu_1\mu_2\mu_3}$ and working on shell, where the auxiliary fields M_1 and b_μ vanish, the bosonic action for three-form supergravity becomes

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\left(\mathcal{R} + F^2\right)d^4x$$

Varying the action with respect to the metric, and noting that $\sqrt{-g}F^2 = \sqrt{-g}g^{\mu_1\sigma_1}\cdots g^{\mu_4\sigma_4}F_{\sigma_1\cdots\sigma_4}F^{\mu_1\cdots\mu_4}$, we find:

$$\delta\left(\sqrt{-g}\mathcal{R} + \sqrt{-g}F^2\right) = \delta\left(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}\right) + \delta\left(F^2\sqrt{-g}\right)$$
$$= \delta g^{\mu\nu}R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g} + \mathcal{R}\delta\left(\sqrt{-g}\right)$$
$$+4F^\mu_{\rho_1\rho_2\rho_3}F^{\nu\rho_1\rho_2\rho_3}\delta g_{\mu\nu}\sqrt{-g} + \frac{1}{2}\sqrt{-g}g^{\mu\nu}F^\mu_{\rho_1\rho_2\rho_3}F^{\nu\rho_1\rho_2\rho_3}\delta g^{\mu\nu}g^{\mu\nu}$$
$$= -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda}R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\left(\nabla_\kappa\delta\Gamma^\kappa_{\nu\mu} - \nabla_\nu\delta\Gamma^\kappa_{\kappa\mu}\right)\sqrt{-g} + \frac{1}{2}R\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$$

$$+4F^{\mu}_{\rho_1\rho_2\rho_3}F^{\nu\rho_1\rho_2\rho_3}\delta g_{\mu\nu}\sqrt{-g}+\frac{1}{2}F^2 g^{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}$$

The second term in the fourth line vanishes as it can be written as a total divegence and so does not contribute to the variation. Imposing the requirement that $\delta S = 0$, we find:

$$\delta S = \int \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 4F_{\mu}^{\rho_1\rho_2\rho_3}F_{\nu\rho_1\rho_2\rho_3\rho_3} + \frac{1}{2}F^2 g_{\mu\nu}\right)\sqrt{-g}\delta g_{\mu\nu} = 0$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -4F_{\mu}^{\rho_1\rho_2\rho_3}F_{\nu\rho_1\rho_2\rho_3\rho_3} - \frac{1}{2}F^2 g_{\mu\nu}$$

Plugging $F_{\mu\nu\rho\sigma} = \lambda\varepsilon_{\mu\nu\rho\sigma}$ into the field equations gives:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -4F_{\mu}^{\rho_1\rho_2\rho_3}F_{\nu\rho_1\rho_2\rho_3\rho_3} - \frac{1}{2}F^2 g_{\mu\nu}$$

$$= -4\left(\lambda\varepsilon_{\mu}^{\rho_1\rho_2\rho_3}\right)\left(\lambda\varepsilon_{\nu\rho_1\rho_2\rho_3}\right) - \frac{1}{2}\left(\lambda^2\varepsilon_{\mu\rho\sigma\tau}\varepsilon^{\mu\rho\sigma\tau}\right)g_{\mu\nu}$$

$$= -4\lambda^2\varepsilon^{\alpha\rho_1\rho_2\rho_3}\varepsilon_{\nu\rho_1\rho_2\rho_3}g_{\mu\alpha} - \frac{1}{2}\lambda^2\varepsilon_{\mu\rho\sigma\tau}\varepsilon^{\mu\rho\sigma\tau}g_{\mu\nu}$$

$$= -4\lambda^2\cdot 3!\cdot\delta_{\nu}^{\alpha}g_{\mu\alpha} - \frac{1}{2}\lambda^2\cdot 4!\cdot g_{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -12\lambda^2 g_{\mu\nu}$$

Note that we have used $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\mu} = 3!\cdot\delta_{\mu}^{\delta}$ and $\varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\mu\nu\sigma\tau} = 4!$ in the fourth line. We see that this is simply Einstein's equation for empty space with a negative cosmological constant $\Lambda = -12\lambda^2$. This simple model has the nice property that it shares some similarities with some of the more interesting supergravity theories. Ovrum and Waldram write, "It is interesting to note that the action [above] has the same general structure as higher-dimensional supergravity actions, such as type IIA supergravity in $d = 10$ or $d = 11$ supergravity, with Einstein gravity coupled to Neveu-Schwarz and Ramond forms" [38]. This model provides a good starting point for constructing supergravity using the generalized geometry framework.

Part 3. **Mathematical Construction of** $E^2 \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M}$

Having outlined some key properties of generalized geometry in Part 1, we now wish to apply the formalism to the theory of supergravity we have developed in Part 2. In this case, we must consider an extension of generalized geometry; that is, instead of considering $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$, we consider $E^p \simeq T\mathcal{M} \oplus \wedge^p T^*\mathcal{M}$ for $p = 2$. Before we get into the precise construction we desire, we summarize some key results from differential geometry that will help to motivate the path that we take (we encourage the reader to see the appendices for more detailed accounts of what follows).

Consider the following important concepts that arise in the study of differential geometry. Let $\mathcal{M}$ be an m-dimensional manifold. First, the group of symmetries that we are interested in is $\mathit{Diff}(\mathcal{M})$ - the group of diffeomorphisms on the manifold. This serves us particularly well in the context of general relativity. The fact that general relativity is a diffeomorphism invariant theory allows us to convey the important concept that the theory is free of any preferred geometry for spacetime. The notion that there exists no preferred coordinate system is at the heart of Einstein's ideas and is an obviously desirable feature of the theory. Second, the vital piece of structure that arises in the study of manifolds is the tangent space $T(\mathcal{M})$. $T(\mathcal{M})$ is in fact the fiber of the tangent bundle $T\mathcal{M}$. We may choose tangent vectors $v \in \Gamma(T\mathcal{M})$. Choosing an appropriate basis $\hat{e}_a \in \Gamma(T\mathcal{M})$, we may express the tangent vectors as $v = v^a \hat{e}_a$, where v^a are the components of the vector. On the overlap of two charts U_i and U_j on the manifold, we may switch between coordinates. This is given by

$$v_i^\mu = (M_{ij})^\mu_{\ \nu} v_j^\nu$$

where $M_{ij} = \frac{\partial x_i^\mu}{\partial x_j^\nu} \in GL(m, \mathbb{R})$. That is, we require a natural "patching" group action on our manifold. In this case it is given by $GL(m, \mathbb{R})$. Finally, we have the notion of a derivative between two vectors on the manifold that is given by the Lie derivative and the Lie bracket. More specifically, $\mathcal{L}_v w = [v, w]$ for $v, w \in T(\mathcal{M})$.

Linear Structure

We now wish to generalize these concepts in the context of three-form supergravity. Let us consider them one by one. Firstly, supergravity not only requires the usual diffeomorphism invariance present in general relativity, but requires more symmetry. In particular, since supergravity purports to unify general relativity with Maxwell's equations, we also need to accomodate the gauge symmetry present in electromagnetism. Thus we now define the generalized tangent space to be $E^2 \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M}$. Intuitively, $T\mathcal{M}$ incorporates the diffeomorphism invariance while $\wedge^2 T^*\mathcal{M}$ incorporates the gauge symmetry. A generalized vector, $V \in \Gamma\left(E^2\right)$ is given by

$$V = v + \lambda = \begin{pmatrix} v \\ \lambda \end{pmatrix}$$

In our four-dimensional theory, the generalized vector will have ten components; four of which are given by $v \in T\mathcal{M}$ and six of which are given by $\lambda \in \wedge^2(T^*\mathcal{M})$. We may choose a generalized basis, $\hat{E}_A \in \Gamma\left(E^2\right)$ for $A = 1, \ldots, 10$ such that we can write $V = V^A \hat{E}_A$ for any generalized vector $V \in \Gamma\left(E^2\right)$. An example of a generalized basis is given by the "split basis." Let $\hat{e}_a$ be the conventional basis on

$T(\mathcal{M})$. This defines a dual basis $\hat{e}^a$ on $T^*(\mathcal{M})$. (Note that this also defines a basis $\hat{e}_{ab} = e_a \wedge e_b$ for two-forms). Then we define

$$\hat{E}_A = \begin{cases} \hat{e}_a + i_a B, \ A = a \\ \hat{e}_{ab}, \ A = [ab] \end{cases}$$

where B is a three-form and i_a is the interior derivative. We require that the three-form B be patched in a particular way; that is, $B_{(i)} = B_{(j)} + d\lambda_{(ij)}$ on $U_i \bigcap U_j$ for some two-form λ_{ij}.

In general we wish to "patch" a generalized vector component-wise on the overlap of two charts in the following way. Let U_i and U_j be two overlapping charts on the manifold. Let $V_i = v_i + \lambda_i$ on U_i and let $V_j = v_j + \lambda_j$ on U_j. Then on $U_i \bigcap U_j$, we require

$$v_i = v_j$$
$$\lambda_i = \lambda_j + i_{v_i} d\lambda_{ij}$$

This says an interesting thing about the generalized vector $V = v + \lambda = \begin{pmatrix} v \\ \lambda \end{pmatrix}$: these equations reflect the fact that globally, v remains a vector; however, λ does not remain a two-form globally, and is twisted by v in some non-trivial topological way.

At this point, we can begin to think about the patching group on the generalized tangent space. In general we would like to specify a group action that satisfies $V_i = M_{ij} V_j$ where

$$M_{ij} = \begin{cases} (M_{ij})^\mu_\nu \in GL(4, \mathbb{R}), \ v \\ (d\lambda_{ij})_{\mu\nu\rho} \end{cases}$$

Interestingly, we can achieve this by invoking a $GL(5, \mathbb{R})$ group action on E^2. We define the $GL(5, \mathbb{R})$ group action on E^2 by the following infinitesimal transformations:

(1) $\delta v^a = m^a_b v^b$ for $m^a_b \in \mathfrak{gl}(4, \mathbb{R})$
(2) $\delta\lambda_{ab} = -m^c_a \lambda_{cb} - m^c_b \lambda_{ac}$ for $m^a_b \in \mathfrak{gl}(4, \mathbb{R})$
(3) "Rescaling": $\delta v^a = c v^a$ and $\delta\lambda_{cb} = c\lambda_{ab}$ for some $c \in \mathbb{R}$
(4) "a-shift": $\delta v^a = 0$ and $\delta\lambda_{ab} = v^c a_{cab}$ for some $a_{cab} \in \wedge^3 T^* \mathcal{M}$. In coordinate free notation, this is given by $\delta v = 0$ and $\delta\lambda = i_v a$
(5) "α-shift": $\delta v^a = \frac{1}{2}\alpha^{abc}\lambda_{bc}$ and $\delta\lambda_{ab} = 0$ for some $\alpha_{abc} \in \wedge^3 T^* \mathcal{M}$. In coordinate free notation, this is given by $\delta v = i_\alpha \lambda$ and $\delta\lambda = 0$

This set of transformations forms a Lie algebra that is isomorphic $\mathfrak{gl}(5, \mathbb{R})$ [bullets 1 and 2 give us $\mathfrak{gl}(4, \mathbb{R})$, while the remainder give us the additional transformations needed to form $\mathfrak{gl}(5, \mathbb{R})$]. Notice that bullets 4 and 5 give us precisely the patching $a = d\lambda_{ij}$ we wanted for our group as defined above. In particular, the a-shifts correspond to the patching of the two forms shown above. The commutation relations for the algebra are given by:

$$[m, m'] = m^a_c m^{'c}_b - m^{'a}_c m^c_b$$
$$[a, \alpha'] = -\frac{1}{2}\alpha'^{dab} a_{eab} + \frac{1}{2}\alpha'^{abc} a_{abc} \delta^d_e$$
$$[m, \alpha'] = \frac{1}{2} m^d_a \alpha'^{abc} + \frac{1}{2} m^c_a \alpha'^{eab} + \frac{1}{2} m^c_b \alpha'^{fab}$$
$$[m, a'] = -m^d_a a'_{dbc} - m^e_b a'_{aec} - m^f_c a'_{abf}$$

$$
\begin{aligned}
[a, a'] &= 0 \\
[\alpha, \alpha'] &= 0 \\
[c, m'] &= 0 \\
[c, \alpha'] &= 0 \\
[c, a'] &= 0 \\
[c, c'] &= 0
\end{aligned}
$$

We give an example for closure of the algebra. Consider the following in the adjoint representation: $[\delta M, \delta M']\, V = \delta M'' \Longrightarrow [M, M'] = M''$. We choose $M = a$ and $M' = \alpha'$. The commutator $[a, \alpha']\,(v, \lambda) = \left(-\frac{1}{2}\alpha^{dab}a_{cab}v^c, \frac{1}{2}\alpha^{abc}a_{ade}\lambda_{bc}\right)$ acts on both components of the generalized vector $V = (v, \lambda)$; so we expect to write $[a, \alpha']$ in terms of m and c, since those transformations also act on both components of the generalized vector. That is, $[a, \alpha'] = m'' + c''$ for some m'' and c''. In particular, we may write component-wise,

(1) $[a, \alpha']\, v^d = -\frac{1}{2}\alpha^{dab}a_{cab}v^c = {m''_c}^{d} v^c + c'' v^c$
(2) $[a, \alpha']\, \lambda_{de} = \frac{1}{2}\alpha^{abc}a_{ade}\lambda_{bc} = -{m''_d}^{c}\lambda_{ce} - {m''_e}^{c}\lambda_{dc} + c\lambda_{de}$

From (1), we write $-\frac{1}{2}\alpha^{dab}a_{cab}v^c = ({m''_c}^{d} + c''\delta^d_c)v^c$. We define $-\frac{1}{2}\beta^d_c \equiv m^d_c + c\delta^d_c$, where $\beta^d_c \equiv \alpha^{dab}a_{cab}$. Now consider $a_{[ade}\lambda_{bc]} = 0$. This is zero since there are five indices antisymmetrized over four dimensions. We introduce some "bar" notation to indicate which indices are antisymmetrized together.

$$
\begin{gathered}
a_{[ade}\lambda_{bc]} = \frac{1}{10}\left(a_{\underline{ade}}\lambda_{\overline{bc}} + 6a_{\underline{ad}\bar{b}}\lambda_{\bar{c}\underline{e}} + 3a_{\underline{a}\overline{bc}}\lambda_{\underline{de}}\right) = 0 \\
\Rightarrow a_{\underline{ade}}\lambda_{\overline{bc}} = -6a_{\underline{ad}\bar{b}}\lambda_{\bar{c}\underline{e}} - 3a_{\underline{a}\overline{bc}}\lambda_{\underline{de}}
\end{gathered}
$$

Substituting into (2),

$$
\begin{aligned}
&\frac{1}{2}\alpha^{abc}\left[-6a_{\underline{ad}\bar{b}}\lambda_{\bar{c}\underline{e}} - 3a_{\underline{a}\overline{bc}}\lambda_{\underline{de}}\right] \\
&= \frac{1}{2}\alpha^{abc}\left[-2\left(a_{ad\bar{b}}\lambda_{\bar{c}e} + a_{de\bar{b}}\lambda_{\bar{c}a} + a_{ea\bar{b}}\lambda_{\bar{c}d}\right) - \left(a_{a\overline{bc}}\lambda_{de} + a_{d\overline{bc}}\lambda_{ea} + a_{e\overline{bc}}\lambda_{ad}\right)\right] \\
&= \frac{1}{2}\alpha^{abc}\left[-a_{adb}\lambda_{ce} - a_{adc}\lambda_{be} - 2a_{de\bar{b}}\lambda_{\bar{c}a} - a_{eab}\lambda_{cd} - a_{eac}\lambda_{bd} - \frac{1}{2}a_{abc}\lambda_{de} + \cdots\right]
\end{aligned}
$$

Comparing terms with (2), we find that

$$
\begin{aligned}
&-\frac{1}{2}\alpha^{abc}a_{adb}\lambda_{ce} - \frac{1}{2}\alpha^{abc}a_{eab}\lambda_{cd} - \frac{1}{4}\alpha^{abc}a_{abc}\lambda_{de} \\
&= \frac{1}{2}\alpha^{cab}a_{dab}\lambda_{ce} + \frac{1}{2}\alpha^{cab}a_{eab}\lambda_{dc} - \frac{1}{4}\alpha^{abc}a_{abc}\lambda_{de} \\
&= \frac{1}{2}\beta^c_d\lambda_{ce} + \frac{1}{2}\beta^c_e\lambda_{dc} - \frac{1}{4}\gamma\lambda_{de}
\end{aligned}
$$

can be expressed in the form we desire from (2) based on our definitions from (1). Note that we define $\gamma \equiv \alpha^{abc}a_{abc}$ in the last line. We have shown that the commutator $[a, \alpha']$ can be written in terms of other transformations in the algebra (namely m and c transformations). We have thus shown an example of closure in the algebra. Similarly, it can be shown that all the commutators above close in the algebra.

We may form a subalgebra of $\mathfrak{gl}(5, \mathbb{R})$ by imposing the following conditions:

(1) $m_{ab} = -m_{ba}$
(2) $a_{abc} = \alpha_{abc}$
(3) $c'' = 0$

Recall that (1) defines the usual $SO(3,1) \subseteq GL(4,\mathbb{R})$ group action and is closed under antisymmetry. Similarly, one can show that condition (2) also closes by antisymmetry in the algebra. This subalgebra forms an $SO(3,2)$ group action on our geometry.

Differential Structure

We now wish to define an analog of the Lie derivative in differential geometry for our extension of generalized geometry. Recall that our bundle is given by $E^2 \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M}$. The generalized vector $V = (v_i, \lambda_i)$ is patched according to

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + \lambda_{(j)} + i_{v_{(j)}} d\lambda_{(ij)}$$

There is also a $GL(5,\mathbb{R})$ group action on the pair (v_i, λ_i). Let $V = v + \lambda \in \Gamma\left(E^2\right)$ and $W = w + \mu \in \Gamma\left(E^2\right)$ be sections on the bundle. We define the Courant bracket by

$$[V,W]_C = [v,w] + \left(\mathcal{L}_v\mu - \mathcal{L}_w\lambda - \frac{1}{2}d\left(i_v\mu - i_w\lambda\right)\right)$$

The Courant bracket satisfies two key properties. First, it is easy to see that it satisfies diffeomorphism invariance. Second, the bracket is also invariant under the following set of automorphisms:

$$\lambda \longrightarrow \lambda + i_v d\omega$$

$$\mu \longrightarrow \omega + i_w \omega$$

for some 3-form ω. This shows that the bracket is defined consistently across patches and is compatible with the way we patched above. We define the generalized Lie derivative, known as the Dorfman derivative, as:

$$\mathbb{L}_V W = \mathcal{L}_v w + \mathcal{L}_v \mu - i_w d\lambda$$

The Dorfman derivative is also compatible with the patching described above. The following property is satisfied by the Courant bracket and Dorfman derivative (the proof is shown in Part 1):

$$[V,W]_C = \frac{1}{2}\left(\mathbb{L}_V W - \mathbb{L}_W V\right)$$

Metric Structure

The next piece of structure we wish to introduce on our bundle is a metric; that is, we would like to define an inner product $G(V,V)$ taking sections on the bundle, $V \in \Gamma(E^2)$, and giving a real number that represents distance. One obvious way to introduce such an object is to do so using the ordinary metric on spacetime:

$$G_0(V,V) = a g_{mn} v^m v^n - b g^{mn} g^{pq} \lambda_{mp} \lambda_{nq}$$

for some constants a and b. A natural question that arises is: what group of $GL(5,\mathbb{R})$ transformations leave G_0 invariant? It turns out to be the $SO(3,2)$ subgroup that we defined above. To motivate why this is so, consider $\delta G_0(V,V) = 2G_0(V,\delta V) = 0 \;\forall\; V$. Let us consider an m-type transformation as defined above. Then $\delta v^a = m^a_b v^b$. We consider only the first term of the variation of G_0:

$$\delta G_0(V,V) = 2a g_{ab} v^a (m^b_c v^c) + \cdots = 0$$

$$= 2a v^a v^c (g_{ab} m^b_c) + \cdots = 0$$

The last term is symmetric in a and c. If this is to equal zero, we require that the symmetric part be zero; that is,

$$g_{ab}m_c^b + g_{cb}m_a^b = 0$$
$$\Rightarrow m_{ab} + m_{ba} = 0$$
$$\Rightarrow m_{ab} = -m_{ba}$$

This is precisely condition (1) in our definition of the $SO(3,2)$ subalgebra defined above. Similarly, it can be shown that the other conditions are satistfied, and thus $SO(3,2)$ transformations leave G_0 invariant. This provides a nice motivation for defining the $SO(3,2)$ group action on our geometry above.

Unfortunately, the metric G_0 is fraught with several problems. We thus seek to define another more suitable metric on our geometry.

Up to this point, we have considered a ten-dimensional representation of $GL(5,\mathbb{R})$ on the generalized bundle $E^2 \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M}$ with generalized vector $V = v + \lambda$. We now construct a five-dimensional representation of $GL(5,\mathbb{R})$ by introducing the following bundle:

$$U \simeq \sqrt{det(T\mathcal{M})} \otimes \left(T\mathcal{M} \oplus \wedge^4 T\mathcal{M}\right)$$

The infinitesimal transformation properties of vectors and one-forms in the representation is given by:

$$\delta v^\mu = m^\mu_\nu v^\nu$$
$$\delta\alpha_\mu = -m^\nu_\mu \alpha_\nu$$

The large transformations corresponding to these are given by:

$$v^\mu \longrightarrow v^{\mu'} = a^\mu_\nu v^\nu$$
$$\alpha_\mu \longrightarrow \alpha'_\mu = \left(a^{-1}\right)^\mu_\nu \alpha_\nu$$

where $a^\mu_\nu = (\mathbf{1} + m^\mu_\nu + \ldots)$. An object $w \in (det\,(T\mathcal{M}))^p \otimes T\mathcal{M}$ has the following transformation property:

$$w^\mu \longrightarrow w^{\mu'} = (det(a))^p a^\mu_\nu w^\nu$$

where the matrix a^μ_ν is the usual transformation matrix between vectors. Similarly an object $w \in (det\,(T^*\mathcal{M}))^p \otimes T\mathcal{M}$ transforms as:

$$w^\mu \longrightarrow w^{\mu'} = \frac{1}{(det(a))^p} a^\mu_\nu w^\nu$$

Consider $H^A \in U$, with $A = 0,\ldots,4$; that is, $H^A = (h^\mu, \varepsilon)$, where $H^\mu = h^\mu$ and $H^4 = \varepsilon^{0123}$. The transformation properties of the components of $H^A \in U$ are given by:

$$h^\mu \longrightarrow h^{\mu'} = \frac{1}{\sqrt{det(a)}} a^\mu_\nu h^\nu$$
$$\varepsilon^{\mu\nu\rho\sigma} \longrightarrow \varepsilon^{\mu\nu\rho\sigma'} = \frac{1}{\sqrt{det(a)}} a^\mu_{\mu'} \cdots a^\sigma_{\sigma'} \varepsilon^{\mu'\nu'\rho'\sigma'}$$

Given this five-dimensional representation, we may think of the objects $V^{[AB]} \in E^2$ with ten components as "bi-vectors" of $GL(5,\mathbb{R})$. More specifically, we can say $E^2 \simeq \wedge^2 U$. To see this, we note that there exists an isomorphism $\wedge^p\,(T^*\mathcal{M}) \simeq (det(T^*\mathcal{M})) \otimes \wedge^{d-p}\,(T\mathcal{M})$. Let $\lambda_{\mu_1\cdots\mu_p} \in \wedge^p\,(T^*\mathcal{M})$ be a p-form and $\widetilde{\lambda}^{[\mu_1\cdots\mu_{d-p}]} \in$

$det(T^*\mathcal{M})\otimes\wedge^{d-p}(T\mathcal{M})$ be a $(d-p)$-vector density. Then the isomorphism is given by:

$$\lambda_{\mu_1\cdots\mu_p} = \frac{1}{(d-p)!}\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{d-p}}\widetilde{\lambda}^{\nu_1\cdots\nu_{d-p}}$$

Similarly, there is an isomorphism $(det(T\mathcal{M}))^{-1}\otimes\wedge^p T\mathcal{M}\simeq\wedge^{4-p}(T^*\mathcal{M})$. Making use of this, we observe:

$$\wedge^2 U \simeq (det(T\mathcal{M}))^{-1}\otimes\left(\wedge^2 T\mathcal{M}\oplus\left(T\mathcal{M}\otimes\wedge^4 T\mathcal{M}\right)\right)$$
$$\simeq\left[\wedge^2 T\mathcal{M}\otimes(det\,(T\mathcal{M}))^{-1}\right]\oplus\left[T\mathcal{M}\otimes\wedge^4 T\mathcal{M}\otimes(det\,(T\mathcal{M}))^{-1}\right]$$
$$\wedge^2 T^*\mathcal{M}\oplus T\mathcal{M}\simeq E^2$$

Thus the representation we have constructed is precisely equivalent to our original representation.

Let $H^A=(h^\mu,\varepsilon)\in U$. The infinitesimal transformations for the algebra are given by:

$$\delta_m h^\mu = m^\mu_\nu h^\nu - \frac{1}{2}m^\nu_\nu h^\mu$$
$$\delta_m\varepsilon^{\mu\nu\rho\sigma} = -m^\mu_{\mu'}\varepsilon^{\mu'\nu\rho\sigma}-\cdots-m^\sigma_{\sigma'}\varepsilon^{\mu\nu\rho\sigma'}-\frac{1}{2}m^\lambda_\lambda\varepsilon^{\mu\nu\rho\sigma}$$
$$\delta_a h^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}a_{\nu\rho\sigma}$$
$$\delta_a\varepsilon = 0$$
$$\delta_\alpha h = 0$$
$$\delta_\alpha\varepsilon^{\mu\nu\rho\sigma} = 4h^{[\mu}\alpha^{\nu\rho\sigma]}$$
$$\delta_c h^\mu = \frac{1}{2}ch^\mu$$
$$\delta_c\varepsilon^{\mu\nu\rho\sigma} = \frac{1}{2}c\varepsilon^{\mu\nu\rho\sigma}$$

Having constructed an appropriate representation, we are now ready to define a generalized metric. We require that the generalized metric be a function assigning a real value to the inner product of two generalized vectors; that is, $G: U\otimes_S U\longrightarrow\mathbb{R}$, where $\otimes_S$ is the symmetric tensor product and $G(H,H)=G_{AB}H^AH^B$. Let $g_{\mu\nu}$ be the conventional Lorentzian metric. Then we define the components of the generalized metric G_{AB} to be:

$$\frac{g_{\mu\nu}h^\mu h^\nu+\frac{1}{24}\lambda^2\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}+\frac{1}{3}A_{\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}h_\mu}{\sqrt{-g}}$$

One can immediately observe that the last term captures the kind of structure we are interested in and studied in Part 2.

It is interesting to consider the special case where $A_{\nu\rho\sigma}=0$. Then $G(H,H)=\frac{1}{\sqrt{-g}}\left[g_{\mu\nu}h^\mu h^\nu+\frac{1}{24}\lambda^2\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}\right]$, and we may write the components of the metric as:

$$G_{AB}=\begin{pmatrix} g_{\mu\nu} & 0\\ 0 & \frac{1}{24}\lambda^2 det(g)\end{pmatrix}$$

A natural question we may ask is, what is the signature of the metric in this case? The signature of the first term is given by the standard signature of the Lorentizan metric $g_{\mu\nu}$, $(3,1)$. The signature of the second term is $(0,1)$. This arises from the fact that $\varepsilon_{\mu\nu\rho\sigma}=-(-det(g))\varepsilon^{\mu\nu\rho\sigma}$. Thus, the total signature of G_{AB} is given by $(3,1)+(0,1)=(3,2)$.

Part 4. Closing Remarks

In this paper, we have examined generalized geometry and begun to apply it to a four dimensional model of supergravity. In particular, in the context of three-form supergravity, we have established the basic differential geometry and metric structure required to better understand the theory. In this short section, we outline some of the steps required to complete the task of completely formulating three-form supergravity in terms of generalized geometry. We also provide some suggestions for pursuing further research in this area.

Up to this point, we have defined a generalized metric on our bundle. The next piece of structure in the mathematical construction of the generalized bundle $E^2 \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M}$ that we wish to define is some kind of notion of a derivative on the bundle. Recall that on a manifold, in order to take the directional derivative of a tensor, we introduce an affine connection map $\nabla : \mathcal{J}_0^1(\mathcal{M}) \times \mathcal{J}_0^1(\mathcal{M}) \to \mathcal{J}_0^1(\mathcal{M})$ and specify the map by its action on basis vectors. That is, taking a chart $(U_i,\, \psi_i)$ with coordinate basis $\{e_\mu\} = \left\{\frac{\partial}{\partial x^\mu}\right\}$, we define

$$\nabla_\mu e_\nu \equiv \nabla_\mu v = \Gamma^\alpha_{\mu\nu} e_\alpha$$

where $\left\{\Gamma^\alpha_{\mu\nu}\right\}$ are the connection components. Using the components with the properties of the connection we can then calculate the derivative of a tangent vector:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\lambda} v^\lambda$$

Similarly, on our generalized bundle, we wish to define a derivative D_A with connection coefficients given by $\Omega_{A,}{}^B_C$ such that its action on the generalized vector V^B with $B = 1, \ldots, 10$ is given by:

$$D_A V^B = \partial_A V^B + \Omega_{A,}{}^B_C\, V^C$$

We need to specify what the partial derivative action ∂_A means in this context. We define $\partial_A V^B = \partial_\mu v^\mu$ for $B = \mu$ and $\partial_A V^B = 0$ for $B = [\mu\nu]$. Finally, we restrict and choose the connection components to be compatible with the ten-dimensional $GL(5,\mathbb{R})$ algebra transformations we defined previously. That is, in our ten-dimensional representation, we defined several kinds of transformations. Roughly, these were given by four different objects: m^μ_ν, $a_{\mu\nu\lambda}$, $\alpha^{\mu\nu\lambda}$ and c. We may think of four corresponding actions that the connection will have:

(1) $\Omega_A \longrightarrow (\Omega_A)^\mu_\nu$
(2) $\Omega_A \longrightarrow (\Omega_A)_{\mu\nu\lambda}$
(3) $\Omega_A \longrightarrow (\Omega_A)^{\mu\nu\lambda}$
(4) $\Omega_A \longrightarrow (\Omega_A)^{(c)}$

Thus, the action of the connection $\Omega_{A,}{}^B_C$ on on the generalized vector V^C is given by:

$$\Omega_{A,}{}^B_C\, V^C = \begin{pmatrix} (\Omega_A)^\mu_\nu\, V^\nu + \frac{1}{2}\,(\Omega_A)^{\mu\nu\rho}\,\lambda_{\nu\rho} + \Omega_A^{(c)} v^\mu \\ -\,(\Omega_A)^\sigma_\mu\,\lambda_{\sigma\nu} - (\Omega_A)^\sigma_\nu\,\lambda_{\mu\sigma} + (\Omega_A)_{\mu\nu\lambda}\, v^\lambda + \Omega_A^{(c)}\lambda_{\mu\nu} \end{pmatrix}$$

Now that we have defined a generalized connection, we wish to define the analog of the Levi-Civita connection on the generalized tangent bundle. Recall that the Levi-Civita connection is the unique metric-compatible and torsion-free connection on the manifold. Thus, we want our generalized connection to satisfy metric-compatibility and to be torsion-free. Let us consider these conditions separately. Recall that for a given metric g, a connection ∇ is metric compatible if at any point $p \in \mathcal{M}$ it obeys

$$\nabla_X g = 0 \,\forall\, X \in T_p(\mathcal{M})$$

Thus in the generalized case, we require that for a generalized metric G, the generalized connection satisfy $D_A G_{BC} = 0$. The torsion-free condition is slightly more tricky than metric-compatibility. Recall that we may define the torsion map for two vectors X and Y in the following elegant way: $\mathcal{T}(X,\, Y) \equiv \nabla_X Y - \nabla_Y X - [X,\, Y]$. Equivalently, we may write $\mathcal{T}(X,\, Y) = \left(\mathcal{L}^{\nabla}_X - \mathcal{L}_X\right) Y$. Now generalizing this to our bundle, we define the torsion for two generalized vectors X and Y to be:

$$\mathcal{T}(X,Y) = [X,Y]^D - [X,Y]_C$$

There remain a couple of subtleties here. Firstly, we defined the Courant bracket in coordinate-free notation. Thus, we need to write the bracket (and subsequently the torsion) in a $GL(5,\mathbb{R})$ covariant way to proceed further. Secondly, there is a fundamental ambiguity in the way in which we defined the "generalized partial derivative." That is, for the vector components of the generalized vector V^A, we defined $\partial_\mu \longrightarrow D_A = D_\mu$; however, for the remaining components we set $D_A = 0$. In principle, though, we are defining $\partial_\mu \longrightarrow D_A = D^{\mu\nu}$ for these remaining indices, and it is unclear what the object $D^{\mu\nu}$ precisely is. This is an area that warrants further investigation.

Assuming one is able to define the torsion in a $GL(5,\mathbb{R})$ invariant way (this in fact can be done!), then one can determine a generalized Levi-Civita connection on the geometry. In fact, there always exists a torsion-free, metric-compatible generalized connection D; however, it is not unique [37]. Despite the fact that such a connection is not unique, one can still construct unique measures of curvature. More precisely, one can construct generalized analogs of the Ricci tensor and scalar, denoted $R_{(AB)}$ and S respectively. From this one may define a scalar action in terms of the generalized Ricci scalar. This action turns out to be virtually identical to the action we defined for three-form supergravity in Part 2. Varying this action will yield an equation of motion in terms of the generalized Ricci tensor $R_{(AB)}$; and this equation unifies both Einstein's equation with $\nabla_{\mu_1} F^{\mu_1\mu_2\mu_3\mu_4} = 0$ (which we discussed in Part 2) into a single object. This then completes the task of rewriting three-form supergravity in terms of generalized geometry.

We have already commented that several loose ends need to be tied in terms of the torsion-free condition imposed on the Levi-Civita connection. There also remains one more area that merits further investigation. Recall that when we defined the generalized metric, we included an extra constant λ. This was an extra degree of freedom used to match the indices in the generalized metric, and we did not precisely define its role in the metric. It turns out that its role is quite ambiguous at best; however, since it is simply a scaling factor, we may "get rid" of it in a sense by defining equivalence classes of the metric in terms of λ. That is, we may define $G_{AB} \sim \lambda G_{AB}$ and so on. What we are doing here is parametrizing the coset $\frac{GL(5,\mathbb{R})}{SO(3,2)\times\mathbb{R}^+}$. Again, the exact role of the constant λ should be defined more precisely, and this is another problem worth studying more deeply.

A different approach one may take in formulating supergravity in terms of generalized geometry is as follows. Rather than proceeding as we have in this paper, one may introduce frames on the generalized bundle and then adopt the Cartan formalism for expressing the geometrical structure required. This is the route we take in the appendix on gauge theories to describe gravity. When applying this to supergravity, this in fact turns out to be slightly cleaner than what we have described above.

Another area that we have chosen not to consider is the generalization of the spin bundle. For a complete theory of supergravity, spinor and fermionic fields need to be accounted for in the mathematical framework, and the appropriate mathematical structures are found in Gualtieri's paper [34]. Additionally, one may consider some more exotic structures of generalized geometry such as the twisted Courant bracket [34] (which has the same set of symmetries as the bracket we introduced in Part 1), or generalized tensors and consider how one may begin to formulate supergravity using these objects.

Perhaps the most remarkable aspect of supergravity and generalized geometry is its connection to the symmetries of string theory. One example of this is the connection that the $O(d, d)$ group action has with T-duality in the description of non-geometric backgrounds [43]. Another example is given by extensions of generalized geometry which we have remarked about earlier. In particular, the $SO(d, d)$ group is a subset of the much larger U-duality group in string theory. This suggests extending generalized geometry to the U-duality in group, as has been done by Hull [39] and Waldram [36]. Hull writes, "M-theory has a similar structure in which there is a metric G and three-form C on an n-dimensional manifold $\mathcal{M}$ with a natural action of E_n, and again one might expect a generalization of generalized geometry with a three-form C playing a central role" [39]. The generalized geometry framework has already proved to be extremely effetive in describing low-energy reductions of the full eleven-dimensional supergravity theory. Thus, extending generalized geometry to express eleven dimensional supergravity theories and M-theory in a suitable mathematical framework may help to shed new light on string theory and may prove to be a key development in supergravity theory. Ultimately, the goal of studying supergravity is to better understand string theory. While the study of string theory merits no justification in itself (the subject is beautiful for its own sake), it currently is our best guess at a quantum theory of gravity. It goes without saying that understanding the intricate workings of string theory and its applicability to nature should be a top priority of physicists, as this could potentially lead to a working theory of quantum gravity. Undoubtedly, work in the area of generalized geometry applied to strings and supergravity should long continue and may turn out to play a vital role in the development of our understanding of string theory and physics at its most fundamental level.

Part 5. Appendix A: Manifolds

One cannot begin to overestimate the role of differentiable manifolds in mathematics and physics. The theory of analysis in $\mathbb{R}^n$ is vital to obtaining results of interest and we may wish to perform operations such as differentation or integration to spaces that are more topologically complicated than $\mathbb{R}^n$. Thus we introduce the notion of a manifold - a structure that corresponds to a complicated topological space but that locally resembles $\mathbb{R}^n$. Heuristically, we wish to construct the manifold by "sewing together" these local subsets of $\mathbb{R}^n$ in a smooth way, thereby endowing the manifold with a dimensionality that must be equal to the dimension of the smooth local regions (in this case n). Then we can very naturally apply our usual notions of calculus to the manifold by considering their actions on Euclidean space. Before providing a rigorous definition of a manifold, we present some preliminary topological definitions.

Topology

Definition 1. A topological space is a pair $(X, \mathscr{T})$ where X is a set and $\mathscr{T} = \{U_\alpha \subseteq X \mid \alpha \in A\}$ is a set of open subsets of X satisfying:

(1) $X \in \mathscr{T}, \emptyset \in \mathscr{T}$
(2) $\bigcup_\alpha U_\alpha \in \mathscr{T}$ for any subset $A' \subseteq A$,and $\alpha \in A'$
(3) $\bigcap_\alpha U_\alpha \in \mathscr{T}$ for any finite subset $A' \subseteq A$ and $\alpha \in A'$. $\square$

Definition 2. A map $f : X \to Y$ between topological spaces is continuous if for every open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open. $\square$

Definition 3. An invertible, continuous map $f : X \to Y$ is a homeomorphism if its inverse $f^{-1} : Y \to X$ is also continuous. $\square$

Intuitively, we may think of a homeomorphism as a map that continuously deforms one space into another.

Definition 4. X is Hausdorff if given any distinct pair x_1, $x_2 \in X$, there exist open sets U_1, U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \bigcap U_2 = \emptyset$. $\square$

Definition 5. A topological space is said to be path connected if for every two points, p, $p' \in X$, $\exists$ a continuous map $f : [0,1] \to X$ such that $f(0) = p$ and $f(1) = p'$. Every path connected space is connected. $\square$

Definition 6. A topological space X is compact of for every open cover of X there exists a finite subset of the open cover that also covers X. $\square$

The notions of connectedness and compactness are very important in mathematics. In the study of calculus, there are three basic theorems regarding continuous functions on which the rest of calculus relies: the *Intermediate Value Theorem,* the *Maximum Value Theorem* and the *Uniform Continuity Theorem.* The first relies heavily on the property of connectedness while the two others depend on the property of compactness. A key result in analysis, and arguably the most important theorem concerning compact subsests of $\mathbb{R}^n$, is the Heine-Borel Theorem:

Theorem. *(Heine-Borel). If a topological space, X, is a subset of* $\mathbb{R}^n$ *such that it is a closed subset and is bounded in extent in* $\mathbb{R}^n$*, then X is compact.* $\square$

We introduce one final topological property - paracompactness - which manifolds must satisfy in order to restrain them from becoming arbitrarily large. An equivalent way to say a space X is compact is the following: a space X is compact if every open covering $\mathcal{A}$ of X has a finite refinement $\mathcal{B}$ that covers X. Given such a refinement $\mathcal{B}$, one can choose for each element of $\mathcal{B}$ an element of $\mathcal{A}$ containing it. In this way, a finite subcollection of $\mathcal{A}$ covering X is obtained.

Definition 7. A space X is paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X. $\square$

An immediate consequence of the definition is that every compact space is paracompact. It can be shown that any Hausdorff topological space which is locally compact and which can be expressed as a coutable union of compact subsets is paracompact. Thus $\mathbb{R}^m$ and S^n (and by the Tychonoff theorem, their products as well) can be verified to be paracompact. For a manifold, paracompactness has some vital consequences. Firstly, it implies that the manifold admits a Riemannian metric. Secondly, it implies that the manifold is second countable (this allows us to cover $\mathcal{M}$ by locally finite charts (ψ_i, U_i) with each U_i compact). Thirdly, with paracompactness, we can show the existence of a partition of unity (this is important for defining integration of differential forms on the manifold).

Homology

We now consider a basic building block for the topological spaces that underly (most) manifolds.

Definition 8. The r-simplex. A set of $(r+1)$ points $\{p_i \mid i = 0, \ldots, r\} \in \mathbb{R}^m$ (take $m \geq r$) defines an r-simplex, which we denote $\langle p_0 p_1 \cdots p_r \rangle$. This is a canonical subset or building block of $\mathbb{R}^m$. The r-simplex $\sigma_r = \langle p_0 p_i \cdots p_r \rangle$ is defined as: $\sigma_r = \{ \vec{x} \in \mathbb{R}^m \mid \vec{x} = \sum_{i=0}^{r} \vec{p}_i c_i,\ c_i \geq 0,\ \sum_{i=0}^{r} c_i = 1\}$. $\square$

Provided the points define a hyperplane in $\mathbb{R}^m$, the vectors defining the points are linearly independent. Such simplices are called nondegenerate. σ_r is a closed subset of $\mathbb{R}^m$ and the interior is homeomorphic to an open set in $\mathbb{R}^r$ with the homeomorphism given by:

$$\sigma_r \subset \mathbb{R}^m \to \mathbb{R}^r$$

$$\vec{x} = \sum_{i=0}^{r} \vec{p}_i c_i \to \{c_i \mid i = 1, \ldots, r\}$$

Taking a $(q+1)$ subset of the $(r+1)$ points, $\{p_{i_0} p_{i_1} \ldots p_{i_q}\}$ with $0 \leq q \lneq r$, so that p_{i_a} are distinct, defines a face of σ_r. This is the q-simplex $\langle p_{i_0} p_{i_1} \ldots p_{i_q} \rangle$. The dimension of an r-simplex is r.

We may give an r-simplex an orientation in the following way. Let $P : \{0, 1, \ldots, r\} \to \{i_0, i_1, \ldots, i_r\}$ be a permutation of points. Now define an oriented r-simplex to be $(p_{i_0} p_{i_1} \cdots p_{i_r}) = sign(P)\,(p_0 p_1 \cdots p_r)$. The even permutations P define an orientation and the odd permutations P define the opposite orientation. Taking a $(q+1)$ subset of the $(r+1)$ points as before allows us to define an oriented face of σ_r given by $sign(P)\left(p_{i_1} p_{i_1} \cdots p_{i_q}\right)$ where P is the permutation,

$P : \{0, 1, \ldots, r\} \to \left\{i_0, i_1, \ldots, 0, 1, \ldots, \hat{i_0}, \ldots, \hat{i_1}, \ldots, \hat{i_q}, \ldots, r\right\}$ and $\hat{i_0}$ indicates that element $\hat{i_0}$ is removed from the sequence.

Definition 9. Simplicial Complex. Let K be a set of simplices in $\mathbb{R}^m$. K is a simpicial complex if:

(1) Any face of any simplex in K also belongs to K
(2) For any two simplices σ_1, $\sigma_2 \in K$, $\sigma_1 \bigcap \sigma_2$ is a face of both σ_1 and σ_2. □

Thus a simplicial complex is simply a set of simplices that fit together "properly." The dimension of K is defined as the maximum dimension of the simplices in K.

Definition 10. The subset of dimension dim$K$ given by the union of all simplices in K, denoted $|K|$, is called the polyhedron of K. □

Definition 11. A topological space X is said to be triangulated by a complex K if there exists a homeomorphism $f : |K| \rightarrow X$. K is a triangulation of X. □

It is now straightfoward to generalize the above to triangulate manifolds. Let K be a triangulation of a connected topological space X. $|K|$ is constructed from sets whose interiors are homeomorphic to $\mathbb{R}^{dimK}$. If all simplices $\sigma_q \in K$ with $q \lneq r$ are faces of other simplices in K, then the space X has the topological structure of a manifold of dimension $m = dimK$. We can use such complexes to explicitly construct the topological spaces that underlie many manifolds.

Consider a manifold of dimension m. Let the complex K triangulate the manifold and suppose that we may assign orientations to all the m-simplices in K. We say that the manifold is orientable if any two m-simplices σ_q, $\sigma'_q \in K$ that intersect non-trivially intersect on a face $\sigma_q \bigcap \sigma'_q = \sigma_{q-1}$, and the orientation induced on σ_{q-1} as a face of σ_q is opposite to that induced on σ_{q-1} as a face of σ'_q. Note that the orientation on the manifold is determined by choice of orientation of the m-simplices in K. The same manifold with opposite orientation is given by reversing the orientations of all the m-simplices in K.

We now take a short digression to discuss the concept of topological invariants. The main purpose of topology is to classify spaces - that is, to say which spaces are equal or different. It is clear that we require some definition of equivalence. In fact, we have already defined this above in *Definition 3.* In topology, we may define two spaces to be equivalent if we can continuously deform one space into the other without tearing them apart or pasting; i.e. two spaces are equivalent if we can find a homeomorphism between them. We say that a topological invariant is a quantity which is conserved under homeomorphism. A topological invariant may be a number (the number of connected components of the space), an algebraic structure (a ring or a group) or a property of the space (connectedness, compactness, Hausdorff property). Now a natural question that follows is how to characterize the equivalence classes of a homeomorphism. One way to address this is using topological invariants. If we know the complete set of topological invariants, we could specify the equivalence class. However, even if we were given all the topological invariants on the spaces , this does not necessarily imply that the spaces are homeomorphic to one another. But, we can say that if two spaces have different topological invariants, then they cannot be homeomorphic to each other. Thus, we can think of topological invariants as characterizing topological spaces.

One of the most useful topological invariants is the Euler characteristic of a surface, which is defined to be: $\chi(X) = v - e + f$, where $X \subseteq \mathbb{R}^3$ is homeomorphic to a polyhedron K; v is the number of vertices in K; e is the number of edges in K; and f is the number of faces in K. We may also construct the Euler characterstic with the machinery developed above. Suppose we have a complex K with dimension

m which triangulates a topological space X. We may label all the r-simplices in this complex as $K_r = \{\sigma_{r,\,i} \in K \mid i = 1,\, 2, \ldots,\, k_r\}$ for $r \leq m$. Then the Euler number $\chi(K)$ is defined as

$$\chi(K) \equiv \sum_{r=0}^{m} (-1)^r\, k_r$$

Assume we have a complex K with dimension m. We choose all simplices in K to be oriented. Let the set of all oriented r-simplices in K be $K_r = \{\sigma_{r,\,i} \in K \mid i = 1,\, 2, \ldots,\, k_r\}$.

Definition 12. An r-chain is the linear combination given by $c = \sum_{i=1}^{k_r} c_i \sigma_{r,\,i}$ for some $c_i \in \mathbb{R}$. □

The space of all r-chains is denoted $C_r(K)$ and is isomorphic to $\mathbb{R}^{k_r}$. We can linearly combine chains:

$$c'' = \alpha c + \beta c' = \sum_i (\alpha c_i + \beta c'_i)\sigma_{r,\,i} \in C_r(K),\ with\ \alpha,\, \beta \in \mathbb{R}$$

Define the vector spaces C_r for $r \lneq 0$ and $r \gneq m$ to be trivial.

Definition 13. Define the boundary operator of an r-simplex to be the map $\partial : K_r \to C_{r-1}(K)$ such that $\partial\sigma_r = \sum_{i=0}^{r} (-1)^i\,(p_0 p_1 \cdots \hat{p}_i \cdots p_r) \in C_{r-1}(K)$. The boundary of a zero simplex is the zero chain $0 = \partial_0\sigma_0$. □

We extend the boundary operator to act linearly on chains:

$$\partial_r\, :\, C_r(K) \to C_{r-1}(K)$$

$$c = \sum c_i \sigma_{r,\,i} \to \partial_r c \equiv \sum c_i (\partial\sigma_{r,\,i})$$

By definition, a boundary has no boundary. Thus an imperative property of ∂ is that $\partial \circ \partial = 0$.

Proof.

$$\begin{aligned}
&\partial^2\,(p_0 \cdots p_r)\\
&= \partial\left(\sum_{i=0}^{r}(-1)^i\,(p_0 p_1 \cdots \hat{p}_i \cdots p_r)\right)\\
&= \sum_{i=0}^{r}(-1)^i\,(\partial(p_0 p_1 \cdots \hat{p}_i \cdots p_r))
\end{aligned}$$

$$= \sum_{i=0}^{r}(-1)^i\left(\sum_{j=0}^{i-1}(-1)^j\,(p_0 p_1 \cdots \hat{p}_j \cdots \hat{p}_i \cdots p_r) + \sum_{j=i+1}^{r}(-1)^{j-1}\,(p_0 p_1 \cdots \hat{p}_i \cdots \hat{p}_j \cdots p_r)\right)$$

$$= \sum_{i \gneq j}(-1)^{i+j}\,(p_0 p_1 \cdots \hat{p}_j \cdots \hat{p}_i \cdots p_r) - \sum_{i \lneq j}(-1)^{i+j}\,(p_0 p_1 \cdots \hat{p}_i \cdots \hat{p}_j \cdots p_r) = 0$$

□

We are now ready to discuss the homology vector space. Recall that the Euler characteristic is a topological invariant and can be computed by the polyhedronization of space. Loosely speaking, homology represents refinements to the Euler characterstic. How do we go about defining homology? Consider two triangles, one solid (shaded) and the other simply the edges of a triangle without interior. Observe that the three edges of the former form a boundary of the solid interior while the edges of the latter do not. Or equivently, the existence of a loop that does not form the boundary of some interior implies that there is a hole in the loop. Thus we will aim to find regions without boundaries which themselves are not a boundary of some region.

Consider a complex K and let $c = C_r(K)$ be an r-chain. Define two new subclasses of r-chains:

Definition 14. c is an r-cycle if $\partial_r c = 0$. □

Definition 15. c is an r-boundary if $c = \partial_{r+1} a$ for some $a \in C_{r+1}(K)$. □

Denote the space of r-cycles and r-boundaries as $Z_r(K)$ and $B_r(K)$ respectively, with $0 \leq r \leq dim(K)$, and $Z_r(K)$ and $B_r(K)$ trivial for any other values of r. Both spaces inherit their linearity form the linearity of $C_r(K)$. Since an r-boundary has no boundary, it follows that every r-boundary is an r-cycle, i.e. $B_r(K) \subset Z_r(K)$.

Definition 16. Two r-cycles z_1, $z_2 \in Z_r(K)$ are equivalent (homologous) if $z_1 - z_2 \in B_r$. We write the equivalence $z_1 \sim z_2$. Denote the set of equivalent r-cycles by $[z] = \{z' \in Z_r(K) \mid z' \sim z\}$ where z is a representative in the equivalence class.

Definition 17. The homology vector space, denoted $H_r(K)$, is given by the set of equivalence classes $\{[z]\}$. It inherits its linear strucutre from $Z_r(K)$ and ∂, and its dimension is given by $\dim(H_r) = \dim(Z_r)$ - $\dim(B_r)$. □

Thinking of ∂_r as a linear map from $C_r \to C_{r-1}$, we have: $\ldots \overset{\partial_{r+2}}{\to} C_{r+1} \overset{\partial_{r+1}}{\to} C_r \overset{\partial_r}{\to} C_{r-1} \overset{\partial_{r-1}}{\to} \ldots$ It then follows that $B_r = Im(\partial_{r+1})$ and $Z_r = Ker(\partial_r)$, and so $dim(H_r) = dim Ker(\partial_r) - dim Im(\partial_{r+1})$.

A manifold may admit infinitely many triangulations, and so in general the vector spaces C_r, Z_r, and B_r may have different dimension. We take the following theorem without proof:

Theorem. *The dimension of the r-th homology vector space, known as the r-th Betti number, $b_r = dim(H_r)$, is a topological invariant.*

The Euler characteristic $\chi(M)$ can be defined in terms of the Betti numbers by the Euler-Poincaré Theorem:

Theorem. *Let K be an m-dimensional simplicial complex that triangulates a manifold M, and let k_r be the number of r-simplices in K. Then*

$$\chi(M) = \sum_{r=0}^{dim(M)} (-1)^r k_r = \sum_{r=0}^{dim(M)} (-1)^r b_r(M)$$

Proof. Recall that for a linear map $f : A \to B$, $dim A = Ker(f) + Im(f)$. Thus, from above, $dim C_r = Ker(\partial_r) + Im(\partial_r)$. Recall that $k_r = dim C_r$ for some K. Then:

$$\chi(M) = \sum_{r=0}^{dim(M)} (-1)^r k_r = \sum_{r=0}^{dim(M)} (-1)^r (Ker(\partial_r) + Im(\partial_r))$$

$$= \sum_{r=0}^{dim(M)} (-1)^r (dim(Z_r) + dim(B_{r+1})) = \sum_{r=0}^{dim(M)} ((-1)^r (dim(Z_r) - (-1)^{r-1} dim(B_{r-1}))$$

$$= \sum_{r=0}^{dim(M)} ((-1)^r dim(Z_r) - (-1)^r dim(B_r)) = \sum_{r=0}^{dim(M)} (-1)^r b_r$$

□

Differential Structure of Manifolds

We have examined the topological structure underlying manifolds. Now we consider the differential structure. The notion of smoothness is vital to the theory of manifolds. Just as topology is fundamentally based on continuity, the theory of manifolds is based on smoothness. Manifolds are generalizations of curves and surfaces to objects of arbitary dimensions. In general, a manifold is a topological space locally homeomorphic to $\mathbb{R}^m$, but possibly different to $\mathbb{R}^m$ globally. The local homeomorphism allows us to introduce a set of numbers that define local coordinates. Since a manifold may be not globally look like $\mathbb{R}^m$ we may need to introduce several local coordinates. It follows naturally that we require the transition from one coordinate to the other to be sufficiently smooth, so that we can apply usual calculus on the manifold. One last useful motivational remark of interest is the following. We would like a notion of "arbitrariness of the coordinate choice" to underlie our theory of manifods. That is, we would like to say that all coordinate systems are equally good - that a physical system behaves in the same way whatever coordinates we wish to use to describe it. Given the above, we take the following formal definition of manifold:

Definition 18. An m-dimensional real manifold $\mathcal{M}$ satisfies the following:

(1) $\mathcal{M}$ is a Hausdorff topological space with open sets J
(2) $\mathcal{M}$ has an atlas: a family of charts $\{(U_i,\, \psi_i)\}$ where $U_i \in J$ such that $\bigcup_i U_i = \mathcal{M}$ and the maps ψ_i are homeomorphisms from $U_i \to U_i' \subseteq \mathbb{R}^m$ with U_i' open
(3) Given $U_i \bigcap U_j \neq 0$, then the transition function $\phi_{ij} = \psi_i \circ \psi_j^{-1}$ from $\mathbb{R}^m \to \mathbb{R}^m$ is C^∞ smooth. □

Definition 19. Two atlases $\{(U_i,\, \psi_i)\}$ and $\{(V_j,\, \varphi_j)\}$ on $\mathcal{M}$ are said to be equivalent if their union is also an atlas on $\mathcal{M}$. □

We can encompass manifolds with boundary by modifying the definition of a manifold so that the maps ψ_i are homeomorphisms into $H^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geq 0\}$. Denote the set of boundary points as $\partial\mathcal{M}$.

The significance of differentiable manifolds lies in the fact that we can use our familiar techniques of analysis in $\mathbb{R}^m$. The additional property of smoothness maintains the important property that the calculus is independent of the coordinates we decide to choose. We now consider some important definitions that really underscore the "differential" in "differential geometry."

Definition 20. Let $f : \mathcal{M} \to \mathcal{N}$ be a map from an m-dimensional manifold $\mathcal{M}$ with atlas $\{(U_i,\, \psi_i)\}$ to an n-dimensional manifold $\mathcal{N}$ with atlas $\{(V_a,\, \varphi_a)\}$, and let $p \in U_i$ and $q = f(p) \in V_a$. Now the transition map $\varphi_a \circ f \circ \psi_i^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth by definition. We say the map f is differentiable at p if the (n-vector of) functions $\varphi_a \circ f \circ \psi_i^{-1}$ are C^∞ smooth in their arguments. □

Definition 21. A differentiable, invertible map $f : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if the inverse map f^{-1} is also differentiable. □

Denote the set of diffeomorphisms on a manifold $\mathcal{M}$ as *Diff* $(\mathcal{M})$. We may think of a diffeomorphism as a smooth relabeling of the points on the manifold. Diffeomorphisms classify spaces into equivalence classes according to whether it is possible to deform one space into another smoothly. Earlier we saw that homeomorphisms classify spaces according to whether we can deform one space into another continuously. Clearly, a diffeomorphism also defines a homeomorphism at the level of the topological spaces $\mathcal{M}$ and $\mathcal{N}$. An interesting question arises when one considers the converse: are there examples of diffeomorphically inequivalent homeomorphisms? In fact, there are highly non-trivial examples of such structures (for example the discovery that $\mathbb{R}^4$ admits an infinite number of differentiable structures due to Donaldson).

Definition 22. Consider a smooth map $f : \mathcal{M} \to \mathcal{N}$ where $dim\mathcal{M} \leq dim\mathcal{N}$. The map f defines an embedding of the manifold $\mathcal{M}$ into $\mathcal{N}$ if it is one-to-one. The image $f(\mathcal{M})$ is a submanifold of $\mathcal{N}$ that is diffeomorphic to $\mathcal{M}$. □

Definition 23. A function f is a differentiable map such that $f : \mathcal{M} \to \mathbb{R}$. □

We may give a coordinate representation of a function as follows. Take a chart (U_i, ψ_i) with coordinates $\{x^\mu\}$. Then for a point $p \in U_i$,

$$f \circ \psi_i^{-1} : \mathbb{R}^m \to \mathbb{R}$$

$$x = \psi_i(p) \to f \circ \psi_i^{-1}(x)$$

Denote the set of functions on $\mathcal{M}$ as $\mathcal{F}(\mathcal{M})$. It inherits the usual ring structure from $\mathbb{R}$.

Definition 24. An open curve C is a smooth map such that $C : \mathbb{R} \to \mathcal{M}$. □

In a chart (U_i, ψ_i) with coordinates $\{x^\mu\}$ we may write,

$$\psi_i \circ C : \mathbb{R} \to \mathbb{R}^m$$

$$\lambda \to x = \psi_i \circ C(\lambda)$$

One can consider closed curves as maps $S^1 \to \mathcal{M}$.

Now that we have defined maps on a manifold, we are ready to define additional geometric structure on the manifold: vectors, dual vectors and tensors. We must modify our notion of a vector when considering curved geometries or complicated topologies. We would still like to maintain the linear vector space strucure that vectors in Euclidean space possess. We can do this by thinking of a vector on a manifold $\mathcal{M}$ in terms of the tangent vector to a curve in $\mathcal{M}$.

Consider an m-dimensional manifold $\mathcal{M}$, a point $p \in \mathcal{M}$ and two curves C_1 and C_2 passing through p such that $p = C_1(0) = C_2(0)$. We can use the fact that the manifold looks locally like $\mathbb{R}^m$ to determine if the two curves have the same direction and speed at p. Taking a chart (U_i, ψ_i) with coordinates $\{x^\mu\}$, we say that the curves have the same direction at p if and only if the curves are tangent to each other at p:

$$\frac{d}{d\lambda} x^\mu(C_1(\lambda)) \mid_{\lambda=0} = \frac{d}{d\lambda} x^\mu(C_2(\lambda)) \mid_{\lambda=0}$$

Define an equivalence at p, $C_1 \sim C_2$, if and only if C_1 and C_2 are tangent at p. Geometrically, we are defining a tangent vector V_p at the point p to be the equivalence class of all curves passing through p that are tangent to each other.

Definition 25. The tangent space at p is the set of all tangent vectors at p. We denote the tangent space $T_p(\mathcal{M})$. □

Since the manifold is equivalent to $\mathbb{R}^m$ locally, it inherits a linear structure. It follows that the tangent space $T_p(\mathcal{M})$ is isomorphic to the vector space $\mathbb{R}^m$ (See Isham for a proof). Thus, we can write any element of the tangent space as $V_p = v^\mu e_{p,(\mu)}$ where $e_{p,(\mu)} \in T_p(\mathcal{M})$ specifies a set of m basis tangent vectors.

For any vector space V we can construct its dual vector space $V^\star$. We can choose a basis $\{e^i\}$ dual to the basis $\{e_i\}$ (i.e. $e^i(e_j) = \delta^i_j$). Bearing this in mind, choose $T^\star_p(\mathcal{M})$ to be the vector space dual to $T_p(\mathcal{M})$, and call it the space of cotangent vectors. For $\omega_p \in T^\star_p(\mathcal{M})$ we can define the bilinear inner product $\langle \cdot, \cdot \rangle : T^\star_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}$ defined by $\langle \omega_p, v_p \rangle \equiv \omega_p(v_p)$.

Tensors are natural generalizations of the maps we just defined. A (q, r) tensor is a linear function acting on q cotangent vectors and r tangent vectors. Taking a basis $\{e_\mu\}$ and a dual basis $\{e^\mu\}$ at p, we can write cotangent vectors as $\omega_{(i)} = \omega_{(i)\mu} e^\mu \mid_p$ and vectors as $v_{(i)} = v^\mu_{(i)} e_\mu \mid_p$. Then the linear tensor map is simply given by:

$$\begin{aligned} &A_p(\omega_{(1)}, \ldots, \omega_{(q)}, v_{(1)}, \ldots, v_{(r)}) \\ &= \omega_{(1)\alpha_1} \ldots \omega_{(q)\alpha_q} v^{\beta_1}_{(1)} \ldots v^{\beta_r}_{(r)} A_p(e^{\alpha_1}, \ldots, e^{\alpha_q}, e_{\beta_1}, \ldots, e_{\beta_r}) \end{aligned}$$

The map A_p is entirely specified by the collection $\{A_p(e^{\alpha_1}, \ldots, e^{\alpha_q}, e_{\beta_1}, \ldots, e_{\beta_r})\}$ which we call the components of the tensor. The set of (q, r) tensors at the point p is denoted $\mathcal{J}^q_{r,p}(\mathcal{M})$. We can defined several useful operations on tensors:

(1) The tensor product is a map $\otimes : \mathcal{J}^{q_1}_{r_1,p}(\mathcal{M}) \times \mathcal{J}^{q_2}_{r_2,p}(\mathcal{M}) \to \mathcal{J}^{q_1+q_2}_{r_1+r_2,p}(\mathcal{M})$ given by:

$$\begin{aligned} C(\omega_{(1)}, \ldots, \omega_{(q_1+q_2)}, v_{(1)}, \ldots, v_{(r_1+r_2)}) \equiv A(\omega_{(1)}, \ldots, \omega_{(q_1)}, v_{(1)}, \ldots, v_{(r_1)}) \times \\ B(\omega_{(q_1+1)}, \ldots, \omega_{(q_1+q_2)}, v_{(r_1+1)}, \ldots, v_{(r_1+r_2)}) \end{aligned}$$

for any $\omega_{(i)} \in T^\star_p(\mathcal{M})$ and $v_{(i)} \in T_p(\mathcal{M})$

(2) Given $A \in \mathcal{J}^{q+1}_{r+1,p}(\mathcal{M})$, we may form a new tensor $B \in \mathcal{J}^q_{r,p}(\mathcal{M})$ by the contraction map:

$$B(\omega_{(1)}, \ldots, \omega_{(q)}, v_{(1)}, \ldots, v_{(r)}) \equiv A(\omega_{(1)}, \ldots, e^\mu, \ldots, \omega_{(q)}, v_{(1)}, \ldots, e_{\mu,\ldots,} v_{(r)})$$

(3) Given any tensor, we can symmetrize (or antisymmetrize) any number of its upper or lower indices. To symmetrize, we take the sum of all permutations of the relevant indices and divide by the number of terms (antisymmetrization comes from the alternating sum - an additional minus sign is given to permutations that are the result of an odd number exchanges of indices).

Definition 26. A directional derivative X_p at a point p is a map $X_p : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ which obeys the properties:

(1) For a constant function c, $X_p[c] = 0$
(2) $X_p[f+g] = X_p[f] + X_p[g]$
(3) $X_p[f \cdot g] = X_p[f] \cdot g + f \cdot X_p[g]$ (Leibnitz Rule). □

The set of all directional derivatives at p is denoted $\mathcal{D}_p(\mathcal{M})$ and has a natural linear structure. We state without proof (see Isham) that $\mathcal{D}_p(\mathcal{M}) \simeq \mathbb{R}^m$ and given a chart on the manifold, we may find a basis $\{(\frac{\partial}{\partial x^\mu}) \mid_p\}$ such that $(\frac{\partial}{\partial x^\mu} \mid_p)[f] \equiv \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}(x)) \mid_{\psi(p)}$, $\forall\, f \in \mathcal{F}(\mathcal{M})$ and $\mu = 1, \ldots, m$. Then any directional derivative at p can be written in terms of the (coordinate) basis vector on the right hand side as

$$X_p = v^\mu \left(\frac{\partial}{\partial x^\mu} \mid_p \right)$$

We can change from one coordinate chart to another as follows. Suppose $p \in U_i \bigcap U_j$ and there are coordinates $\{x^\mu\}$ on U_i' and $\{y^\alpha\}$ on U_j'. Consider a directional derivative $X_p \in \mathcal{D}_p(\mathcal{M})$ acting on a function f. We can write its action as

$$\begin{aligned} X_p\,[f] &= v^\mu \left(\frac{\partial}{\partial x^\mu} \mid_p \right)[f] = v^\mu \frac{\partial f(x)}{\partial x^\mu} \mid_{x(p)} \\ &= v'^\mu \left(\frac{\partial}{\partial y^\mu} \mid_p \right)[f] = v^\mu \frac{\partial f(y)}{\partial y^\mu} \mid_{y(p)} \end{aligned}$$

From the partial derivative chain rule, it follows that on the overlap we have the following relation between components:

$$v^\mu = v'^\alpha \frac{\partial x^\mu(y)}{\partial y \alpha} \mid_{y(p)}$$

Now follows a crucial step in the development of differential geometry. This is the remarkable statement that $\mathcal{D}_p(\mathcal{M}) \simeq T_p(\mathcal{M})$. In order to show this we must observe that the differential structure on the manifold allows tangent vectors to have a natural derivative action on functions. Choose a curve C passing through the point p and a function $f \in \mathcal{F}(\mathcal{M})$. Then $f \cdot C : \mathbb{R} \to \mathbb{R}$. The rate of change of f along C is given by $\frac{d}{d\lambda}(f \cdot C(\lambda))$. Taking a chart (U_i, ψ_i) with coordinates $\{x^\mu\}$, we can write this as

$$\frac{d}{d\lambda}(f \cdot C(\lambda)) \mid_{p=C(\lambda)} = \frac{d}{d\lambda} x^\mu(C(\lambda)) \frac{\partial}{\partial x^\mu} f(x) \mid_p$$

where we have used the chain rule to break up $f \cdot C = (f \cdot \psi^{-1}) \cdot (\psi \cdot C)$. We see that any other curve C' such that $C' \sim C$ at p would give the same rate of change. Hence a tangent vector V_p at p can be thought of as a map from the functions to the reals which gives the rate of change of the function at p along any representative curve.

$\mathcal{D}_p(\mathcal{M}) \simeq T_p(\mathcal{M})$ is quite a remarkable result. One may prove this by considering the map $\chi : T_p(\mathcal{M}) \to \mathcal{D}_p(\mathcal{M})$ taking $[C] \to \chi([C])$, where C is a representative curve passing through the point p. The proof rests on showing that this map is one-to-one and onto. Using the chart above, we can write explicitly:

$$\chi([C]) = v^\mu \left(\frac{\partial}{\partial x^\mu} \mid_p \right),\ v^\mu = \frac{d}{d\lambda}(\psi \cdot C) \mid_{\lambda=0}$$

So given a tangent vector we may uniquely give a directional derivative (showing "one-to-one"). Conversely, for any directional derivative, and hence the components v^μ, we can find a representative curve such that the above holds (showing "onto"). While it is elegant to consider tangent vectors as equivalence classes of curves, it is far more powerful to consider them as directional derivatives. In particular, the "differential" in differential geometry points to the importance of equating tangent vectors to derivatives acting on functions.

We now wish to define a "coordinate basis" for cotangent vectors of the dual space. Observe that differentials transform exactly in the way we would like dual basis vectors to transform. So we identify the dual coordinate basis precisely as the space of differentials at p: $e^\mu \equiv (dx^\mu \mid_p)$. We now have a coordinate basis $\left\{ \left(\frac{\partial}{\partial x^\mu}\right) \mid_p \right\}$ and a dual coordinate basis $\{(dx^\mu \mid_p)\}$ which always satisfy $\left\langle dx^\mu, \frac{\partial}{\partial x^\alpha} \right\rangle = \delta^\mu_\alpha$. Geometrically, we say that a covector at p is the equivalence class of functions that have the same gradient at p (this is coordinate independent as required).

Finally, we may write a general (q, r) tensor at p as

$$T_p = T^{\alpha_1 \cdots \alpha_q}_{\beta_1 \cdots \beta_r} \frac{\partial}{\partial x^{\alpha_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_q}} \oplus \cdots \otimes dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_r} \mid_p$$

Under a change of coordinates, the components of the tensor transform as:

$$T'^{\alpha_1 \cdots \alpha_q}_{\beta_1 \cdots \beta_r} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_q}}{\partial x^{\mu_q}} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\nu_r}}{\partial y^{\beta_r}} T^{\mu_1 \cdots \mu_q}_{\nu_1 \cdots \nu_r}$$

If a vector is assigned smoothly to each point of $\mathcal{M}$, it is called a vector field. Similarly we define a tensor field of type (q, r) by a smooth assignment of an element of $\mathcal{J}^q_{r,p}(\mathcal{M})$ at each point $p \in \mathcal{M}$.

Induced Maps, Flows and Lie Derivatives

In the previous section we discussed maps on manifolds and how they could be composed. We now investigate this in a little more detail. Suppose we have a map $f : \mathcal{M} \to \mathcal{N}$. We may ask what happens to the tangent space $T_p(\mathcal{M})$. A map is induced by f on $T_p(\mathcal{M})$, denoted $f_\star$, and is called the push-forward. We define $f_\star : T_p(\mathcal{M}) \to T_{f(p)}(\mathcal{N})$ such that $V \to f_\star V$ by first considering a function on $\mathcal{N}$. For some $g \in \mathcal{N}$ we can define the pull-back of the function g onto $\mathcal{M}$ as $g \circ f : \mathcal{M} \to \mathbb{R}$. Recall that a vector $V \in T_p(\mathcal{M})$ can then act on the function $g \circ f$ to give its rate of change $V[g \circ f]$. Likewise the vector $f_\star V \in T_{f(p)}\mathcal{N}$ can act on g to give its rate of change $f_\star V[g]$. We define the push-forward map by equating these rates of change. That is, the push-forward map $f_\star$ satisfies $f_\star V[g] \equiv V[g \circ f]$ for any $g \in \mathcal{F}(\mathcal{M})$. Using a chart (U_i, ψ_i) on $\mathcal{M}$ with coordinates $\{x^\mu \mid_p\}$, a chart (V_j, φ_j) on $\mathcal{N}$ with coordinates $\{y^\alpha \mid_p\}$, and the chain rule, one can see that the components of the push-forward $f_\star V$ transform as

$$\omega^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} \mid_p v^\mu$$

The push-forward map generalizes to tensors of type $(q, 0)$. The behavior of a vector under a pushforward bears a resemblance to the vector transformation law under a change of coordinates. In fact, it is a generalization since when $\mathcal{M}$ and $\mathcal{N}$ are the same manifold, the constructions will be identical. But in general, the indices α and μ have different allowed values and the matrix $\frac{\partial y^\alpha}{\partial x^\mu} \mid_p$ may not be invertible.

Since cotangent vectors are dual to vectors, one should not be surprised to find that cotangent vecotrs can be pulled back (but in general not pushed forward). We define the pull-back map $f^\star : T^\star_p(\mathcal{M}) \leftarrow T^\star_{f(p)}(\mathcal{N})$ as follows. Take a vector $V \in T_p(\mathcal{M})$ so that $f_\star V \in T_{f(p)}\mathcal{N}$. Then take a covector $\omega \in T^\star_{f(p)}\mathcal{N}$. For any v, ω we require $\langle f^\star \omega, v \rangle = \langle \omega, f_\star v \rangle$. As for the push-forward map, the pull-back map $f^\star$ naturally extends to $\mathcal{J}^0_{r,p}$ tensors.

Now we focus on the case when the two manifolds are the same - they are related by a diffeomorphism. The reason why we could not pull-back or push-forward tensors in general is linked to the fact that the map f may not be invertible. However, if the map $f \in \mathit{Diff}(\mathcal{M})$, we can use $(f^{-1})_\star$ to pull-back a vector or $(f^{-1})^\star$ to push-forward a cotangent vector. The beauty of diffeomorphisms is that they allow us push-forward or pull-back arbitrary tensors and thus provide a way of comparing tensors at different points on a manifold. Given a diffeomorphism and a tensor field, we can form the difference between the value of the tensor at some point p and the value of the pull-back of the tensor at p. This suggests that we can define a derivative operator on tensor fields - one that categorizes the rate of change of the tensor along the flow of the diffeomorphism. However, in order to do this, we require not just a single diffeomorphism, but a one-paramenter family of diffeomorphisms. These one parameter families of diffeomorphisms arise from the notion of flows.

Consider a vector field V in $\mathcal{M}$. An integral curve $x(\lambda)$ of V is a curve in $\mathcal{M}$ whose tangent vector at $x(\lambda)$ is $V\mid_x$. Taking a chart $(U_i,\, \psi_i)$ with coordinates $\{x^\mu\}$, we write this as:

$$\frac{dx^\mu}{d\lambda} = V^\mu(x(\lambda))$$

In other words, finding the integral curve of a vector field V is equivalent to solving the autonomous system of ordinary differential equations shown above. The initial condition $x_0^\mu = x^\mu(0)$ corresponds to the coordinates of an integral curve at $\lambda = 0$. The existence and uniqueness theorem of ODEs guarantees a unique solution locally (in a neighborhood of $\lambda = 0$ - although it can be shown that for a compact manifold, one can extend $\lambda \to \pm\infty$). The solution, denoted $\sigma_V(\lambda,\, p_0)$, gives the flow defined by the vector field V. For a fixed λ, the flow σ_V gives a diffeomorphism that has Abelian group structure. This is the group of active coordinate transformations generated by V.

Now that we have a family of diffeomorphisms parametrized by λ, we can ask how fast a tensor changes as we travel along integral curves.

Definition 27. The Lie derivative of a vector field, denoted $\mathcal{L}_\mathcal{V}$, is defined to be

$$\mathcal{L}_\mathcal{V}\left[Y\right]\mid_p \equiv \lim_{\epsilon\to 0}\left(\frac{\sigma_V\left(-\epsilon\right)_\star\, Y\mid_{p'} - Y\mid_p}{\epsilon}\right) \in T_p(\mathcal{M}),\ p' = \sigma_V(\epsilon)\cdot p$$

In a chart $(U_i,\, \psi_i)$ with coordinates $\{x^\mu\}$,

$$L_V\, Y\mid_p = \left(V^\mu(x)\frac{\partial}{\partial x^\mu}\, Y^\nu(x) - Y^\mu(x)\frac{\partial}{\partial x^\mu}\, V^\nu(x)\right)\frac{\partial}{\partial x^\nu}\mid_{x=x(p)}.$$

The Lie bracket is a map $[\cdot,\, \cdot] : \mathcal{J}_0^1 \times \mathcal{J}_0^1 \to \mathcal{J}_0^1$ satisfying

$$[X,\, Y]\,[g] \equiv X\,[Y\,[g]] - Y\,[X\,[g]]\,,\ \forall\, g \in \mathcal{F}(\mathcal{M})$$

One can check that the Lie derivative is simply expressed in terms of the Lie bracket as $\mathcal{L}_V\, Y = [V,\ Y]$. The Lie bracket is bilinear, skew-symmetric and satisfies the Jacobi identity.

Something we will require later is the transformation properties of the Lie bracket under a diffeomorphism. Consider a diffeomorphism $f \in \mathit{Diff}(\mathcal{M})$. Then for two vector fields X and Y the following holds:

$$f_\star\left([X,\, Y]\mid_p\right) = [f_\star X,\, f_\star Y]\mid_{f(p)}$$

Proof. Recall $(f_\star X\,[g])\cdot f = X\,[g\cdot f]$ for a map $f : \mathcal{M} \to \mathcal{N}$ and a function $g \in \mathcal{F}(\mathcal{M})$.
$X\,[Y\,[g\cdot f]] = X\,[(f_\star Y)\,[g]\cdot f] = (f_\star X)\,[(f_\star Y)\,[g]]$.
And hence, $f_\star\left([X,\ Y]\right)[g]\cdot f = [X,\, Y]\,[g\cdot f] = X\,[Y\,[g\cdot f]] - Y\,[X\,[g\cdot f]]$
$= (f_\star X)\,[(f_\star Y)\,[g]]\cdot f - (f_\star Y)\,[(f_\star X)\,[g]]\cdot f = [f_\star X,\, f_\star Y]\,[g]\cdot f$ □

Geometrically, the Lie bracket shows the non-commutativity of two flows.

The Lie derivative of a general tensor field is obtained from the following proposition which we state without proof (see Nakahara).

Proposition. *The Lie derivative satisfies* $\mathcal{L}_V(t_1 + t_2) = \mathcal{L}_V t_1 + \mathcal{L}_V t_2$ *where* t_1 *and* t_2 *are tensor fields of the same type and* $\mathcal{L}_V(t_1 \otimes t_2) = (\mathcal{L}_V t_1) \otimes t_2 + t_1 \otimes (\mathcal{L}_V t_2)$ *where* t_1 *and* t_2 *are tensor fields of arbitrary type.* □

This proposition enables us to calculate the Lie derivative of a general tensor field.

Differential Forms

Flanders gives a nice introduction outlining the need for differential forms. He writes, "In a great many situations, particularly those dealing with symmetries, tensor methods are very natural and effective. However, in many other situations the use of the exterior calculus, often combined with the method of moving frames of É. Cartan, leads to decisive results in a way which is very difficult with tensors alone." Why should we care about differential forms? Simply, because forms can be both differentiated and integrated without the help of any additional geometric structure. These objects generalize the notion of differentials in calculus and are extremely useful.

Definition 28. A differential form of order r is a totally antisymmetric tensor in $\mathcal{J}^0_{r,p}(\mathcal{M})$. □

Definition 29. The Cartan wedge product $\wedge$ is given by $dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r} \equiv \sum_P sign(P)\, dx^{\mu_{P_1}} \otimes dx^{\mu_{P_2}} \otimes \cdots \otimes dx^{\mu_{P_r}}$ where the sum is over all possible permutations $P : 1, \ldots, r \to P_1, \ldots, P_r$. □

From the definition it follows that $dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r} \equiv sign(P)\, dx^{\mu_{P_1}} \otimes dx^{\mu_{P_2}} \otimes \cdots \otimes dx^{\mu_{P_r}}$ and this product forms a coordinate basis for the antisymmetric tensor r-forms. Thus we may write an r-form ω in a coordinate basis as

$$\omega = \frac{1}{r!}\omega_{\mu_1\mu_2\cdots\,\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}$$

$$\omega_{\mu_1\mu_2\cdots\,\mu_r} = \omega\left(\frac{\partial}{\partial x^{\mu_1}}, \frac{\partial}{\partial x^{\mu_2}}, \ldots, \frac{\partial}{\partial x^{\mu_r}}\right)$$

Denote the space of r-forms at the point p as $\Omega^r_p(\mathcal{M})$. It inherits the linearity of $\mathcal{J}^0_{r,p}(\mathcal{M})$.

Definition 30. We can use the wedge product on two basis covectors to construct the "exterior product," $\wedge : \Omega^{r_1}_p(\mathcal{M}) \times \Omega^{r_2}_p(\mathcal{M}) \to \Omega^{r_1+r_2}_p(\mathcal{M})$. For

$$\omega = \frac{1}{r_1!}\omega_{\mu_1\mu_2\cdots\,\mu_{r_1}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{r_1}}$$

$$\xi = \frac{1}{r_2!}\xi_{\mu_1\mu_2\cdots\,\mu_{r_2}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{r_2}}$$

we define the exterior product as

$$\omega \wedge \xi \equiv \frac{1}{r_1!}\frac{1}{r_2!}\omega_{\mu_1\mu_2\cdots\,\mu_{r_1}}\xi_{\nu_1\nu_2\cdots\,\nu_{r_2}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{r_1}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{r_2}}$$

The exterior product is bilinear, associative and satisfies "graded commutativity" ($\xi \wedge \eta = (-1)^{qr}\eta \wedge \xi$ for $\xi \in \Omega^q$ and $\eta \in \Omega^r$).

Definition 31. A form field smoothly assigns an r-form to every point on the manifold. Denote the set of r-form fields as $\Omega^r(\mathcal{M})$. □

Perhaps the most important feature of a form is that there is a very natural notion of differentiation of a form field - the exterior derivative, $d : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$.

Definition 32. For $\omega = \frac{1}{r!}\omega_{\mu_1\mu_2\cdots\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}$, define

$$d\omega = \frac{1}{r!}(\frac{\partial}{\partial x^\sigma}\omega_{\mu_1\mu_2\cdots\mu_r})\, dx^\sigma \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}$$

This is an interesting derivative for the following reason. Previously, when we differentiated a function we required a vector (a directional derivative); however, here we have a derivative operator that has no "direction" built in.

The exterior derivative satisfies a graded Leibnitz rule - for $\xi \in \Omega^q$ and $\omega \in \Omega^r$, $d(\xi \wedge \omega) = (d\xi) \wedge \omega + (-1)^q \xi \wedge (d\omega)$

An important property of the exterior derivative is that it is nilpotent (this follows from symmetry). A basic relation is the Poincaré Lemma: $d(d\omega) = 0$.

Suppose we have a map $f : \mathcal{M} \to \mathcal{N}$. We can pull-back forms in the usual way using $f^\star$. Additionally it can be shown that

$$f^\star(\xi \wedge \omega) = (f^\star\xi) \wedge (f^\star\omega)$$

$$d(f^\star\omega) = f^\star(d\omega)$$

The exterior product and derivative increase the degree of a form. A vector field X provides a natural map that reduces the degree of a form. So we define the interior product $i_X : \Omega^r \to \Omega^{r-1}$ by

$$(i_X\omega)(V_1, \ldots, V_{r-1}) \equiv \omega(X, V_1, \ldots, V_{r-1})$$

Using coordinates, $\omega = \frac{1}{r!}\omega_{\mu_1\mu_2\cdots\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}$, we have

$$i_X\omega = \frac{1}{(r-1)!} X^\nu \omega_{\nu\mu_1\cdots\mu_{r-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{r-1}}$$

Analogously to d the interior product i_X is nilpotent, i.e. $i_X^2 = 0$.

A nice and important result is that the Lie derivative of a form can conveniently be written using the interior and exterior products as,

$$\mathcal{L}_X\omega = (di_X + i_X d)\omega$$

Definition 33. An r-form $\omega \in \Omega^r$ is closed if $d\omega = 0$. Denote the space of closed r-forms as $Z^r(\mathcal{M})$. □

Definition 34. An r-form $\omega \in \Omega^r$ is exact if $\omega = d\sigma$ where $\sigma \in \Omega^{r-1}$. Denote the space of exact r-forms as $B^r(\mathcal{M})$. □

Z^r and B^r inherit the linear structure from Ω^r.

Earlier we saw how to see if a manifold is orientable in the context of the topological spaces underlying the manifold. Now we examine the consequences of orientation in the context of differentiability. Consider a chart with coordinate basis vectors $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^m}\right\}$. We are interested in the order of the list of these basis vectors because locally the order defines an orientation. If we take a permutation $P : \{1, 2, \ldots, m\} \to \{i_1, i_2, \ldots, i_m\}$, then the ordered set of basis vectors $\left\{\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \cdots, \frac{\partial}{\partial x^{i_m}}\right\}$ defines the same/opposite orientation if P is even/odd.

Definition 35. A manifold is orientable if there exists an atlas with charts (U_i, ψ_i) and coordinates $x^\mu_{(i)}(p) = \psi_i(p)$ such that on all overlaps $U_i \bigcap U_j \neq 0$,

$$J_{ij} = det \left(\frac{x^\mu_{(i)}}{x^\nu_{(j)}} \right) \gneq 0 \; \forall \, p \in U_i \bigcap U_j$$

Note that this definition of orientability is easily seen to be consistent with our earlier one involving triangulations with simplices. Since it uses the differential structure of the manifold as opposed to its topological structure, it is a less basic definition. But it is easier to work with in practice.

Definition 36. Take an orientable m-dimensional manifold $\mathcal{M}$ with an atlas such that $J_{ij} \gneq 0$. A volume form, V, is an m-form on $\mathcal{M}$ such that for any point $p \in \mathcal{M}$, V can be written in the coordinate bais of a chart containing p and satisfies

$$V = v(x_{(i)})\, dx^1_{(i)} \wedge \cdots \wedge dx^m_{(i)} \; with \; v(x_{(i)}) \gneq 0$$

A volume form V defines an orientation on $\mathcal{M}$. Note that if V is a volume form, then $-V$ is also a volume form but with the opposite orientation.

We are now ready to define integration on the manifold.

Definition 37. Consider an orientable manifold with a volume for V and choose an atlas so that $J_{ij} = det \left(\frac{x^\mu_{(i)}}{x^\nu_{(j)}} \right) \gneq 0$ and the component of the volume form in any chart is positive definite. Then we define the integral of a function f with respect V over any chart U_i to be

$$\int_{U_i} f\, V \equiv \int_{U'_i} d^m x_{(i)} f(\psi^{-1}(x_{(i)})) v_{(i)}(\psi^{-1}(x_{(i)}))$$

The right hand side of this equation is simply the usual multidimensional integral on $\mathbb{R}^m$ from analysis. We require the form V to be a volume form rather than just an m-form so that its component is positive definite, and thereby nowhere vanishing. This defines a positive definite integration measure. so that $\int_{U_i} f\, V \gneq 0$ for any positive definite function f.

We may use a volume form to provide a measure to integrate a function on a manifold by using a partition of unity on the manifold. Take an atlas $\{(U_i, \psi_i)\}$ on $\mathcal{M}$ so that any point p is found in a finite number of the sets U_i. Take smooth functions $\epsilon_i(p)$ on $\mathcal{M}$ such that:

(1) $0 \leq \epsilon_i(p) \leq 1$
(2) $\epsilon_i(p) = 0$ if $p \notin U_i$
(3) $\sum_i \epsilon_i(p) = 1 \; \forall \, p \in \mathcal{M}$

Then any smooth function $f \in \mathcal{F}(\mathcal{M})$ can be decomposed as $f(p) = \sum_i \epsilon_i(p) f(p) = \sum_i f_i(p)$ where $f_i(p) = \epsilon_i(p) f(p)$ are new smooth functions that are only non-zero in the open set U_i. Finally using the volume form V and the partition of unity we can integrate a function f on $\mathcal{M}$:

$$\int_M f\, V = \sum_i \int_{U_i} f_i\, V$$

Stokes' Theorem

Recall that a smooth map $f\,:\,\mathcal{M}\to\mathcal{N}$ where $dim\mathcal{M}\leq dim\mathcal{N}$ defines an embedding of the manifold $\mathcal{M}$ into $\mathcal{N}$ if it is one-to-one. The image $f(\mathcal{M})$ is a submanifold of $\mathcal{N}$ that is diffeomorphic to $\mathcal{M}$. Now suppose $\mathcal{M}$ is orientable and we can choose a volume form V on $\mathcal{M}$ which defines an orientation on $\mathcal{M}$. The pushforward of a tangent vector in $\mathcal{M}$ will give a set of vectors which define the orientation of the submanifold $f(\mathcal{N})$. Given such an r-dimensional oriented submanifold of $\mathcal{N}$, we can naturally integrate an r-form ω over it using the pull-back:

$$\int_{f(\mathcal{N})}\omega\equiv\int_{\mathcal{N}}(f^{\star}\omega)=\int_{\mathcal{N}}g\,V$$

where the function $g\in\mathcal{F}(\mathcal{M})$ satisfies $(f^{\star}\omega)=gV$.

Recall that we may trianguate our m-dimensional manifold $\mathcal{M}$ by a simplicial complex K so that the set of points in the manifold is the polyhedron $\mathcal{M}=|K|$. Take the simplices in the triangulation to be oriented. Now we can view an oriented r-simplex as an oriented r-dimensional oriented submanifold of $\mathcal{M}$.

An r-simplex is defined as $\sigma_r=\{\,\vec{x}\in\mathbb{R}^m\mid\vec{x}=\sum_{i=0}^{r}\vec{p}_i c_i,\ c_i\geq 0,\ \sum_{i=0}^{r}c_i=1\}$. We can view an r-simplex σ_r as a one-to-one map f_σ from $\mathbb{R}^r$ to $\mathcal{M}$:

$$f_\sigma\,:\,\widetilde{\sigma}_r\subset\mathbb{R}^r\to\sigma_r\subset|K|\equiv\mathcal{M}$$

$$\{c_i\mid i=1,\,\ldots,\,r\}\to\vec{x}(c_i)$$

where $\widetilde{\sigma}_r$ is the standard r-simplex in $\mathbb{R}^r$. Thus the r-simplex σ_r is a submanifold on $\mathcal{M}$. The volume form $V=dx^1\wedge\cdots dx^r$ on $\mathbb{R}^r$ defines the orientation of $\widetilde{\sigma}_r$. The map f_σ then determines the orientation of the submanifold. We may now define integration of an r-form ω on $\mathcal{M}$ over an oriented r-simplex σ_r in the triangulation of $\mathcal{M}$:

$$\int_{\sigma_r}\omega=\int_{\widetilde{\sigma}_r}(f_\sigma)^{\star}\,\omega$$

where we may evaluate this using the volume form $V=dx^1\wedge\cdots dx^r$ on $\mathbb{R}^r$ from above.

Recall that an r-chain is given by $c=\sum_i c_i\sigma_{r,\,i}$. We simply define integration of an r-form ω over the chain c as

$$\int_c\omega\equiv\sum_i c_i\int_{\sigma_{r,i}}\omega$$

What then is the value of this long story on chains, boundaries and integrals? It provides the foundations for us to define one of the most elegant and powerful theoretical tools in mathematics. The general result we now establish includes all known formulas which transform an integral into one over a one-higher dimension spread.

Theorem. *(Stokes' Theorem). For $\omega\in\Omega^{r-1}(\mathcal{M})$ and an r-chain $c\in C_r(\mathcal{M})$ with boundary given by the chain $\partial c\in C_{r-1}(\mathcal{M})$,*

$$\int_c d\omega=\int_{\partial c}\omega$$

This is quite a remarkably elegant result. We will not prove Stoke's Theorem here (see Nakahara). One must prove this is true for the standard simplex $\widetilde{\sigma}_r$ and then extend the result to general simplices.

De Rham Cohomology

If a topological space $\mathcal{M}$ is a manifold, we may define the dual of the homology vector space in terms of differential forms defined on $\mathcal{M}$. Earlier we defined the spaces of closed and exact forms. Since the exterior derivative is nilpotent it follows that for $B^r \subset Z^r$. Thus a natural question arises: are there closed forms that are not exact?

Definition 38. Two closed forms z_1, $z_2 \in Z^r(\mathcal{M})$ are equivalent (cohomologous) if $z_1 - z_2 \in B^r(\mathcal{M})$. We denote the set of equivalent closed r-forms by$[z]$ and $\{[z]\} = \{z' \in Z^r(\mathcal{M}) \mid z' \sim z\}$ is the set of equivalence classes. □

$\{[z]\}$ forms the cohomology vector space $H^r(\mathcal{M})$ which inherits its linear structure from Z^r and the d operator. While the spaces of closed and exact forms are infinite dimensional vector spaces, the cohomology vector space may be finite dimensional - there are only a finite number of inequivalent closed r-forms on a manifold.

Theorem. *(deRham's First Theorem). If $\mathcal{M}$ is compact then $H^r(\mathcal{M})$ is finite dimensional. Furthermore dim($H^r(\mathcal{M})$)=$b_r(\mathcal{M})$.*

This implies that the dimension of the cohomology vector space and the dimension of the set of inequivalent non-trivial r-cycles (the homology vector space) are the same and equal to the Betti numbers. Hence the number of homology classes is equal to the number of cohomology classes. This is a very profound statement as it links the topological structure and differential structure on the manifold.

Define a map $\Lambda : H_r(\mathcal{M}) \times H^r(\mathcal{M}) \to \mathbb{R}$ given by $\Lambda([c], [\omega]) \equiv \int_c \omega$. One can show (using Stokes' Theorem) that the equation is independent of the representatives of the homology and cohomology classes chosen.

Theorem. *(deRham's Second Theorem). Given a set of r-cycles $\{c_i\}$ defining a basis $\{[c_i]\}$ of $H_r(\mathcal{M})$ for i=1,..., b_r, we may choose a dual set of closed r-forms $\{\omega_i\}$ which define a basis $\{[\omega_i]\}$ of $H^r(\mathcal{M})$ for i=1,..., b_r such that $\Lambda\left([c_i], \left[\omega^j\right]\right) \equiv \int_{c_i} \omega^j = \delta^i_j$.*

Hence each class of r-cycles is paired with a class of closed (but not exact) r-forms and we may think of these as voume forms on the manifold for these r-cycles.

We now discuss a few more interesting results regarding cohomology. Take an open set $U \subset \mathcal{M}$ that can be smoothly contracted to a point. More precisely, let there be a map $f : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such that $U = f(0, U)$, $p_0 = f(1, U)$ and f is C^0. Consider forms on the open set U (which itself is a manifold that is diffeomorphic to $\mathbb{R}^{dim(\mathcal{M})}$). Then $Z^r(U)$ and $B^r(U)$ are the space of closed and exact forms defined over U. We state the following without proof.

Lemma. *(Poincaré's Lemma). For contractible $U \subset \mathcal{M}$, any closed form over U is also exact.*

We say that any closed form is locally exact (where locally implies on a contractible open set). We may therefore think of deRham cohomology as the failure of closed forms being globally exact on the manifold.

We discuss one last interesting consequence of cohomology - Poincaré duality. Take an m-dimensional compact manifold $\mathcal{M}$. We can define an inner product

$\langle\cdot,\cdot\rangle : H^r(\mathcal{M}) \times H^{m-r}(\mathcal{M}) \to \mathbb{R}$ given by $\langle[\omega],[\alpha]\rangle \equiv \int_{\mathcal{M}} \omega \wedge \alpha$. This inner product is bilinear in both arguments and independent of the representatives of the cohomology classes. We state the following theorem without proof.

Theorem. *(Poincaré). The inner product*$\langle\cdot,\cdot\rangle$ is non-degenerate.

It follows that $H^{m-r}(\mathcal{M})$ is the dual vector space to $H(\mathcal{M})$. This implies the following relation between the Betti numbers: $b_r = b_{m-r}$. □

Part 6. Appendix B: Riemannian Geometry

In the chapter on manifolds we started with the basic notion of a set and introduced a topology on the set, promoting it to a topological space. We then asserted that the topological space looks locally like a subset of $\mathbb{R}^n$. Analysis on the space is assured by the existence of smooth coordinate systems. Taking these definitions, the topological space is upgraded to a manifold. The manifold naturally contains a lot of rich mathematical structure. It comes equipped naturally with a tangent bundle, tensor bundles of various ranks, the ability to take exterior derivatives, and so on. The problem we now wish to deal with is regarding the inner geometry of a manifold which is not part of a Euclidean space. Flanders notes, "If the manifold were part of Eucliden space, it would naturally inherit a local Euclidean geometry (distance function) from that of the including space...However, [if] it is not part of Euclidean space, we must postulate the existence of a local distance geometry. What we do in effect is to presuppose that each tangent space possesses an inner product which is smooth." Thus a manifold may carry further structure if we proceed to endow it with a metric tensor - the natural generalization of the inner product between two vectors in a tangent space $T_p(\mathcal{M})$. It also provides us with a way of comparing a vector at a point $p \in \mathcal{M}$ with another vector at another point $p' \in \mathcal{M}$ using a "connection."

The Metric Tensor

Definition 39. Let $\mathcal{M}$ be a manifold. Let the point $p \in \mathcal{M}$ be on the manifold and take any two vectors $U,\ V \in T_p(\mathcal{M})$. The Riemannian metric is a symmetric, positive $(0, 2)$-tensor field $g \in \mathcal{J}^0_2$ satisfying

(1) $g(U, V) = g(V, U)$
(2) $g(U, U) \geq 0$ with equality if and only if $U = 0$. $\square$

The manifold $\mathcal{M}$ together with a metric g, $(\mathcal{M}, g)$, form a Riemannian manifold. A pseudo-Riemannian (Lorentzian) manifold is a manifold together with a pseudo-Riemannian metric. A pseudo-Riemannian metric is a symmetric $(0, 2)$-tensor field $g \in \mathcal{J}^0_2$ satisfying $g(U, V) = 0\ \forall\ U \in T_p(\mathcal{M}) \implies V = 0$ instead of (2) above. Taking a chart with coordinates $\{x^\mu\}$, we can express the metric as $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$. The positivity of the Riemannian metric (and also for the pseudo-Riemannian metric) ensures that the metric is non-degenerate. We define the inverse metric components as $g^{\mu\nu}$ so that $g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta$.

Definition 40. For a d-dimensional manifold $\mathcal{M}$, we define the signature for a Riemannian metric and pseudo-Riemannian metric to be $(0, d)$ and $(1, d-1)$ respectively. $\square$

Definition 41. The metric tensor defines a non-degenerate inner product at the point p, $g :\ T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}$. $\square$

It thus provides a map for raising and lowering indices which can be extended to general (q, r) tensors.

Definition 42. The metric g on an orientable m-dimensional manifold gives a canonical volume form (independent of coordinates) Ω_g defined as,

$$\Omega_g \equiv \sqrt{|det g_{\mu\nu}|} dx^1 \wedge \cdots \wedge dx^m$$

The volume element is called a pseudo-tensor as the sign of the tensor changes if we change the orientation of the coordinate charts covering the manifold.

Definition 43. We define the epsilon symbol $\epsilon_{\mu_1\mu_2\cdots\,\mu_m}$ as follows:

(1) $\epsilon_{\mu_1\mu_2\cdots\,\mu_m} = 1$ for an even permutation $(\mu_1, \mu_2, \dots, \mu_m) \to (1, 2, \dots, m)$
(2) $\epsilon_{\mu_1\mu_2\cdots\,\mu_m} = -1$ for an odd permutation $(\mu_1, \mu_2, \dots, \mu_m) \to (1, 2, \dots, m)$
(3) $\epsilon_{\mu_1\mu_2\cdots\,\mu_m} = 0$ otherwise. □

We may use the metric to raise indices to give

$$\epsilon^{\nu_1\cdots\,\nu_r}_{\mu_{r+1}\cdots\,\mu_m} = g^{\nu_1\mu_1}\cdots\, g^{\nu_r\mu_r}\epsilon_{\mu_1\mu_2\cdots\,\mu_m}$$

Tensors possess a certain beauty and simplicity; however, in many situations it is useful to consider non-tensorial objects. By definition, the epsilon symbol has the same components specified above in any coordinate system (upto overall orientation), and so we note that the epsilon symbol does not transform as a tensor. In fact the epsilon symbol is said to transform as a tensor density, namely as the components of a tensor up to multipication by $g = detg_{\mu\nu}$.

Definition 44. We define a tensor density $T^{a\cdots\,b}_{c\cdots\,d}$ to be a tensor which can be expressed in the form $T^{a\cdots\,b}_{c\cdots\,d} = \sqrt{|g|}\,\widetilde{T}^{a\cdots\,b}_{c\cdots\,d}$. □

Differential Forms and Hodge Theory

Recall that on an m-dimensional manifold $\Omega^r(\mathcal{M})$ is isomorphic to $\Omega^{m-r}(\mathcal{M})$. If $\mathcal{M}$ is endowed with a metric, we can define an isomorphism between them known as the Hodge star operation.

Definition 45. Define the Hodge star map $\star : \Omega^r(\mathcal{M}) \to \Omega^{m-r}(\mathcal{M})$ acting on $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$ by

$$\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}) \equiv \frac{1}{(m-r)!}\sqrt{|g|}\epsilon^{\mu_1\cdots\,\mu_r}_{\mu_{r+1}\cdots\,\nu_m} dx^{\mu_{r+1}} \wedge \cdots \wedge dx^{\mu_m}$$

The volume element can be written as $\Omega_g = \star 1$.

Definition 46. For $\omega \in \Omega^r$,

(1) $\star\star\,\omega = +(-1)^{r(m-r)}\omega$ for a Riemannian manifold
(2) $\star\star\,\omega = -(-1)^{r(m-r)}\omega$ for a Lorentzian manifold. □

We now define an inner product on the space of r-forms. Let $\alpha,\, \beta \in \Omega^r(\mathcal{M})$ be two r-forms such that $\alpha = \frac{1}{r!}\alpha_{\mu_1\cdots\,\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$ and $\beta = \frac{1}{r!}\beta_{\mu_1\cdots\,\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$. Then $\alpha \wedge (\star\beta) = \beta \wedge (\star\alpha) = \frac{1}{r!}(\alpha_{\mu_1\cdots\,\mu_r}\beta^{\mu_1\cdots\,\mu_r})\Omega_g$. This is a natural object to use to integrate over $\mathcal{M}$. We use it to define the symmetric, positive-definite inner product $(\cdot,\,\cdot) : \Omega^r(\mathcal{M}) \times \Omega^r(\mathcal{M}) \to \mathbb{R}$ given by

$$(\alpha,\, \beta) \equiv \int_{\mathcal{M}} \alpha \wedge (\star\beta)$$

Definition 47. We define the adjoint exterior derivative $d^\dagger : \Omega^r(\mathcal{M}) \to \Omega^{r-1}(\mathcal{M})$ as

(1) $d^\dagger\omega \equiv -(-1)^{m(r+1)} \star d \star \omega$ for a Riemannian manifold
(2) $d^\dagger\omega \equiv +(-1)^{m(r+1)} \star d \star \omega$ for a Lorentzian manifold. □

Observe that $d^\dagger$ is nilpotent. This follows from the fact that $d^\dagger d^\dagger = \star d \star \star d \star = \pm \star d^2 \star = 0$ since $d^2 = 0$. Why all the difficult signs in the definition? They are there to ensure that we can define the following nicely.

Definition 48. Let $(\mathcal{M},\, g)$ be a compact orientable manifold with metric and let $\alpha \in \Omega^r(\mathcal{M})$ and $\beta \in \Omega^{r-1}(\mathcal{M})$. Then

$$(d\beta, \alpha) = (\beta, d^\dagger \alpha)$$

Hence $d^\dagger$ is the adjoint of d with respect to the inner product $(\cdot,\, \cdot)$. One can show the relation above using Stokes' Theorem.
We are now ready to write down one of the most important differential operators - the Laplacian.

Definition 49. The Laplacian map $\triangle :\, \Omega^r(\mathcal{M}) \to \Omega^r(\mathcal{M})$ operates on $\omega \in \Omega^r(\mathcal{M})$ as

$$\triangle\omega \equiv (d + d^\dagger)\omega = (dd^\dagger + d^\dagger d)\omega$$

An r-form ω is called harmonic if $\triangle\omega = 0$, coclosed if $d^\dagger\omega = \mathit{0}$ and coexact if it is written globally as $\omega_r = d^\dagger\beta_{r+1}$ where $\beta_{r+1} \in \Omega^{r+1}(\mathcal{M})$. We denote the set of harmonic r-forms on $\mathcal{M}$ by $Harm^r(\mathcal{M})$, the set of exact forms by $d\mathit{\Omega}^{r-\mathit{1}}(\mathcal{M})$, and the set of coexact r-forms by $d^\dagger \mathit{\Omega}^{r+\mathit{1}}(\mathcal{M})$. We now state a few interesting theorems in Hodge theory (without proof! See Nakahara).

Theorem. *An r-form ω is harmonic if and only if ω is closed and coclosed.*

Theorem. *(Hodge Decomposition Theorem). Let $(\mathcal{M}, g)$ be a compact orientable Riemannian manifold without boundary. Then $\Omega^r(\mathcal{M})$ is uniquely decomposed as*

$$\Omega^r(\mathcal{M}) = d\mathit{\Omega}^{r-\mathit{1}}(\mathcal{M}) \oplus d^\dagger \mathit{\Omega}^{r+\mathit{1}}(\mathcal{M}) \oplus Harm^r(\mathcal{M})$$

Theorem. *(Hodge's Theorem). On a compact orientable Riemannian manifold $(\mathcal{M}, g)$, $H^r(\mathcal{M}) \cong Harm^r(\mathcal{M})$.*

The isomorphism is given by identifying $[\omega] \in H^r(\mathcal{M})$ with $P\omega \in Harm^r(\mathcal{M})$ where $P :\, \mathit{\Omega}^r(\mathcal{M}) \to Harm^r(\mathcal{M})$ is a projection operator to the space of harmonic r-forms. In particular, we have $dimHarm^r(\mathcal{M}) = dimH^r(\mathcal{M}) = b^r$ where b^r is the Betti number. So, we can write the Euler characteristic as

$$\chi(\mathcal{M}) = \sum(-1)^r b^r = \sum(-1)^r dimHarm^r(\mathcal{M})$$

This is a beautiful result as it connects a topological quantity on the left hand side to an analytical quantity on the right hand side given by the eigenvalue problem of the Laplacian.

Connections, Torsion and Curvature

We cannot overemphasize the vitality of the role played by the metric in the theory of manifolds. Perhaps the most important feature of the metric is that it allows us to give a geometry to the manifold. At a point p, the metric gives the inner product on the tangent space and (on a manifold with Riemannian signature) it defines the dot product of tangent vectors (and hence determines their lengths, the angle between them and so forth just as in $\mathbb{R}^m$). Consider a curve C on the manifold given explicitly by $\{x^\mu(\lambda)\}$. The metric allows us to associate a geometric distance to a portion of the curve $C' \subset C$ by the integral $\int_{C'} ds = \int_{C'} \frac{ds}{d\lambda} d\lambda$. For $\mathcal{M} = \mathbb{R}^m$ with a Euclidean metric, this agrees with the usual notion of distance. For a general manifold, the metric provides a generalization of it.

Definition 50. Given a manifold $\mathcal{N}$ with metric g, consider an embedded submanifold diffeomorphic to $\mathcal{M}$ defined by the smooth map $f : \mathcal{M} \to \mathcal{N}$. We may induce a metric $g_{\mathcal{M}}$ on $\mathcal{M}$ via the pull-back $g_{\mathcal{M}} = f^{\star} g$. This is called the induced metric on the manifold $\mathcal{M}$. □

Recall that given a tangent vector V at a point p, we can take the directional derivative of a function g by taking the natural action of the tangent vector on the function $V[g]$. We cannot, however, take a directional derivative of a vector field (or a tensor field for that matter). This is because the Lie derivative is independent of the direction of the tensor at the point). In order to take the directional derivative of a tensor, we define an affine connection map $\nabla : \mathcal{J}^{1}_{0}(\mathcal{M}) \times \mathcal{J}^{1}_{0}(\mathcal{M}) \to \mathcal{J}^{1}_{0}(\mathcal{M})$ satisfying the following properties:

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
(2) $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$
(3) $\nabla_{fX} Y = f \nabla_X Y$
(4) $\nabla_X(fY) = X[f]\, Y + f\nabla_X Y$ for $X, Y, Z \in \mathcal{J}^{1}_{0}(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$

The first two properties show the linear structure of the affine connection map. Of the latter two properties, the former shows that the derivative is directional and the latter shows that the map obeys a Leibnitz rule. We may specify the map by its action on basis vectors. Taking a chart (U_i, ψ_i) with coordinate basis $\{e_\mu\} = \left\{\frac{\partial}{\partial x^\mu}\right\}$, we define

$$\nabla_\mu e_\nu \equiv \nabla_\mu v = \Gamma^{\alpha}_{\mu\nu} e_\alpha$$

where $\left\{\Gamma^{\alpha}_{\mu\nu}\right\}$ are the connection components. Using the components with the properties of the connection we can then calculate

$$\nabla_X(Y) = X^\mu\left(\frac{\partial}{\partial x^\mu} Y^\nu + \Gamma^{\nu}_{\mu\alpha} Y^\alpha\right) e_\nu$$

The connection components do not quite transform as a tensor. On an overlapping chart with coordinates $\{y^\mu\}$ corresponding to coordinate basis $\{\widetilde{e}_\mu\} = \left\{\frac{\partial}{\partial y^\mu}\right\}$, they transform according to

$$\widetilde{\Gamma}^{\rho}_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\frac{\partial x^\rho}{\partial y^\sigma}\Gamma^{\sigma}_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu}\frac{\partial y^\rho}{\partial x^\alpha}$$

Suppose we have two connections ∇ and ∇' which have coordinates $\left\{\Gamma^{\alpha}_{\mu\nu}\right\}$ and $\left\{\tilde{\Gamma}^{\alpha}_{\mu\nu}\right\}$ in some chart. Then the difference of their components $\delta\Gamma^{\alpha}_{\mu\nu}$ forms a $(1, 2)$ tensor satisfying $\widetilde{\delta\Gamma}^{\rho}_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\frac{\partial x^\rho}{\partial y^\sigma}\delta\Gamma^{\sigma}_{\alpha\beta}$. Next notice that given a connection we can immediately specify another connection by permuting the lower indices. We decompose the connection ∇ into symmetric and antisymmetric parts, $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}(S^{\rho}_{\mu\nu} + T^{\rho}_{\mu\nu})$ with $S^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\nu\mu}$ and $T^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}$.

Definition 51. The antisymmetric components $T^{\rho}_{\mu\nu}$ form a $(1, 2)$ tensor, $T = T^{\rho}_{\mu\nu}\frac{\partial}{\partial x^\rho} \otimes dx^\mu \otimes dx^\nu$, called the Torsion tensor. □

From the definition it follows that the symmetric components $S^{\rho}_{\mu\nu}$ transform as a connection ∇_S. So, we can think of a general connection as being defined by a symmetric connection together with a torsion tensor. There is an elegant basis invariant definition for the torsion map:

Definition 52. The torsion map $\mathcal{T} : \mathcal{J}^{1}_{0}(\mathcal{M}) \times \mathcal{J}^{1}_{0}(\mathcal{M}) \to \mathcal{J}^{1}_{0}(\mathcal{M})$ is given by $\mathcal{T}(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y]$. □

The torsion map is bilinear, antisymmetric and satisfies

$$\mathcal{T}(aX, bY) = ab\mathcal{T}(X, Y)\ \forall\ a, b \in \mathcal{F}(\mathcal{M})$$

Consider a vector field $X \in \mathcal{J}^1_0(\mathcal{M})$ and a curve $C \subset \mathcal{M}$ given by $p(\lambda)$. We may wish to investigate how the vector field changes at a point on the curve.

Definition 53. We say the vector $X\mid_{p(\lambda)}$ is parallel-transported (does not change) along the curve if $\nabla_{\frac{d}{d\lambda}} X\mid_{p(\lambda)}= 0$.

Definition 54. Consider a vector field $X \in \mathcal{J}^1_0(\mathcal{M})$. Let C be an integral curve of V. Then if $\nabla_V X\mid_p= 0\ \forall\ p \in C$ then the curve is said to be a geodesic.

Heuristically, we can think of a geodesic as the generalization of a straight line in Euclidean space. So a geodesic is a curve that is "straight" with respect to the connection as its tangent is parallel-transported along the curve. An interesting remark is that the geodesic curves of a connection only depend on the symmetric part of the connection. So two connections that differ by a torsion will have exactly the same geodesic curves. Knowing all the geodesics of a connection allows us to construct the symmetric components $S^\rho_{\mu\nu}$ of a connection ∇. Given a geodesic, we may think of torsion as a measure of how a vector orthogonal to the tangent changes when parallel-transported along a curve. Starting at a point p with two vectors $X, Y \in T_p(\mathcal{M})$, we can flow by a small parameter $\delta\lambda$ along the geodesic of X to the point p_x. Now we can parallel-transport Y along this curve to obtain $Y' \in T_{p_x}(\mathcal{M})$ and then flow by the same parameter $\delta\lambda$ along the geodesic generated by Y' starting at p_x to reach the point p_{yx}. Equivalently, to reach the point p_{xy} we can repeat the same operations while interchanging X and Y. The torsion then measures how close the points p_{yx} and p_{xy} are to each other (or how near the parallelogram formed by the trajectories closes). In classical Euclidean geometry, the torsion naturally vanishes.
We now define a directional derivative ∇_X ($X \in \mathcal{J}^1_0(\mathcal{M})$) which acts on a general tensor field.

Definition 55. The covariant derivative is the bilinear map $\nabla :\ J(\mathcal{M}) \times \mathcal{J}^q_r(\mathcal{M}) \to \mathcal{J}^q_r(\mathcal{M})$ satisfying

(1) $\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$ where T_1 and T_2 are tensor fields.
(2) $(\nabla_X T)(\ldots, e^\mu, \ldots, e_\mu, \ldots) = \nabla_X(T(\ldots, e^\mu, \ldots, e_\mu, \ldots))$
(3) $\nabla_X f \equiv X[f]$ for some function $f \in \mathcal{F}(\mathcal{M})$. $\square$

In coordinates we find that the action of the covariant derivative is given by

(1) $\nabla_\alpha \frac{\partial}{\partial x^\mu} = +\Gamma^\nu_{\alpha\mu} \frac{\partial}{\partial x^\nu}$ for a basis vector
(2) $\nabla_\alpha dx^\mu = -\Gamma dx^\nu$ for a basis covector. $\square$

Definition 56. For a given metric g, a connection ∇ is a metric connection if at any point $p \in \mathcal{M}$ it obeys

$$\nabla_X g = 0\ \forall\ X \in T_p(\mathcal{M})$$

In a coordinate basis this implies $\partial_\mu g_{\alpha\beta} - \Gamma^\rho_{\mu\alpha} g_{\rho\beta} - \Gamma^\rho_{\mu\beta} g_{\alpha\rho} = 0\ \forall\ \alpha, \beta, \mu$. Taking cyclic permutations and subtracting, we obtain

$$\Gamma^\rho_{(\alpha\beta)} g_{\rho\mu} - \Gamma^\rho_{[\mu\alpha]} g_{\rho\beta} - \Gamma^\rho_{[\mu\beta]} g_{\rho\alpha} = C^\rho_{\alpha\beta} g_{\rho\mu}$$

$$C^{\rho}_{\alpha\beta} \equiv \frac{1}{2} g^{\rho\sigma}(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})$$

$C^{\rho}_{\alpha\beta}$ is known as the Christoffel connection. We see that the symmetric part of the connection can be written in terms of the Christoffel Connection and the torsion tensor: $\frac{1}{2}S^{\rho}_{\alpha\beta} = C^{\rho}_{\alpha\beta} + T^{\rho}_{\alpha\beta}$ where we have used the metric to raise and lower indices on the torsion tensor defined from above. So a metric connection is determined entirely by the metric (which determines the Christoffel connection) and the torsion tensor. In the case of vanishing torsion there is a unique metric connection $\Gamma^{\rho}_{\alpha\beta} = C^{\rho}_{\alpha\beta}$ (we won't prove that here). This connection is symmetric and is called the Levi-Civita connection. The Levi-Civita connection arises in several interesting contexts - in the variation of the Einstein-Hilbert action and when computing geodesic curves by functional variation, to name a few.

Definition 57. We define the curvature map $R : \mathcal{J}^1_0(\mathcal{M}) \times \mathcal{J}^1_0(\mathcal{M}) \times \mathcal{J}^1_0(\mathcal{M}) \to \mathcal{J}^1_0(\mathcal{M})$ by

$$R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_Z \nabla_X Z - \nabla_{[X,Y]} Z$$

The map is antisymmetric, trilinear and satsfies

$$R(aX, bY, cZ) = abcR(X, Y, Z) \; \forall \; a, b, c \in \mathcal{F}(\mathcal{M})$$

This implies the map defines a *(1, 3)* tensor $R^{\alpha}_{\mu\nu\rho} e_\alpha \otimes e^\mu \otimes e^\nu \otimes e^\rho$ called the Riemann curvature tensor. Taking the basis to be a coordintae basis one finds

$$R^{\alpha}_{\mu\nu\rho} = \frac{\partial}{\partial x^\mu} \Gamma^{\alpha}_{\nu\rho} - \frac{\partial}{\partial x^\nu} \Gamma^{\alpha}_{\mu\rho} + \Gamma^{\beta}_{\nu\rho} \Gamma^{\alpha}_{\mu\beta} - \Gamma^{\beta}_{\mu\rho} \Gamma^{\alpha}_{\nu\beta}$$

The Riemann tensor satisfies several interesting properties:

(1) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
(2) $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
(3) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$
(4) $R_{\rho[\sigma\mu\nu]} = 0$
(5) $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$ (Bianchi Identity)

Definition 58. The Ricci tensor is given by $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ and the Ricci scalar is given by $R = g^{\mu\nu} R_{\mu\nu}$. □

Killing Vector Fields

We think of a manifold $\mathcal{M}$ as possessing a symmetry if the geometry is invariant under a transformation that maps $\mathcal{M}$ to itself. In other words, we would like to say that a manifold possesses a symmetry if the metric is the same (in some sense) at different points on the manifold. Symmetries of the metric are called isometries. We say a diffeomorphism ϕ is a symmetry of some tensor T if $\phi^\star T = T$ holds. It is common to have a one-parameter family of symmetries ϕ_t. If the family is generated by a vector field V^μ, then the condition that the pull-back of the tensor must equal the tensor is equivalent to $\mathcal{L}_V T = 0$ (see Nakahara for a proof). One implication of a symmetry is that if a tensor is symmetric under a one-parameter family of diffeomorphisms, then we can find a coordinate system in which the components of the tensor are independent of the integral curve coordinates of the vector field. The converse is also true. The most important symmetries are those of the metric. A diffeomorphism satisfying $\phi^\star g_{\mu\nu} = g_{\mu\nu}$ for the metric $g_{\mu\nu}$ is called an isometry. If the vector field $V^\mu(x)$ generates a one-parameter family of isometries, then $V^\mu(x)$

is known as a Killing vector field. Thus the condition for V^μ being a Killing vector is

$$\mathcal{L}_V g_{\mu\nu} = 0$$

A particularly useful formula for the Lie derivative of the metric is

$$\mathcal{L}_V g_{\mu\nu} = V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} = \nabla_\mu V_\nu + \nabla_\nu V_\mu$$

$$\Rightarrow \mathcal{L}_V g_{\mu\nu} = 2\nabla_{(\mu} V_{\nu)}$$

So the condition that V^μ is a Killing vector is equivalently given by Killing's Equation:

$$\nabla_{(\mu} V_{\nu)} = 0$$

We may ask, "how symmetric can a space possibly be?" An example of a space with the highest possible degree of symmetry is Minkowski space. In n-dimensional Minkowski spacetime, there are $\frac{n(n+1)}{2}$ Killing vector fields, n of which generate translations, $(n-1)$ of which generate boosts and $\frac{(n-1)(n-2)}{2}$ of which generate space rotations. Those spaces which admit $\frac{n(n+1)}{2}$ Killing vector fields are called maximally symmetric spaces.

Non-Coordinate Bases

In studying manifolds, we chose bases for the tangent spaces that were derived from the coordinate system. We now investigate how to use sets of basis vectors that are not derived from any coordinate system. This will give us a different point of view on the connection and curvature. Until now we have taken the partial derivative with respect to the coordinates at a point to be a natural basis for the tangent space. Similarly we have taken the gradients of the coordinate functions to be a basis for the cotangent space. We now introduce at each point on the manifold a set of basis vectors $\hat{e}_{(a)}$ that are orthonormal in a sense that is appropriate to the signature of the manifold. That is, if the metric is written as η_{ab}, then we require $g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}$. This set of orthonormal vectors is known as a tetrad or vielbein. We can express our old basis vectors, say $\hat{e}_{(\mu)} = \frac{\partial}{\partial x^\mu}$, in terms of the new ones by $\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)}$. The components e^a_μ form a $GL(n, \mathbb{R})$ matrix. We can also define the inverse components e^μ_a which satisfy $e^\mu_a e^a_\nu = \delta^\mu_\nu$ and $e^a_\mu e^\mu_b = \delta^a_b$. In a similar fashion, we can set up an orthonormal basis of one forms $\hat{\theta}^{(a)}$ in the cotangent space which are chosen to be compatible with $\hat{\theta}^{(a)} \hat{e}_{(a)} = \delta^a_b$. An immediate consequence of the definition is $\hat{\theta}^{(\mu)} = e^\mu_a \hat{\theta}^{(a)}$ and $\hat{\theta}^{(a)} = e^a_\mu \hat{\theta}^{(\mu)}$. Any other vector can be expressed in terms of its components in the orthonormal basis. For instance, if a vector V is written as $V^\mu \hat{e}_{(\mu)}$ in the coordinate basis and as $V^a \hat{e}_{(a)}$ in the orthonormal basis, the sets of components are related by $V^a = e^a_\mu V^\mu$. The components of the metric tensor in the orthonormal basis are just those of the flat metric: $g_{\mu\nu} = e^\mu_a e^\nu_b \eta_{ab}$.

Now that we have noncoordinate bases we may ask how these objects transform. The only restriction we have is that the orthonormality of the basis must remain. We know what kind of transformations preserve the flat metric. These are orthogonal transformations for Riemannian signature and Lorentz transformations for Lorentzian signature. We therefore consider a change of basis of the form

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda^a_{a'}(x) \hat{e}_{(a)}$$

where $\Lambda^a_{a'}(x)$ give transformations which leave the canonical form of the metric unaltered: $\Lambda^a_{a'} \Lambda^b_{b'} \eta_{ab} = \eta_{a'b'}$. We now have the freedom to perform a local Lorentz transformation (or orthogonal transformation depending on the signature of the manifold) at every point. We also still have the freedom to perform a general coordinate transformation at each point. This generalizes to a general mixed transformation law involving both Λ matrices and the usual Jacobian matrices.

In the ordinary formalism, the covariant derivative of a tensor gives its partial derivative plus corrections terms involving the connection coefficients. The same will be true now in a noncoordinate basis except that the ordinary connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ get replaced by the spin connection, $\omega^{a}_{\mu b}$. The name spin connection is derived from the fact that $\omega^{a}_{\mu b}$ can be used to calculate the covariant derivative of spinors. It is not surprising that the spin connection does not transform tensorially. Under a local Lorentz transformation, the spin connection transforms as

$$\omega^{a'}_{\mu b'} = \Lambda^{a'}_{a}\Lambda^{b'}_{b}\omega^{a}_{\mu b} - \Lambda^{c}_{b'}\partial_{\mu}\Lambda^{a'}_{c}$$

We gain several advantages by adopting this new formalism. Firstly, it allows us to describe spinor fields (we won't describe this further here). But secondly, it allows us to view tensors as tensor-valued differential forms. For instance, consider X^{a}_{μ}. We may think of this as a $(1,1)$ tensor with mixed indices. But we can also think of it as a one form taking a vector value for each lower index. Similarly a tensor $D_{\mu\nu}{}^{a}{}_{b}$ can be thought of as a "$(1,1)$ tensor-valued two form." In general, we may think of any tensor with antisymmetric lower indices as a differential form taking on values in the tensor bundle. This is useful when we consider taking the exterior derivative. The tensor transforms in the right way (as a form) under general coordinate transformations; however, it does not transform as a vector under local Lorentz transformations. To remedy this we introduce the spin connection. So,

$$(dX)^{a}_{\mu\nu} + (\omega\wedge X)^{a}_{\mu\nu} = \partial_{\mu}X^{a}_{\nu} - \partial_{\nu}X^{a}_{\mu} + \omega^{a}_{\mu b}X^{b}_{v} + \omega^{a}_{\nu b}X^{b}_{\mu}$$

transforms correctly as a tensor.

We can apply this formalism to express the torsion and the curvature. The torsion can be thought of as a vector-valued antisymmetric two form $T^{a}_{\mu\nu}$ while the curvature can be thought of as a $(1,1)$ tensor-valued antisymmetric two form $R^{a}_{b\mu\nu}$. We can express these in terms of the basis one forms (note that we change notation defining $e^{a} = \hat{\theta}^{(a)}$) as

$$e^{a} = e^{a}_{\mu}dx^{\mu}$$

and the spin connection one-forms

$$\omega^{a}_{b} = \omega^{a}_{\mu b}dx^{\mu}$$

The defining relations for the torsion and curvature are given by

$$T^{a} = de^{a} + \omega^{a}_{b}\wedge e^{b}$$

$$R^{a}_{b} = d\omega^{a}_{b} + \omega^{a}_{c}\wedge\omega^{c}_{b}$$

These are known as the Cartan structure equations. We can also express identities obeyed by these tensors:

$$dT^{a} + \omega^{a}_{b}\wedge T^{b} = R^{a}_{b}\wedge e^{b}$$

$$dR^{a}_{b} + \omega^{a}_{c}\wedge R^{c}_{b} - R^{a}_{c}\wedge\omega^{c}_{b} = 0$$

The first of these is a generalization of $R^{\rho}_{[\sigma\mu\nu]} = 0$ and the second is the Bianchi Identity, $\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0$.

All of the above holds for general connections. We now discuss what happens in the special case of a Christoffel connection. The torsion free condition implies

$T^a = 0$. Metric compatibility gives $\nabla g = 0$. In the orthonormal basis where the metric has components η_{ab}, this gives

$$\nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega^c_{\mu a}\eta_{cb} - \omega^c_{\mu b}\eta_{ac} = -\omega_{\mu ab} - \omega_{\mu ba}$$
$$\Rightarrow \omega_{\mu ab} = -\omega_{\mu ba}$$

Metric compatibility is equivalent to the antisymmetry of the spin connection in its Latin indices. These two conditions allow us to express the spin connection in terms of the tetrads. In practice, though, it is easier to simply solve the torsion-free condition $-de^a = \omega^a_b \wedge e^b$.

Part 7. Appendix C: Fiber Bundles

A manifold is a toplogical space which looks locally like $\mathbb{R}^m$ on which we apply conventional calculus techniques. Intuitively, a fiber bundle is a topological space which looks locally like a direct product of two topological spaces. We begin by defining the tangent bundle $T\mathcal{M}$ on an m-dimensional manifold $\mathcal{M}$ to be the collection of all the tangent spaces of $\mathcal{M}$:

$$T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$$

$\mathcal{M}$ is called the base space. Let $\{U_i\}$ be an open covering of $\mathcal{M}$ with coordinates $\{x^\mu\}$. Then an element of $TU_i = \bigcup\limits_{p\in U_i} T_p\mathcal{M}$ is specified by a point $p \in \mathcal{M}$ and a vector $V^\mu \frac{\partial}{\partial x^\mu}\mid_p \in T_p\mathcal{M}$. Since U_i and $T_p\mathcal{M}$ are both homeomorphic to $\mathbb{R}^m$, it follows that TU_i can be identified with the direct product $\mathbb{R}^m \times \mathbb{R}^m$. So if we pick a point $u \in TU_i$, we can decompose it into a point and a vector. Thus we are naturally led to the concept of projection $\pi : TU_i \to U_i$. For a point $u \in TU_i$, $\pi(u)$ is a point $p \in U_i$ at which the vector is defined. We see that $\pi^{-1}(p) = T_p\mathcal{M}$. $T_p\mathcal{M}$ is called the fiber at p. Suppose now that we are on the overlap of two charts with two sets of coordinates. A vector on the overlap may be expressed in either set of coordinates. We know that the components must be related to each other by the Jacobian matrix of the coordinate functions evaluated at the point. For the coordinates to be good cordinates systems, this matrix must be nonsingular. That is, it must be an element of $GL(m,\mathbb{R})$. This group is called the structure group of $T\mathcal{M}$. We see that the tangent bundle gives us interesting mathematical structure on the manifold. It is an example of a more general framework known as a fiber bundle.

Definition 59. A differentiable fiber bundle $(E, \pi, \mathcal{M}, F, G)$ consists of the following:

(1) A differentiable manifold E called the total space
(2) A differentiable manifold $\mathcal{M}$ called the base space
(3) A differentiable manifold F called the fiber
(4) A surjection $\pi : E \to \mathcal{M}$ called the projection
(5) A Lie group G called the structure group which acts on F on the left
(6) An open covering $\{U_i\}$ of $\mathcal{M}$ with a diffeomorphism $\varphi : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \varphi_i(p, f) = p$.
(7) The map $\varphi_i(p, f) : F \to F(p)$ is a diffeomorphism. On $U_i \bigcap U_j \neq \emptyset$, we require $t_{ij}(p) \equiv \varphi_i^{-1}(p) \circ \varphi_j(p) : F \to F$ be an element of G. The maps t_{ij} are called transition functions. □

Strictly, the definition should be independent of the open covering of the manifold. The definition above is used to define a coordinate bundle. A fiber bundle is then defined as an equivalence class of coordinate bundles.

Definition 60. If all the transition functions can be taken to be identity maps, then the fiber bundle is called a trivial bundle. A trivial bundle is a direct product $\mathcal{M} \times F$. □

Definition 61. Let $E \xrightarrow{\pi} \mathcal{M}$ be a fiber bundle. A section $s : \mathcal{M} \to E$ is a smooth map which satisfies $\pi \circ s = id_{\mathcal{M}}$. Denote the set of sections on $\mathcal{M}$ as $\Gamma(\mathcal{M}, F)$. A local section is defined only on U. □

Definition 62. Let $E \xrightarrow{\pi} \mathcal{M}$ and $E' \xrightarrow{\pi'} \mathcal{M}'$ be fiber bundles. A smooth map $\bar{f} : E' \to E$ is called a bundle map if it maps each fiber F'_p of E' onto F_q of E. Then $\bar{f}$ naturally induces a smooth map $f : \mathcal{M}' \to \mathcal{M}$ such that $f(p) = q$. $\square$

Definition 63. Two bundles $E \xrightarrow{\pi} \mathcal{M}$ and $E' \xrightarrow{\pi'} \mathcal{M}'$ are equivalent if there exists a bundle map $\bar{f} : E' \to E$ such that $f : \mathcal{M}' \to \mathcal{M}$ is the identity map and $\bar{f}$ is a diffeomorphism. $\square$

Definition 64. A vector bundle $E \xrightarrow{\pi} \mathcal{M}$ is a fiber bundle whose fiber is a vector space. $\square$

Definition 65. The cotangent bundle $T^\star\mathcal{M} = \bigcup\limits_{p\in\mathcal{M}} T^\star_p\mathcal{M}$ is defined similarly to the tangent bundle. On a chart U_i with coordinates x^μ, the basis of $T^\star_p\mathcal{M}$ is taken to be $\left\{dx^1, \ldots, dx^m\right\}$. $\square$

The cotangent bundle can be extended to the general case. Given a vector bundle $E \xrightarrow{\pi} \mathcal{M}$ with a fiber F, we may define its dual bundle $E^\star \xrightarrow{\pi} \mathcal{M}$. The fiber $F^\star$ of $E^\star$ is the set of linear maps of F to $\mathbb{R}$. We define the dual basis $\left\{\theta^{(\alpha)}(p)\right\}$ of $F^\star_p$ by $\left\langle\theta^{(\alpha)}(p), e_\beta(p)\right\rangle = \delta^\alpha_\beta$.

Definition 66. Let $E \xrightarrow{\pi} \mathcal{M}$ and $E' \xrightarrow{\pi'} \mathcal{M}'$ be vector bundles on $\mathcal{M}$. The tensor product bundle $E \otimes E'$ is obtained by assigning the tensor product of fibers $F_p \otimes F'_p$ to each point $p \in \mathcal{M}$. If $\{e_\alpha\}$ and $\{f_\beta\}$ are bases of F and F', $F \otimes F'$ is spanned by $\{e_\alpha \otimes f_\beta\}$ and so $dim(E \otimes E') = dimE \times dimE'$. $\square$

Definition 67. A principal bundle has a fiber F which is identical to the structure group G. A principal bundle $P \xrightarrow{\pi} \mathcal{M}$ is denoted $P(\mathcal{M}, G)$ and is called a G bundle over $\mathcal{M}$. $\square$

Given a principal fiber bundle $P(\mathcal{M}, G)$ we may construct an associated fiber bundle. Let G act on a manifold F on the left. Define an action of $g \in G$ on $P \times F$ by $(u, f) \to (ug, g^{-1}f)$ with $u \in P$ and $f \in F$.

Definition 68. The associated fiber bundle $(E, \pi, \mathcal{M}, G, F, P)$ is an equivalence class $\frac{P\times F}{G}$ in which two points (u, f) and $(ug, g^{-1}f)$ are identified. $\square$

Let $E \xrightarrow{\pi} \mathcal{M}$ be a vector bundle whose fiber is $\mathbb{R}^k$. On chart U_i, the piece $\pi^{-1}(U_i)$ is trivial and so we choose k linearly independent sections $\{e_1(p), \ldots, e_k(p)\}$ over U_i. These sections are said to define a frame $u = \{X_1, \ldots, X_m\}$ over U_i. Associated with a tangent bundle $T\mathcal{M}$ over an m-dimensional manifold $\mathcal{M}$ is a principal bundle called the frame bundle. It is denoted $L\mathcal{M} \equiv \bigcup\limits_{p\in\mathcal{M}} L_p\mathcal{M}$ where $L_p\mathcal{M}$ is the set of frames at the point p. Take a chart U_i with coordinates $\{x^\mu\}$. The tangent space has a natural basis on U_i given by $\left\{\frac{\partial}{\partial x^\mu}\right\}$. Then a frame $u = \{X_1, \ldots, X_m\}$ at the point p is expressed as $X_\alpha = X^\mu_\alpha \frac{\partial}{\partial x^\mu}\,|_p$, $1 \le \alpha \le m$ where $X^\mu_\alpha \in GL(m, \mathbb{R})$. Define the local trivialization $\phi_i : U_i \times GL(m, \mathbb{R}) \to \pi^{-1}(U_i)$ by $\phi^{-1}(u) = (p, (X^\mu_\alpha))$. The bundle structure of $L\mathcal{M}$ is given by the following:

(1) Let $u = \{X_1, \ldots, X_m\}$ be a frame at the point p. Then define $\pi_L : L\mathcal{M} \to \mathcal{M}$ by $\pi_L(u) = p$
(2) $GL(m, \mathbb{R})$ acts transitively on $L\mathcal{M}$. The action of of $a = a^i_j \in GL(m, \mathbb{R})$ on the frame $u = \{X_1, \ldots, X_m\}$ is given by $(u, a) \mapsto ua$ where ua is given by

$Y_\beta = X_\alpha a^\alpha_\beta$. Conversely, for any two frames $\{X_\alpha\}$ and $\{Y_\beta\}$, we can find an element of $GL(m,\mathbb{R})$ such that the previous relation is satisfied

(3) Take two overlapping charts U_i and U_j with coordintes $\{x^\mu\}$ and $\{y^\mu\}$ respectively. On the overlap, we see that the transition function must be $t^L_{ij}(p) = \left(\frac{\partial x^\mu}{\partial y^\nu}\right)_p \in GL(m,\mathbb{R})$. So, the frame bundle has the same transition functions as the tangent bundle.

The frame bundle has important applications in Einstein's theory of general relativity. The right action corresponds to local Lorentz transformations while the left actions corresponds to general coordinate transformations. The frame bundle is a natural framework to accomodate these transformations.
We can define a spin bundle by introducing a frame bundle whose structure group is $Spin(m)$ - the universal covering group of $SO(m)$.

Connections on Fiber Bundles

In the chapter on Riemannian Geometry, we introduced an affine connection to allow us to compare vectors in different tangent spaces on a manifold. We now try to abstract this to the case of fiber bundles. Before beginning we briefly outline some key concepts from the theory of Lie groups which will be useful.

- A Lie group G is a differentiable manifold which is endowed with group structure on its points. The group operations $\cdot : G\times G \to G$ and $^{-1} : G \to G$ are defined to be smooth
- Let G be a Lie group. The left action L_g and the right action R_g are defined by $L_g g = gh$ and $R_g g = hg$ for $g, h \in G$
- L_g induces a map $L_{g^\star} : T_h(G) \to T_{gh}(G)$. A left-invariant vector field V satisfies $L_{g^\star} V \mid_h = V \mid_{gh}$
- Left-invariant vector fields form a Lie algebra of G, denoted $\mathfrak{g}$. There exists a vector isomorphism $T_e G \cong \mathfrak{g}$
- The Lie algebra $\mathfrak{g}$ is closed under the Lie bracket $[T_\alpha, T_\beta] = f^\gamma_{\alpha\beta} T_\gamma$ where $\{T_\alpha\}$ are the generators of the algebra and $f^\gamma_{\alpha\beta}$ are the structure constants
- The adjoint action, $ad : G \to G$, is given by $ad_g h \equiv ghg^{-1}$

Let $u \in P(\mathcal{M}, G)$ where $P(\mathcal{M}, G)$ is a principal bundle, and let G_p be the fiber at $p = \pi(u)$. The vertical subspace $V_u P$ is the subspace of $T_u P$ that is tangent to G_p at u. The horizontal subspace $H_u P$ is the complement of $V_u P$ in $T_u P$ and is uniquely specified if a connection is defined in P.

Definition 69. Let $P(\mathcal{M}, G)$ be a principal bundle. A connection on $P(\mathcal{M}, G)$ is a unique separation of the tangent space $T_u P$ into the vertical subspace $V_u P$ and the horizontal subspace $H_u P$ such that

(1) $T_u P = H_u P \oplus V_u P$
(2) A smooth vector field V on P is separated into smooth vector fields $V^H \in H_u P$ and $V^V \in V_u P$ as $V = V^H + V^V$
(3) $H_{ug} P = R_{g^\star} H_u P$ for any $u \in P$ and $g \in G$. □

In computations we can separate $T_u P$ into the vertical subspace $V_u P$ and the horizontal subspace $H_u P$ by introducing a Lie-algebra valued one-form $\omega \in \mathfrak{g} \otimes T^\star P$ known as the connection one-form.

Definition 70. A connection one-form $\omega \in \mathfrak{g} \otimes T^\star P$ is a projection of $T_u P$ onto the vertical component $V_u P \simeq \mathfrak{g}$. □

Suppose $\{U_i\}$ is an open covering of $\mathcal{M}$ and σ_i is a local section defined on each of the subsets of the open cover. We can introduce a Lie-algebra valued one form $\mathcal{A}_i$ on U_i by $\mathcal{A}_i \equiv \sigma_i^\star \omega \in \mathfrak{g} \otimes \Omega^1(U_i)$. We state the following theorem without proof (see Nakahara):

Theorem. *Given a one-form $\mathcal{A}_i$ on U_i and a local section $\sigma_i : U_i \to \pi^{-1}(U_i)$, then there exists a connection one-form ω such that $\mathcal{A}_i = \sigma_i^\star \omega$.*

For ω to be defined uniquely on P, we must have $\omega_i = \omega_j$ on $U_i \bigcap U_j$. For this condition to be true, the local forms $\mathcal{A}_i$ have to satisfy the following compatibility condition that we do not prove here:

$$\mathcal{A}_j = t_{ij}^{-1} \mathcal{A}_i t_{ij} + t_{ij}^{-1} dt_{ij}$$

In gauge theories, we identify $\mathcal{A}_i$ with the gauge potential.
We now generalize the exterior derivative operation so that we can dfferentiate a vector-valued r-form $\phi \in \Omega^r(P) \otimes V$ where V is a k-dimensional vector space.

Definition 71. Let $\phi \in \Omega^r(P) \otimes V$ and $X_1, \ldots, X_{r+1} \in T_u P$. The covariant derivative of ϕ is defined by

$$D\phi(X_1, \ldots, X_{r+1}) \equiv d_P \phi(X_1^H, \ldots, X_{r+1}^H)$$

where $d_P \phi \alpha \otimes e_\alpha$. □

Definition 72. The curvature two-form Ω is the covariant derivative of the connection one-form ω, $\Omega \equiv D\omega$. □

We state the following proposition and theorem without proof (see Nakahara if interested).

Proposition. *Ω satisfies $R_a^\star \Omega = a^{-1} \Omega a,\ a \in G$.*

Theorem. *Let $X, Y \in T_u P$. Then Ω and ω satisfy the Cartan Structure Equation,*

$$\Omega(X, Y) = d_P \omega(X, Y) + [\omega(X), \omega(Y)]$$

$$\Omega = d_P \omega + \omega \wedge \omega$$

The proof of the theorem consists of considering three cases - two vectors in the horizontal subspace, two vectors in the vertical subspace and one in each - and showing that the Cartan structure equation is satisfied in all three cases. The details are shown in Nakahara's text.
The local form of the curvature Ω is defined by $\mathcal{F} \equiv \sigma^\star \Omega$ where σ is a local section defined on a chart of the manifold.
$\mathcal{F}$ is expressed in terms of the gauge potential $\mathcal{A}$ as $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ where d is the exterior derivative.

Proof. Note that $\mathcal{A} = \sigma^\star \omega$, $\sigma^\star d_P \omega = d\sigma^\star \omega$ and $\sigma^\star(\zeta \wedge \eta) = \sigma^\star \zeta + \sigma^\star \eta$. From the Cartan structure equation, we find

$$\mathcal{F} = \sigma^\star(d_P \omega + \omega \wedge \omega) = d\sigma^\star \omega + \sigma^\star \omega \wedge \sigma^\star \omega$$

$$= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

□

The action of $\mathcal{F}$ on vectors of $T\mathcal{M}$ is given by $\mathcal{F}(X,Y) = d\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)]$. We can write this in components on a chart U with coordinates $\{x^\mu\}$. Then $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ is the gauge potential. If we write $\mathcal{F} = \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu$, then

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

We identify $\mathcal{F}$ with the field strength. Since $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ are Lie-algebra valued functions, they can be expanded in terms of the basis $\{T_\alpha\} \in \mathfrak{g}$ as $\mathcal{A}_\mu = A^\alpha_\mu T_\alpha$ and $\mathcal{F}_{\mu\nu} = F^\alpha_{\mu\nu} T_\alpha$. Since the basis vectors satisfy the Lie algebra commutation relations, we can write

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^\alpha_{\beta\gamma} A^\beta_\mu A^\gamma_\nu$$

Since ω and Ω are Lie algebra valued, we may expand them in terms of the basis of the Lie algebra $\mathfrak{g}$. If we write $\omega = \omega^\alpha T_\alpha$ and $\Omega = \Omega^\alpha T_\alpha$, then the Cartan structure equation gives

$$\Omega^\alpha = d_P \omega^\alpha + f^\alpha_{\beta\gamma} \omega^\beta \wedge \omega^\gamma$$

Taking the exterior derivative gives

$$d_P \Omega^\alpha = f^\alpha_{\beta\gamma} d_P \omega^\beta \wedge \omega^\gamma + f^\alpha_{\beta\gamma} \omega^\beta \wedge d_P \omega^\gamma$$

Noting that $\omega(X) = 0$ for a horizontal vector X we find

$$D\Omega(X, Y, Z) = d_P \Omega(X^H, Y^H, Z^H) = 0$$

where X, Y, $Z \in T_u P$. This proves the Bianchi identity:

$$D\Omega = 0$$

A local form for the Bianchi identity may be given using $\sigma^\star$:

$$\begin{aligned}\sigma^\star(d_P\omega \wedge \omega + \omega \wedge d_P\omega) &= d\sigma^\star\omega \wedge \sigma^\star\omega - \sigma^\star\omega \wedge d\sigma^\star\omega \\ &= d\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge d\mathcal{A} = \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F}\end{aligned}$$

So the Bianchi identity is

$$D\mathcal{F} = d\mathcal{F} + \mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A} = d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$$

Part 8. Appendix D: Gauge Theories

The language of geometry and topology that we have developed can be applied beautifully to gauge theories of physics and provide some nice insights into the geometrical character of physical theories. In this section we explore some of the more geometrical formulations of theories of electromagnetism and gravity.

Electromagnetism

In their classical form, Maxwell's equations describe the behavior of the electromagnetic field defined through space. We take "space" here to mean $\mathbb{R}^3$, and we note that the equations are also time-dependent. The electric and magnetic fields also depend on the electric charge density and the electric current density which are given by ρ and $\vec{j}$ respectively. We define all functions to be real-valued and all vetor fields to be smooth. In units where $c = 1$, Maxwell's equations are given by:

$$\nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

We can use the modern language of manifolds and differential forms to rewrite Maxwell's equations in an elegant way. Instead of treating the electric and magnetic fields as vector fields, we treat them as a 1-form and 2-form respectively:

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

$$E = E_x dx + E_y dy + E_z dz$$

We think of the electric and magnetic fields as living on Minkowski spacetime, $\mathbb{R}^4$. Taking coordinates (t, x, y, z) we combine both fields into a unified electromagnetic field F (which is a 2-form) defined as follows:

$$F = B + E \wedge dt$$

The components of $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ are given by the matrix

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_x & B_y & -B_x & 0 \end{pmatrix}$$

This allows us to unify the electric and magnetic fields elegantly into a single equation which gives us the sourceless Maxwell equations:

$$dF = 0$$

We now consider the sourced equations. We use the Hodge star to perform a dual symmetry on the electric and magnetic fields, $E_i \mapsto -B_i$, $B_i \mapsto E_i$. The components of the electromagnetic field 2-form now become

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -B_z & 0 & E_x \\ -B_x & E_y & -E_x & 0 \end{pmatrix}$$

We also introduce the current 1-form $j = j_x dx + j_y dy + j_z dz - \rho dt$. Now we can write the sourced Maxwell equations beautifully as

$$d^\dagger F = j$$

Applying the adjoint exterior derivative to both sides of this equation (and noting that it is nilpotent) gives an important result:

$$(d^\dagger)^2 F = d^\dagger j = 0$$

This is the continuity equation that expresses the important concept of local charge conservation. It is a simple consequence of Maxwell's equations in the general modern formalism.

In electrostatics, we may obtain solutions to one of Maxwell's equations more easly if the electric field arises from a scalar potential. Similarly, in magnetostatics, we may simplify a lot of the work if the magnetic field arises from a vector potential. The equations of magnetostatics are given (from above) by $dB = 0$ and $d^\dagger B = j$ (with the magnetic field B a two-form and the current density j a one-form on space). If B is exact, then we can write

$$B = dA$$

for some one-form A - a vector potential. The first equation is automatically true and the second becomes

$$d^\dagger dA = 0$$

A is not uniquely determined because we can add any closed one-form to A without changing dA. In particular, we can change A_μ to $A_\mu \to A_\mu + df$ for any function f without changing B. This is called a gauge transformation. We can apply the same principle to the electromagnetic field F. The electromagnetic field two-form satisfies $dF = 0$ and $d^\dagger F = j$. We say A is a vector potential for F if $dA = F$. Maxwell's equations now reduce to

$$d^\dagger dA = j$$

We now try to generalize these ideas using the machinery developed in the chapter on fiber bundles. Maxwell's theory of electromagnetism is described by the $U(1)$ gauge group. We take the base space to be four dimensional Minkowski spacetime. The principal bundle is given by $P = \mathbb{R}^4 \times U(1)$ and the gauge potential is $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. Since we expect the field strength two-form to be exact (by Poincaré's Lemma), we may write $\mathcal{F} = d\mathcal{A}$. Note that the gauge potential (and field strength two-form) differs from the usual vector potential (field strength tensor) by the relation $\mathcal{A} \equiv iA_\mu$ ($\mathcal{F} \equiv iF_{\mu\nu}$). In components, the field strength two-form is given by $\mathcal{F}_{\mu\nu} = \frac{\partial \mathcal{A}_\nu}{\partial x^\mu} - \frac{\partial \mathcal{A}_\mu}{\partial x^\nu}$ and $\mathcal{F}$ satisfies the Bianchi identity:

$$d\mathcal{F} = \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} = 0$$

To describe the dynamics of the theory we need to specify the action. The Maxwell action is a functional of $\mathcal{A}$ and is given by

$$S \equiv -\frac{1}{4}\int_{\mathbb{R}^4} \mathcal{F} \wedge \star\mathcal{F} = \frac{1}{4}\int_{\mathbb{R}^4} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} d^4x = -\frac{1}{4}\int_{\mathbb{R}^4} F_{\mu\nu}F^{\mu\nu} d^4x$$

Varying the Maxwell action with respect to $\mathcal{A}_\mu$ gives the equation of motion

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0$$

Yang-Mills Theory

Yang-Mills Theory generalizes the Maxwell theory to non-abelian gauge groups. Consider $SU(2)$ gauge theory on $\mathbb{R}^4$. This gauge theory is described by the $P(\mathbb{R}^4, SU(2))$ principal bundle. Since $\mathbb{R}^4$ is contracible, by Poincaré's Lemma, we can write the gauge potential as

$$\mathcal{A} = A_\mu^\alpha T_\alpha dx^\mu$$

where $\left\{T_\alpha = \frac{\sigma_\alpha}{2i}\right\}$ are the generators of the $\mathfrak{su}(2)$ algebra given by $[T_\alpha, T_\beta] = \epsilon_{\alpha\beta\gamma} T_\gamma$. The field strength is

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \frac{1}{2}\mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] = F_{\mu\nu}^\alpha T_\alpha$$

$$F_{\mu\nu}^\alpha = \partial_\mu A_{\nu\alpha} - \partial_\nu A_{\mu\alpha} + \epsilon_{\alpha\beta\gamma} A_{\mu\beta} A_{\nu\gamma}$$

The field strength satisfies the Bianchi identity, $D\mathcal{F} = d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$. To specify the dynamics, we state the Yang-Mills action,

$$S \equiv \frac{1}{2}\int_{\mathcal{M}} tr(\mathcal{F} \wedge \star\mathcal{F}) = -\frac{1}{4}\int_M \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

Variation of the action with respect to $\mathcal{A}_\mu$ gives the equation of motion

$$D_\mu \mathcal{F}^{\mu\nu} = 0$$

Gravity

The general theory of relativity is Einstein's theory of space, time and gravitation - and arguably one of the most beautiful theories in physics. Einstein proposed the following principles to construct his theory:

(1) Principle of General Relativity: All laws of physics take the same form in any coordinate system
(2) Principle of Equivalence: There exists a coordinate system in which the effect of the gravitational field vanishes locally

Any theory of gravity must reduce to Newton's theory in the weak field limit. There are two basic elements - an equation for the gravitational field as influenced by matter and an equation for the response of matter to this field. The former is given by Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho$$

while the latter is given by $\mathbf{a} = \nabla\Phi$. Equivalently, we can see this in terms of the conventional Newtonian statement of these rules: $\mathbf{F} = \frac{GMm}{r^2}\mathbf{e}_{(r)}$ and $\mathbf{F} = m\mathbf{a}$. These equations serve to define Newtonian gravity. According general relativity, what we experience as gravity can be thought of as the curvature of spacetime. Einstein's equation describes the response of spacetime curvature to the presence of matter and energy:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

The Ricci tensor and scalar on the left hand side describe the curvature of spacetime while the stress-energy tensor on the right hand side is a measure of the energy and momentum of matter. (Note that G is Newton's gravitational constant). The equation that governs the response of matter to spacetime curvature is based on the concept that free particles move along paths of shortest possible distance. In curved spacetime, this is given by the geodesic equation

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt} = 0$$

The general theory of relativity describes the dynamics of the geometry of spacetime - more precisely, the dynamics of the metric $g_{\mu\nu}$. We now describe some appropriate Lagrangians that recover Einstein's equations by the action principle. We require that the Lagrangian be a scalar. By the equivalence principle, we know that the metric can be set equal to its canonical form and its first derivatives equal to zero at any point (the so called Riemann normal coordinate frame). Therefore any nontrivial scalar that could serve as a Lagrangian must involve at least second derivatives of the metric. An obvious candidate for this is the Ricci scalar R. We define the Einstein-Hilbert action by

$$S \equiv \frac{1}{16\pi G}\int R\sqrt{-g}d^4x$$

We now show that $\delta S = 0$ leads to the vacuum Einstein equation.

Proposition 73. *Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Under the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $g_{\mu\nu}$, g* and $R_{\mu\nu}$ change as

(1) $\delta g^{\mu\nu} = -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda}$
(2) $\delta g = gg^{\mu\nu}\delta g_{\mu\nu}$, $\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}$
(3) $\delta R_{\mu\nu} = \nabla_\kappa \delta\Gamma^\kappa_{\nu\mu} - \nabla_\nu\delta\Gamma^\kappa_{\kappa\mu}$ (Palatini Identity)

We prove each part separately:

(1) From $g_{\kappa\lambda}g^{\lambda\nu} = \delta^\nu_\kappa$, it follows that $0 = \delta(g_{\kappa\lambda}g^{\lambda\nu}) = \delta g_{\kappa\lambda}g^{\lambda\nu} + g_{\kappa\lambda}\delta g^{\lambda\nu}$. Multiplying by $g^{\mu\kappa}$ we find $\delta g^{\mu\nu} = -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda}$.
(2) First note the matrix identity $ln(det g_{\mu\nu}) = tr(ln g_{\mu\nu})$. This can be shown by diagonalizing the metric. Under the variation $\delta g_{\mu\nu}$, the left hand side becomes $\delta g \cdot g^{-1}$ and the right hand side becomes $g^{\mu\nu}\cdot\delta g_{\mu\nu}$. Hence $\delta g = gg^{\mu\nu}\delta g_{\mu\nu}$. The rest follows from this.
(3) Let Γ and $\tilde{\Gamma}$ be connections. Their difference is a tensor. We take Γ to be associated with g and $\tilde{\Gamma}$ to be associated with $g+\delta g$. Then working in normal coordinates in which $\Gamma \equiv 0$, we find $\delta R_{\mu\nu} = \partial_\kappa\delta\Gamma^\kappa_{\nu\mu} - \partial_\nu\delta\Gamma^\kappa_{\kappa\mu} = \nabla_\kappa\delta\Gamma^\kappa_{\nu\mu} - \nabla_\nu\delta\Gamma^\kappa_{\kappa\mu}$. $\square$

Under the variation $g \to g + \delta g$ such that $\delta g \to 0$ as $|x| \to 0$, the integrand of the action changes as

$$\delta(R\sqrt{-g}) = \delta(g^{\mu\nu}R_{\mu\nu}\sqrt{-g}) = \delta g^{\mu\nu}R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g} + R\delta(\sqrt{-g})$$
$$= -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda}R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}(\nabla_\kappa\delta\Gamma^\kappa_{\nu\mu} - \nabla_\nu\delta\Gamma^\kappa_{\kappa\mu})\sqrt{-g} + \frac{1}{2}R\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$$

Note that the second term can be written as a total divergence,

$$\nabla_\kappa(g^{\mu\nu}\delta\Gamma^\kappa_{\nu\mu}\sqrt{-g}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\kappa_{\kappa\mu}\sqrt{-g})$$
$$= \partial_\kappa(g^{\mu\nu}\delta\Gamma^\kappa_{\nu\mu}\sqrt{-g}) - \partial_\nu(g^{\mu\nu}\delta\Gamma^\kappa_{\kappa\mu}\sqrt{-g})$$

and so does not contribute to the variation. From the remaining terms we have

$$\delta S = \frac{1}{16\pi G}\int(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\sqrt{-g}\delta g_{\mu\nu}d^4x$$

Requiring $\delta S = 0$ under any variation δg, we are left with the Einstein equation in vacuum,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

where we define the Einstein tensor $G_{\mu\nu}$ to be $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$.
So far we have considered the gravitational field only. Now we consider an action of the form

$$S = S_{EH} + S_M$$

where S_{EH} is the Einstein-Hilbert action and S_M is the matter action. Following through the same procedure as above gives

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0$$

We define the energy-momentum tensor to be

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$

This allows us to recover the complete the Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

We now try to consider general relativity as a gauge theory. We can start by asking what symmetry group we can take for general relativity. We would like to use *Diff* $(\mathcal{M})$, but it is not a Lie group. The equivalence principle tells us that the gauge theory we would like to construct should be locally Lorentz invariant. So instead, we take the Lorentz group $SO(3,1)$ to be our gauge group. The frame bundle is a natural framework to accomodate the transformations of general relativity. We take a non-coordinate (tetrad) basis of sections of the frame bundle according to the Cartan formalism (as described in the sections on fiber bundles and Riemannian geometry). Then the independent objects of the theory are the connection one-forms

$$\omega^a_b = \omega^a_{\mu b}dx^\mu$$

and the tetrad basis one-forms

$$e^a = e^a_\mu dx^\mu$$

The Lie algebra of $SO(3,1)$ is given by antisymmetric matrices, and so we can write a covariant derivative as

$$(D_\mu)_{ab} = \partial_\mu \delta_{ab} + A^i_\mu t^i_{ab}$$

for $\left\{t^i\right\}$, the generators of the Lie algebra. We can rewrite the covariant derivative in terms of the connection as

$$D^a_{\mu b} = \partial_\mu + \omega^a_{\mu b}$$

The defining relations for the torsion and curvature are given by the Cartan structure equations:

$$T^a = de^a + \omega^a_b \wedge e^b$$

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

In general relativity we are most concerned with the Christoffel connection - the unique torsion-free and metric-compatible connection on the manifold. The torsion free condition implies

$$T^a = De^a = de^a + \omega^a_b \wedge e^b = 0$$

The field strength is given in terms of the curvature by the second Cartan structure equation:

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

The field strength obeys the Bianchi identity

$$dR^a_b + \omega^a_c \wedge R^c_b - R^a_c \wedge \omega^c_b = 0$$

To specify the dynamics, we give the Palatini-Kibble action. It is covariant and gauge invariant (under local Lorentz transformations) and is given by

$$S\left[e^a, \omega^a_b\right] = \frac{1}{32\pi} \int e^a \wedge e^b \wedge \star R_{ab}$$

Variation of the action with respect to ω^a_b gives the torsion-free equation and variation of the action with respect to e^a gives the Einstein equation for the field strength. Using the properties of the Levi-Civita connection, one can show that the Einstein-Hilbert action and the Palatini-Kibble action are equivalent.

References

[1] Nakahara, M. (2003) *Geometry, Topology and Physics.* Abingdon, Taylor & Francis Group.

[2] Carroll, S. (2004) *Spacetime and Geometry: An Introduction to General Relativity.* San Francisco, Addison Wesley.

[3] Flanders, H. (1989) *Differential Forms with Applications to the Physical Sciences.* New York, Dover Publications, Inc.

[4] Baez, J. & Muniain, J.P. (1994) *Gauge Fields, Knots & Gravity.* Singapore, World Scientific Publishing Co. Pte. Ltd.

[5] Isham, C.J. (2001) *Modern Differential Geometry for Physicists.* Singapore, World Scientific Publishing Co. Pte. Ltd.

[6] Kobayashi, S. & Nomizu, K. (1996) *Foundations of Differential Geometry Volume 1.* New York, John Wiley & Sons, Inc.

[7] Wald, R.M. (1984) *General Relativity.* Chicago, The University of Chicago Press.

[8] Misner, C.W., Thorne, K.S., & Wheeler, J.A. (1973) *Gravitation.* San Francisco, W.H. Freeman and Company.

[9] Weinberg, S. (1972) *Gravitation and Cosmology.* New York, John Wiley & Sons, Inc.

[10] Munkres, J. (2000) *Topology.* Upper Saddle River, Prentice Hall, Inc.

[11] Gilmore, R. (2005) *Lie Groups, Lie Algebras and Some of their Applications.* New York, Dover Publications, Inc.

[12] Penrose, R. (2004) *The Road to Reality.* London, Jonathan Cape.

[13] Müller-Kirsten, H. & Wiedemann, A. (2010) *Introduction to Supersymmetry.* Singapore, World Scientific Publishing Co. Pte. Ltd.

[14] Krippendorf, S., Quevedo, F., &s Schlotterer, O. "Cambridge Lectures on Supersymmetry and Extra Dimensions." arXiv: 1011.1491v1 [hep-th].

[15] Singer, A. (2009) "ABC of SUSY." [Online] *J. Phys.*, G36: 073002. arXiv: 0905.4630v1 [hep-ph].

[16] Reed, M. & Simon B. (1980) *Modern Mathematical Physics.* San Diego, Academic Press, Inc.

[17] Eguchi, T., Gilkey, P., & Hanson, A. (1980) "Gravitation, Gauge Theories and Differential Geometry." *Physics Reports (Review Section of Physics Letters)*, 66, No. 6, 213-393.

[18] Ferrara, S., Freedman, D., & Van Nieuwenhuizen, P. (1976) "Progress toward a Theory of Supergravity." *Phys. Rev.*, D13, 3214.

[19] Deser, S., & Zumino, B. (1976) "Consistent Supergravity." *Phys. Lett.*, B62, 335.

[20] Chamseddine, A., & West, P. (1977) "Supergravity as the Gauge Theory of Supersymmetry." *Nucl. Phys.*, B129, 39.

[21] Van Proeyen, A. (2002) "Structure of Supergravity Theories." *Proceedings of the XI Fall Workshop on Geometry and Physics.* arXiv: 0301005 [hep-th].

[22] Howe, P., & Stelle, K. (1989) "The Ultraviolet Properties of Supersymmetric Field Theories." *Int. J. Mod. Phys.,* A4, 1871.

[23] Van Nieuwenhuizen, P. (1981) "Supergravity." *Phys. Rep.,* 68, 189.

[24] Nilles, H. (1984) "Supersymmetry, Supergravity and Particle Physics." *Phys. Rep.,* 110, 1.

[25] Arnowitt, R., & Nath, P. "Supersymmetry and Supergravity: Phenomenology and Grand Unification." [Online] arXiv: 9309277 [hep-th].

[26] Green, M., Schwarz, J., & Witten, E. (1987) *Superstring Theory Volumes I and II.* New York, Cambridge University Press.

[27] West, P. (1997) "Supergravity, Brane Dynamics and String Duality." In: Olive, D.I. & West, P.C. *(eds.) Duality and Supersymmetric Theories.* New York, Cambridge University Press.

[28] Nahm, W. (1978) "Supersymmetries and their Representations." *Nucl. Phys.,* B135, 149.

[29] Cremmer E., Julia, B., & Scherk, J. (1978) "Supergravity Theory in 11 Dimensions." *Phys. Lett.*, B76, 409.

[30] Duff, M., Nilsson, B., & Pope, C. (1986) "Kaluza-Klein Supergravity." *Phys. Rep.,* B242, 377.

[31] Polchinski, J. (2005) *String Theory Volume I: An Introduction to the Bosonic String Theory.* New York, Cambridge University Press.

[32] Zwiebach, B., (2004) *A First Course in String Theory.* Cambridge, Cambridge University Press.

[33] Hitchin, N. (2003) "Generalized Calabi-Yau Manifolds." *Quart. J. Math. Oxford,* Ser. 54, 281. [arXiv: Math.DG/0209099].

[34] Gualtieri, M. (2004) "Generalized Complex Geometry." *Oxford University DPhil Thesis.* [arXiv: Math.DG/0401221] and [arXiv: Math.DG/0703298].

[35] Gabella, M., et al. (2010) "AdS_5 Solutions of Type IIB Supergravity and Generalized Complex Geometry." *Commun. Math. Phys.,* 299, 365-408. [arXiv: 0906.4109v2].

[36] Pacheco, P., & Waldram, D. (2008) "M-Theory, Exceptional Generalized Geometry and Superpotentials." arXiv: 0804.1362v2 [hep-th].

[37] Coimbra, A., Strickland-Constable, C., & Waldram, D. (2011) "Supergravity as Generalized Geometry I: Type II Theories." arXiv: 1107.1733v1 [hep-th].

[38] Ovrut, B., & Waldram, D. (1997) "Membranes and Three-Form Supergravity." *Nucl. Phys.,* B506, 236-266. [arXiv: hep-th/9704045v1].

[39] Hull, C. (2007) "Generalized Geometry for M-Theory." *JHEP,* 0707, 079. [arXiv: hep-th/0701203v1].

[40] Zabzine, M. (2006) "Lectures on Generalized Complex Geometry and Supersymmetry." *Archivum Mathematicum,* 42, 119-146. [arXiv: hep-th/0605148v4].

[41] Baraglia, D. (2007) "Generalized Geometry." *Masters Thesis, University of Adelaide.*

[42] Gates, S., Hull, C., & Rocek, M. (1984) "Twisted Multiplets and New Supersymmetric Non-Linear Sigma Models." *Nucl. Phys.,* B248, 157.

[43] Grana, M., Minasian, R., Petrini, M., & Waldram, D. (2009) "T-Duality, Generalized Geometry and Non-Geometric Backgrounds." *JHEP,* 0904, 075. [arXiv: hep-th/0807.4527].