Black Holes and String Theory

Hussain Ali Termezy

Submitted in partial fulfilment of the requirements
for the degree of Master of Science of Imperial College London

September 2012

Contents

Introduction

The study of black holes has been an intense area of research for many decades now, as they are a very useful theoretical construct where theories of quantum gravity become relevant. There are many curiosities associated with black holes, and the resolution of some of the more pertinent problems seem to require a quantum theory of gravity to resolve. With the advent of string theory, which purports to be a unified quantum theory of gravity, attention has naturally turned to these questions, and have remarkably shown signs of progress. In this project we will first review black hole solutions in GR, and then look at how a thermodynamic description of black holes is made possible. We then turn to introduce string theory and in particular review the black Dp-brane solutions of type IIB supergravity. Lastly we see how to compute a microscopic account of the Bekenstein entropy is given in string theory.

Chapter 1

Black Holes in General Relativity

1.1 Black Hole Solutions

We begin by reviewing some the basics of black holes as they arise in the study of general relativity. Let us first consider the Schwarzschild solution where we encounter our simplest example.

Schwarzschild Solution

The Schwarzschild metric is, by Birkhoff's theorem, the unique, spherically symmetric solution to the vacuum Einstein equations, $R_{\mu\nu} = 0$, with mass distribution M, given in coordinates (t, r, θ, ϕ)

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. This metric is valid outside and up to the surface of the spherically symmetric source (e.g. star); is asymptotically flat, i.e. is Minkowski in the limit $r \to \infty$; and admits the timelike Killing vector $k^\mu = (1, 0, 0, 0)$, i.e. is stationary (in fact *static*).

The metric has two *singular points* (i.e. at which $\det g_{\mu\nu} = 0$ or parts of the metric blow up): one at $r = 0$ and the other at $r = 2G_N M$. Singularities can be due to a failure of one's coordinate system to cover all points on the manifold and are removable by changing system, but can be also feature of the spacetime itself. In the latter case, this means that there will be geodesics which cannot be continued through all values of their affine parameter, hence it is said to suffer from *geodesic incompleteness*. We can test for such singularities by building scalars from the curvature tensor, which if blow up in the limit of the singularity, imply geodesic incompleteness of the spacetime. (However, the implication doesn't run the other way).

Let us characterise the Schwarzschild singularities. It can be shown that

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 G_N M^2}{r^6}. \tag{1.2}$$

This is the *Kretschmann* scalar which blows up as $r \to 0$, hence we conclude that $r = 0$ is a spacetime singularity. However, it turns out that $r = 2G_N M$ doesn't react similarly with any such scalar. This can be seen by changing coordinates to those of Kruskal-Szekeres

$$u = \left(\frac{r}{2G_N M} - 1\right)^{1/2} e^{r/4G_N M} \cosh\left(\frac{t}{4G_N M}\right) \tag{1.3}$$

$$v = \left(\frac{r}{2G_N M} - 1\right)^{1/2} e^{r/4G_N M} \sinh\left(\frac{t}{4G_N M}\right) \tag{1.4}$$

so that the metric takes the form

$$ds^2 = \frac{32G_N^3 M^3}{r} e^{-r/2m}(-dv^2 + du^2) + r^2 d\Omega^2 \tag{1.5}$$

where r is given implicitly by

$$u^2 - v^2 = \left(\frac{r}{2G_N M} - 1\right) e^{r/2m}. \tag{1.6}$$

This is the maximal analytic extension to the Schwarzschild metric. In particular, the metric now covers the region between $r = 2G_N M$ and $r = 0$, and we see that the surface $r = 2G_N M$, (now at $v = \pm u$), is no longer singular, though far from unremarkable. A handy perspective of spacetime is offered by a *Carter-Penrose* or *conformal* diagram, since the diagram is obtained by conformally transforming the metric, i.e. $g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $\Omega = \Omega(x^\mu)$. The diagram's significance lies in 'compacting' an otherwise infinite spacetime picture into a finite one whilst maintaining its causal structure. The CP diagram for Schwarzschild is

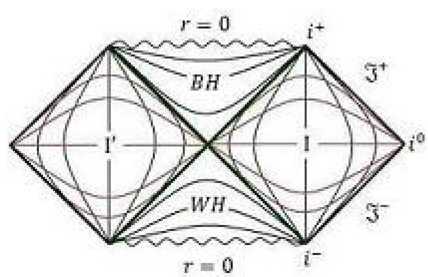

Figure 1.1: (eternal) Schwarzschild space-time: red (black) lines are constant in r (t), the central diagonals are event horizons ($r = 2G_N M$), and the top and bottom are singularities. Only (t, r) plane drawn, so each point is a two-sphere. BH and WH mean black hole and white hole. $\Im^+, \Im^-$ mean future and past null infinity respectively, and i^+, i^- mean future and past timelike infinity whilst i^0 is spatial infinity. Labels are left-right symmetric.

Interpretation of the diagram and event horizon- example of null hypersurface. Cosmic censor

Reissner-Nordström Solution

The next black hole solution we need to consider is the Reissner-Nordström solution, which is also static and incorporates charge. It is a solution to Einstein-Maxwell (EM coupled to GR) action and will play a key role when we come to address string theory. The metric is given by

$$ds^2 = -\left(1 - \frac{2G_NM}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2G_NM}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2 \tag{1.7}$$

The solutions for which the metric is singular are $r = r_\pm = G_NM \pm \sqrt{G^2M^2 - Q^2}$. We will be interested in the special case of the extremal black hole given by $|Q| = G_NM$, and the two horizons coincide. This will link to a similar relation occuring in supersymmetry: the saturation of the BPS bound. Rewriting this using the coordinate $r = \rho + Q$ gives[1]

$$ds^2 = -\left(1 + \frac{Q}{\rho}\right)^{-2}dt^2 + \left(1 + \frac{Q}{\rho}\right)^{2}d\rho^2 + \rho^2d\Omega^2 \tag{1.8}$$

in Cartesian-like coordinates, $\rho = |\bar{x}_3|$, $\bar{x}_3 = (x, y, z)$, $d\bar{x}_3^2 = d\rho^2 + \rho^2d\Omega^2$ we can write this as

$$ds^2 = -H^{-2}dt^2 + H^2d\bar{x}_3^2 \tag{1.9}$$

where $H = 1 + \frac{Q}{|\bar{x}_3|}$. The function, H, is harmonic in the three-dimensional Euclidean space and belongs to the *Majumdar-Papapetrou* class of solutions which solve the Einstein equations

$$H = 1 + \sum_{i=1}^{N}\frac{Q_i}{|\bar{x}_3 - \bar{x}_{3,i}|} \tag{1.10}$$

These are in fact supersymmetric solutions (c.f. brane solutions later) where the gravitational attraction is cancelled by electrostatic repulsion. Going back to (1.8) the near horizon geometry can be examined close to $\rho = 0$, this becomes

$$ds^2 = -\frac{\rho^2}{Q^2}dt^2 + \frac{Q^2}{\rho^2}d\rho^2 + Q^2d\Omega^2 \tag{1.11}$$

writing $z = Q\ln\rho$ we have an $AdS_2 \times S^2$ geometry

$$ds^2 = -e^{-2z}dt^2 + dz^2 + Q^2d\Omega^2 \tag{1.12}$$

ADM mass

In d-dimensions, for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$h_{00} \simeq \frac{16\pi G_N^{(d)}}{(d-2)\omega_{d-2}}\frac{M}{r^{d-3}} \tag{1.13}$$

$$M = \frac{(d-2)\omega_{d-2}}{16\pi G_N^{(d)}}, \qquad \omega_n = \int_{S^n}d\omega^n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \tag{1.14}$$

Law	Thermodynamics	Black Holes
Zeroth	T constant throughtout body in thermal equlilbrium	κ constant over horizon of stationary black hole
First	$dE = TdS +$ work terms	$dM = \frac{\kappa}{8\pi G_N} dA + \Omega_H dJ + \Phi_H dQ$
Second	$\delta S \geq 0$ in any process	$\delta A \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $\kappa = 0$ by a physical process

1.2 Black Hole Thermodynamics

The four laws of black hole mechanics are, using [8]: Therefore there is an explicit analogy suggesting that black holes may in fact be thermodynamic objects: $S \leftrightarrow A/8\pi\alpha$, $E \leftrightarrow M$, and $T \leftrightarrow \alpha\kappa$. Entropy doesn't scale as volume. The Planck-Nernst version of the third law, i.e. that $S \to 0$ as $T \to 0$ is not satisfied, since e.g. extremal black holes have an area with $\kappa = 0$. This is not concerning since, in any case, the third law is not seen to be 'fundamental' in the sense that the other laws are, and there exist examples of simple quantum systems violating it: see [9]. The stumbling block preventing us making an identification between these sets of laws is that black holes only absorb and emit no radiation, i.e. T is not physically given by κ (hence S is not given by A). This problem was in fact resolved, and the identification made precise, by Hawking in 1974 who showed that thermal emission did indeed occur with respect to black holes at temperature

$$T = \frac{\hbar\kappa}{2\pi} \Longrightarrow S = \frac{A}{4\hbar}$$

as a **blackbody**. This is a very nice result, but the rabbit hole gets deeper. Problems with entropy: what are the microstates, since we know the origin of temperature and things thrown in vanish (overall entropy decrease- however, black hole radiation has associated and entropy and infalling mass causes area to increase); how to define entropy (entanglement entropy);

Generalised second law (total entropy never decreases)

$$S_{total} = S + S_{BH} \qquad \delta S_{total} \geq 0 \tag{1.15}$$

Information problem- will come to later. These problems are related see [11]. GSL seems to work well challenges to and protection of [10] We prove the second

The Second Law: Hawking's area theorem

We need to introduce some machinery in order to arrive at proof, due to Hawking [5]

Definition 1.1. Given a manifold M be an oriented manifold of dimension n and S another with dimension $p < n$. An *embedding* is given by a one-to-one map $\phi : S \to M$ and any point $p \in S$ has an open neighbourhood, O, such that $\phi^{-1} : \phi[O] \to S$ is smooth. We say $\phi[S]$ is an *embedded submanifold* of M. An embeddded submanifold of dimension $n - 1$ is called a ***hypersurface***.

[1] following [3]

A tangent vector $t \in T_p(M)$ is said to be *tangent to* $\phi[S]$ *at* p if it is the tangent to a curve in $\phi[S]$ at p. Given such a t, the *normal to* $\phi[S]$ *at* p, $l \in T_p^*(M)$ if $l^\mu t_\mu = 0$.
A hypersurface in a Lorentzian manifold is said to be *timelike*, *null* or *spacelike* if any normal l to it is *spacelike* ($g_{\mu\nu}l^\mu l^\nu > 0$), *null* ($g_{\mu\nu}l^\mu l^\nu = 0$) or *timelike* ($g_{\mu\nu}l^\mu l^\nu < 0$) respectively.

Let S be a smooth function of coordinates x^μ and consider the hypersurface $S(x) = 0$, denoted by $\mathcal{H}$, with normal given by $l^\mu = f(x)(g^{\mu\nu}\partial_\nu S)$ where f is an arbitrary non-zero function. A tangent vector, t, to $\mathcal{H}$ by definition has $t \cdot l = 0$ and therefore l is itself tangent as $l \cdot l = 0$ from above. Hence we may write:

$$l^\mu = \frac{dx^\mu}{d\lambda}$$

In fact, x^μ above will be null geodesics, which we now show:

$$\begin{aligned}
l^\rho \nabla_\rho l^\mu &= (l^\rho \partial_\rho f) g^{\mu\nu} \partial_\nu S + f g^{\mu\nu} l^\rho \nabla_\rho \partial_\nu S = (l \cdot \partial \ln f) l^\mu + f g^{\mu\nu} l^\rho \nabla_\nu \partial_\rho S \\
&= (\frac{dx^\rho}{d\lambda}\frac{\partial}{\partial x^\rho} \ln f) l^\mu + f l^\rho \nabla^\mu (f^{-1} l_\rho) = (\frac{d}{d\lambda} \ln f) l^\mu + l^\rho \nabla^\mu l_\rho - (\partial^\mu \ln f) l^2 \\
&= (\frac{d}{d\lambda} \ln f) l^\mu + \frac{1}{2}\nabla^\mu (l^\rho l_\rho) - (\partial^\mu \ln f) l^2 = (\frac{d}{d\lambda} \ln f) l^\mu + \frac{1}{2}\partial^\mu l^2 - (\partial^\mu \ln f) l^2
\end{aligned}$$

Note $l^2|_{\mathcal{H}} = 0$ does not mean $\partial_\mu l^2|_{\mathcal{H}} = 0$, so

$$l^\rho \nabla_\rho l^\mu|_{\mathcal{H}} = (\frac{d}{d\lambda} \ln f) l^\mu + \frac{1}{2}\partial^\mu l^2|_{\mathcal{H}}$$

But,

$$l^2|_{\mathcal{H}} = \text{const} \Rightarrow \frac{d}{d\lambda} l^2 = 0 \Rightarrow t^\mu \partial_\mu l^2|_{\mathcal{H}} = 0 \Rightarrow \partial_\mu l^2|_{\mathcal{H}} \propto l^\mu$$

for any t^μ tangent on $\mathcal{H}$. So, finally, $l^\rho \nabla_\rho l^\mu|_{\mathcal{H}} = \kappa l^\mu$, $\kappa \in \mathbb{R}$ and can be set to vanish by choosing f in which case λ is an affine parameter.

Hence we see that the $x^\mu(\lambda)$'s, λ affine, are geodesics which we will call the *generators of* $\mathcal{H}$.

By Frobenius' theorem, a vector field, l^μ, is hypersurface orthogonal iff

$$l_{[\alpha}\partial_\beta l_{\gamma]} = 0$$

In particular, given a Killing vector ξ, we define a *Killing horizon* of ξ as a null hypersurface such that $\xi|_{\mathcal{H}}$ is normal, i.e. $\xi = fl$, for l normal to $\mathcal{H}$ in the affine parametrisation. Hence we obtain a formula for *surface gravity*:

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)\Big|_{\mathcal{H}}$$

and κ is constant on orbits of ξ.

Killing horizon coincides with event horizon for static spacetimes. Simple example of bifurcate horizon is Rindler.

Having acquainted ourselves with hypersurfaces we can now go on to define what exactly we mean by a black hole, which we have, up to now, regarded as a 'region of no escape', but can also think of as the impossibility of escaping to $\Im^+$. Before we can arrive at that, we will need to write down a flurry of definitions which will also be needed in understanding the statement of the area theorem.

The first definition that will concern us will allow us to distinguish between *future*(*past*), i.e. forwards (backwards) in time, as we have in special relativity, in a general setting. (Note, there are spacetimes which do not admit such a demarcation continuously, but we will not refer to them).

Definition 1.2. Given (M, g), we say this spacetime is *time orientable*, i.e. has a *time orientation*, if there exists a smooth nowhere vanishing timelike vector field, T, on M. Given a causal (i.e. timelike or null) vector, V, we say that V is *future directed* if $V \cdot T < 0$ and *past directed* if $V \cdot T > 0$, and a curve λ whose tangent is such a V, is also, respectively, called *future* or *past directed*.

Definition 1.3. A future (past) directed causal curve $\lambda : I \to M$, where I an open interval in $\mathbf{R}$, is said to have a *future (past) endpoint*, $p \in M$, if for each of its neighbourhood's, O, there is a $t_0 \in I$ such that for all $t > t_0$ we have $\lambda(t) \in O$.

If such a point does not exist, we say the curve is *future (past) inextendible*.

Note that a curve (as above) cannot have more than one future or past endpoint, and that the point p need not lie on it. This definition allows us to separate cases where a curve terminates due to not being defined to extend further, or e.g. due to 'hitting' a singularity.

The next few definitions are crucial for understanding the notion of 'predictability' of general spacetimes.

Definition 1.4. Given $S \subset M$, define the *future domain of dependence of S* as the set

$$D^+(S) = \{p \in M \big| \text{ Every past, inextendible, causal curve through } p \text{ intersects } S\}$$

The *past domain of dependence of S*, $D^-(S)$, is defined similarly, replacing "future" by "past". If S is such that

$$D(S) =: D^+(S) \cup D^-(S) = M$$

then S is said to be a *Cauchy surface*, and (M, g), a *globally hyperbolic* spacetime.

A globally hyperbolic spacetime is a particularly powerful concept as it allows one to determine uniquely solutions to hyperbolic differential equations globally from the initial data on S, where S is a Cauchy surface.

The next definition needed in this discussion will use the idea of an 'asymptotically flat' spacetime. Whilst it is possible to formulate a precise definition of the same, so we know exactly what we want'at infinity', we omit it here and refer the reader to pg. 276 of [8] where it is given in full. The main difficulty in defining an asymptotically flat spacetime is that it is not clear what we mean by taking 'limits to infinity' in general coordinate systems such as to retain coordinate independence. The solution, roughly, is that we can conformally embed our spacetime $(M, g_{\mu\nu})$ into another $(\widetilde{M}, \widetilde{g}_{\mu\nu})$ where the latter has properties similar to Minkowski space.

Given an asymptotically flat spacetime, $(M, g_{\mu\nu})$, define the **causal past** of the subset $\Im^+ \subset M$ by $J^-(\Im^+)$. Define also the [topological] closure of this by $\bar{J}^-(\Im^+)$ and its boundary by $\dot{J}^-(\Im^+) = \bar{J}^-(\Im^+) - J^-(\Im^+)$.

Definition 1.5. (Wald).
Given an asymptotically flat spacetime (M, g) and its associated conformal embedding $(\widetilde{M}, \widetilde{g})$, we say that the former is *strongly asymptotically predictable* if an open region $\widetilde{V} \subset \tilde{M}$ satisfying $\overline{M \cap J^-(\Im^+)} \subset \widetilde{V}$, with the closure taken in $\widetilde{M}$ exists such that $(\widetilde{V}, \widetilde{g})$ is globally hyperbolic.

Note the closure manifests the utility of the embedding we spoke of earlier, in particular $i^0 \in \widetilde{V}$. The crucial point here is that $J^-(\Im^+)$ needn't contain the whole of M, i.e. there can exist regions in M not visible to $J^-(\Im^+)$, so that a black hole can be defined as

Definition 1.6. A strongly asymptotically predictable spacetime (M, g) is said to contain a *black hole* if $M \not\subseteq J^-(\Im^+)$ and the *(future) event horizon* is given by $\mathcal{H}^+ = M \cap \dot{J}^-(\Im^+)$

It is relevant here to mention that a globally hyperbolic spacetime forbids observers outside or on the horizon from detecting a black hole in finite time, meaning that *naked singularities*, i.e. ones singularities not 'covered' by a black hole, are precluded- apart from say an initial singularity. This point forms the basis of the *cosmic censorship hypothesis*, which roughly states: All 'physical' spacetimes are globally hyperbolic, i.e. nature forbids naked singularities. Note this has not been proven to date and is one of the major outstanding problems in classical GR.

Properties of future event horizons:

1. null hypersurface
2. *achronal*, i.e. no two points on the horizon can be connected by a timelike curve
3. does not contain i^0 or $\Im^-$
4. can have past endpoints
5. cannot have future endpoints (Penrose's theorem)

Having introduced null hypersurfaces, we next need the idea of *congruences of curves*: a family of curves, with exactly one passing each point. We call it a *geodesic congruence* if the curves are geodesics. In particular, the integral curves generated by a continuous vector field are a congruence of curves. In the following, we will be following [2] closely. We will be concerned with null geodesic congruences, so to start, consider the null tangent field in the affine parametrisation, ξ^α, generating a congruence. We can define the tensor

$$B_{\mu\nu} = \nabla_\mu \xi_\nu$$

so that $B_{\mu\nu}\xi^{\nu} = B_{\mu\nu}\xi^{\mu} = 0$. Consider a 'displacement' vector η^{α}, that can be chosen to commute with ξ (by appropriately choosing coordinates, c.f. commutability of coordinate basis) so that

$$\mathcal{L}_{\xi}\eta^{\mu} = 0 \Rightarrow \xi \cdot \nabla\eta = \eta \cdot \nabla\xi = B^{\mu}_{\ \nu}\eta^{\nu}. \tag{1.16}$$

$B^{\mu}_{\nu}\eta^{\nu}$ will thus measure *geodesic deviation*, i.e. the failure to parallel transport η^{α}. Hence, given a geodesic, we can describe geodesics nearby by a displacement vector, however such a displacement vector will not be unique since adding a multiple of ξ will give another such vector. To fix this, we impose the conditions

$$\eta \cdot \xi = 0 = \eta \cdot n \qquad n \cdot \xi = -1 \qquad n^2 = 0$$

where n is another vector that has been introduced for the following reason: the space of vectors orthogonal to ξ also includes ξ as it is null, hence the condition $\eta \cdot \xi = 0$ is insufficient to fix the gauge. Thus we choose a vector, n, not orthogonal to ξ, so that the condition $\eta \cdot n$ will uniquely specify the displacement vectors in the two-dimensional space we are interested in. Note the choice $n \cdot \xi = -1$ is arbitrary. Consistency in our choice requires that $t \cdot \nabla n = 0$- (parallel transport). Write a projector $P^{\mu}_{\nu} = \delta^{\mu}_{\nu} + t^{\mu}n_{\nu} + n^{\mu}t_{\nu}$, so that in particular $P^{\mu}_{\nu}\eta^{\nu} = \eta^{\mu}$. Hence, P projects into the two-dimensional tangent space spanned by the η vectors. Project B into the two-dimensional so that it is orthogonal to n as well, i.e. so there are no components in the direction of n. Decompose $\hat{B}$ into algebraic irreducible parts. $\hat{B}^{\mu}_{\ \nu} = P^{\mu}_{\rho}B^{\rho}_{\ \sigma}P^{\sigma}_{\nu}$

It is readily checked that $\hat{B}$ also satisfies a relation similar to (1.16) i.e.

$$\hat{B}^{\mu}_{\ \nu}\eta^{\nu} = t \cdot \nabla\eta^{\mu} \tag{1.17}$$

Define the *expansion*, *shear* and *twist*, respectively, as:

$$\theta = \hat{B}^{\mu}_{\ \nu} \tag{1.18}$$

$$\hat{\sigma}^{\mu}_{\ \nu} = \hat{B}_{(\mu\nu)} - \frac{1}{2}P^{\mu}_{\ \nu}\hat{B}^{\rho}_{\ \rho} \tag{1.19}$$

$$\hat{\omega}^{\mu}_{\ \nu} = \hat{B}_{[\mu\nu]} \tag{1.20}$$

which (algebraically) decomposes $\hat{B}$ as

$$\hat{B}^{\mu}_{\ \nu} = \frac{1}{2}\theta P^{\mu}_{\ \nu} + \hat{\sigma}^{\mu}_{\ \nu} + \hat{\omega}^{\mu}_{\ \nu} \tag{1.21}$$

A relation which will be of great use to us is the Raychaudhuri equation (for null geodesics), which is also of key importance in establishing the singularity theorems of Hawking and Penrose. We arrive at it from the following considerations. First,

$$\frac{d\theta}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\frac{\partial}{\partial x^{\mu}}\theta = \xi \cdot \nabla\theta = \xi \cdot \nabla\hat{B}^{\mu}_{\ \mu}$$

using $\xi \cdot \nabla\xi = 0$ and $\xi^2 = 0$

$$\hat{B}^{\mu}_{\ \nu} = B^{\mu}_{\ \nu} + \xi^{\mu}(n_{\sigma}B^{\sigma}_{\ \nu} + n_{\sigma}B^{\sigma}_{\ \rho}n^{\rho}\xi_{\nu}) + (B^{\mu}_{\ \rho}n^{\rho})\xi_{\nu} \tag{1.22}$$

Since $B^\mu_{\ \nu}\xi_\mu = 0$, we have (again with $\xi \cdot \nabla\xi = 0$ and $\xi^2 = 0$),

$$\hat{B}^\mu_{\ \mu} = B^\mu_{\ \mu} = B^\mu_{\ \nu} P^\nu_{\ \mu}$$

Recall that $\xi \cdot \nabla n = 0$. This means $\xi \cdot \nabla P^\nu_{\ \mu} = 0$. Hence,

$$\begin{aligned}
\frac{d\theta}{d\lambda} &= \xi \cdot \nabla (B^\mu_{\ \nu} P^\nu_{\ \mu}) \\
&= P^\nu_{\ \mu} \xi \cdot \nabla B^\mu_{\ \nu} \\
&= P^\nu_{\ \mu} \xi^\rho \nabla_\rho \nabla_\nu \xi^\mu \\
&= P^\nu_{\ \mu} \xi^\rho \nabla_\nu \nabla_\rho \xi^\mu + P^\nu_{\ \mu} \xi^\rho \left[\nabla_\rho, \nabla_\nu\right] \xi^\mu
\end{aligned}$$

Note $\nabla(\xi \cdot \nabla\xi) = (\nabla\xi)(\nabla\xi) + \xi\nabla\nabla\xi = 0 \implies \xi\nabla\nabla\xi = -(\nabla\xi)(\nabla\xi) = -BB$, and $\left[\nabla_\mu, \nabla_\nu\right] \xi^\alpha = R^\alpha_{\ \beta\mu\nu} \xi^\beta$

$$\begin{aligned}
\frac{d\theta}{d\lambda} &= -P^\nu_{\ \mu} B^\mu_{\ \rho} B_\nu^{\ \rho} + P^\nu_{\ \mu} \xi^\rho R^\mu_{\ \sigma\rho\nu} \xi^\sigma \\
&= -P^\nu_{\ \mu} B^\mu_{\ \rho} B^\rho_{\ \nu} + \delta^\nu_\mu \xi^\rho R^\mu_{\ \sigma\rho\nu} \xi^\sigma \\
&= -\hat{B}^\mu_{\ \rho} \hat{B}^\rho_{\ \mu} - R_{\sigma\rho} \xi^\sigma \xi^\rho
\end{aligned}$$

where the last equality holds due to the trace, and made manifest by e.g. copious use of the identities.

Using $P^{\mu\nu} P_{\nu\mu} = \mathrm{tr}\delta + 4(n \cdot \xi) + 2(n \cdot \xi)(n \cdot \xi) = 4 - 4 + 2 = 2$, and $P^{\mu\nu} \hat{B}_{\nu\mu} = \mathrm{tr}\hat{B} = \theta$, which again is obtained from the identities. This gives

$$P^{\mu\nu} \sigma_{\nu\mu} = P^{\mu\nu} \hat{B}_{\nu\mu} - \frac{1}{2}\theta P^{\mu\nu} P_{\nu\mu} = \theta - \theta = 0$$

Hence we can write

$$\begin{aligned}
-\hat{B}^\mu_{\ \rho} \hat{B}^\rho_{\ \mu} &= -(\frac{1}{2}\theta P^\mu_{\ \rho} + \hat{\sigma}^\mu_{\ \rho} + \hat{\omega}^\mu_{\ \rho})(\frac{1}{2}\theta P^\rho_{\ \mu} + \hat{\sigma}^\rho_{\ \mu} + \hat{\omega}^\rho_{\ \mu}) \\
&= -(\frac{1}{4}\theta^2 P^\mu_{\ \rho} P^\rho_{\ \mu} + \frac{1}{2}\theta P^\mu_{\ \rho} \hat{\sigma}^\rho_{\ \mu} + \frac{1}{2}\theta \hat{\sigma}^\mu_{\ \rho} P^\rho_{\ \mu} + \hat{\sigma}^\mu_{\ \rho} \hat{\sigma}^\rho_{\ \mu} + \hat{\omega}^\mu_{\ \rho} \hat{\omega}^\rho_{\ \mu}) \\
&= -(\frac{1}{2}\theta^2 + \hat{\sigma}^{\mu\rho} \hat{\sigma}_{\mu\rho} - \hat{\omega}^{\mu\rho} \hat{\omega}_{\mu\rho})
\end{aligned}$$

where antisymmetry has been used to write the last line in a prettier way. Finally, we can write Raychadhuri's equation:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\rho} \hat{\sigma}_{\mu\rho} + \hat{\omega}^{\mu\rho} \hat{\omega}_{\mu\rho} - R_{\sigma\rho} \xi^\sigma \xi^\rho \tag{1.23}$$

In particular, let us consider this equation given the *weak energy condition*, $T_{\mu\nu} \xi^\mu \xi^\nu \geq 0$, and the expansion of θ of the null geodesic generator of a null hypersurface, so that

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 - 8\pi T_{\mu\nu} \xi^\mu \xi^\nu \leq -\frac{1}{2}\theta^2 \iff \frac{d}{d\lambda}\theta^{-1} \geq \frac{1}{2} \iff \theta^{-1} \geq \theta_0^{-1} + \frac{1}{2}\lambda \tag{1.24}$$

where we have used the positivity of $\hat{\sigma}^2$, the vanishing of omega (since ξ here is normal to the null hypersurface andusing (1.22) with Frobenius' theorem) and that ξ is null in Einstein's equation in the first inequality. Here θ_0 is the initial value of θ. Suppose that $\theta_0 < 0$, then $\theta \to -\infty$ within an affine length of $2/|\theta_0|$. This is easy to see, since $\theta \leq \theta_0/(1 + (\lambda\theta_0/2))$ and the denominator is zero at $\lambda = 2/|\theta_0|$. This tells us that if there is convergence in the congruence at any point, then there will exist caustics which we can think of as singularities in the congruence. Conjugate points? $\theta = 0$ for stationary spacetimes.

We define the area element: $a = \varepsilon^{\mu\nu\rho\sigma}\xi_\mu n_\nu \eta'_\rho \eta''_\sigma$ then

$$\frac{da}{d\lambda} = \xi \cdot \nabla a = \varepsilon^{\mu\nu\rho\sigma}\xi_\mu n_\nu(\xi \cdot \nabla(\eta'_\rho)\eta''_\sigma + \eta'_\rho \xi \cdot \nabla \eta''_\sigma) = \varepsilon^{\mu\nu\rho\sigma}\xi_\mu n_\nu \hat{B}_\rho{}^\lambda(\eta'_\lambda \eta''_\sigma - \eta''_\lambda \eta'_\sigma) \tag{1.25}$$

$$\begin{aligned}
4\varepsilon^{\mu\nu\rho\sigma}B_\rho{}^\lambda \xi_{[\mu}n_\nu \eta'_\lambda \eta''_{\sigma]} &= \varepsilon^{\mu\nu\rho\sigma}B_\rho{}^\lambda(\xi_\lambda n_{[\mu}\eta'_\nu \eta''_{\sigma]} + n_\lambda \xi_{[\mu}\eta'_{nu}\eta''_{\sigma]} + \xi_{[\mu}n_\nu \eta''_{\sigma]}\eta'_\lambda - \xi_{[\mu}n_\nu \eta'_{\sigma]}\eta''_\lambda) \\
&= \varepsilon^{\mu\nu\rho\sigma}B_\rho{}^\lambda(\xi_{[\mu}n_\nu \eta''_{\sigma]}\eta'_\lambda - \xi_{[\mu}n_\nu \eta'_{\sigma]}\eta''_\lambda)) \qquad \text{(Using } B \cdot \xi = B \cdot n = 0) \\
&= \varepsilon^{\mu\nu\rho\sigma}\xi_\mu n_\nu \hat{B}_\rho{}^\lambda(\eta'_\lambda \eta''_\sigma - \eta''_\lambda \eta'_\sigma) \qquad \text{(equation (1.25))} \\
&= \frac{4}{4!}\varepsilon_{\mu\nu\lambda\sigma}\varepsilon^{\mu\nu\rho\sigma}B_\rho{}^\lambda \xi_{\mu_1}n_{\mu_2}\eta'_{\mu_3}\eta''_{\mu_4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4} \\
&= \frac{4!}{4!}\delta^\rho_\lambda B_\rho{}^\lambda \xi_{\mu_1}n_{\mu_2}\eta'_{\mu_3}\eta''_{\mu_4}\varepsilon^{\mu_1\mu_2\mu_3\mu_4} \\
&\implies \frac{da}{d\lambda} = \theta a
\end{aligned} \tag{1.26}$$

Theorem 1.1 (Hawking's area theorem). *Let (M, g) be a strongly predictable asymptotically flat spactime satisfying $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ for all null ξ^μ. The area, $\mathcal{A}$, of the future event horizon doesn't decrease with time.*

Proof. The requirement of a strong asymptotically flat spacetime is equivalent to the requirement that a family of Cauchy surfaces, $\Sigma(\lambda)$, exist such that $\Sigma(\lambda') \subset D^+(\Sigma(\lambda))$. We know that $\mathcal{H}^+$ is a null hypersurface, so we can use the affine parameter, λ if its generator to parametrise the Cauchy surfaces, so that the area of the event horizon is given by $\mathcal{A}(\lambda) = \mathcal{H}^+ \cap \Sigma(\lambda)$- i.e. the area of the horizon at affine point λ. Hence we seek to prove $\mathcal{A}(\lambda') \geq \mathcal{A}(\lambda)$ for $\lambda' > \lambda$.

In order to do this, it suffices to consider the area elements above. Recall

$$\theta \geq 0 \Rightarrow \frac{da}{d\lambda} = \theta a \geq 0 \tag{1.27}$$

so that all we need to show is $\theta \geq 0$ everywhere on $\mathcal{H}^+$. Suppose it's not. Then $\theta \to -\infty$ in finite affine parameter. Since generators of the horizon do not have future endpoints, by theorem 4.5.12 in Hawking and Ellis, which states that if a point r lies in the interval of two points p and q, such that r is a 'conjugate' point to p along a geodesic γ, then there is a timelike curve connecting p to q. Here r conjugate to p along γ means that $\theta \to -\infty$ at r. But if there is such a timelike curve for the generators of the horizon then the achronicity property of the horizon is violated. Hence $\theta \geq 0$. Hence

$$\frac{d\mathcal{A}}{d\lambda} \geq \int \theta da \geq 0 \tag{1.28}$$

and the proof is complete. $\square$

Hawking Radiation

Particle creation in curved spacetime

In order to show that black holes radiate, and are indeed thermodynamic in their own right, we will adopt the approach of Unruh, which captures the main ideas of the original caluclation by Hawking [6], and has the advantage of not relying on transplankian modes. The latter used what is known as the *geometrical optics approximation* to tackle the question. These are semi-classical calculations wherein the black hole is treated as classical and the fields are quantum mechanincal, i.e. gravity is not quantised. This however does not appear to detract from the validity of the result, since curvature is small in the region where these particles are created.

Consider the generally covariant Klein-Gordon equation[2]:

$$(\Box - m^2)f_i = 0$$

which has solutions $\{f_i\}$. For a pair of solutions, we introduce the "inner product"[3] (endowing the space of solutions with a symplectic structure):

$$\begin{aligned}(f_1, f_2) &\equiv i \int d\Sigma^\mu \, (f_2^* \partial_\mu f_1 - f_1 \partial_\mu f_2^*) \\ &\equiv i \int d\Sigma^\mu \, (f_2^* \overset{\leftrightarrow}{\partial_\mu} f_1)\end{aligned}$$

where $d\Sigma^\mu$ is the volume element pointing normal to a Cauchy hypersurface, i.e. we consider globally hyperbolic spacetimes. This will guarantee that the space of global solutions corresponds is determined by inital data on a Cauchy surface. It is not hard to show that this definition is independent of the choice of hypersurface, i.e. that

$$(f_1, f_2)_{\Sigma_1} = (f_1, f_2)_{\Sigma_2}$$

To do this, take two hypersurfaces Σ_1 and Σ_2 where functions on these surfaces vanish at spatial infinity (in case they're non-compact). Let V be the four-volume bounded by these hypersurfaces and if necessary, time-like surfaces on which the functions vanish. Then we have

$$\begin{aligned}(f_1, f_2)_{\Sigma_1} - (f_1, f_2)_{\Sigma_2} &= i \oint_{\partial V} d\Sigma^\mu \, (f_2^* \overset{\leftrightarrow}{\partial_\mu} f_1) = i \int_V dV \, \nabla^\mu (f_2^* \overset{\leftrightarrow}{\partial_\mu} f_1) \\ &= i \int_V dV \, \nabla^\mu (f_2^* \partial_\mu f_1 - f_1 \partial_\mu f_2^*) \\ &= i \int_V dV \, (f_2^* \Box f_1 - f_1 \Box f_2^*) = i \int_V dV \, m^2 (f_2^* f_1 - f_1 f_2^*) \\ &= 0\end{aligned}$$

[2] In general we will have $(\Box - m^2 - \xi R)f_i = 0$ and we take the "minimal coupling" prescription $\xi = 0$, although this is not essential.

[3] Not positive definite since $(f, f) = -(f^*, f^*)$.

where dV is the four-volume element, and in the second line the quadratic terms cancel the Klein Gordon equation gives the final step. Talk about volumes- see jacobson. We can choose the basis $\{f_i, f_i^*\}$ such that the following relations are satisfied,

$$(f_i, f_j) = \delta_{ij}, \quad (f_i^*, f_j^*) = -\delta_{ij}, \quad (f_i, f_j^*) = 0 \tag{1.29}$$

A solution of the Klein-Gordon (KG)equation can be expanded as:

$$\varphi(x) = \sum_k (a_k f_k(x) + a_k^* f_k^*(x))$$

which gives the quantized field

$$\hat{\varphi}(x) = \sum_k (\hat{a}_k f_k(x) + \hat{a}_k^\dagger f_k^*(x)) \tag{1.30}$$

where hats are placed to denote operators. We can write,

$$\hat{a}_i = (\hat{\varphi}, f_i) \qquad \hat{a}_i^\dagger = -(\varphi, f_i^*) \tag{1.31}$$

and the commutation relation, $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ - all else vanish. In flat space, the field operator, φ, can be decomposed into Fourier modes

$$\hat{\varphi} = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega V}} (\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t})$$

where $\omega = \sqrt{|\mathbf{k}|^2 + m^2}$, V is the volume on which φ is defined, the vacuum state is defined as

$$\hat{a}_{\mathbf{k}} \left| 0 \right\rangle = 0$$

i.e. the state annihilated by all $\hat{a}_{\mathbf{k}}$'s, and the $\hat{a}_{\mathbf{k}}^\dagger$'s create particles. The above expansion gives $f \sim e^{-i\omega t}$- the *positive frequency* modes. This is a canonical choice that we make in Minkowski spacetime: we have a natural split of positive and negative norm solutions whence we can define the vacuum uniquely (using Poincaré invariance) and hence the Fock space, no matter the frame in which t is taken to be the time coordinate. Explicitly, the *positive* Fourier modes $u(t, \mathbf{x})_{\mathbf{k}}$ are the set of orthonormal functions satisfying the KG equation, with

$$ik^\mu \partial_\mu u_{\mathbf{k}} = \omega_{\mathbf{k}} u_{\mathbf{k}} \tag{1.32}$$

and $k^\mu = (1, 0, 0, 0)$ is a timelike Killing vector, and we can go on to define our Fock space etc.

However, splitting positive and negative frequency modes in curved spacetime is coordinate dependent and since, *prima facie*, there is no rule for choosing from the many bases, $\{f_i, f_i^*\}$, there is no unique vacuum. This means it is unclear how to define a particle in curved spacetime, rather such a definition is coordinate dependent. We would like to understand this 'non-uniqueness' further. A Fock space can be constructed from (1.30) by taking the 'vacuum' as the state satisfying[4]

$$\hat{a}_k \left| 0 \right\rangle_a = 0 \qquad \left\langle 0 | 0 \right\rangle_a = 1$$

[4] The inner product, $\langle \, | \, \rangle$, is positive definite

for all k, and Fock space, H, has basis $\{|0\rangle_a, a_i^\dagger|0\rangle_a, a_i^\dagger a_j^\dagger|0\rangle_a, \dots\}$. Consider now two bases of complex solutions to the KG equation: $\{f_i, f_i^*\}$, taken as before, with time coordinate t and vacuum $|0\rangle_a$; and $\{h_i, h_i^*\}$, with time coordinate t' and vacuum $|0\rangle_b$, analogously defined, so we can expand as

$$\hat{\varphi}(x) = \sum_k (\hat{a}_k f_k(x) + \hat{a}_k^\dagger f_k^*(x)) = \sum_k (\hat{b}_k h_k(x) + \hat{b}_k^\dagger h_k^*(x)) \tag{1.33}$$

Since we have two complete bases, we can write

$$f_i = \sum_j \alpha_{ij} h_j + \beta_{ij} h_j^* \qquad f_i^* = \sum_j \alpha_{ij}^* h_j^* + \beta_{ij}^* h_j \tag{1.34}$$

These expressions go by the name of a *Bogolubov transformation*, and the coefficients α_{ij}, β_{ij} are called *Bogolubov coefficients*, which, using (1.29) satisfy

$$\sum_k \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* = \delta_{ij} \tag{1.35}$$

and we can write operators using (1.31) as

$$\hat{a}_i = \sum_j \alpha_{ij}\hat{b}_j + \beta_{ij}^*\hat{b}_j^\dagger \quad \hat{b}_i = \sum_j \alpha_{ij}^*\hat{a}_j - \beta_{ij}^*\hat{a}_j^\dagger \tag{1.36}$$

Of particular use is the particle number operator, defined for the f_i-th mode

$$N_i^a = a_i^\dagger a_i \tag{1.37}$$

where the superscript labels the basis. Also,

$$\langle 0|N_i^a|0\rangle_b = \langle 0|a_i^\dagger a_i|0\rangle_b = \sum_{j,k}\langle 0|(b_j\beta_{ji})(\beta_{ik}^* b_k^\dagger)|0\rangle_b = \sum_j |\beta_{ji}|^2 \tag{1.38}$$

where we have used the commutation relation for the b-operators to pick up a delta in the last step.

This poses the following problem: an observer using operators $\hat{a}_i$ will not regard the state $|0\rangle_b$ of an observer using operators $\hat{b}_i$ as the vacuum, rather will find particles therein. Thus, the vacuum in one setting appears to have a multitude of particles when observed in another. In cases where the spacetime is asymptotically flat, it may be suggested to use the natural particle definitions of this region to bypass the above ambiguity. However, modes travelling through curved regions may be found to contain particles despite not starting out with any. This is the particle creation effect and it occurs when $\beta \neq 0$.

Unruh Effect

Having established the phenomena of particle creation in curved spacetime, let us now make contact with black holes and radiation. Before we can do that though, we need to first review particle creation in Minkowski spacetime as seen by an accelerated observer, i.e. Rindler spacetime, where we will find that the Minkowski vaccum appears thermal:

the Unruh effect. We will work in (1+1)-dimensional spacetime for simplicity, but this easily generalises.

The metric in two-dimensional Rindler space is related to the Minkowski metric as

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) = -dt^2 + dx^2 \tag{1.39}$$

where $x = \frac{1}{a}e^{2\xi}\cosh(a\eta)$ and $t = \frac{1}{a}e^{2\xi}\sinh(a\eta)$ gives the relation to Minkowski space. These coordinates only cover the region $x > |t|$- 'right wedge'. By defining the null coordinates $u = \eta + \xi$ and $v = \eta - \xi$ we see

$$ds^2 = -2e^{2a\xi}dudv$$

where the patch covered by these coordinates is $u < 0$, $v > 0$. If we further define U and V as exponentials of u and v respectively, we see there is a close analogy existing between transformations taking us from Minkowski to Rindler, and Schwarzschild to Kruskal: A spacetime with a horizon usually displays such an exponential relationship between two coordinate systems. The horizon in Rindler space is an *acceleration horizon*, $u = 0 = v$, which chops up our spacetime into wedges (right and left) such that they are causally disconnected. The horizon is in fact a bifurcate Killing horizon so we can use this to capture the essential aspects of an arbitrary Killing horizon.

We have Cauchy surfaces given by $\eta = const$, so the Rindler wedge is globally hyperbolic, but it does not cover all of Minkowski space. Hence we are left with the problem that these Cauchy surfaces are not Cauchy in Minkowski space, so modes defined on a Rindler patch do not provide a complete set into which a general Minkowski solution can be expanded. Thus we seek to seek to extend Rindler coordinates and in fact it is sufficient to cover the region $|x| > |t|$, i.e. right and left wedges. To do this, note that η gives us the timelike Killing vector

$$\frac{\partial}{\partial\eta} = ax\frac{\partial}{\partial t} + at\frac{\partial}{\partial x} \tag{1.40}$$

which is equally good in both left and right wedges (it will run backwards to time in the left wedge ($t \sim -\sinh\eta$))[5].

The KG-equation (massless for simplicity) has postive frequency solutions[6]:

$$f^R(k) = \begin{cases} e^{-i\omega\eta + ik\xi} & \text{right wedge} \\ 0 & \text{left wedge} \end{cases} \tag{1.41}$$

$$f^L(k) = \begin{cases} 0 & \text{right wedge} \\ e^{i\omega\eta + ik\xi} & \text{left wedge} \end{cases} \tag{1.42}$$

where we have explicitly indicated the support for our solutions, and $|\omega| = k$.. We can write the field by expanding as

$$\hat{\varphi} = \sum_k \hat{b}^R_k f^R_k + \hat{b}^{\dagger R}_k f^{*R}_k + \hat{b}^L_k f^L_k + \hat{b}^{\dagger L}_k f^{*L}_k \tag{1.43}$$

[5] So we are using a boost timelike Killing vector field, as opposed to a time translation one.

[6] use $\nabla^2\phi \sim \partial(\sqrt{g}g\partial\phi)$. Note also that the two dimensional case is special

Our task is to find the particle content as seen by a Rindler observer. Thus, we need to expand Minkowski in a basis of Rindler and then compute the Bogolubov transformation in order to achieve this. Bogolubov will relate Minkowski vacuum to Rindler whose vacuum is $|0\rangle_R = |0\rangle^R \otimes |0\rangle^L$ so that it covers the full Cauchy slice in Minkowski. Recall the Minkowski expansion is given by

$$\hat{\varphi} = \sum_k \hat{c}_k h_k + \hat{c}_k^\dagger h_k^* \tag{1.44}$$

Unruh essential insight: work with any complete set of Minkowski modes (will give same vacuum). Any positive freq. set will do so long as it is (a) pos. freq. (b) well defined on timeslice (c) can be analytically continued. Now writing, which is easily verifiable

$$(a(x-t))^{i\omega/a} = \begin{cases} f^R(k) & \text{right wedge} \\ e^{-\pi\omega/a} f^{*L}(-k) & \text{left wedge} \end{cases} \tag{1.45}$$

we can see the similarity with Schwarzschild in the geometrical optics approach. Then

$$h(k) = N(f^R(k) + e^{-\pi\omega/a} f^{*L}(-k)) \tag{1.46}$$

where N is a normalisation constant. Bogolubov transformation is then (dropping left and right notation)

$$b = N(c + e^{-\pi\omega/a} c^\dagger) \tag{1.47}$$

hence we find for N

$$b = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}(e^{\pi\omega/a} c + e^{-\pi\omega/a} c^\dagger) \tag{1.48}$$

Number operator gives: (note c's kill mink. vacua)

$$\langle N \rangle_R = \langle 0|b^\dagger b|0\rangle_M = \frac{1}{e^{2\pi\omega/a} - 1} \tag{1.49}$$

and Planck distribution.

$$kT = \frac{a}{2\pi}\frac{\hbar}{c} \tag{1.50}$$

replacing dimensionality constants. Note that essentially what happened was that horizon split region; frequency mismatch; observer can see pair production at horizon without violating energy cons.

Black Hole Thermality

In order to analyse particle creation by a black hole[7], we can use our knowledge of Minkowski to Rindler, to relate Schwarzschild to Kruskal, although black holes occur naturally arise due to gravitational collapse, and an eternal black hole radiating is somewhat unphysical nonetheless provides a useful way of approaching the problem. So we work analogously to the Rindler case. We quantise the right hand (region I) of Kruskal by taking the Cauchy surface $\mathcal{H}^- \cup \Im^-$ and finding positive modes here. There are two Killing

[7] this follows [17].

vectors at $\mathcal{H}^-$, one given by Kruskal-Szekeres, and the other by Schwarzschild time. We choose affine (akin to geometrical optics) which is given by Kruskal-Szekeres. For $\Im^-$ take the null past coordinate. Then we seek to relate the positive frequency solutions from the past to the future region $\mathcal{H}^+ \cup \Im^+$ by computing the Bogolubov transformation. Again, by analogy with Rindler, we do this by extending our solutions to other regions. In the end we find

$$T = \frac{1}{8\pi M} \tag{1.51}$$

Information Paradox

The radiation Hawking found in 1974, and we have been discussing, raises very big problems which theorists have been trying to resolve ever since. Here we discuss what is known as the 'information' paradox, or 'loss of quantum coherence.' This is

To begin, Hawking discovered that black holes radiation is exactly thermal, in the semiclassical approximation (which is not exact, in particular backreaction effects- backscattering of states affecting the geometry around the black hole- give rise to *greybody factors*[8] that are deviations away from perfect thermality), and independent of the details of the collapsing object, i.e. the initial state, characterised by a variety of parameters, collapses to a black hole described by only a few. This last statement is the *no-hair conjecture* which loosely reads: stationary black holes are characterised by their mass, angular momentum and (electric or magnetic) charge- where these quantities are in fact global charges that can be measured at infinity. There already appears to be loss of information. However, the initial state is just hiding behind the horizon and part of a quantum system along with the outgoing radiation, so everything seems to be alright at this point. But let's probe this situation deeper.

First let us recall some basic facts about states in quantum mechanics. A basic principle in quantum theory is known as the 'quantum xerox principle', or 'no-cloning theorem'. make a duplicate of state, and (schematically) goes like this. Suppose we have a state that we insert into a machine which outputs the state and its duplicate

$$|\psi\rangle \to |\psi\rangle \times |\psi\rangle \tag{1.52}$$

then

$$|\psi_1\rangle + |\psi_2\rangle \to (|\psi_1\rangle + |\psi_2\rangle) \times (|\psi_1\rangle + |\psi_2\rangle)$$

where $|\psi\rangle = |\psi_1\rangle + |\psi_2$. This is wrong by the rule of linear evolution of state vectors which asserts

$$|\psi_1\rangle + |\psi_2\rangle \to |\psi_1\rangle \times |\psi_1\rangle + |\psi_2\rangle + |\psi_2\rangle \tag{1.53}$$

so we see if one copy of information were to be outside a black hole we cannot have the same inside and vice versa. Recall also the notion of pure and mixed state: the former being described by linear combinations of state vectors, and the latter a statistical ensemble of pure states, e.g. an entangled state, which we can describe by a density matrix ρ whence

[8] Here we are using the word 'thermal' to denote a planckian spectrum of radiation. Greybody factors are then a feature of any warm body whose wavelength is similar to its size.

the von-Neumann entropy, $S = -\mathrm{tr}\ \rho \ln \rho$, can be computed. What we need to take home is: "unitarity demands pure states do not evolve to mixed states".

By Stefan's law, as black holes radiate they will lose mass, at a rate $dM/dt \sim M^{-2}$, which gives a lifetime of $\tau \sim M^{3}$[9], although what exactly happens to the black hole at late times (when the hole is Planck sized) requires a quantum theory of gravity to tell us. As the black hole evaporates we have three possibilities

1. we are left with a singularity
2. we are left with a stable remnant
3. the black hole radiates away all its mass and disappears.

Option 1 is not satisfactory as it means the theory is incomplete since quantum mechanics will not have explained the singularity problem. Option 2 seems to imply an infinite number of possible states since black holes can be arbitrary sized, so an arbitrary amount of information is stored in this remnant. This doesn't seem to be correct as an infinite number of states within a finite bounded region is very problematic, e.g. how could we keep the remnant stable? Option 3 means the black hole has to somehow get its information out before it vanishes in order for information loss not to occur. The last option presents two main difficulties, which are in essence the information paradox:

1. the initial state is uncorrelated with the outgoing radiation: how does it transfer information? If the initial state starts of pure, how does it end in a mixed state?[10]
2. by a Schwinger process, we can create an entangled pair (in mixed state). If this pair straddles the horizon, where one falls in and the other shoots off to infinity, then when the black hole disappears, the quanta at infinity will be in mixed state with nothing to mix with!

These two problems have been vigorously attacked, but the information seems to be lost unless we change fundamental notions like locality. Recently however, there has come a proposal from string theory, developed by Mathur and others [23][24], which seems to have a solution in what is called 'fuzzballs'. The key point here is that the horizon is not empty space: the matter making the hole exists up to the horizon, so that information is radiated out.

[9] this is longer than the age of the universe for a solar mass sized black hole!
[10] This is a point in principle not in practice where information is often lost.

Chapter 2

String Theory Background

2.1 Strings

A string is a one dimensional object sweeping out a *world-sheet* in space time akin to the point particle's world-line. Hence, to write an action describing the string we consider the relativistic point particle first. We will see that the action for the point particle is proportional to the length of its world-line; generalising, the string action should thus be proportional to the world-sheet area.
The action for a (massive) point particle in D dimensions is,

$$S = -m \int d\tau \sqrt{-\dot{X}^2} \tag{2.1}$$

Here we have used $\dot{X}^\mu = \dfrac{dX^\mu}{d\tau}$ and $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$. The vector $X^\mu(\tau)$ describes the position of the particle along its world-line, parametrised by τ which we take to be proper time. Also, we are working in D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$ with signature $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, ..., +1)$.

Varying, we have:

$$\delta S = m \int_{\tau_2}^{\tau_1} d\tau \frac{\partial}{\partial \tau} \left[\frac{-\dot{X}_\mu}{\sqrt{-\dot{X}^2}} \right] \delta X^\mu + \underbrace{m \left[\frac{\dot{X}_\mu \delta X^\mu}{\sqrt{-\dot{X}^2}} \right]_{\tau_2}^{\tau_1}}_{\text{surface term}} \tag{2.2}$$

Where the 'surface' term vanishes by imposing suitable boundary conditions. This yields the equation of motion:

$$\frac{\partial}{\partial \tau} \frac{-m\dot{X}_\mu}{\sqrt{-\dot{X}^2}} = 0 \tag{2.3}$$

There are a couple of unsatisfactory things with the above action. First off, there is a square root which likes to make life hard. Second, it is clear this action is useless to describe massless particles. Let us do better by writing an action which holds for massless particles too, and is easier to deal with. Consider

$$S = \frac{1}{2} \int d\tau \, (e^{-1}\dot{X}^2 - em^2) \tag{2.4}$$

where $e = e(\tau)$.
The equations of motion, varying with respect to X^μ and then by e, are:

$$\frac{\partial}{\partial\tau}(-e^{-1}\dot{X}^\mu) = 0 \tag{2.5}$$

$$\dot{X}^2 + e^2 m^2 = 0 \tag{2.6}$$

We can show equivalence with the previous action by solving (2.6) for e,

$$e = \frac{\sqrt{-\dot{X}^2}}{m}$$

from which we recover [the equation of motion] (2.3) by substituting into (2.5). Plugging this expression for e in the action (2.4) gets us back (2.1), the action we started with.

The action has symmetry under both Poincaré transformations and reparametrisations. Recall the first of these is given by the transformation

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu$$

where c^μ is a constant translation and Λ is a Lorentz matrix satisfying $\eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu = \eta_{\mu\nu}$. The second can be shown by realising that $e(\tau)$ in fact acts as an 'einbein'. Writing $e = \sqrt{-g_{\tau\tau}}$, where $g_{\tau\tau}$ is the 1d metric on the particle's world-line and $g_{\tau\tau}g^{\tau\tau} = 1$, we get reparametrisation invariance by

$$e(\tau)d\tau = \sqrt{g_{\tau\tau}}d\tau = \sqrt{g_{\tau\tau}d\tau^2} \equiv \sqrt{g_{\tilde{\tau}\tilde{\tau}}d\tilde{\tau}^2} = e(\tilde{\tau})d\tilde{\tau}$$

$\therefore$ $d\tilde{\tau}/d\tau = e/\tilde{e}$, where $\tilde{e} \equiv e(\tilde{\tau})$. Since $dX/d\tau = (dX/d\tilde{\tau})(d\tilde{\tau}/d\tau)$ we have invariance as required, i.e.

$$S = \frac{1}{2}\int d\tilde{\tau}\,(\tilde{e}^{-1}\left(\frac{dX}{d\tilde{\tau}}\right)^2 - \tilde{e}m^2) \tag{2.7}$$

Nambu-Goto Action

In order to arrive at an action for a string, we parametrise its world-sheet by two coordinates, $\sigma^a = (\tau, \sigma)$, timelike and spacelike respectively where $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \pi$. The world-sheet is mapped onto the D-dimensional Minkowski spacetime background, or *target space*, via $X^\mu(\sigma, \tau)$. Strings can be **open** or **closed**, for which we have a curved world-sheet in the former case and a curved cylinder in the latter. Our action- in keeping with our earlier comments about proportionality with the world-sheet area- is hence written,

$$\begin{aligned} S = -T\int dA \;&= -T\int d\tau d\sigma\,\sqrt{-\det\left(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right)} \\ &= -T\int d^2\sigma\,\sqrt{-\det\gamma_{ab}} \end{aligned} \tag{2.8}$$

where γ_{ab} is the *induced* metric on the world-sheet and T is for tension.

$$T = \frac{1}{2\pi\alpha'}, \quad \alpha' = l_s^2$$

where we have defined the "string length" l_s and α' is the "universal Regge slope". It is easily checked that all of the above is dimensionally consistent. This is the basic scale of the theory. Alternatively, by evaluating the determinant, we can write

$$S = -T \int d^2\sigma \, \sqrt{-(\dot{X} \cdot X')^2 + \dot{X}^2 X'^2}$$

This is the "Nambu-Goto action". Again, as with the point particle action (2.1), we have a square root which is bothersome (in particular for purposes of quantisation). Thankfully, we have an alternative, (classically) equivalent action:

Polyakov action

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{2.9}$$

where $g_{\alpha\beta}$ is the worldsheet metric. The symmetries of this action include those of Nambu-Goto, but with an extra addition: Weyl invariance.

$$g_{\alpha\beta} \to e^{2\phi(\sigma)} g_{\alpha\beta}$$

Conformal gauge

$$g_{\alpha\beta} = e^{2\phi(\sigma)} \eta_{\alpha\beta}$$

In this gauge, the action becomes

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \cdot \partial^\alpha X$$

variation:

$$\delta S = \frac{1}{2\pi\alpha'} \int d^2\sigma \, (\partial_\alpha \partial^\alpha X) \cdot \delta X + \frac{1}{2\pi\alpha'} \left[\int_0^\pi d\sigma \, \dot{X} \cdot \delta X \right]_{\tau=\tau_i}^{\tau=\tau_f} \tag{2.10}$$

$$- \frac{1}{2\pi\alpha'} \left[\int_{\tau_i}^{\tau_f} d\tau \, X' \cdot \delta X \right]_{\sigma=0}^{\sigma=\pi} \tag{2.11}$$

In order for the last term to vanish- like the second term above- and equation of motion reduce to

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{2.12}$$

we need

$$X'^\mu \delta X_\mu \big|_{\sigma=0,\pi} = 0$$

we have boundary conditions (not unique but physically relevant):
"Neumann" boundary conditions:

$$X'^\mu \big|_{\sigma=0,\pi} = 0 \tag{2.13}$$

ends are freely placed in spacetime. "Dirichlet" boundary conditions:

$$\delta X^{\mu}|_{\sigma=0,\pi} = 0 \tag{2.14}$$

$\rightarrow$ end points fixed at $X^{\mu} = c^{\mu}$ where c^{μ} is constant. Considering Neumann boundary conditions (at both ends) for some coordinates $X^{a} = 0, ..., p$, and Dirichlet (at both ends) for $X^{I} = p+1, ..., D-1$ gives

$$SO(1, D-1) \rightarrow SO(1, p) \times SO(D-p-1)$$

so that the ends of the (open) string are fixed in a $(p+1)$-dimension hypersurface- a *D-brane* or *Dp-brane* where p is the spatial dimension of the brane. This hypersurface is in fact a dynamical object due to the momentum from open strings ending on it. Note that the equation of motion (2.12) is just the two-dimensional Laplace equation which has solutions

$$X^{\mu} = X^{\mu}_{+}(\sigma^{+}) + X^{\mu}_{-}(\sigma^{-}) \tag{2.15}$$

where $\sigma^{\pm} = \tau \pm \sigma$ are lightcone coordinates and X^{μ}_{+} and X^{μ}_{-} are *left*- and *right*-moving waves.

D-brane action: Born-Infeld action

$$S = -T_{D_p} \int d^{p+1}\xi \, \sqrt{-\det(\gamma_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \tag{2.16}$$

generalisation on Nambu-goto, ξ runs from zero to p. T_{D_p} is D-brane tension. $F_{\mu\nu} = 2\partial_{[\mu}V_{\nu]}$ field strength, V_{μ} is BI vector.

$$T_p = \frac{1}{g_s (2\pi)^p l_s^{p+1}} \tag{2.17}$$

Dp-branes couple to p+1 RR gauge fields just like electric charge to gauge potential in Maxwell theory. D-branes exert zero force (see remark about Majumdar-Papapetrou). Note here along with gravitational have dilatonic attraction. Low-energies massless modes are relevant. For Q D-branes, weak field theory is $U(Q)$ super Yang-Mills. Tension proportonal to $1/g_s$ means d-branes non-perturbative excitations of string theory becoming heavy in weak coupling limit. We have introduced the coupling constant which counts loops in string amplitudes (the interaction[1] of strings an S-matrix summing over topologies). Note $g^2_{open} = g_s$. D-branes preserve 1/2 supersymmetry. Note, D(-1) is instanton. Not all type II branes are Dirichlet, e.g. NS5 brane, F1. Each electric p-brane has magnetic dual (in 10 dimensions):

$$\star \, dC^{p+1} = d\tilde{C}^{7-p} \tag{2.18}$$

So far we have only described the bosonic string. When we allow for fermions we have what is known as a *superstring theory*, due obviously to the supersymmetry induced and is realised by the inclusion of spinors. The number of consistent superstring theories is then constrained to five, all of which are in ten dimensions: $SO(32)$ Heterotic, $E_8 \times E_8$ Heterotic, Type I, Type IIA and Type IIB. These theories are all in fact related by *duality*

[1] interactions occur by joining and splitting of the string.

transformations. In addition, there is an eleven-dimensional theory, *M-theory*, which is approached by the type IIA and $E_8 \times E_8$ Heterotic theories at strong coupling. However, this last theory is not well understood past its low-energy limit, (N=1 eleven dimensional supergravity) and no strings are found there either. We will only concern ourselves with the type II theories ($N = 2$ supersymmetry) in ten dimensions.

Type IIA: obtained from 11D sugra by KK, non-chiral NS-NS sector:

$$g_{\mu\nu} \quad B_{\mu\nu} \quad \Phi \tag{2.19}$$

RR-sector

$$C^{(1)} \quad C^{(3)} \quad C^{(5)} \quad C^{(7)} \tag{2.20}$$

Type IIB:chiral, not from reduction, related by dualities-will not go into.

$$C^{(0)} \quad C^{(2)} \quad C^{(4)} \quad C^{(6)} \quad C^{(8)} \tag{2.21}$$

Similarly we have the Type IIA non-chiral fermions

$$\psi_\mu \quad \lambda \tag{2.22}$$

and IIB

$$\zeta_\mu^{i(\mp)}, \quad \chi^{i(\pm)} \tag{2.23}$$

D-branes and BPS states, the (extended) supersymmetry algebra is given schematically

$$\{Q, Q\} \sim (C\Gamma) \cdot P + (C\Gamma^{(p)}) \cdot Z_p \tag{2.24}$$

where Γ^p is a totally antisymmetric product of Dirac matrices (see later) and Z_p is a p-form 'central'[2] charge, and P is the momentum vector. This is the charge for what forms do these couple to-like EM. If we now sandwich the above anti-commutator between states we see (in the rest frame and dropping indices):

$$0 \leq \langle phys|\{Q, Q\}|phys\rangle = (M - a|Z|) \tag{2.25}$$

$$\Rightarrow M \geq a|Z| \tag{2.26}$$

Equation (2.26) is known as the Bogomolny-Prasad-Sommerfield (BPS) bound. Saturated bound (BPS state) means some generators annihilate states, number of states doesn't change under adiabatic variations. D-branes are BPS with 2^8 states instead of maximal 2^{16}.

2.2 Supergravity

Supergravity theories describe interactions of massless fields, i.e. the low-energy dynamics, of a given string theory, and are the supersymmetric extensions of general relativity.

Action in d=10 dimensions for the ns-ns sector of the supergravity

$$S = \frac{1}{2\kappa^2} \int d^d x \, \sqrt{-g} e^{-2\Phi} \left[R - 4(\partial\Phi)^2 + \frac{1}{2 \cdot 3!} H^2 \right] \tag{2.27}$$

[2] Although does not commute with Lorentz transformations

note that indices not shown means complete antisymmetrisation. Phi is known as the dilaton, and its vacuum expectation value gives the string coupling: $g_s = e^{\phi_0}$ Low energy effective NS-NS sector common to all sugras where $H = dB_2$ and $d = 10, 26$. This action is written in the *string frame*, with string metric $g_{\mu\nu}$ but we can also write the action under a conformal rescaling, $g_{\mu} = e^{\frac{4}{d-2}\Phi} g_{E\mu\nu}$, where $g_{E\mu\nu}$ is the *Einstein frame* metric. We will be working mostly in the string frame. Varying the action is easy but note the metric isn't as straightforward as usual since we have coupling with the dilaton, but the field equation is obtained essentially by integrating by parts twice. It is given in full in section 4.2 of Ortin.

$$\nabla^{\mu}(e^{-2\Phi} H_{\mu\nu\rho}) = 0 \tag{2.28}$$

$$\nabla^2 \Phi = (\partial \Phi)^2 + \frac{1}{4} R + \frac{1}{48} H^2 \tag{2.29}$$

$$R_{\mu\nu} = 2\nabla_{\mu}\partial_{\nu}\Phi + \frac{1}{4} H_{\mu}^{\ \rho\sigma} H_{\nu\rho\sigma} \tag{2.30}$$

Chapter 3

Type IIB and Dp-brane solutions

Type IIB supergravity contains a self-dual five-form field strength which makes writing a classical action for it problematic, although can be dealt with, e.g. see [29], and we will write one down after to emphasise that it is these equations which are primary. It is best therefore to work at the level of the equations of motion. Note, here $F_5 = \partial C_4 + \frac{3}{4}B_2\partial C_2 - \frac{3}{4}C_2\partial B_2$, $F_5^+ = \partial C_4$, $H = \partial B_2$, $\tilde{H} = \partial C_2$, and $C_0 = \ell$. For the metric, we have

$$\begin{aligned} R_{\mu\nu} &= 2\nabla_\mu\partial_\nu\Phi - \frac{9}{4}H_\mu^{\rho\sigma}H_{\nu\rho\sigma} - \frac{1}{2}e^{2\Phi}\left[\partial_\mu\ell\partial_\nu\ell - \frac{1}{2}g_{\mu\nu}(\partial\ell)^2\right] \\ &+ \frac{9}{4}e^{2\Phi}\left[(\widetilde{H} - \ell H)_{(\mu}^{\rho\sigma}(\widetilde{H} - \ell H)_{\nu)\rho\sigma} - \frac{1}{6}g_{\mu\nu}(\widetilde{H} - \ell H)^2\right] + \frac{25}{6}e^{2\Phi}F_{\mu\rho\sigma\lambda\kappa}F_\nu^{\rho\sigma\lambda\kappa} \end{aligned} \tag{3.1}$$

the gauge fields give,

$$\nabla^\mu\left[(\ell^2 + e^{-2\Phi})\partial B_2 - \ell\partial C_2\right]_{\mu\sigma\lambda} = \frac{10}{3}F_{\mu\nu\rho\sigma\lambda}(\partial C_2)^{\mu\nu\rho} \tag{3.2}$$

$$\nabla^\mu\left[\partial C_2 - \ell\partial B_2\right]_{\mu\sigma\lambda} = -\frac{10}{3}F_{\mu\nu\rho\sigma\lambda}(\partial B_2)^{\mu\nu\rho} \tag{3.3}$$

$$F_5^+ = *F_5^+ \tag{3.4}$$

and finally, the varying the scalars yields,

$$\nabla^2\ell = -\frac{3}{2}\partial B_2(\partial C_2 - \ell\partial B_2) \tag{3.5}$$

$$\nabla^2\Phi = (\partial\Phi)^2 + \frac{1}{4}R_G + \frac{3}{16}(\partial B_2)^2 \tag{3.6}$$

Type IIB supergravity action:

$$S_{\text{IIB}} = \int d^{10}x\,\sqrt{-g}\left\{e^{-2\Phi}\left[-R + 4(\partial\Phi)^2 - \frac{3}{4}(\partial B_2)^2\right] - \frac{1}{2}(\partial\ell)^2 - \frac{3}{4}(\partial C_2 - \ell\partial B_2)^2 - \frac{5}{6}F_5^2\right\} + S_{CS} \tag{3.7}$$

where S_{CS} is the Chern-Simons part of the action.
We now turn to the subject of black-brane solutions to the Type IIB theory. Recall that the symmetry for BPS D-branes is given by the broken Lorentz symmetry $SO(d,1) \to$

$SO(d-p) \times SO(p,1)$. The factors describe the symmetry transverse and parallel to the brane respectively, and since we also have translational symmetry along the brane, the Lorentz group can be upgraded to a Poincaré$_4$ symmetry. We will also have some preserved supersymmetry which we will discuss later. For now, note that the solutions we will look at are analogues of the extremal Reissner-Nordström solution[1] and we will not be discussing non-BPS systems. We seek to verify the Dp-brane solutions table 3 (where $|x_\perp| = r$): These solutions were proposed in [30]. We see that the dilaton is constant only

$ds^2 = H_p^{-1/2}(r)dx_\parallel^2 + H_p^{1/2}(r)dx_\perp^2$
$e^\Phi = g_s H_p^{(3-p)/4}$
$C_{01\ldots p} = -(1 - H_p^{-1})$
$H_p = 1 + \left(\frac{r_p}{r}\right)^{7-p}$

Table 3.1: Dp-brane solutions

for $p = 3$. The constant one in the harmonic function, H_p, is the asymptotically flat part of the geometry. We can see that, for $p < 3$, as $r \to 0$ the coupling becomes large such that the system is now in the nonpertubative regime, and the solution is not reliable. The near horizon geometry for the D3-brane is $AdS_5 \times S^5$. The singularity and the horizon for solutions $p \neq 3$ is at $r = 0$. This may seem like a naked singularity, but we demand that causal geodesics are not able to hit the horizon in finite affine parameter; however, for $p = 6$ this is actually a naked singularity. For $p = 3$, examining the curvature invariants gives that the spacetime everywhere nonsingular, so that there are two asymptotically flat regions separated by the horizon.

For black Dp-brane solutions, we need only retain the R-R field C_{p+1} with field strength F_{p+2}. The NS-NS two-form is also dropped. We are left with the following equations of motion for the only three cases we need to verify, since the others are related by dualising: $p = 3$ brane:

$$R_{\mu\nu} = \frac{25}{6} F_{\mu\alpha\beta\sigma\rho} F_\nu{}^{\alpha\beta\sigma\rho} \tag{3.8}$$

$$F_5 = *F_5 \tag{3.9}$$

$$R_G = 0 \tag{3.10}$$

$p = 1$ brane:

$$\begin{aligned} R_{\mu\nu} &= 2\nabla_\mu \partial_\nu \Phi - \frac{9}{4} e^{2\Phi} \left(\frac{1}{6} G_{\mu\nu} H^2 \right) \\ \nabla^2 \Phi &= (\partial\Phi)^2 + \frac{R_G}{4} \\ \nabla^\mu H &= 0 \end{aligned} \tag{3.11}$$

[1] One reason for this is that in the extremal case we need not worry about Hawking radiation which causes quantum mechanical instabilities for the black p-brane solution.

$p = -1$ brane[2]:

$$R_{\mu\nu} = 2\nabla_\mu \partial_\nu \Phi - \frac{e^{2\Phi}}{2}\left(\partial_\mu \ell \partial_\nu \ell - \frac{1}{2}G_{\mu\nu}(\partial \ell)^2\right) \tag{3.12}$$
$$\nabla^2 \ell = 0$$

We will concentrate on the $p = 3$ brane and remark that the verification follows similarly for the other cases. To compute the Ricci we use the vielbein:

$$\begin{aligned}
&e^a = H_p^{-1/4} dx^a \qquad a = 0, ..., p \\
&e^A = H_p^{1/4} dx^A \qquad A = p+1, ..., 9 \\
\longrightarrow\ &de^a = -\frac{1}{4}H_p^{-5/4}(\partial_C H_p)\ dx^C \wedge dx^a = -\frac{1}{4}H_p^{-5/4}(\partial_C H_p)\ e^C \wedge e^a \\
&de^A = \frac{1}{4}H_p^{-3/4}(\partial_A H_p)\ dx^C \wedge dx^A = \frac{1}{4}H_p^{-5/4}(\partial_C H_p)\ e^C \wedge e^A
\end{aligned}$$

Cartan's first equation reads

$$de^{\tilde{\mu}} = -w^{\tilde{\mu}}{}_{\tilde{\nu}} \wedge e^{\tilde{\nu}} \tag{3.13}$$

where $\tilde{\mu}$ is the vielbein index from 0 to 9. So

$$\begin{aligned}
de^a &= -(w^a{}_b \wedge e^b + w^a{}_B \wedge e^B) \\
de^A &= -(w^A{}_b \wedge e^b + w^A{}_B \wedge e^B)
\end{aligned}$$

suggests only non-zero components: ($w^a{}_b = 0$ by antisymmetry)

$$\begin{aligned}
&w^{aB} = -\frac{1}{4}H_p^{-3/2}(\partial^B H_p)\, dx^a \\
&:= -w^{Ba} \\
&w^{AB} = \frac{1}{2}H_p^{-1}\partial^{[B} H_p\, dx^{A]} \\
\longrightarrow\ &dw^a{}_B = -\frac{1}{4}\partial_C(H_p^{-3/2}\partial_B H_p)\, dx^C \wedge dx^a \\
&dw^A{}_B = \frac{1}{2}\delta_{BD}\partial_C(H_p^{-1}\partial^{[D|} H_p) dx^C \wedge dx^{|A]}
\end{aligned}$$

Cartan's second equation for the curvature two-form is

$$R^{\tilde{\mu}}{}_{\tilde{\nu}} = dw^{\tilde{\mu}}{}_{\tilde{\nu}} + w^{\tilde{\mu}}{}_{\tilde{\rho}} \wedge w^{\tilde{\rho}}{}_{\tilde{\nu}} \tag{3.14}$$
$$= \frac{1}{2}R^{\tilde{\mu}}{}_{\tilde{\nu}\tilde{\sigma}\tilde{\rho}}\, e^{\tilde{\sigma}} \wedge e^{\tilde{\rho}} \tag{3.15}$$

[2] see Ortin section 19.2.7 for a full discussion of the D(-1)-brane solution

So we have the components:

$$
\begin{aligned}
R^a{}_b &= w^a{}_C \wedge w^C{}_b = -\frac{1}{16} H_p^{-3} \partial_C H_p \partial^C H_p \, dx^a \wedge dx_b \\
&= -\frac{1}{16} H_p^{-3} (\partial H_p)^2 \, dx^a \wedge dx_b
\end{aligned}
$$

$$
\begin{aligned}
R^A{}_B &= dw^A{}_B + w^A{}_C \wedge w^C{}_B \\
&= \frac{1}{2}\delta_{BD}\partial_C (H_p^{-1}\partial^{[D|}H_p)\, dx^C \wedge dx^{|A]} + \frac{1}{4}H_p^{-2}\delta_{BD}\delta_{CE}\partial^{[E}H_p dx^{A]} \wedge \partial^{[D}H_p dx^{C]}
\end{aligned}
$$

$$
\begin{aligned}
R^a{}_B &= dw^a{}_B + w^a{}_C \wedge w^C{}_B \\
&= -\frac{1}{4}\partial_C (H_p^{-3/2}\partial_B H_p)\, dx^C \wedge dx^a - \frac{1}{8}\delta_{BD} H_p^{-5/2}\partial_C H_p \, dx^a \wedge \partial^{[D}H_p \, dx^{C]}
\end{aligned}
$$

$$
\begin{aligned}
R^A{}_b &= dw^A{}_b + w^A{}_C \wedge w^C{}_b \\
&= \frac{1}{4}\partial_C (H_p^{-3/2}\partial^A H_p)\, dx^C \wedge dx_b + \frac{1}{8}H_p^{-5/2}\partial^{[C}H_p \, dx^{A]} \wedge \partial_C H_p \, dx_b
\end{aligned}
$$

using equation (3.15), we can read off the Riemann tensor components:

$$
\begin{aligned}
R^a{}_{bcd} &= -(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc})\frac{1}{16}H_p^{-5/2}\partial_B H_p \partial^B H_p \\
R^A{}_{bCd} &= \frac{1}{4}\eta_{bd}\partial_C (H_p^{-3/2}\partial^A H_p) + \frac{1}{16}\eta_{bd}\delta^A_C H_p^{-5/2}\partial_E H_p \partial^E H_p - \frac{1}{16}\eta_{bd}H_p^{-5/2}\partial_C H_p \partial^A H_p \\
R^a{}_{BCd} &= -\frac{1}{4}\delta^a_d \partial_C (H_p^{-3/2}\partial_B H) + \frac{1}{16}\delta^a_d H_p^{-5/2}\partial_C H_p \partial_B H_p - \frac{1}{16}\delta_{BC}\delta^a_d H_p^{-5/2}\partial_E H_p \partial^E H_p \\
R^A{}_{BCD} &= \frac{1}{2}\delta^A_{[D} H_p^{-1/2}\partial_{C]}(H_p^{-1}\partial_B H_p) - \frac{1}{2}\delta_{B[D}H_p^{-1/2}\partial_{C]}(H_p^{-1}\partial^A H_p) \\
&- \frac{1}{8}\delta^A_{[D}H_p^{-5/2}\partial_{C]}H_p \partial_B H_p + \frac{1}{8}\delta_{B[D}H_p^{-5/2}\partial_{C]}H_p \partial^A H_p \\
&- \frac{1}{16}(\delta^A_C \delta_{BD} - \delta^A_D \delta_{BC})H_p^{-5/2}\partial_E H_p \partial^E H_p
\end{aligned}
$$

and so the Ricci:

$$
R_{bd} = -\eta_{bd}\frac{2(p-1)}{16}H_p^{-5/2}\partial_B H_p \partial^B H_p + \eta_{bd}\frac{1}{4}H_p^{-3/2}\partial_C \partial^C H_p \tag{3.16}
$$

$$
\begin{aligned}
R_{BD} &= \delta_{BD}\frac{2(p-1)}{16}H_p^{-5/2}\partial_C H_p \partial^C H_p + \frac{28-12p}{16}H_p^{-5/2}\partial_D H_p \partial_B H_p \\
&- \delta_{BD}\frac{1}{4}H_p^{-3/2}\partial_C \partial^C H_p + \frac{p-3}{2}H_p^{-3/2}\partial_B \partial_D H_p
\end{aligned} \tag{3.17}
$$

Finally, the Ricci scalar is

$$
R = R^{\tilde\mu}{}_{\tilde\mu} = R^a{}_a + R^A{}_A = \frac{-(p-3)(p+1)}{4}H_p^{-5/2}\partial_B H_p \partial^B H_p + \frac{2p-7}{2}H_p^{-3/2}\partial_B \partial^B H_p \tag{3.18}
$$

The case of $p = 3$ reduces to $R = -\frac{1}{2} H_3^{-3/2} \partial_B \partial^B H_3$ which indeed vanishes for H_p harmonic.

For the field strength we have, by symmetry (Poincaré$_4$ $\times$ $SO(6)$), the only surviving components are F_{Xuvwz} and F_{UVWXY}. We can write

$$\begin{aligned}
F_{Xuvwz} &= \frac{1}{4!} \varepsilon_{uvwz} \tilde{F}_X \; dx^X \wedge dx^u \wedge dx^v \wedge dx^w \wedge dx^z \\
&= \frac{1}{4!} \varepsilon_{uvwz} \varepsilon^{uvwz} \tilde{F}_X \; dx^X \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&= \frac{4!}{4!} \tilde{F}_X \; dx^X \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&= \tilde{F}_X \; dx^X \wedge \text{(w.s) vol. form}
\end{aligned}$$

Bianchi identity $dF = 0$ gives:

$$d\tilde{F} = 0 \Rightarrow \tilde{F} = dK$$

where $K = K(x_\perp)$ is a scalar. So,

$$\begin{aligned}
*F &= \frac{1}{5!} \epsilon^X{}_{MNQST} \tilde{F}_X \; dx^M \wedge dx^N \wedge dx^Q \wedge dx^S \wedge dx^T \\
&= \frac{1}{\sqrt{-g} 5!} \varepsilon^{XUVWYZ} g_{UM} g_{VN} g_{WQ} g_{YS} g_{ZT} \tilde{F}_X \; dx^M \wedge dx^N \wedge dx^Q \wedge dx^S \wedge dx^T \\
&= \frac{1}{H_p^{1/2} 5!} \varepsilon^{XUVWYZ} H_p^{5/2} \tilde{F}_X \; dx_U \wedge dx_V \wedge dx_W \wedge dx_Y \wedge dx_Z \\
&= \frac{1}{5!} \varepsilon^{XUVWYZ} H_p^2 \tilde{F}_X \; dx_U \wedge dx_V \wedge dx_W \wedge dx_Y \wedge dx_Z
\end{aligned}$$

By the self duality of the five-form field strength, we have $d * F = 0$, so:

$$\begin{aligned}
d * F &= \frac{1}{5!} \varepsilon^{XABCDE} d(H_p^2 \partial_X K) \wedge dx_A \wedge dx_B \wedge dx_C \wedge dx_D \wedge dx_E \\
&= \frac{1}{5!} \varepsilon^{XABCDE} \partial_Y (H_p^2 \partial_X K) \; dx^Y \wedge dx_A \wedge dx_B \wedge dx_C \wedge dx_D \wedge dx_E
\end{aligned}$$

Using a harmonic ansatz $K(r) = (H_p)^q$ we see $d * F = 0$ for $q = -1$. Hence we have:

$$F = dH_p^{-1} \wedge \text{(w.s.) vol. form}$$

Finally, we show that (3.8) is satisfied.

$$\begin{aligned}
F_{ap_1p_2p_3M} F_b^{p_1p_2p_3M} &\sim (H^{-1/4} \cdot H \cdot H^{-2})^2 \eta_{bd} \varepsilon_{ac_1c_2c_3} \varepsilon^{dc_1c_2c_3} \partial^C H \partial_C H && (3.19) \\
&\sim \eta_{ab} H^{-5/2} \partial^C H \partial_C H && (3.20)
\end{aligned}$$

Since the field strength, F, is only determined up to some constant factor, we needn't be precise. Then

$$R_{ab} - F_{ac_1c_2c_3C} F_b^{c_1c_2c_3C} \sim \eta_{ab} \partial^2 H \tag{3.21}$$

and again for H harmonic, our equations are satisfied.

$$R_{AB}F_{AM_1M_2M_3M_4}F_B^{M_1M_2M_3M_4} + F_{Am_1m_2m_3m_4}F_B^{m_1m_2m_3m_4} \tag{3.22}$$
$$\sim (H\cdot H^{-1/4}\cdot H^{-2})^2\epsilon_A^{X\cdots}\partial_X H\epsilon_B^{Y\cdots}\partial_Y H + H^{-5/2}\partial_A H\partial_B H \tag{3.23}$$
$$\sim \delta_{AB}H^{5/2}\partial_C H\partial^C H + H^{-5/2}\partial_A H\partial_B H \tag{3.24}$$

so we see $(R - FF) \sim \partial\partial H$ as required. The fact that the same function H appears in the field strength as well as both 'parts' of the metric is no coincidence. We now show that this is in fact necessary- for the particular case of the three-brane, but others follow similarly- due to supersymmetry. We will be following [33] closely. We start by the making the following ansatz

$$x^\mu = \{x^m, y^M\} \qquad \text{Poincaré}_4\times SO(6)\,\text{split} \tag{3.25}$$
$$ds^2 = e^{2X}\eta_{mn}dx^m dx^n + e^{2Y}\delta_{MN}dy^M dy^N \tag{3.26}$$
$$C^{(4)}_{abcd} = -\frac{1}{\sqrt{-g_{\shortparallel}}}\epsilon_{abcd}e^Z \qquad C^{(4)}_{0123} = -e^Z \tag{3.27}$$

Here X, Y, Z depend only on y^M, and by $SO(6)$ symmetry, only on $r = \sqrt{\delta_{MN}y^M y^N}$. Similarly $\phi = \phi(r)$ is the dilaton ansatz. Note also the determinant of the 'parallel' space metric $g_{\shortparallel}$.

Supersymmetry preservation requires that there exists Killing spinor κ satisfying

$$\delta\psi_\mu = \left(\tilde{\nabla}_\mu + \frac{i}{4^2\cdot 5!}\Gamma^{\mu_1\ldots\mu_5}\Gamma_\mu F_{\mu_1\ldots\mu_5}\right)\kappa = 0 \tag{3.28}$$
$$\delta\lambda = i\Gamma^\mu\kappa^* P_\mu = 0 \tag{3.29}$$

Here $P_\mu = (\partial_\mu\Phi)(1-\Phi^*\Phi)^{-1}$ and $\tilde{\nabla}_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{\tilde{\alpha}\tilde{\beta}}\Gamma_{\tilde{\alpha}\tilde{\beta}}$. Here the vielbein indices are given, using the same split, as $\tilde{\alpha} = \{a, A\}$. The vielbeins and spin connection are, (similar to our previous calculation)

$$e_m^{\,a} = e^X\delta_m^a \qquad\qquad e_M^{\,A} = e^Y\delta_M^A \tag{3.30}$$
$$(\omega_m)^a{}_B = e^{X-Y}\partial_B X\delta_m^a \qquad (\omega_M)_{BA} = 2\partial_{[B}Y\delta_{A]M} \tag{3.31}$$

A basis for the gamma matrices preserving our symmetry is

$$\Gamma_{\tilde{\mu}} = \{\gamma_a\otimes 1, \gamma_5\otimes\Sigma_A\}$$

where $\Gamma_\mu = e_\mu^{\tilde{\nu}}\Gamma_{\tilde{\nu}}$, $\Gamma_{\mu_1\ldots\mu_n} = \Gamma_{[\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_n]}$, where anitsymmetrisation is taken with weight unity and γ_a and Σ_A are the Dirac matrices for $d = 4$ and $d = 6$ respectively. We will also need

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad \text{and} \quad \Gamma_7 = -i\Sigma^4\Sigma^5\Sigma^6\Sigma^7\Sigma^8\Sigma^9$$

so $\gamma_5^2 = 1 = \Gamma_7^2$. The most general spinor respecting $SO(1,3)\times SO(6)$ is written

$$\kappa(x,y) = \zeta\otimes\eta$$

where ζ is a constant spinor and $\eta = \eta(r)$ for $SO(1,3)$ and $SO(6)$ respectively. We decompose these into their chiral eigenstates using $(1 \pm \gamma_5)$ and $(1 \pm \Gamma_7)$, again respectively. Note also the chirality condition

$$\Gamma_{11}\kappa = \kappa \qquad \Gamma_{11} =: \gamma_5 \otimes \Gamma_7$$

hence the chiralities are correlated. Let us plug these facts into our transformation rule, and also note the identities

$$\{\Gamma_a, \Gamma_A\} = 0 \tag{3.32}$$

$$\Gamma^{\mu_1 \dots \mu_n \alpha} = \Gamma^{\mu_1 \dots \mu_n}\Gamma^{\alpha} - n\Gamma^{[\mu_1 \dots \mu_{n-1}}\eta^{\mu_n]\alpha} \tag{3.33}$$

$$\Gamma^{\mu_1 \dots \mu_n} = -\frac{1}{(10-n)!}(-1)^{n(n-1)/2}\epsilon^{\mu_1 \dots \mu_{10}}\Gamma_{\mu_{n+1} \dots \mu_{10}}\Gamma_{11} \tag{3.34}$$

hence,

$$\begin{aligned}\delta\psi_m = (\partial_m + \frac{1}{4}\omega_m^{\ aB}(\Gamma_a\Gamma_B - \Gamma_B\Gamma_a)\kappa + \frac{i}{4^2 \cdot 5!}(\Gamma^{n_1 n_2 n_3 n_4 P}\Gamma_m F_{n_1 n_2 n_3 n_4 P})\kappa \\ + \frac{i}{4^2 \cdot 5!}(\Gamma^{N_1 N_2 N_3 N_4 N_5}\Gamma_m F_{N_1 N_2 N_3 N_4 N_5})\kappa\end{aligned} \tag{3.35}$$

Take this expression term by term:

$$\begin{aligned}\frac{1}{4}\omega_m^{\ aB}(\Gamma_a\Gamma_B - \Gamma_B\Gamma_a) &= \frac{1}{2}(\omega_m)^a_{\ B}\Gamma_a\Gamma^B = \frac{1}{2}e^{X-Y}\partial_B X\delta^a_m\Gamma_a\Gamma^B = \frac{1}{2}e^{X-Y}\partial_M X\delta^M_B\delta^a_m\Gamma_a\Gamma^B \\ &= \frac{1}{2}e^{X-Y}\partial_M X(e^{-X}e^{Y}e^a_m e^M_B)\Gamma_a\Gamma^B \\ &= \frac{1}{2}\gamma_a \otimes \Sigma^M\gamma_5\partial_M X\end{aligned}$$

where the vielbeins have been inserted to convert to world indices. Next consider the term

$$\begin{aligned}\frac{i}{4^2 \cdot 5!}(\Gamma^{n_1 n_2 n_3 n_4 P}\Gamma_m F_{n_1 n_2 n_3 n_4 P}) &= \frac{i}{4^2 \cdot 5!}\Gamma^{n_1 n_2 n_3 n_4}\Gamma^P\Gamma_m F_{n_1 n_2 n_3 n_4 P} \\ &= -\frac{5!}{4^2 \cdot 5!}i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^P\Gamma_m g_{\shortparallel}^{-1}\partial_P e^Z \\ &= -\frac{1}{16}\gamma_5^2\Sigma^P\gamma_m g_{\shortparallel}^{-1}\partial_P e^Z \\ &= -\frac{1}{16}\Sigma^P\gamma_m g_{\shortparallel}^{-1}\partial_P e^Z\end{aligned} \tag{3.36}$$

and the final term goes as

$$\begin{aligned}\frac{i}{4^2 \cdot 5!}\Gamma^{N_1 N_2 N_3 N_4 N_5}\Gamma_m F_{N_1 N_2 N_3 N_4 N_5} &= -\frac{i}{4^2 \cdot 5!5!}\epsilon^{n_1 \dots n_4 N_1 \dots N_6}\Gamma_{n_1 \dots n_4 N_6}\Gamma_{11}\Gamma_m(\star F_{N_1 \dots N_5}) \\ &= -\frac{1}{16}\Sigma^P\gamma_m g_{\shortparallel}^{-1}\partial_P e^Z\end{aligned} \tag{3.37}$$

where the last line is obtained by playing with epsilon identities and reducing to the previous term. Altogether,

$$\delta\psi_m = \partial_m\kappa + \frac{1}{2}\gamma_m \otimes \Sigma^M(\gamma_5\partial_M X - \frac{1}{4}e^{-4X}\partial_M e^Z)\kappa \tag{3.38}$$

The next component we need to consider is

$$\delta\psi_M = (\partial_M + \frac{1}{4}\omega_M^{AB}\Gamma_{AB})\kappa + \frac{i}{4^2\cdot 5!}(\Gamma^{Pn_1n_2n_3n_4}\Gamma_M F_{Pn_1n_2n_3n_4} + \Gamma^{N_1N_2N_3N_4N_5}\Gamma_M F_{N_1N_2N_3N_4N_5})\kappa \tag{3.39}$$

This essentially goes as before, but we use $\Gamma_{AB} = 2\eta_{AB} - \Gamma_{BA}$ in the $(\omega_M)^{AB}\Gamma_{AB}$ term. We arrive at

$$\delta\psi_M = \partial_M\kappa + \frac{1}{2}\partial_M Y\kappa - \frac{1}{2}\gamma_5 \otimes \Sigma^P\Sigma_M(\gamma_5\partial_P Y + \frac{1}{4}e^{-4X}\partial_P e^Z)\kappa \tag{3.40}$$

Let us analyse our results. Starting with

$$\delta\lambda = i\Gamma^\mu\kappa^* P_\mu = 0 \tag{3.41}$$

we see that this is satisfied for $\Phi =$ constant. In equation (3.38), the derivative on κ vanishes by independence from the parallel directions. Using $(1-\gamma_5)\zeta = 0$ we see that $Z = 4A$. Finally, equation (3.40) is solved by $Y = -X$, using $(1-\Gamma_7)\eta_0 = 0$, where η_0 is a constant spinor and $\partial\kappa = \frac{1}{2}\partial Y\kappa$. The final solution therefore is $\kappa = e^{X/2}\zeta \otimes \eta_0$. We have verified that in fact $e^{X/2} = H^{1/8}$ so that in the end we have

$$\kappa = H^{1/8}\zeta_0 \otimes \eta_o \tag{3.42}$$

We can write

$$H_p = 1 + \left(\frac{r_p}{r}\right)^{7-p}, \qquad \left(\frac{r_p}{l_s}\right)^{7-p} = (2\sqrt{\pi})^{5-p}\Gamma\left(\frac{7-p}{2}\right) g_s N_p \tag{3.43}$$

Also for the field strength,

$$F_{p+2} = Q \star \omega_{8-p} \tag{3.44}$$

so that integrating over an $(8-p)$-sphere gives Q as the charge. But for the $p = 3$ case we need to incorporate duality, so

$$F_5 = Q(\omega_5 + \star\omega_5) \tag{3.45}$$

Chapter 4

Black Holes in String Theory

4.1 Entropy Counting

The entropy counting of black holes, i.e. a statistical mechanical counting of the microstates giving rise to the entropy, in string theory has been one of the main successes to date for this particular programme for quantizing gravity, first achieved by Strominger and Vafa in [34] following earlier papers examining the subject such as Susskind [36], Sen [37] and others. The calculation was for an extreme, non-rotating five dimensional black hole and was simplified by Callan and Maldacena in [35]. Following this breakthrough, calculations were performed for non-extremal [39], spinning [38], four dimensional [40] and far-from-extremal [41] black holes. Let us discuss in some more detail how black holes have been treated in string theory.

As early back as the '60's, people were considering the idea of black holes as fundamental strings. This approach was made more robust by Sen and Susskind, as we've already mentioned, and led to what is known as the *correspondence principle*, formulated by Horowitz and Polchinski in [41]. The principle asserts that since we have an infinite tower of massive states for the quantized string, and since for a Schwarzschild black hole, the radius is proportional to the mass of the hole, then for large enough black hole masses- enough that the radius is larger than the string scale- there will be a string state with equivalent mass such that its length scale will be smaller than its Schwarzschild radius, thus forming a black hole. (Recall in four dimensions $G \sim g_s^2 l_s^2$ so that G increases as we turn up the string coupling, and spacetime becomes flat by in the null limit). If we consider here the low energy effective action, then we know the fundamental string couples to the NS gauge field and is electrically charged. Susskind proposed this should hold the other way: black holes at weak coupling should be described by strings. The transition point is known as the 'correspondence point.' Should this be so, the entropies of the two pictures should be comparable. In fact this turns out to be the case, but works modulo exact coefficients.

Another approach, that we have been alluding to up till now, is given by exploiting the supersymmetry of string theory. The advantage here is that we need not renormalise mass since we use the BPS property of the states, the degeneracy of which does not change at least for ≥ 16 supercharges, so that we compute this degeneracy in the weak coupling limit whence we find S_{micro} (microscopic entropy) using the logarithm and then turn up the coupling so that the state forms a black hole and we can use the Bekenstein-Hawking entropy formula. The focus is then shifted from strings to the nonperturbative solitonic states coupling to the RR sector, the Dp-branes we've been looking at, since they are BPS preserving 1/2 supersymmetry, valid for small curvature[1]. We do not need the analogue of the correspondence point since the mass is directly related to the charge. When we compactify the theory down to d-dimensions, and wrap the brane around the compactfied directions, the system will appear as a d-dimensional pointlike object.

Finally note that D-branes aren't the end of the story: M-theory uses its symmetry to map between black hole systems and thereby provides another method for entropy calculations. In addition there have been recent developments in the subject such as the attractor mechanism and AdS/CFT correspondence.

Let us expand further on this BPS (supersymmetric) approach by seeing how it works. First off, how do we put BPS branes together so that we can make a black hole. This is given by the 'intersection rules', as is known in the literature. For supersymmetric intersections, we will get an extremal black hole. Here, we will only be looking at Dp-branes, and supersymmetry along with duality relations give (for $p \leq 6$, and restriciting to pairwise intersections)[2]:

$$\mathrm{D}m \parallel \mathrm{D}m+4(m), \;\; m=0,1,2 \longrightarrow \mathrm{D}p \perp \mathrm{D}p'(m), \;\; p+p'=4+2m, \quad W \parallel \mathrm{D}p \qquad (4.1)$$

where the brackets tell us the dimension of the intersection. In this section we will study the case of the $d=5$ black hole with three charges. For $d>5$, the black hole horizon has a non-zero radius only by including higher order corrections to the curvature tensor in Einstein-Hilbert action. We choose $d=5$ here as it is the simplest example to consider, and take the approach described by [35]. The system we consider is $D1-D5-PP$, where PP is a gravitational wave on $D1$.

We begin by compactification of Type IIB on a T^5 so that the noncompact directions are $(x^0,\ldots,x^4)$, then wrap the D5 on the whole of T^5; the D1 on x^5 with momentum in this direction also. This system preserves 1/8 of the supersymmetries of the vacuum. Harmonic function rule, gives ansatz for metric, p+1 form potential, and dialton (based on solutions describing the extreme branes independently), proposed in for example [42][43]. It goes as

1. take individual solutions, e.g. table 3, smear over relative transverse directions
2. metric remains diagonal and is a superposition of the (metric) solutions
3. do the same to the dilaton
4. sum for the form potentials

A system for d=5 is given by D1-D5-W: regular. To see how the harmonic function rule works, it is instructive to first ignore the gravitational wave and concentrate on the

[1] small compared to $1/l_s$ in the string-frame. This is because the low energy apporximation is only valid here.

[2] they can intersect at angles to, but this is more complicated.

	0	1	2	3	4	5	6	7	8	9
D1	$-$	$-$	$\sim$	$\sim$	$\sim$	$\sim$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
D5	$-$	$-$	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
W	$-$	$\rightarrow$	$\sim$	$\sim$	$\sim$	$\sim$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table 4.1: D1-D5-PP system: the symbols $-$, $\cdot$, $\sim$ represent extended, pointlike and smeared directions of the brane, whilst $\rightarrow$ indicates the direction in which the wave moves.

D1-D5 system. Smearing the dependence in some directions by harmonic rule to constrain the dependence of the solution.

$$\begin{aligned} ds^2 = {} & H_{D1}(r)^{-1/2}H_{D5}(r)^{-1/2}(-dt^2+dx_1^2)+H_{D1}(r)^{1/2}H_{D5}(r)^{1/2}(dr^2+d\Omega_3^2) \\ & +\sum_{k=2}^{4}H_{D1}(r)^{1/2}H_{D5}(r)^{-1/2}dx_k dx_k \end{aligned} \tag{4.2}$$

and dilaton

$$e^{-2\Phi}=\frac{H_{D5}}{H_{D1}} \tag{4.3}$$

The black hole just constructed has zero entropy, since after compactifying, we can compute its area (in a calculation like the one below) and find

$$A=\lim_{r\to 0}2\pi^2\left(r^3\sqrt{\frac{r_{D1}r_{D5}}{r^4}}\right)=0 \tag{4.4}$$

So in order to get a solution with non-zero horizon we need a BPS superposition with another object. The solution is the D1-D5-PP described above. To put a wave on the system, we boost the solution to obtain

$$ds^2=-\lambda^{-2/3}dt^2+\lambda^{1/3}(dr^2+r^2d\Omega_3^2) \tag{4.5}$$

$$\lambda=\prod_i H_i \qquad H_i=1+\left(\frac{r_i}{r}\right)^2, \;\; i=\text{D1, D5, W}$$

Note that this reduces to the $d=5$ RN black hole in the case $r_{\text{D1}}=r_{\text{D5}}=r_{\text{W}}$, although this equality is not attained in general. Using arraying, we can write[3]

$$r_{\text{D1}}^2=\frac{g_sN_{\text{D1}}\ell_s^6}{V}, \qquad r_{\text{D5}}^2=g_sN_{\text{D5}}\ell_s^2, \qquad r_{\text{W}}^2=\frac{g_s^2N_{\text{W}}\ell_s^8}{R^2V} \tag{4.6}$$

For an arbitrary black hole in d dimensions the entropy formula is (note conventions $\hbar=k_B=c=1$)

$$S=\frac{A_d}{4G_d}$$

Newton's constant in $d<10$ dimensions is obtained using the ten dimensional one, as (in the case of $d=5$)

$$G_d=\left(\frac{G_{10}}{(2\pi)^{10-d}V_{10-d}}\right)\Bigg|_{d=5}=\frac{\pi g_s^2\ell_s^8}{4RV}, \qquad G_{10}=(\sqrt{2}\pi)^6g_s^2\ell_s^8$$

[3] As we remarked earlier, in order for the curvature invariants to be small, we need $r_{D1,D5,W}\gg l_s$.

The area of the $d = 5$ dimensional black hole is given by the 3-sphere volume $= 2\pi^2 r^3$. The horizon of this black hole is $r = 0$. Close to this limit, $H_i = 1 + \left(\frac{r_i}{r}\right)^2 \to \left(\frac{r_i}{r}\right)^2$ we have for the angular piece of the metric

$$\begin{aligned}\lim_{r\to 0} \lambda^{1/3} r^2 d\Omega_3^2 &= \left(\frac{r_{D1}^2 r_{D5}^2 r_W^2}{r^6}\right)^{1/3} r^2 d\Omega_3^2 \\ &= \left(r_{D1}^2 r_{D5}^2 r_W^2\right)^{1/3} d\Omega_3^2\end{aligned}$$

Hence we can read off the horizon radius as $r_H = (r_{D1}^2 r_{D5}^2 r_W^2)^{1/6}$, whence we compute the entropy as

$$\begin{aligned}S &= \frac{A_5}{4G_5} = \frac{2\pi^2 r_H^3}{4(\pi g_s^2 \ell_s^8/4RV)} = \frac{2\pi VR(r_{D1}^2 r_{D5}^2 r_W^2)^{1/2}}{g_s^2 \ell_s^8} \\ &= \frac{2\pi VR}{g_s^2 \ell_s^8}\left(\frac{N_{D1}N_{D5}N_W g_s^4 \ell_s^{16}}{V^2 R^2}\right)^{1/2} \\ &= 2\pi\sqrt{N_{D1}N_{D5}N_W}\end{aligned}$$

The ADM mass can be written $M = M_{D1} + M_{D2} + M_W$ due to the BPS condition where charges are additive, and can also be made apparent by expanding the coefficient of dt^2 in the limit $r \to \infty$

$$H_i^{-2/3} = \left(1 + \left(\frac{r_i}{r}\right)^2\right)^{-2/3} = 1 - \frac{2}{3}\left(\frac{r_i}{r}\right)^2 + \ldots \Longrightarrow g_{tt} \sim -\left(1 - \frac{2}{3}\left(\frac{r_{D1}^2}{r^2} + \frac{r_{D5}^2}{r^2} + \frac{r_W^2}{r^2}\right)\right)$$

and $M_i = \frac{\pi r_i^2}{4G_5}$ (c.f. RN BH) so that

$$M = \frac{N_{D1}R}{g_s \ell_s^2} + \frac{N_{D5}RV}{g_s \ell_s^6} + \frac{N_W}{R}$$

We now need to count the states of the system described, i.e. give a statistical derivation of the entropy. To do this in detail is beyond the scope of this dissertation, however we will outline the approach given in [35]. Recall that the configuration of the $N = 1$, $d = 5$- preserving 4 of the 32 type IIB supercharges- system we are studying: Type IIB theory on $T^5 = T^4 \times S^1$ such that N_{D5} D5 branes are wrapped on the whole of T^5; N_{D1} D1 branes wrap the S^1 of length $2\pi R$ and momentum N_W/R is carried along the S^1. The system is in a bound state with zero binding energy. Recall that we add a gravitational wave since the D1-D5 system itself has zero entropy, probably because it's in the ground state, thus we seek excited BPS states realised by adding a gravity wave to the D1-brane. More specifically, we consider plane fronted gravitational waves with parallel rays (pp-waves), which carry either left or right moving momentum and are 1/2 BPS states. In the world volume theory right movers are in ground state, left movers carry N_W modes. The entropy then counts how many states this momentum can be distributed between excitations of the system. We look for massless excitations since the wave moves at the speed of light, and otherwise the BPS mass formula is violated.

We have not yet identified the degrees of freedom carrying the momentum, so we first do this. Since, our symmetry group is $SO(1,1) \times SO_{\|}(4) \times SO_{\perp}(4)$, we cannot carry momenta on the parallel directions of the (rigid) D-branes. However, excitations on the brane are described by massless open strings, bosonic and fermionic, so that we can make them carry the momentum. Several types of open strings can feature: (1,1) i.e. open strings from D1 to D1, and similarly (5,5), (1,5) and (5,1). The last two are distinguished by orientation of the string. The total momentum, N_W/R is carried by bosonic and fermionic strings in quanta of $1/R$.

The maximum number of massless strings will give us the highest entropy, but exciting some causes others to become massive. Now, take R large and dimensionally reduce on T^4 so that the theory is 1+1 dimensional and the energy carried by individual excitations is small. The theory is (4,4) superconformal, i.e. four left and four right moving supersymmetric generators. The (1,1) and (5,5)strings are gauge bosons of the $U(N_1)$ and $U(N_5)$ gauge groups. Similarly, the (1,5) and (5,1) strings are fundamental and antifundamental of $U(N_1) \times U(N_5)$. We are interested in the IR limit Higgs branch, as opposed to the Coulomb branch of the theory, essentially because the latter does not allow the bound state we're after. Here, we can drop the (1,1) and (5,5) strings from our counting since the Higgs field makes the vector multiplets describing them massive. Counting the number of massless degrees of freedom now gives $4Q_1Q_5$ bosonic and $4Q_1Q_5$ fermionic, and so the total central charge, characterising the conformal theory is given by

$$c = n_{\text{bose}} + \frac{1}{2} n_{\text{fermi}} = (1 + \frac{1}{2}) 4 N_{D1} N_{D5} \tag{4.7}$$

where bosons contribute one and fermions a half. The degeneracy is given by Cardy's formula

$$d(c, N_W) \sim \exp \sqrt{\frac{1}{3}\pi c E L} = \exp \left(2\pi \sqrt{\frac{c}{6} E R} \right) \tag{4.8}$$

and plugging the total energy $E = N_W/R$ and volume $L = 2\pi R$ in gives

$$S_{\text{micro}} = \log d(c, N_W) = 2\pi \sqrt{N_{D1} N_{D5} N_W} \tag{4.9}$$

Bibliography

[1] S.M. Carroll, “Lecture Notes on General Relativity”, gr-qc/9712019.

[2] P.K. Townsend, “Black Holes”, gr-qc/9707012.

[3] N. Lambert, “Advanced General Relativity”, Lecture notes, Dept. of Mathematics, King’s College London.

[4] E. Papantonopoulos, “Physics of Black Holes: A Guided Tour”, Lecture Notes in Physics 769, Springer, Berlin, 2009.

[5] S. Hawking, “Black Holes in General Relativity”, Commun. Math. Phys. 25, 152-166, (1972).

[6] S. Hawking, “Particle Creation by Black Holes”, Commun. Math. Phys. 43, 199-220, (1975)

[7] S. Hawking and G. Ellis, “The large scale structure of space-time”, Cambridge University Press, 1973.

[8] R.M. Wald, “General Relativity”, The University of Chicago Press, 1974.

[9] R.M. Wald, “The ‘Nernst Theorem’ and Black Hole Thermodynamics”, gr-qc/9704008.

[10] C. Eling and J.D. Bekenstein, “Challenging the generalized second law”, gr-qc/0810.5255.

[11] J.B. Bekenstein,“Do we understand black hole entropy?”gr-qc/9409015

[12] G. Compère, “An introduction to the mechanics of black holes”, gr-qc/0611129.

[13] L.H. Ford, “Quantum Field Theory in Curved Spacetime”, gr-qc/9707062.

[14] T. Jacobson, “Introduction to Quantum Fields in Curved Spacetime and the Hawking Effect”, gr-qc/0308048.

[15] S.F. Ross, “Black Hole Thermodynamics”, hep-th/0502195

[16] P. Duffell, “Quantum Field Theory in Curved Spacetime”, New York University, 2009.

[17] C. Krishnan, “Quantum Field Theory, Black Holes and Holography”, hep-th/1011.5875

[18] D. Biggati and L. Susskind, “TASI lectures on the Holographic Principle”, hep-th/0002044

[19] S.D. Mathur, “The information paradox: a pedagogical introduction”, hep-th/0909.1038

[20] J. Preskill, “Do Black Holes Destroy Information?”, hep-th/9209058

[21] D. Page, “Black Hole Information”, hep-th/9305040

[22] L. Smolin, “The black hole information paradox and relative locality”, gr-qc/1108.0910

[23] K. Skenderis and M. Taylor, “The fuzzball proposal for black holes”, hep-th/0804.0552

[24] S.D. Mathur,“The fuzzball proposal for black holes: an elementary review”, hep-th/050205

[25] D. Tong, “Lectures on String Theory”, hep-th/0908.0333.

[26] R.J. Szabo, “An Introduction to String Theory and D-Brane Dynamics”, Imperial College Press, London, 2004.

[27] K. Becker, M. Becker, J.H. Schwarz, “String Theory and M-Theory: A Modern Introduction”, Cambridge University Press, 2007.

[28] T. Ortín, “Gravity and Strings”, Cambridge University Press, 2004.

[29] E. Bergshoeff, H.J. Boonstra and T. Ortín, hep-th/9508091.

[30] G.T. Horowitz and A. Strominger, “Black strings and p-branes”, Nucl. Phys. B360, 197 (1991).

[31] A.W. Peet, “TASI lectures on Black Holes in String Theory”, hep-th/0008241

[32] A.W. Peet, “The Bekenstein Formula and String Theory (N-brane Theory)”, hep-th/9712253

[33] M.J. Duff and J.X. Lu, “THE SELF-DUAL TYPE IIB SUPERTHREEBRANE”, Nucl. Phys. B357 (1991) 534

[34] A. Strominger and C. Vafa, “Microscopic Orgin of the Bekenstein-Hawking Entropy”, hep-th/9601029

[35] C.G. Callan and J.M Maldacena, “D-brane Approach to Black Hole Quantum Mechanics”, hep-th/9602043

[36] L. Susskind, “Some Speculations about Black Hole Entropy in String Theory” , hep-th/9309145

[37] A. Sen, “EXTREMAL BLACK HOLES AND ELEMENTARY STRING STATES”, hep-th/9504147

[38] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, "D-branes and Spinning Black Holes", hep-th/9602065

[39] Some relevant papers include:
C.G. Callan and J.M Maldacena, "D-brane Approach to Black Hole Quantum Mechanics", hep-th/9602043.
G.T. Horowitz and A. Strominger, "Counting States of Near-Extremal Black Holes", hep-th/9602051.
J.C. Breckenridge, D.A. Lowe, R.C. Myers, A.W. Peet, A. Strominger and C. Vafa, "Macroscopic and Microscopic Entropy of Near-Extremal Spinning Black Holes", hep-th/9603078.

[40] See for example: J.M. Maldacena and A. Strominger, "Statistical Entropy of Four-Dimensional Extremal Black Holes", hep-th/9603060

[41] G.T. Horowitz and J. Polchinski, "A Correspondence Principle for Black Holes and Strings", hep-th/9612146

[42] A.A. Tseytlin, "Harmonic superpositions of M-branes", hep-th/9604035

[43] J.P. Gauntlett, D.A. Kastor, J. Traschen, "Overlapping Branes in M-Theory", hep-th/9604179

[44] G.T. Horowitz, "The Origin of Black Hole Entropy in String Theory", gr-qc/9604051

[45] G.T. Horowitz, "Quantum States of Black Holes", gr-qc/9704072

[46] J.M. Maldacena, "Black holes and D-branes", hep-th/9705078

[47] T. Mohaupt, "Black Holes in Supergravity and String Theory", hep-th/0004098

[48] D. Klemm, "Black holes and singularities in string theory", hep-th/0410040

[49] K. Skenderis, "Black holes and branes in string theory", hep-th/9901050

[50] S.R. Das and S.D. Mathur, "The Quantum Physics of Black Holes: Results from String Theory", gr-qc/0105063

[51] A. Sen, "An Introduction to Non-perturbative String Theory", hep-th/9802051

[52] K.S. Stelle, "BPS Branes in Supergravity", hep-th/9803116