Introduction to loop quantum gravity

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Abstract

Loop quantum gravity is an attempt to formulate a quantum theory of general relativity. The quantisation is performed in a mathematically rigorous, non-perturbative and background independent manner and standard matter couplings are considered. This dissertation considers the foundations required to build the theory, general relativity as a Hamiltonian system and quantisation of gravity in the spin network basis. Some applications and results of loop quantum gravity are presented: loop quantum cosmology and the removal of singularities, black hole entropy, and the modern spin-foam formalism. The current status of loop quantum gravity research is discussed.

Contents

1 Introduction

The quantum nature of three of the four forces, electromagnetism, weak and strong interactions, suggest that gravitational force too should have quantum properties at Planck scales. Loop quantum gravity (LQG) is an attempt to quantise gravity in a non-perturbative and background independent way. This section introduces the reader to the ideas behind loop quantum gravity.

Why must one quantise? Quantum mechanics has an external time variable, t in the Schrödinger equation, or alternatively in quantum field theory a fixed non-dynamical background. On the other hand, in general relativity the metric is a smooth deterministic dynamical field. But in quantum mechanics any dynamical field is quantised, hence the gravitational field should be quantised.

General relativity has two main properties: diffeomorphism invariance and background independence. General relativity could be viewed as a field theory from the form of the Einstein-Hilbert action but this form does not influence the notions of space and time. Only diffeomorphism invariance and background independence are important. The task is to find a background independent quantum field theory or a general relativistic quantum field theory.

There are several attempts at quantising gravity, the most developed of which are loop quantum gravity and string theory. Other approaches are dynamical triangulations, noncommutative geometry, Hartle's quantum mechanics of spacetime, Hawking's Euclidean sum over geometries, quantum Regge calculus, Penrose's twistor theory, Sorkin's causal set, t'Hooft's deterministic approach and Finkelstein's theory.

The main competing theory of loop quantum gravity, namely string theory, assumes that elementary objects are extended rather than point-like. String theory is successful in the sense that it contains a lot of phenomenology but on the cost of having to introduce supersymmetry, extra dimensions and an infinite number of fields with arbitrary masses and spins. The aim of string theory is broader than that of loop quantum theory, since it unifies gravity with the other forces. Loop quantum gravity does not attempt to unify, only find a background independent quantum field theory. While string theory is perturbative, loop quantum gravity is non-perturbative. It leads to a discrete structure of spacetime at the Planck scale. Loop quantum gravity is UV divergence free; the UV divergences of QFT are a consequence of the assumption that the background geometry is smooth.

How does one think in loop quantum gravity? Loop quantum gravity is based on the assumption that quantum mechanics and general relativity are correct. It assumes background independence and does not attempt to unify forces, only to quantise gravity. Spacetime is assumed to be four-dimensional and no supersymmetry is required but the possibility of supersymmetry is not excluded.

One must step away from the idea that the world inhabits space and evolves in time. This is because spacetime itself is dynamical so spacetime is constructed from the quanta of the gravitational field. The absence of spacetime is what is called background independence. Background independence is manifested as diffeomorphism invariance of the action. Diffeomorphism invariance means that the action is invariant under coordinate change and that it lacks a non-dynamical background field.

Loop quantum gravity although looking for a general relativistic quantum field theory does not use conventional quantum field theory because quantum field theory is defined in a background dependent way. Instead loop quantum gravity uses the Hilbert space of states, operators and transition amplitudes of traditional quantum mechanics. The canonical algebra of fields with positive and negative frequency components is replaced by an algebra of matrices of parallel transport along closed curves. These matrices are called holonomies or Wilson loops.

The holonomies are the essence of loop quantum gravity as they, on quantisation, become operators that create loop states. A loop state transforms under an infinitesimal transformation into an equivalent representation of the same state. Finite transformations change the state into a different one. This is because only the relative position of the loop with respect to other loops is significant.

The states in loop quantum gravity are solutions of the generalised Schrödinger equation called the Wheeler-DeWitt equation

$$H\Psi = 0, \tag{1}$$

where H is the relativistic Hamiltonian or Hamiltonian constraint and Ψ is the space of solutions to the equation. The right-hand side is zero since space and time are on an equal footing in loop quantum gravity, so there is no external time variable to differentiate by.

Construction of the Hamiltonian is a significant issue in the theroy and there is more than one version of the constraint.

The space of solutions can be expressed in terms of an orthogonal basis of spin network states. Spin network states are finite linear combinations of loop states. H acts only on the nodes of a spin network, hence a loop or a set of non-intersecting loops solves the Wheeler-DeWitt equation.

In this spin network basis, a set of quantum operators can be defined. The area and volume operators have discrete spectra. The eigenvalues of the area operator are

$$\mathbf{A} = 8\pi\gamma\hbar G \sum_i \sqrt{j_i(j_i+1)} \tag{2}$$

where j_i are labels on the spin network and γ is the Immirzi parameter. Hence the size of a quantum of space is determined to some extent by the Immirzi parameter.

Loop quantum gravity has been applied to cosmology with some interesting results. The discreteness of physical space implies that the Big Bang singularity of classical cosmology is removed and replaced instead by the Big Bounce where evolution can be continued past the classical singularity. A universe that contracts bounces back to an expanding phase and vice versa under this result. Loop quantum cosmology is successful in describing flat and homogeneous models but investigating more complicated models requires more developed numerical methods.

As well as cosmology, loop quantum gravity can be applied to the study of black holes. Loop quantum gravity is consistent with the Bekenstein-Hawking entropy formula and it predicts a logarithmic quantum correction to the entropy formula.

An extension of traditional loop quantum gravity is the path integral formalism, which is based on spinfoams. Spinfoams are like the world-surface of a spin network evolving in time. The spinfoam formalism is a very active area of research and a major result is that the graviton propagator, or Newton's law in the classical limit, has been found.

While loop quantum gravity has produced lots of results, there are gaps to fill in. One of them is the low-energy limit: does loop quantum gravity have general relativity as the correct low-energy limit. At present loop quantum gravity, like all theories of quantum gravity, lacks experimental verification. It may be possible to get predictions from loop cosmology that could be verified by experiment, though this is still far from being the case.

There are a few books that present the theory of loop quantum gravity and its history. A comprehensive one is Rovelli (2004) [1], and from a slightly different point of view Gambini and Pullin (1997) [2]. They do not include many technical details, however Thiemann (2007) [3] presents thorough derivations. A new book that is accessible to undergraduates is Gambini and Pullin (2011) [4]. A reader of popular science books may find Smolin's Three roads to quantum gravity [5] interesting. For applications to loop quantum cosmology there are good review articles, for example [6] and [7]. Spinfoam formalism is introduced for example in [8] The next sections delve into the details of what has been said in this section.

2 General relativity

This section gives an introduction to the background material needed for the theory of loop quantum gravity following [1].

2.1 Tetrad formalism

The tetrad formalism is a standard formalism of GR and it was used as a basis for the later formalisms. This section contains standard material which is covered thoroughly in Wald's book [9] and briefly in [1].

In the tetrad formalism the gravitational field, or tetrad field, is a one-form $e^I(x) = e^I_\mu(x)\mathrm{d}x^\mu$ where μ is a spacetime tangent index and I are components of a Minkowski vector raised and lowered with the Minkowski metric. The spin connection $\omega^I_J(x) = \omega^I_{\mu J}(x)\mathrm{d}x^\mu$, with I, J being Lorentz indices, is a one-form with values in the Lie algebra of the Lorentz group. The covariant partial derivative is defined by acting with it on a vector field $D_\mu v^I = \partial_\mu v^I + \omega^I_{\mu J} v^J$.

A gauge covariant derivative on a one-form is $\mathrm{D}u^I = \mathrm{d}u^I + \omega^I{}_J \wedge u^J$ and the torsion is $T^I = \mathrm{D}e^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J$. The torsion is zero for a torsion-free spin connection. Curvature is given by $R^I{}_J = \mathrm{d}\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J$. The torsion and curvature expressions are the famous Cartan structure equations.

The action that gives rise to the Einstein equations and determines that torsion is zero is the action

$$S[e,\omega] = \frac{1}{16\pi G}\int \epsilon_{IJKL}(\frac{1}{4}e^I \wedge e^J \wedge R[\omega]^{KL} - \frac{1}{12}\lambda e^I \wedge e^J \wedge e^K \wedge e^L), \qquad (3)$$

where λ is the cosmological constant and G is Newton's constant.

As well as the gravitational field there is matter - Yang-Mills fields, fermion fields and scalar fields. These can be described with the following actions. For a connection A and curvature F the Yang-Mills action is

$$S_{YM}[e,A] = \frac{1}{4}\int \mathrm{Tr}[F^\star \wedge F], \qquad (4)$$

with the trace being the trace on the algebra. Scalars and fermions can be added similarly.

Varying the total action including the matter terms with respect to e leads to the Einstein equations

$$\epsilon_{IJKL}(e^I \wedge R^{JK} - \frac{2}{3}\lambda e^I \wedge e^J \wedge e^K) = 2\pi G T_L \qquad (5)$$

with the usual definition of the energy-momentum three-form

$$T_I(x) = \frac{\delta S_{matter}}{\delta e^I(x)}. \qquad (6)$$

2.2 Gauge invariance

Gauge invariance plays a crucial role in loop quantum gravity so the gauge transformations of the Standard Model are listed here. The equations of motion derived from the $G = SU(3) \times SU(2) \times U(1)$ model action are invariant under local Yang-Mills transformations, Lorentz transformations and diffeomorphisms. These three maps transform a solution of the equations of motion into another solution. These transformations leave physical quantities invariant. If a local quantity depends on a fixed point it is not diffeomorphism invariant. The maps are the following: Local Yang-Mills transformations are given by the map $\lambda : \mathcal{M} \to G$

$$\begin{aligned} \lambda : \phi(x) &\mapsto R_\phi(\lambda(x))\phi(x) \\ \psi(x) &\mapsto R_\psi(\lambda(x))\psi(x) \\ A_\mu(x) &\mapsto R(\lambda(x))A_\mu(x) + \lambda(x)\partial_\mu\lambda^{-1}(x) \\ e^I_\mu(x) &\mapsto e^I_\mu(x) \\ \omega^I_{\mu J}(x) &\mapsto \omega^I_{\mu J}(x), \end{aligned} \tag{7}$$

where R is in the adjoint representation of the group G and R_ϕ and R_ψ are in the representations of G that contain ϕ and ψ respectively.

Local Lorentz transformations are give by the map $\lambda : \mathcal{M} \to SO(3,1)$

$$\begin{aligned} \lambda : \phi(x) &\mapsto \phi(x) \\ \psi(x) &\mapsto S(\lambda(x))\psi(x) \\ A_\mu(x) &\mapsto A_\mu(x) \\ e^I_\mu(x) &\mapsto \lambda^I_J(x)e^J_\mu(x) \\ \omega^I_{\mu J} &\mapsto \lambda^I_K(x)\omega^K_{\mu L}(x)\lambda^L_J(x) + \lambda^I_K(x)\partial_m u\lambda^K_J(x), \end{aligned} \tag{8}$$

where $\lambda^I_J \in SO(3,1)$.

A diffeomorphism is a smooth invertible map $\phi : \mathcal{M} \to \mathcal{M}$

$$\begin{aligned} \phi : \varphi(x) &\mapsto \varphi(\phi(x)) \\ \psi(x) &\mapsto \psi(\phi(x)) \\ A_\mu(x) &\mapsto \frac{\partial\phi^\nu}{\partial x^\mu}A_\nu(\phi(x)) \\ e^I_\mu(x) &\mapsto \frac{\partial\phi^\nu}{\partial x^\mu}e^I_\nu(\phi(x)) \\ \omega^I_{\mu J} &\mapsto \frac{\partial\phi^\nu}{\partial x^\mu}\omega^I_{\nu J}(\phi(x)), \end{aligned} \tag{9}$$

Any physical quantity must be invariant under these transformations.

2.3 $3+1$ decomposition - splitting of spacetime

The Hamiltonian formalism of general relativity is a step towards quantising general relativity. In order to combine general relativity with standard quantum mechanics, spacetime must be split into spatial slices and time. This is required as the Hamiltonian formalism gives the evolution of fields with respect to a time variable. Although picking a time direction breaks covariance, the choice of the time direction was entirely arbitrary and hence covariance is restored later. The Hamiltonian formulation of general relativity was created by Arnowitt, Deser and Misner, whose work appeared in [10]. Details about this section are in [2] and [3].

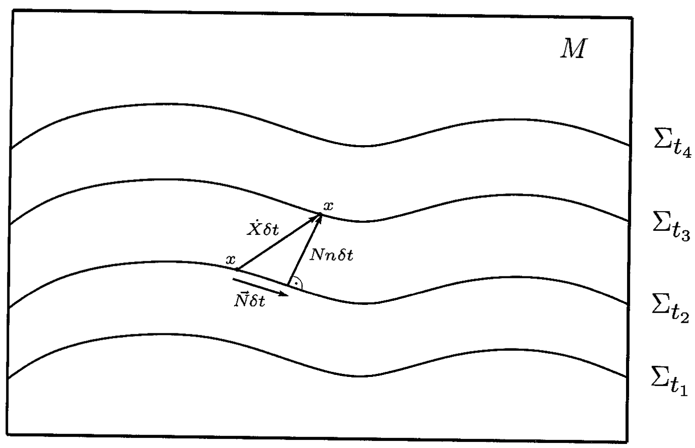

Figure 1: Split of spacetime into spatial slices of constant t and the lapse N and shift N_a indicated in the figure from [3]

To make the split into space and time assume that the four-dimensional spacetime manifold $\mathcal{M}$ with a metric g_{ab} can be split as $\mathcal{M} \cong \Sigma \times R$, where Σ is a three-dimensional spatial slice. Let t label the constant spatial slices Σ and n^μ be a normal to Σ. Then define the spatial three-metric q_{ab} as

$$q_{ab} \equiv g_{ab} + n_a n_b. \tag{10}$$

It is easiest to think that in this expression $a, b = 1, 2, 3$. Then one can decompose a vector t^a into components normal and tangential to Σ as

$$t^a = N n^a + N^a. \tag{11}$$

N is called the lapse and is related to moving between spatial slices, while N^a is called the shift vector which transports along the spatial slice Σ (see figure 1). Here t has indices 0 to 3 but only N not N^a contributes to the zeroth component.

On the four-dimensional manifold one can now choose coordinates (t, x^a), with $a = 1, 2, 3$ coordinates on the spatial slice, with which the metric becomes

$$ds^2 = (-N^2 + N_a N^a)dt^2 + 2N_a dt dx^a + q_{ab} dx^a dx^b. \tag{12}$$

It will be useful later to consider the extrinsic curvature of the spatial slices. The extrinsic curvature measures the change in the three-dimensional metric moving from one slice to another. The extrinsic curvature is defined as

$$K_{ab} \equiv q_a^c q_b^d \nabla_c n_d = \frac{1}{2}\mathcal{L}_n q_{ab}. \tag{13}$$

Also,

$$\dot{q}_{ab} \equiv \mathcal{L}_{\vec{t}} q_{ab} = 2NK_{ab} + \mathcal{L}_{\vec{N}} q_{ab}. \tag{14}$$

The extrinsic curvature is related to the conjugate momentum of the metric, which alongside with the metric will be a canonical variable.

3 Constraint mechanics

Ordinary non-relativistic Hamiltonian mechanics relies on having a time variable to describe evolution. In general relativity however, there is not an external time variable so ordinary mechanics is not broad enough to describe general relativistic systems. The aim is to find a formulation which is as close to quantum mechanics as possible. Let's start with non-relativistic mechanics. This section follows [1].

3.1 Non-relativistic mechanics

The dynamics of a non-relativistic system is specified by a lagrangian $L(q^i, v^i) = L(q^i(t), \frac{\mathrm{d}q^i(t)}{\mathrm{d}t})$ where i runs over the number of degrees of freedom m of the system and q_i are values in the configuration space. The allowed motion is then given by an extremum of the action

$$S[q] = \int_{t_1}^{t_2} L(q^i(t), \frac{\mathrm{d}q^i(t)}{\mathrm{d}t})\mathrm{d}t. \tag{15}$$

The extrema are given by the Euler-Lagrange equations or equivalently Hamilton's equations. The Hamilton-Jacobi equation, which can be derived by taking the classical limit of the Schrödinger equation, is

$$\frac{\partial S(q^i,t)}{\partial t} + H_0(q^i, \frac{\partial S(q^i,t)}{\partial q^i}) = 0, \tag{16}$$

where H_0 is the non-relativistic Hamiltonian and $S(q^i,t)$ is the action evaluated along the classical trajectory with end-point (q^i,t).

Solutions of the Hamilton-Jacobi equation can be found in the form $S(q^i,Q^i,t) = Et - W(q^i,Q^i)$, where E is a constant and W, the characteristic Hamilton-Jacobi function, satisfies

$$H_o(q^i, \frac{\partial W(q^i,Q^i)}{\partial q^i}) = E. \tag{17}$$

$S(q^i,Q^i,t)$ is called the principal Hamilton-Jacobi function. Once those have been found, the solutions of the equation of motion q_i can be found by first calculating

$$P^i(q^i,Q^i,t) = -\frac{\partial S(q^i,Q^i,t)}{\partial Q^i}, \tag{18}$$

and then inverting the equation to $q^i(t) = q^i(Q^i, P_i, t)$. P_i are integration constants.

Another form of solution of the Hamilton-Jacobi equation is the Hamilton function $S(t_1, q_1^i, t_2, q_2^i)$, a function in the configuration space between (t_1, q_1^i) and (t_2, q_2^i) given by

$$S(t_1, q_1^i, t_2, q_2^i) = \int_{t_1}^{t_2} \mathrm{d}t L(q^i(t), \dot{q}^i(t)), \tag{19}$$

where q_i again minimises the action. The Hamilton function has the quantum propagator as its classical limit.

3.2 Relativistic mechanics

The Hamilton-Jacobi formulation can be extended to include relativistic systems.

All relativistic systems excluding quantum effects can be described by the following set:

1. Relativistic configuration space C of partial observables
2. Relativistic phase space Γ of relativistic states
3. Evolution equation $f = 0$ for the map to a linear space V $f : \Gamma \times \mathcal{C} \to V$

The relativistic Hamilton-Jacobi equation is given by

$$H(q^a, \frac{\partial S(q^a)}{\partial q^a}) = 0 \tag{20}$$

where q^a are the observables. Notice that this is simpler than (16) since an external time variable cannot be specified. The evolution equation is

$$f^i(q^a, P_i, Q^i) \equiv \frac{\partial S(q^a, Q^i)}{\partial Q^i} + P_i = 0. \tag{21}$$

4 Ashtekar variables

General relativity can be expressed in terms of a three dimensional $SU(2)$ connection A^i_a and a real three dimensional momentum conjugate, the densitised triad $\tilde{E}^a_i = \sqrt{\det q} E^a_i$ with

$$E^a_i = \frac{\partial S[A]}{\partial A^i_a}. \tag{22}$$

Both sets of indices run from 1 to 3. Indices a denote vector indices in curved space and i are internal indices raised and lowered with the flat metric δ_{ij}. This new formulation of GR was introduced by Abhay Ashtekar in his paper [11] in 1986. The tildes will be dropped for convenience; E^a_i now refers to the densitised triad. The variables in his formulation have the Poisson bracket

$$\{A^i_A(x), E^b_j(y)\} = 8\pi \mathrm{G} \gamma \delta^a_b \delta^i_j \delta^3(x-y) \tag{23}$$

where γ is a complex constant called Barbero-Immirzi parameter. Theories with different γ's are related by canonical transformation variables. The importance of the Immirzi parameter for quantum theory was pointed out in [12].

In Ashtekar formulation the spatial metric can be written in terms of the triad as

$$q^{ab} = E^a_i E^b_j \delta^{ij}. \tag{24}$$

The connection A^i_a is related to the spin connection $\Gamma^i_a = \Gamma_{ajk}\epsilon^{jki}$ and the extrinsic curvature by

$$A^i_a = \Gamma^i_a + \gamma \mathrm{K}^i_a \tag{25}$$

with K the extrinsic curvature $K^i_a = K_{ab}E^{ai}/\sqrt{\det(q)}$. The definition of K_{ab} was given in (13).

In terms of these variables the Einstein-Hilbert action,

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-\det(g)}R, \tag{26}$$

with R the Ricci scalar, becomes (see e.g. [1])

$$L = \frac{1}{8\pi \mathrm{G}\gamma}\int \mathrm{d}^3x(E^a_i\dot{A}^i_a + N\epsilon_{ijk}E^a_iE^b_jF^k_{ab} + N^aE^b_iF^i_{ab} + \lambda^i(D_aE^a)^i) \tag{27}$$

where γ has been set to i and D_a is the covariant derivative with A^i_k and F^i_{ab} is the curvature. Explicitly,

$$\begin{aligned} D_a v_i &= \partial_a v_i + \epsilon_{ijk}A^j_a v^k \\ F^i_{ab} &= \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk}A^j_a A^k_b \end{aligned} \tag{28}$$

In the Lagrangian (27), the 00 and $0i$ components of the metric g have been replaced by the shift and lapse functions (see equation 11), as $g^{00} = 1/N^2$ and $g^{0i} = N^i/N^2$. The lapse N and shift N^a and the gauge parameter λ^i are Lagrange multipliers. This means that there are seven constraints given by [13], [14]

$$\begin{aligned} G^i &= D_aE^a_i = 0, \\ V_b &= E^a_iF^i_{ab} = 0, \\ H &= \epsilon_{ijk}E^a_iE^b_jF^k_{ab} = 0, \end{aligned} \tag{29}$$

G is called the Gauss' law, V is the momentum or vector constraint and H is the Hamiltonian constraint. The Hamiltonian constraint generates the time evolution in terms of the zeroth component of the x-coordinate. The total Hamiltonian of GR is a linear combination of these constraints as can be seen from the Lagrangian.

It is useful to smear the constraints so that the Gauss law becomes

$$G[\lambda] = -\int \mathrm{d}^3 x \lambda^i D_a E_i^a = \int \mathrm{d}^3 x D_a \lambda^i E_i^a = 0, \tag{30}$$

The diffeomorphism constraint, the generator of pure diffeomorphisms, is a combination of Gauss' law and momentum constraint $C_a = V_a - A_a^i (D_b E_i^b)$ with the smeared form being

$$C(\vec{N}) = \int \mathrm{d}^3 x N^a C_a. \tag{31}$$

The Poisson bracket of the diffeomorphism constraint with a function of the canonical variables is just the Lie derivative along $\vec{N}$ [4]

$$\{C(\vec{N}), f(E, A)\} \sim \mathcal{L}_{\vec{N}} f \tag{32}$$

For the Hamiltonian constraint, which is a density of weight two, it is convenient to divide it by a density of weight one to enable the integration to be defined. Using the volume

$$\mathbf{V} = \int \mathrm{d}^3 x \sqrt{|\det(E(x))|} \tag{33}$$

and its commutator with the Ashtekar connection ($\gamma = i$)

$$\{\mathbf{V}, A_a^i(x)\} = (8\pi i G) \frac{E_j^b(x) E_k^c(x) \epsilon_{abc} \epsilon^{ijk}}{4\sqrt{|\det E(x)|}} \tag{34}$$

the Hamiltonian constraint can be written in the following useful form [15]:

$$H[N] = \int N \ \mathrm{Tr}(F \wedge \{\mathbf{V}, A\}) = 0. \tag{35}$$

On a further note, GR is a totally constrained system with the Hamiltonian written explicitly in Ashtekar variables i.e.

$$H = \int \mathrm{d}^3x \{N\epsilon_{ijk}E_i^aE_j^bF_{ab}^k + N^aE_i^bF_{ab}^i + \lambda^i(D_aE^a)^i\}. \tag{36}$$

This is for Barbero-Immirzi parameter $\gamma = i$. If one restricts γ to a real number then one obtains Lorentzian GR where everything else stays the same except the Hamiltonian constraint is then [16]

$$H = \epsilon_{ijk}E_i^aE_j^bF_{ab}^k + 2\frac{\gamma^2+1}{\gamma^2}(E_i^aE_j^b - E_j^aE_i^b)(A_a^i - \Gamma_a^i)(A_b^j - \gamma_b^j) = 0. \tag{37}$$

Details about the new second term in the Hamiltonian and its implications are in [3].

It is important to consider the constraint algebra because the constraints of the theory must be constant over time. This means that their Poisson bracket with the total Hamiltonian must be zero. The total Hamiltonian is a linear combination of constraints, hence the Poisson brackets among constraints must be proportional to constraints [4]. Consider the smeared versions of the constraints, the smeared Hamiltonian constraint being

$$H(N) = \int \mathrm{d}^3x N \frac{E^{ai}E^{bj}F_{ab}^k\epsilon_{ijk}}{\sqrt{det(q)}}, \tag{38}$$

the Gauss constraint (30) and diffeomorphism constraint (31). The constraint algebra is then [4]

$$\{G(\lambda), G(\mu)\} = G([\lambda, \mu]), \tag{39}$$

with $[\lambda, \mu]^i = \lambda_j\mu_k\epsilon^{ijk}$, i.e. the commutator of two smearing constants or Lagrange multipliers. Most of the other Poisson bracket brackets are simple, the diffeomorphism simply shifts the smearing so

$$\{C(\vec{N}), C(\vec{M})\} = C(\mathcal{L}_{\vec{N}}M), \tag{40}$$

$$\{C(\vec{N}), G(\lambda)\} = G(\mathcal{L}_{\vec{N}}\lambda), \tag{41}$$

$$\{G(\lambda), H(M)\} = 0, \tag{42}$$

$$\{C(\vec{N}), H(M)\} = H(\mathcal{L}_{\vec{N}}M), \tag{43}$$

$$\{H(N), H(M)\} = C(\vec{K}). \tag{44}$$

with

$$K^a = E_i^a E^{bi}(N\partial_b M - M\partial_b N)/\det(q). \tag{45}$$

The last Poisson bracket, the bracket of two Hamiltonian constraints is different because it depends on the canonical variables, which will become quantum operators.

5 Quantum theory

Loop quantum gravity builds on holonomies, which are introduced below. These are crucial for quantising the gravitational theory. The states of the theory, spin networks, are formed from closed loops. Quantum operators act on these and the quantum structure of spacetime turns out to be discrete. Further, the constraints are quantised and matter can also be included.

5.1 Holonomies

This section discusses the concept of holonomies following [2]. Holonomies are important because all observables that are functions of the connection only, can be expressed in a basis of holonomies. A holonomy $H(\gamma)$ is the parallel transport along a closed curve γ with a basepoint. The holonomy has the same information in it as the curvature: knowing all holonomies of the one-form connection A_a^i defines the connection uniquely [17]. The holonomy for any closed curve implies the connection at any point modulo a gauge transformation.

Two closed curves are equivalent if one can be continuously deformed to the other. All loops which are equivalent form an equivalence class. The equivalence classes of closed curves form a group structure, and holonomies can be thought of as a map from this group onto a Lie group G. Technically the equivalence classes of closed curves are called loops and the group a group of loops. Functions of the elements in the group of loops are called wavefunctions and these form the loop representation. Explanation of all of this in more detail follows.

The group of loops is really a semi-group. Consider the set of closed curves with a start and end point at o. The identity element is a null curve, $i(s) = 0$ for any parametrisation.

The composition law for two curves is given by $(l_1, l_2) \to l_1 \circ l_2$. The opposite curve l^{-1} is not a group inverse since $l \circ l^{-1} \neq i$.

Parallel transport around a closed curve l is given by the path ordered exponential

$$H_A(l) = \mathrm{P}\exp \int_l A_a(y)\mathrm{d}y^a, \tag{46}$$

where A_a is the connection. H is the holonomy and it is an element of the group G with the group properties for curves l_1 and l_2 and for the basepoint o

$$\begin{aligned} H_A(l_1 \circ l_2) &= H_A(l_1)H_A(l_2) \\ \hat{o} \to \hat{o}' = \hat{o}g \implies H'_a(l) &= g^{-1}H_A(l)g \end{aligned} \tag{47}$$

One can introduce an equivalence relation where one identifies all closed curves that lead to the same holonomy for all smooth connections. There are several ways of defining these equivalence classes [2] - an example of one is the following. Let p_1, p_2 and q be open curves and l_1 and l_2 closed curves. Then if $l_1 = p_1 \circ p_2$ and $l_2 = p_1 \circ q \circ q^{-1} \circ p_2$ then $l_1 \sim l_2$. These equivalence classes are called loops and they are identified with Greek letters The inverse is defined as the curve travelled in the opposite direction: the inverse γ^{-1} of the loop γ satisfies $\gamma \circ \gamma^{-1} = \imath$ with $\imath$ being the set of closed curves equivalent to the null curve.

With this definition of a loop, the holonomy has the properties $H(\gamma_1 \circ \gamma_2) = H(\gamma_1)H(\gamma_2)$ and $H(\gamma^{-1}) = (H(\gamma))^{-1}$.

5.2 Wilson loops

Any gauge invariant quantity involving the connection A can be written in terms of traces of holonomies or Wilson loops

$$W_A(\gamma) = \mathrm{Tr}\ [P\exp(i \oint_\gamma \mathrm{d}y^a A_a)]. \tag{48}$$

They have vanishing Poisson brackets which means that they are observables too.

Wilson loops have two useful properties which together imply that Wilson loops are an overcomplete basis of the Gauss' law constraint. There properties are called Mandelstam identities and the reconstruction property. Mandelstam identities were first introduced in 1968 [18]. They follow from trace identities of $N \times N$ matrices and reflect the structure of the gauge group in consideration. The first type of Mandelstam identity follows directly from the cyclicity of traces. For any gauge group of any dimension

$$W(\gamma_1 \circ \gamma_2) = W(\gamma_2 \circ \gamma_1). \tag{49}$$

The second type of identity is a restriction which guarantees that the Wilson loops are traces of $N \times N$ matrices. These identities can be derived by considering the vanishing of an $N+1$ dimensional antisymmetric matrix in N dimensions with matrix representation indices A and B,

$$\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\dots\delta^{A_{N+1}}_{B_{N+1}]} = 0. \tag{50}$$

Contracting this with $N+1$ holonomies

$$H(\gamma_1)^{B_1}_{A_1}\dots H(\gamma_{N+1})^{B_{N+1}}_{A_{N+1}} \tag{51}$$

gives a vanishing sum of trace products of holonomy products. For example in the important case of $U(1)$ with $(N=1)$

$$W(\gamma_1)W(\gamma_2) - W(\gamma_1 \circ \gamma_2) = 0. \tag{52}$$

For $SU(2)$

$$\begin{aligned} &W(\gamma_1 \circ \gamma_2) = W(\gamma_2 \circ \gamma_1), \\ &W(\gamma_1)W(\gamma_2) = W(\gamma_1 \circ \gamma_2^{-1}) + W(\gamma_1 \circ \gamma_2), \\ &W(\gamma) = W(\gamma^{-1}) \end{aligned} \tag{53}$$

In general $W(\imath) = N$ and $|W(\gamma)| \leq |W(\imath)| = N$. There is a recurrence relation which allows the calculation of the Mandelstam identities of this second kind, which can be found in [2].

The reconstruction property describes whether one can reconstruct the holonomy given a function of loops that satisfy the Mandelstam identities. It was proved in [17] that given a function $W(\gamma)$ satisfying the Mandelstam constraints one can reconstruct the holonomy. Wilson loops satisfy the Mandelstam identities so they uniquely define the holonomy. The details can be found in [2].

Having defined the Wilson loops, wavefunctions ψ can be expressed in the basis of the loops by the following expansion,

$$\psi(\gamma) = \int \mathrm{d}A W_A^*(\gamma)\psi[A]. \tag{54}$$

The holonomy constructed from Wilson loops is a representation of the group of loops and the traces of the representation satisfy the Mandelstam identities. Any gauge invariant function can be expressed as a combination of products of Wilson loops.

Loops are a solution of the Hamiltonian constraint in Ashtekar variables [19], [20], [21]. This was an important discovery in the history of loop quantum gravity. However, the loop basis is an overcomplete basis because of the Mandelstam identities. In order to remove the overcompleteness one can use spin network states [22], which are a basis of states for the quantum theory.

5.3 Spin network states

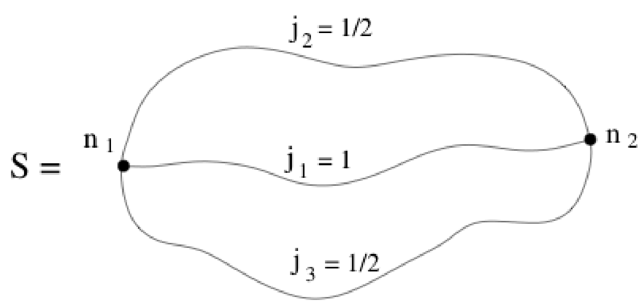

Figure 2: A spin network with nodes n_i and links j_i. Figure from [1]

The definition of a spin network state is as follows [22, 1]. Consider a set of curves or "links" that only overlap at the ends of the curves or "nodes". The curves are oriented, and each node has a multiplicity labelled by m, which denotes the number of curves coming into and going out of the node. The set of curves forms a graph Γ. In addition, the links carry a non-trivial irreducible representation j_i, and the nodes carry an intertwiner n_k. An intertwiner is an invariant tensor in the product space of a set of representations: in this case the intertwiner n_k is in the product space that the representations of the adjacent links form. Then a spin network is simply this construction $S = (\Gamma, j_i, n_k)$, illustrated by figure 2.

Spin networks are an orthonormal basis for gauge invariant functions, or the kinematical state space of loop quantum gravity. They have an inner product defined 1 if two spin network states are related by a diffeomorphism and 0 otherwise. It must be noted that spin networks are a linear combination of loop states - see figure 3.

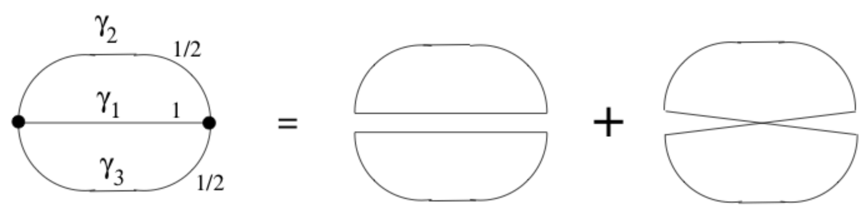

Figure 3: A spin network decomposed into loop states. Figure from [1]

Spin network states are not diffeomorphism invariant but they are used to define a diffeomorphism invariant state called a spin-knot or s-knot [1]. A diffeomorphism changes a spin network state to an orthogonal one or changes the orientation of the links. There is a finite number of states related by changes in link orientation. Let them all be equivalent. If two spin network states are knotted in a different way then they are orthogonal. In order to make them orthonormal, a diagonalisation has to be done. So then, the type of knot and the labelling of the links and nodes form the spin-knot.

5.4 Loop representation of operators

Following [1], the quantum theory in the loop basis is constructed along these lines. All quantities can be constructed from the connection A_a^i and the conjugate E_i^a. Their quantum equivalents are defined on the functionals or states $\Psi[A]$ as [1]

$$\begin{aligned} \hat{A}^i_a \Psi[A] &= A^i_a \Psi[A], \\ \frac{1}{8\pi G}\hat{E}^a_i \Psi[A] &= -i\hbar \frac{\delta}{\delta A^i_a}\Psi[A]. \end{aligned} \tag{55}$$

In other words the action of A on a state is simply multiplication and the conjugate is a functional derivative. The factor $\frac{1}{8\pi G}$ can be set to one. Notice that the right hand sides do not live in the same space as $\Psi[A]$ so use holonomies as the operators instead of A and E [1].

The connection can be replaced by the holonomy $U(A,\gamma)$ (46). Then the action of the holonomy on the state Ψ is multiplication. The momentum E requires a bit more work and in fact the action of E is to pick out intersections of the holonomy with a two-dimensional surface:

$$\hat{E}_i(\mathcal{S})U(A,\gamma) = \pm i\hbar U(A,\gamma_1)\tau_i U(A,\gamma_2) \tag{56}$$

i.e. $\hat{E}$ on a holonomy is equivalent to inserting a matrix $\pm i\hbar\tau_i$ where E and the holonomy curve intersect. The holonomy γ is then split into γ_1 and γ_2. This comes from the fact that E can be rewritten in smeared over a two-dimensional plane as

$$E_i(\mathcal{S}) \equiv -i\hbar \int_{\mathcal{S}} \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \epsilon_{abc} \frac{\partial x^b(\sigma)}{\partial \sigma^1}\frac{\partial x^c(\sigma)}{\partial \sigma^2} \tag{57}$$

where σ are coordinates on the two-dimensional surface. The derivative of U by A is given by

$$\frac{\delta}{\delta A^i_a(x)}U(A,\gamma) = \int \mathrm{d}s\dot{\gamma}^a(s)\delta^3(\gamma(s)x)[U(A,\gamma_1)\tau_i U(A,\gamma_2)] \tag{58}$$

with γ_1 and γ_2 being the two parts into which the point s separates the curve γ and τ the Pauli matrices [1].

The operators $\mathrm{Tr}(U)$ (a closed loop) and E give the unique representation, the loop representation, of a quantised diffeomorphism invariant theory. This is the LOST theorem (for Lewandowski, Okolow, Sahlmann and Thiemann) [23].

Equation (58) is crucial in what comes next. It turns out that the area of any physical surface is quantised. This is the main result of loop quantum gravity: spacetime is quantised. The area and volume operators were first defined by Rovelli and Smolin in [24] and the first results of the disrcetisation of space were presented in this paper, later corrected by Loll [25]. The full spectrum of the area and volume operators was calculated by Ashtekar and Lewandowski in [26]. The argument for the discreteness of the area operator goes as follows.

The area of a 2d surface $\mathcal{S}$ in 3d space is given by

$$\mathbf{A} = \int_{\mathcal{S}} \mathrm{d}x^1 \mathrm{d}x^2 \sqrt{\det q_{2d}}, \tag{59}$$

which using Ashtekar variables can be written as

$$\mathbf{A} = \int_{\mathcal{S}} \mathrm{d}x^1 \mathrm{d}x^2 \sqrt{E_i^3 E^{3i}}, \tag{60}$$

where the i in E^{3i} has been raised with $-2\mathrm{Tr}(\tau_i \tau_j)$. Then one could use 55 to quantise but this would be troublesome with two $E \propto \frac{\delta}{\delta A}$ and the square root. Instead, promote A_Σ to a quantum operator by smearing the triad as in (57). First observe that the gauge dependent quantity

$$E^2(\mathcal{S}) \equiv \sum_i E_i(\mathcal{S}) E_i(\mathcal{S}) \tag{61}$$

acts on a spin network state at the intersections. If there is only one intersection then the action is to insert two matrices τ according to equation (56). As two τ's form the Casimir operator $j(j+1) \times \mathbb{1}$ then

$$E^2(\mathcal{S})|S\rangle = \hbar^2 j(j+1)|S\rangle. \tag{62}$$

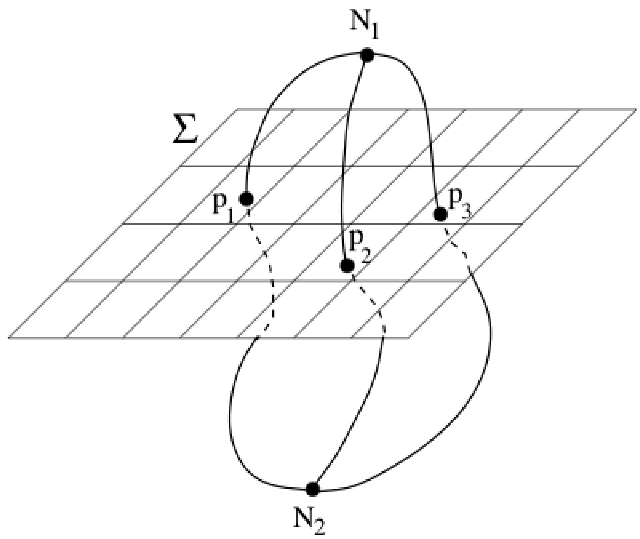

Figure 4: The spin network intersects with the surface which is divided into section. Each intersection gives a contribution to the area eigenvalue. Figure from [1]

To get the result of the area operator on the whole spin network as per figure 4, split the area into small sections so that each section only intersects with one line of the spin network. Then the action of the area operator (eq. 60) is

$$\mathbf{A}(\mathcal{S})|S\rangle = \hbar \sum_i \sqrt{j_i(j_i+1)}|S\rangle \tag{63}$$

for the intersection points i and the representation j_i of the spin network line at the point i. Putting physical units back in and using the Barbero connection instead of the real connection [4],

$$\mathbf{A} = 8\pi\gamma\hbar G c^{-3} \sum_i \sqrt{j_i(j_i+1)} = 8\pi l_P^2 \gamma \sum_i \sqrt{j_i(j_i+1)} \tag{64}$$

Hence, the smallest quantum of area from $j = 1/2$ for $\gamma = 1$ is $4\sqrt{3}\pi\hbar G c^{-3} \sim 10^{-66}\text{cm}^2$ [1]. This discreteness is a result of quantisation in LQG rather than postulated. The area expression contains the Barbero-Immirzi parameter which classically did not affect the theory but in the quantum theory it affects the physics.

It was assumed in the treatment above that the spin network and area intersect only on the links not the nodes. This allowed one to obtain the main part of the spectrum. If the assumption is dropped then according to [26] one obtains the full spectrum

$$\mathbf{A}(\mathcal{S}) = \frac{4\pi\hbar G\gamma}{c^3} \sum_i \sqrt{2j_i^u(j_i^u+1) + 2j_i^d(j_i^d+1) - j_i^t(j_i^t+1)} \tag{65}$$

where $j_i = (j_i^u, j_i^d j_i^t)$ is the triplet at the intersection with a node. The intersections with links are contained in this for $j_i^u = j_i^d$.

It must be noted that the area operator and the volume operator below are not gauge-invariant operators. Hence, they cannot be directly taken to represent physical quantities. However, for certain reasons they are thought to imply that spacetime is discrete [27]: Firstly, it is true that the area and volume operators have discrete spectra in any gauge. As the discreteness depends on commutators of the operators and not the precise description of the operators or observables, for any area the discreteness prevails. Also, discreteness is independent of the dynamics of the system. This gives the basis for saying that LQG predicts a discrete area and volume.

There is also a volume operator, of which there exist variants summarised in [28]. For a full derivation of the quantum volume operator see [3]. Essentially the aim is to make the following expression for the volume $\mathbf{V}$ for a region R into a quantum expression:

$$\mathbf{V}(R) = \int_R \mathrm{d}^3x \sqrt{\det q} = \frac{1}{6} \int_R \mathrm{d}^3x \sqrt{|\epsilon_{abc}\epsilon^{ijk} E_i^a E_j^b E_k^c|}. \tag{66}$$

The volume operator only acts on nodes. More precisely the volume operator vanishes if the region R does not contain a node of four or more spin network lines. The end result is that the action of the volume operator on a spin network state is of the form

$$\mathbf{V}(R)|\Gamma, j_l, i_1 \dots i_N\rangle = (16\pi\hbar\gamma G)^{3/2} \sum_n \mathcal{V}_{i_n}^{i'_n} |\Gamma, j_l, i_1 \dots i_n \dots i_N\rangle, \tag{67}$$

where $|\Gamma, j_l, i_1 \dots i_N\rangle = |s\rangle$ is a spin network state (Γ graph, j_l irreducible representations labelling links and i_n intertwiners of nodes) and $\mathcal{V}_{i_n}^{i'_n}$ are matrices computed for some cases in [29] and all cases included in [30]. These matrices depend on the nodes of the spin network. The conclusion is that the volume operator has a discrete spectrum like the area operator.

How can continuous smooth geometry arise from a quantised geometry? The concept of a weave is the key to this understanding. Weaves were studied in [31] before the spin network states and the discreteness of spacetime of loop quantum gravity were found. The quick introduction below follows [1].

A 3d metric $g_{ab}(x) = e^i_a(x)e_{ib}(x)$ can be approximated by a spin network state for lengths much larger than the Planck length. A spin network state is called a weave if it satisfies

$$\begin{aligned} V(R)|S\rangle &= (V[e,R] + \mathcal{O}(l_P/l))|S\rangle, \\ \mathbf{A}(\mathcal{S})|S\rangle &= (\mathbf{A}[e,\mathcal{S}] + \mathcal{O}(l_p/l))|S\rangle \end{aligned} \tag{68}$$

with $V[e,R]$ and $\mathbf{A}[e,\mathcal{S}]$ denoting classical expressions of volume and area in terms of the tetrad basis. This definition is not unique so physical space is a quantum superposition of these weave states.

5.5 Quantisation of the Hamiltonian constraint

The Hamiltonian constraint was consistently quantised first by Thiemann in 1996 in [32]. The aim was to make a well-defined quantum Wheeler-DeWitt equation $\hat{H}\Psi = 0$. The quantisation of the Hamiltonian constraint starts from the form of equation (35)

$$H(N) = \int \mathrm{d}^3 N\{\mathbf{V}, A^k_c\} F^k_{ab}\epsilon^{abc}. \tag{69}$$

To proceed with quantisation of the Hamiltonian constraint, introduce a path $\gamma_{x,u}$ that starts at point x with a tangent u and has length ϵ [1]. Then the holonomy can be written as

$$h_{\gamma_{x,u}} = 1 + \epsilon u^a A_a(x) + \mathcal{O}(\epsilon^2). \tag{70}$$

For a closed triangular loop $\alpha_{x,uv}$ with two sides of length ϵ tangent to u, v

$$h_{\alpha_{x,uv}} = 1 + \frac{1}{2}\epsilon^2 u^a v^b F_{ab}(x) + \mathcal{O}(\epsilon^3). \tag{71}$$

Then the Hamiltonian constraint (69) can be written as

$$H = \lim_{\epsilon \to 0} \int N \epsilon^{ijk} \ \mathrm{Tr} \left(h_{\gamma^{-1}_{x,u_k}} h_{\alpha_{x,u_i u_j}} \{ \mathbf{V}, h_{\gamma_{x,u_k}} \} \right) \mathrm{d}^3 x \tag{72}$$

This can be promoted to a quantum operator since the holonomies can be directly promoted and the volume operator was constructed earlier. The Poisson bracket becomes a commutator. Since the volume operator acts only on nodes so does the Hamiltonian constraint. To make the constraint covariant under diffeomorphisms and invariant under internal gauge transformations the choice to make for x, u_i, γ and α is given by the choices in figure 5. The point x sits on a node n and the three tangents are along the links of the spin network. If the node is four-valent or more, one must sum over all ordered permutations of three links l, l', l''. In this case there is also ambiguity over position of the line joining two of the tangents together. This ambiguity must be summed over, denoted by label r. The end form of the quantum Hamiltonian constraint is then

$$H|S\rangle = -\frac{i}{\hbar} \sum_{n,l,l',l'',r} N_n \epsilon_{l,l',l''} \mathrm{Tr} \left(h_{\gamma^{-1}_{x_n,l}} h_{\alpha^r_{x_n,l',l''}} [\mathbf{V}, h_{\gamma_{x_n,l}}] \right) |S\rangle. \tag{73}$$

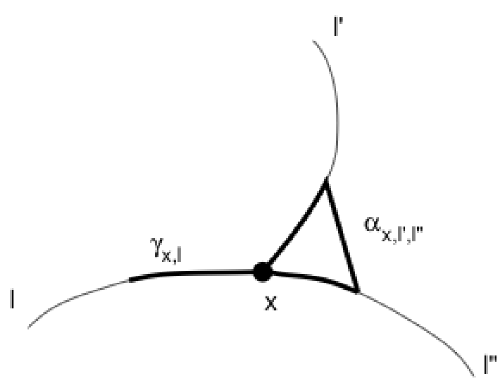

Figure 5: These are the definitions of the labels in the Hamiltonian operator 72. Figure from [1].

The action of the Hamiltonian constraint on a node of a spin network state or more precisely an s-knot state (diffeomorphism invariant spin network) is illustarated in figure 6 and calculated exactly in [33]. It is possible to construct a Hamiltonian constraint with a positive cosmological constant [34].

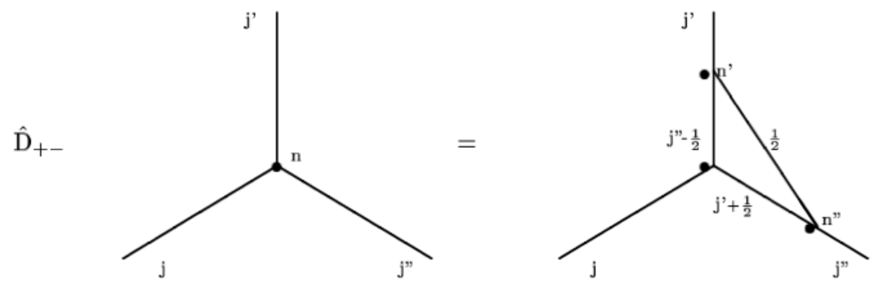

Figure 6: The Hamiltonian acts by adding a line and changing the representations labelling the links. Figure from [1]

5.6 Coupling to matter

While loop quantum gravity does not try to unify matter fields, it is natural to extend the formalism for gravity to matter fields. All kinds of matter fields can be included. For details, derivations and quantisation see [35]. In brief, the s-knot state gets new labels from the Yang-Mills group and the Hamiltonian is a sum of contributions from all matter fields. H_{gravity} is simply 69 and

$$\begin{aligned} H_{\text{YM}} &= \frac{1}{2g_{YM}^2\sqrt{|\det E|}^3}\text{tr}[E_a E_b]\text{Tr}[\mathcal{E}^a\mathcal{E}^b + B^a B^b], \\ H_{\text{Dirac}} &= \frac{1}{2\sqrt{|\det E|}}E_i^a(i\pi\tau^i\mathcal{D}_a\xi + \mathcal{D}_a(\pi\tau^i\xi) + \frac{i}{2}K_a^i\pi\xi + \text{complex conjugate}), \\ H_{\text{Higgs}} &= \frac{1}{2\sqrt{|\det E|}}\left(p^2 + \text{tr}[E^a E^b]\text{Tr}[(\mathcal{D}_a\phi)(\mathcal{D}_b\phi)] + e^2 V(\phi^2)\right), \end{aligned} \tag{74}$$

with p conjugate to the scalar field ϕ , π momentum conjugate to the fermion field ξ and $\mathcal{E}$ conjugate to and B curvature of the Yang-Mills potential. The trace tr is in $SU(2)$ and Tr in G_{YM}. g_{YM} is the Yang-Mills coupling constant and $\mathcal{D}$ is the covariant derivative for $SU(2) \times G_{\text{YM}}$ [1].

6 An application: loop cosmology

The aim of quantum cosmology is to combine general relativity and quantum mechanics in order to understand the universe as a whole. Concentrating on cosmology rather than a general theory of quantum gravity has benefits. Firstly, cosmological symmetries of spacetime simplify calculations. Cosmology is also a good stage for investigating the full theory of quantum gravity and for drawing conclusions for issues such as the problem of time, producing dynamics from frozen formalisms and observables in a background independent theory. As well as being a theoretical playing field, quantum cosmology is the most experimentally accessible part of quantum gravity. A recent review on loop quantum cosmology is [7].

While there are benefits to investigating quantum gravity in cosmological setups, there are also limits. Since symmetries of cosmological spacetime reduce the number of degrees of freedom, the question must be posed: does quantum cosmology portray the main features of quantum gravity? The answer seems to be yes if the extra degrees of freedom are integrated out rather than "frozen" and the construction of the cosmology has the relevant features [7]. It must be emphasised that cosmological models are not an approximation of the full theory but a truncation. Hence, the equivalence of the model to the full theory can only be tested by making the model more complicated and investigating whether the results from the simpler theory carry over.

Loop quantum cosmology is different from other quantum cosmologies by the effects of the quantum geometry. Since the full theory of loop gravity has a discrete spacetime, this is inherited in the more symmetric cosmology models. The discreteness creates a new repulsive force which is present in high-energy regimes in Planck scales. This is the main result of LQC so far. Quantisation of geometry has some major effects in the cosmology context [1]:

1. Singularities are absent. The inverse scale factor a^{-1} that appears in the Freedman equation is bounded. In a sense, the universe has a minumum size and the Big Bang is well behaved.

2. The scale factor is quantised and along with it the volume of the universe is quantised too.

3. Since the scale factor can be interpreted as a cosmological time parameter, the cosmological time is quantised and so the evolution is discrete. The Wheeler-DeWitt equation is a difference equation as will be presented below.

4. There is inflation immediately after the Big Bang which is driven by the quantum properties of the gravitational field.

Key questions of LQC are how close to the big bang is spacetime smooth? Are dynamical equations well-behaved at the singularity? Was there a deterministically connected universe before the Big Bang? Is there an emergent time variable, required by Hamiltonian theory, which governs the evolution of other parameters?

6.1 Isotropic spacetime

This brief introduction to isotropic models follows the presentation in [6].

Isotropy reduces the phase space such that only one variable is free. The metric is

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) \tag{75}$$

where the lapse function $N(t)$ can be absorbed into proper time τ by $\mathrm{d}\tau = N(t)\mathrm{d}t$, and k is -1 0 or 1 for a negative, zero or positive curvature model respectively. Then the evolution equation or Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}a^{-3}H_{matter}(a). \tag{76}$$

The model can be studied with Hamiltonian techniques using the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int \mathrm{d}^3x R[g] \tag{77}$$

where the Ricci scalar for the metric in question is

$$R = 6\left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2} - \frac{\dot{a}}{a}\frac{\dot{N}}{N^3}\right). \tag{78}$$

Substituting this into the action and integrating by parts

$$S = \frac{V_0}{16\pi G}\int \mathrm{d}t N a^3 R = \frac{3V_0}{8\pi G}\int \mathrm{d}t N\left(\frac{-a\dot{a}^2}{N^2} + ka\right) \tag{79}$$

with the definition $V_0 = \int_\Sigma \mathrm{d}^3x$. From this form of the action the momenta are

$$p_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3V_0}{4\pi G}\frac{a\dot{a}}{N}, \qquad p_N = \frac{\partial L}{\partial \dot{N}} = 0. \tag{80}$$

Note that p_N is not a degree of freedom but a Lagrange multiplier since $\dot{N}$ does not appear in the action. N multiplies the Hamiltonian constaint in the action and varying with respect to N forces the Hamiltonian constraint to zero.

The same can be expressed in terms of Ashtekar variables. In Ashtekar variables the isotropic connection and triad are

$$\begin{aligned} A^i_a(x)\mathrm{d}x^a &= c\omega^i, \\ E^a_i\frac{\partial}{\partial x^a} &= pX_i \end{aligned} \tag{81}$$

where ω^i are invariant one-forms and X_i are invariant vector fields. In flat space $\omega^i = \mathrm{d}x^i$ and X_i are derivatives [36]. These two numbers c and p describe the fields. They are related to the variables used above by

$$\begin{aligned} c &= \frac{1}{2}(k - \gamma\dot{a}) \\ |p| &= a^2. \end{aligned} \tag{82}$$

The Poisson bracket is

$$\{\tilde{c}, \tilde{p}\} = \frac{8\pi\gamma G}{3V_0}. \tag{83}$$

A new property is that p can be either negative or positive, corresponding to two different possible orientations of the triad. In order to remove the volume element V_0 from the Poisson bracket, the variables can be rescaled as

$$p = V_0^{2/3}\tilde{p}, \quad c = V_0^{1/3}\tilde{c}, \quad \Gamma = V_0^{1/3}\tilde{\Gamma}. \tag{84}$$

Thus, the Hamiltonian constraint in isotropic Ashtekar variables is

$$H = -\frac{3}{8\pi G}(\gamma^{-2}(c-\Gamma)^2 + \Gamma^2)\sqrt{|p|} + H_{\text{matter}}(p) = 0. \tag{85}$$

which is equivalent to the Friedmann equation.

All states can be constructed by acting on the ground state by the holonomy

$$h_i(c) = \exp(c\tau_i) = \cos(c/2) + 2\tau_i \sin(c/2). \tag{86}$$

An orthonormal basis in this representation is given by [37]

$$\langle c|n\rangle = \frac{\exp(inc/2)}{\sqrt{2}\sin(c/2)}. \tag{87}$$

for integer n.

There are classical singularities at $p = 0$ and $a = 0$ so the evolution equation with scalar matter ϕ

$$H_\phi(a, \phi, p_\phi) = \frac{1}{2}|p|^{-3/2}p_\phi^2 + |p|^{3/2}V(\phi) \tag{88}$$

is singular which means the classical theory is incomplete. As we will see the singularities can be removed as a result of the quantisation process.

Quantisation can be attempted via DeWitt quantisation [38] where states are denoted by $\psi(a)$. The canonical variables act on this, the scale factor as multiplication and its conjugate p_a as derivative. This can be then used to form the Wheeler-DeWitt operator that quantises the Hamiltonian constraint. However, the scale factor has a continuous spectrum in this formalism while quantum gravity implies a discrete spectrum for the volume. Also, the scale factor is unbounded because it appears as a^{-1}. Therefore one should start with the full quantisation and then increase the symmetry to isotropy.

6.2 Discrete scale factor

In loop quantum cosmology, unlike the Wheeler-DeWitt quantisation, the inverse scale factor can in fact be quantised and be found to be finite and to have an upper bound [39]. Its quantisation proceeds as follows [36]. The same trick can be performed on a^{-1} as for the volume operator in eq 34. One way to write a^{-1} is

$$\begin{aligned} a^{-1} &= \left(\frac{1}{2\pi\gamma G}\{c, |p|^{3/4}\}\right)^2 \\ &= \left(\frac{1}{3\pi\gamma G}\sum_i \mathrm{tr}_j(\tau_i h_i(c)\{h_i(c)^{-1}, \sqrt{V}\})\right)^2 \end{aligned} \tag{89}$$

for $j = 1/2$ and trace in the fundamental representation. Now it is straightforward to promote a^{-1} to a quantum operator since this expression contains holonomies and the volume operator. They can be promoted to their quantum versions and the Poisson bracket turned into a commutator so that for an eigenstate $|n\rangle$

$$a^{-1}|n\rangle = \frac{16}{\gamma^2 l_P^4}\left(\sqrt{V_{|n|/2}} - \sqrt{V_{|n|/2-1}}\right)^2 |n\rangle \tag{90}$$

with the volume eigenvalues on an eigenstate $|n\rangle$ [40]

$$V_{(|n|-1)/2} = (\frac{1}{6}\gamma l_P^2)^{3/2}\sqrt{(|n|-1)|n|(|n|+1)} \tag{91}$$

The operator $\hat{a}^{-1}$ has an upper bound $(a^{-1}_{\text{max}}) = \frac{32(2-\sqrt{2})}{3l_{\text{P}}}$ reaching the maximum at $n = 2$ [39]. In the classical limit $l_{\text{P}} \to 0$ the divergence reappears. For large volumes, i.e. large eigenstates, quantum behaviour is very similar to classical behaviour. Figure 7 illustrates the behaviour of the inverse scale factor eigenvalues compared to the classical divergence.

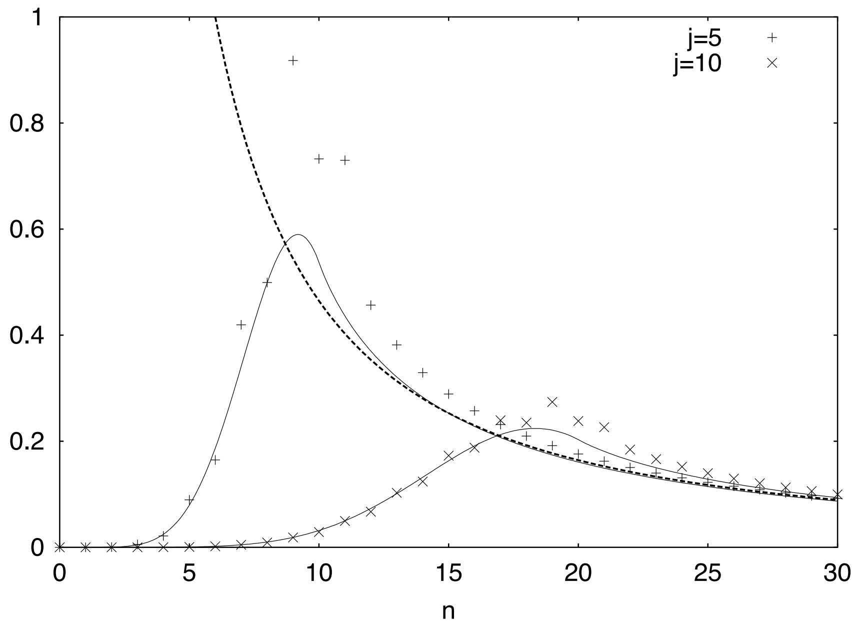

Figure 7: Eigenvalues of the inverse scale factor (× and +), approximations solid lines and classical divergent behaviour dashed line. Figure from [36]

6.3 Dynamics

In order to study the dynamics one must quantise the Hamiltonian constraint, which in loop quantum cosmology becomes a difference equation [43], [44]:

$$\begin{aligned}&(V_{|n+4|/2} - V_{|n+4|/2-1})e^{ik}\psi_{n+4}(\phi) - (2+\gamma^2k^2)(V_{|n|/2} - V_{|n|/2-1})\psi_n(\phi)\\ &+ (V_{|n-4|/2 - V_{|n-4|/2-1}})e^{-ik}\psi_{n-4}(\phi)\\ =& -\frac{8\pi}{3}G\gamma^3 l_P^2 \hat{H}_{matter}(n)\psi_n(\phi)\end{aligned} \tag{92}$$

It has this discrete structure since now due to the discreteness of the scale factor, time can be interpreted as discrete too. The above equation is valid for $k=0$ and $k=1$ only. In the classical limit, i.e. $n \ll 1$ the discrete wavefunction can be appoximated smooth and the difference equation may be Taylor expanded in powers of $p/\gamma l_{\mathrm{P}}^2$. This recovers the classically quantised Wheeler-DeWitt equation

$$\frac{1}{2}\left(\frac{4}{9}l_{\mathrm{P}}^{4}\frac{\partial^2}{\partial p^2}-k\right)\tilde{\psi}(p,\phi)=-\frac{8\pi}{3}G\hat{H}_{\mathrm{matter}}(p)\tilde{\psi}(p,\phi). \tag{93}$$

6.4 Removal of singularity

In the flat $k=0,\Lambda=0$ model there is no Big Bang but there is the Big Bounce [45]. One can evolve the wavefunctions in the evolution equation (92) to negative time, or negative n, so that there is evolution before the classical singularity at $a=0$. This is because the $\psi_0(\phi)$ drops out of the equation since for $n=0$, $V_{|n|/2}-V_{|n|/2-1}$. Also $H_{\mathrm{matter}}|n\rangle=0$ so $\psi_0(\phi)$ is decoupled from all other values of ψ [36]. This can be seen in figure 8 where the classical singular behaviour and the quantum bounce are pictured.

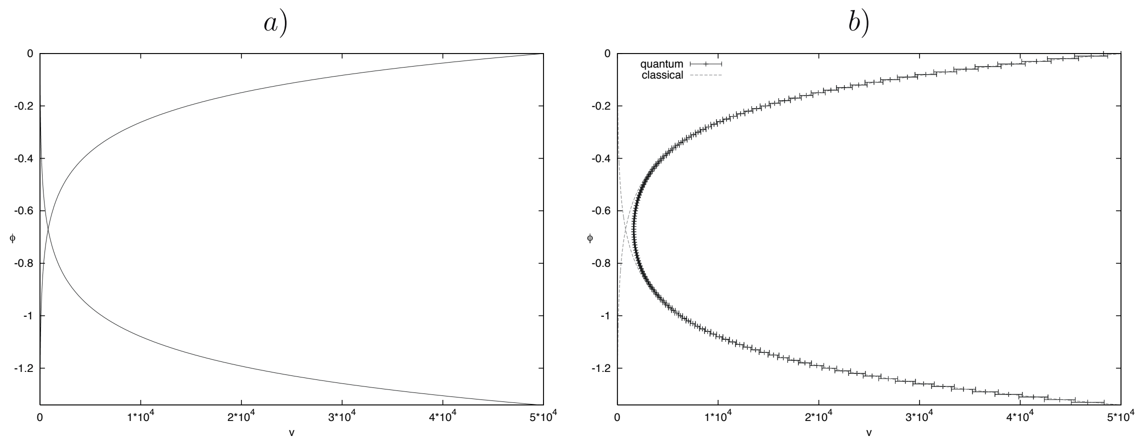

Figure 8: a) Singular behaviour in classical models with $k=0,\Lambda=0$ flat model with a massless scalar field. b) In LQC there is bounce [45] at matter density $\rho\sim 0.41\rho_{\mathrm{Planck}}$. Figure from [7]

6.5 Inflation

Inflation can be realised classically by introducing a state parameter ω and setting it to $\omega<-\frac{1}{3}$. Then the energy density to be used in the Friedmann equation is

$$\rho(a)\propto a^{-3(\omega+1)} \tag{94}$$

and the solutions of the Friedmann equation fall into the following categories

$$a(t) \propto \begin{cases} (t-t_0)^{2/(3\omega+3)} & \text{for } -1 < \omega < -\frac{1}{3} \text{ (power-law inflation)}, \\ \exp(\sqrt{\Lambda t}) & \text{for } \omega = -1 \text{ (standard inflation)}, \\ (t_0-t)^{2/(3\omega+3)} & \text{for } \omega < -1 \text{ (super-inflation)}. \end{cases} \tag{95}$$

In quantum theory, the Friedmann equation is modified since the inverse scale factor is bounded. The boundedness of a^{-1} can be absorbed into the classical equations of motion by replacing a^{-3} with $d_j(a) = a^{-3}p(3a^2/j\gamma l_{\mathrm{P}}^6)$. Then the effective scalar energy density is

$$\rho_{\mathrm{eff}}(a) = \frac{1}{2} x a^{l(a)-3} l_{\mathrm{P}}^{-l(a)-3} p_\phi^2 W(\phi), \tag{96}$$

where x and l have some dependence on quantisation ambiguities. For a usual quantisation the effective ω is then less than -1 [36] so LQC predicts that there is super-inflation in the initial stage of the universe [41]. This is true for any matter. In addition, the inflation ends when the modified energy density starts to decrease [36]. This is a major result of LQC: early universe inflation is predicted and also ended by loop quantum effects.

7 Black hole entropy

Loop quantum gravity provides a way of calculating the entropy of a black hole. It turns out that compared to the semiclassical Hawking-Bekenstein entropy formula there is a logarithmic quantum correction to the entropy of a black hole [46]. Before discussing the loop treatment of black holes, the classical theory will be discussed in order to motivate the search for a quantum description of black hole entropy.

All stationary black holes can be described by three parameters, mass M, charge Q and angular momentum J according to the famous no-hair theorem. Space near a black hole with angular momentum and charge is given by the Kerr-Newman solution of Einstein's equations,

$$c^2 ds^2 = -\left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2 + \left(cdt - \alpha sin^2\theta d\phi\right)^2 \frac{\Delta}{\rho^2} - \left((r^2+\alpha^2)d\phi - \alpha cdt\right)\frac{sin^2\theta}{\rho^2} \tag{97}$$

which is in spherical coordinates and α, ρ and Δ are the following:

$$\begin{aligned}\alpha &= \frac{J}{Mc},\\ \rho^2 &= r^2 + \alpha^2 cos^2\theta,\\ \Delta &= r^2 - \frac{2GM}{c^2}r + \alpha^2 + \frac{Q^2G}{4\pi\epsilon_0 c^4}.\end{aligned} \tag{98}$$

The energy of the black hole can be reduced but only to a finite minimum [47]. On the other hand, Hawking proved that the area of a black hole cannot be decrease [48] and interestingly the area of a black hole horizon is given by the irreducible mass,

$$A = 16\pi M_{irr}^2. \tag{99}$$

Hawking's area theorem inspired Bekenstein to make a connection between entropy and the area of a black hole [49]. The Bekenstein-Hawking entropy of a black hole is

$$S_{BH} = \frac{k_B A}{4l_P^2}. \tag{100}$$

It was argued that since the mass of a Schwarzschild black hole is given by

$$M = \sqrt{\frac{A}{16\pi G^2}}, \tag{101}$$

the thermodynamical equation $T^{-1} = \frac{dS}{dE}$ should imply that there is a black hole temperature. Hawking proved that black holes in fact emit thermal Hawking radiation [50] by considering quantum fields near the horizon. For example for a Schwarzschild black hole the Hawking temperature is

$$T = \frac{\hbar c^3}{8\pi GMk_B}. \tag{102}$$

Hawking radiation is essentially due to a stationary observer near the black hole being non-inertial. This causes the observer to see a different vacuum compared to an inertial observer's vacuum that is usually used as the definition of a vacuum.

Using the Hawking temperature to compute the thermodynamic entropy and comparing it with the Bekenstein-Hawking entropy, it can be argued that the entropy of a black hole is not given by the entropy of the mass of the black hole only [4]. This is a motivation for the search of a quantum description of black hole entropy.

Besides the entropy there is another motivation for finding a full quantum description of black holes. It is the fact that the black hole becomes smaller as it radiates and the curvature near the horizon increases [4]. As the curvature increases the semiclassical picture no longer holds and contradictions arise. An example is the information paradox which asks where does the information trapped in the black hole go as the black hole radiates and decreases in size.

Loop quantum gravity is the only quantum theory of gravity which has managed to reproduce the Bekenstein-Hawking formula for many types of black hole; string theory can be used only to solve extremal or near extremal black holes [1]. Black hole entropy has been investigated in the loop quantum gravity context e.g. in [51]. Most black hole calculations assume that the black hole is large compared to the Planck scale and all ignore Hawking radiation completely [4]. The study of black holes in loop quantum gravity is essentially the study of the interaction between the horizon and the lines of the spin network puncturing the horizon.

The simplest and most thoroughly studied case is the isolated spherical black hole which does not interact with the outside space. Classically any black hole without charge and angular momentum evolves to a Schwarzschild black hole. In quantum theory, however, the Heisenberg uncertainty principle forbids a black hole from being exactly Schwarzschild. The thermal fluctuations in the matter distribution inside the black hole cause inhomogeneities in the gravitational field. These little fluctuations can be ignored [4].

The microstates N_i of the black hole are complicated but they are needed to calculate the entropy $S = k_B \log(N_i)$. There is a major simplification, however, in that the ensemble of microstates required contains only the states of the horizon [1]. This is because a statistical ensemble is defined as the phase space of energy exchanges, which for black holes correspond to changes in the area. Hence, entropy is a function of the area.

First consider the definition of an isolated horizon. There are two conditions for an isolated horizon [4]. Consired the spin connection Γ^i_a defined in the space-time. Denote the pullback of a quantity with a bar. Then define an Abelian connection $W_a = -\bar{\Gamma}^i_a r_i$ on the boundary S, with r_i vector. Using a definition of an antisymmetric tensor $\Sigma^i_{ab}\epsilon^{abc} \equiv E^{ci}$ one can state that the first condition that an isolated horizon satisfies is [52]

$$\partial_a W_b - \partial_b W_a = -2\gamma \bar{\Sigma}^i_{ab} r_i. \tag{103}$$

Secondly, the lapse must vanish for the boundary so that the Hamiltonian constraint does not generate evolution across the boundary.

For calculating the entropy of a spherical black hole, one must count the number of quantum states with the eigenvalue of the area in an infinitesimal range around the black hole area A_0. If a line of a spin network with value j_I punctures the black hole horizon it creates an area $A = 8\pi\gamma l_P^2 \sqrt{j_I(j_I+1)}$ as is familiar from eq. 64. There is also a contribution from $\bar{\Sigma}_{ab}$ which is the E^3 operator with eigenvalues m ranging from $-j_I \le m_I \le j_I$ with a condition $\sum_I m_I = 0$. Including diffeomorphism invariance and all these conditions the Bekenstein-Hawking formula is recovered with a logarithmic correction [46]:

$$S(A) = \frac{A}{4l_P^2} - \frac{3}{2}\log\frac{a}{l_P^2} + \mathcal{O}(1) + \ldots \tag{104}$$

This is an important result since to get this form of the entropy, a particular value for the Barbero-Immirzi parameter must be assumed. The value assumed is from [53], from the solution of the equation

$$1 = \sum_{k=1}^{\infty}(k+1)\exp\left(-\frac{1}{2}\gamma\sqrt{k(k+2)}\right). \tag{105}$$

The solution is $\gamma \approx 0.2375$. It seems dissatisfactory to have to determine a value for γ. However, consistency with the Bekenstein-Hawking formula for several types of black holes including charge, matter or a cosmological constant can be seen to require this value of γ (see [54] and references therein).

The exact value of the Immirzi parameter is still the target of discussion (e.g. [55]). In particular, the current agreement is that

$$S_{loop} = \frac{1}{4}\frac{\gamma}{\gamma_0}\frac{A}{\hbar G} \tag{106}$$

with the most updated value [56, 57] of $\gamma = \gamma_0 = 0.274076...$ It is not understood why the Immirzi parameter must have this particular value and why it should be cancelled.

8 Spinfoam formalism

In ordinary quantum mechanics there are several equivalent formulations, the Schrödinger and Heisenberg pictures and the path integral formalism. Similarly, the Hamiltonian formalism of loop quantum gravity has an alternative called spinfoam formalism. It is a sum-over-histories approach to calculating transition amplitudes of spin network states. The question it aims to answer is what is the transition probability from an initial geometry on a 3d slice to a final geometry. This presentation follows [1]. An introduction to the topic is in [8].

The path integral which measures the transition amplitude and that the theory aims to define is of the form

$$\int D[g_{\mu\nu}(x)]\mathrm{e}^{iS_{GR}[g]} \tag{107}$$

with D being the integration measure which covers all possible metrics [1]. However, it is not known what a non-perturbative integration measure looks like. But the discreteness of space in LQG allows another approach - the sum-over-histories formalism is found from spin networks. Interestingly, several different approaches all lead to the same spinfoam formalism of nonperturbative quantum gravity [1].

The spinfoam formalism was first developed in [58] and in [59]. The key element of the formalism is that instead of an integral, the transition amplitude is given by a sum-over-histories of spin networks. This is where the concept of a spinfoam comes in. A spin foam is a history of spin networks defined as follows. Essentially it is a worldsurface that a spin network sweeps out as a "time"-variable progresses.

In more detail a spinfoam consists of vertices swept out by nodes, edges swept out by links and additionally faces where nodes branch. This is a two-complex structure and it is illustrated in figure 11. A spinfoam has the additional structure that the edges inherit the representation labelling the links and the vertices have the intertwiners from the nodes as labels. It is important to note that a spinfoam is background independent and is defined without reference to a spacetime.

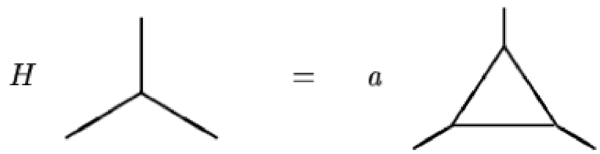

Figure 9: The action of a Hamiltonian is on a node, for example adding lines like this. [1]

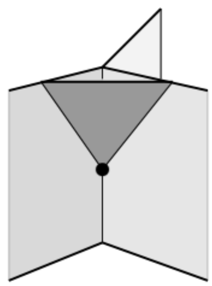

Figure 10: A vertex of a spinfoam caused by the action of the Hamiltonian [1]

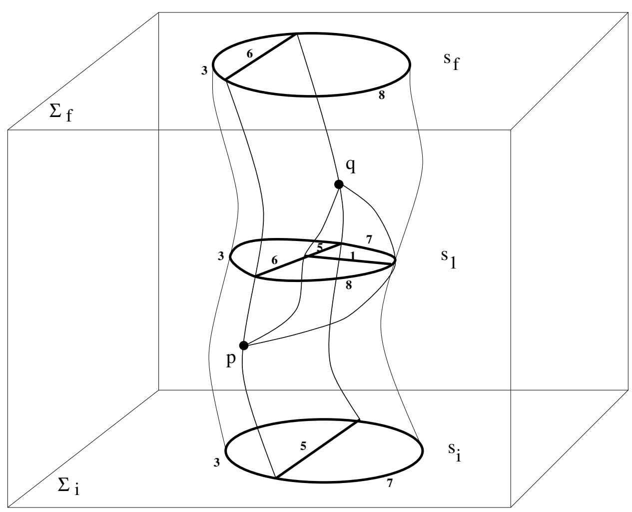

Figure 11: A spinfoam with two vertices. s_i and s_f label the initial and final spin network states. p and q label the vertices. [1]

Formally, a spinfoam W is a sum over amplitudes $A(\sigma)$

$$W(s, s') = \sum_\sigma A(\sigma), \qquad A(\sigma) = \prod_v A_v(\sigma) \tag{108}$$

where s is the initial spin network state and s' is the final spin network state and a single step in the evolution is

$$A_v(\sigma) = \langle s_{n+1} | \mathrm{e}^{-\int d^3 H(x) \mathrm{d}t} | s_n \rangle. \tag{109}$$

For small $\mathrm{d}t$, one can discard higher-order terms in H so that only the linear action of H remains. H acts on the nodes adding lines for example like in figure 11. The action of H is the origin of the vertices in a spinfoam.

A spinfoam model is then defined by the set of two-complexes and weights associated to them, the set of representations and intertwiners, and vertex and edge amplitudes added to the vertices and edges. The most recent spinfoam model is built on a chain of theories of increasing complexity. Mostly the theories are euclidean but the later versions can be made lorentzian [1]. In brief, the first attempt was the Ponzano-Regge model [60] which discretises 3d space with triangles. It comes with infrared divergences so further developments were required. The Ponzano-Regge model has a generalisation to 4d by the Ooguri model. The Ooguri model uses BF-theory[1] instead of GR to help simplify calculations. The next version, the Barrett-Crane model, includes local degrees of freedom so it allows to implement GR in the form of constraints. Then a possible way to implement the two-complexes is for example group field theory [61].

Most new developments in loop quantum gravity come via this formalism. Like with ordinary quantum mechanics, different formulations are useful for different types of calculation and the case is the same in loop quantum gravity. The spinfoam formalism should be equivalent to the Hamiltonian formalism, however, this has only been rigorously proved in 3d in [62].

The question is what can be done with the spinfoam formalism. The amplitudes it allows one to calculate relate to physical measurements via the area and volume operators. Since spin networks are eigenvalues of the area and volume operators, they should in principle give results of measuring 3d surface geometry [1]. One can also compute the Minkowski vacuum via it [63].

[1]For an $SO(4)$ two-form B^{IJ} and connection ω^{IJ}, $S[B, \omega] = \int B_{IJ} \wedge F^{IJ}[\omega]$.

A major result is that spinfoams can be used to calculate n-point functions, which have been defined [64] for loop quantum gravity. The n-point functions can be used to find the graviton propagator or in other words Newton's law without space and time [65]. There is increasing use of spin foam formalism in loop quantum cosmology too [66]

9 Some open problems

Loop quantum gravity has managed to surprise its investigators with its properties. Quantisation of the area and volume was not expected and neither was the resolution of the Big Bang. Success came in the form of a coherent combination of general relativity with quantum mechanics without further assumptions. On the way problems have been solved such as the overcompleteness of the loop basis and the need of producing background dependent quantities from a background independent theory [67]. Several parts of the theory, however require further research as the theory is promising but still incomplete.

There is lack of experimental evidence. This is a problem plaguing theoretical physics in general and quantum gravity is not an exception. There is not a straightforward prediction that the theory makes which could be experimentally tested, though the theory does make predictions such as the quantisation of area.

Loop gravity phenomenology is in general a developing area. This is because the full loop quantum gravity does not provide a derivation of the phenomenological assumptions [69]. A few active directions of research exist: one is the possible Lorentz violation. There is a potential description of this in the form of doubly special relativity where the Planck energy is invariant like the speed of light [70]. For an introduction to the topic see [71].

The idea behind the claim that Lorentz invariance might be violated is easy to understand. Broken Lorentz invariance seems inevitable since space-time is discrete at the Planck scale and there are possible consequences for light dispersion relations. These deviations may be observable in gamma ray bursts in cosmology [72].

A recent claim by Bojowald is that loop gravity can be tested via the inverse tetrad corrections [73]. There are several types of issue to be solved in loop quantum gravity. Investigation needs to be done on the dynamics of inhomogeneous models, the relationship between cosmological models and full quantum gravity and derivation of effective equations for inhomogeneous models [6].

On the application to loop quantum cosmology, as was discussed in section 6, it is unclear whether the singularity resolved in the symmetric case is a property of the full theory. It is possible that even if it is resolved in simple cases, in cases with inhomogeneous matter and other degrees of freedom the singularity returns.

Black hole entropy calculations have been done numerically without the usual assumption of a large area. There are results in [68] which are not yet understood. Therefore, more needs to be done in the black hole direction too.

Rovelli [1] suggests that despite the theory having no ultraviolet divergences there might be infrared divergences. Ultraviolet finiteness can be shown in different ways but for large j there might be infrared divergences [67]. A possible strategy for removing infrared divergences is under investigation [74, 75].

The recovery of the correct classical limit has not been confirmed. A proof has not been found to confirm that general relativity is recovered in the low-energy limit. However, the degrees of freedom and general covariance match general relativity and calculations with n-point functions are successful [67].

The spinfoam formalism needs to be applied further. Parts of the theory have not been solved yet, such as higher order n-point functions.

As well as these areas which require further investigation there is criticism of the existing parts of the theory, most notably argued in [77]. The arguments have been rebutted by Thiemann in [69]. One criticism was that LQG does not reproduce the harmonic oscillator since p and q, the usual variables, are not defined in the loop representation. It is shown in [69] however, that there is a set of observables with energy expectation values $n\hbar$ like for the harmonic oscillator.

There was also criticism of the ambiguities in constructing the Hamiltonian constraint. There are factor ordering ambiguities and freedom of the choice of representation for the holonomies. The factor ordering ambiguity can be removed by demanding a natural result. The choice was whether to put tetrads to the left or right of the holonomies. If the holonomy acts on an arbitrary point of a spin network line so that it produces a vertex, there is a problem. Then the tetrads, or the volume operator that they form, will give a contribution for that vertex when, to start with, there was no vertex [4]. This in the limit of the triangulation going to zero will cause a divergence of the Hamiltonian. Another ambiguity was that the holonomy was chosen to be in the fundamental representation when in fact it could have been in any representation. The criticism of these ambiguities has been discussed in [76].

Related to the Hamiltonian constraint is the constraint algebra which has been under microscope too. It was argued that the constraint algebra does not close off-shell, in particular for two Hamiltonian constraints [77]. This problem is still unanswered, however, attempts are being made with the Master constraint programme [78]. The Master constraint programme combines the diffeomorphism constraint and the Hamiltonian constraint into one, and the correct classical limit of the algebra has been recovered in Algebraic quantum gravity [79].

It has also been shown that standard model matter couplings can be included easily in LQG. The coupling does not lead to anomalies or divergences. The question is the absence of consistency requirements on the matter couplings at the kinematical level. Will there be consistency requirements for the matter in the dynamical or semiclassical limit?

10 Conclusion

Loop quantum gravity is a nonperturbative background independent formulation of quantised general relativity. It builds on the Hamiltonian formalism of general relativity where four-dimensional spacetime is split into three-dimensional spatial slices and a time direction. The action of general relativity has constraints. These constraints can be expressed neatly in terms of the Ashtekar variables which use an $SU(2)$ connection and its momentum conjugate. This introduces a variable called the Barbero-Immirzi parameter which has an important role in the quantum theory.

A basis of solutions for the constraints are the Wilson loops. These can be developed further to define the spin network basis which is a complete basis for the quantum theory. Using the spin network basis, the system of constraints can be quantised. It is found that the area and volume operators of loop quantum gravity have discrete eigenvalues, which implies that spacetime is quantised. The size of the quanta are of Planck scale and proportional to the Immirzi parameter. This dicreteness of space is the reason why loop quantum gravity is UV divergence free.

The quantisation of spacetime has interesting consequences in the context of cosmology. Loop quantum cosmology is a truncated version of the full loop quantum gravity theory and the exact relationship between loop cosmology and the full theory is still under investigation. However, the result that spacetime is quantised implies at least for the FLRW model and homogeneous cosmology models that singularities are resolved. The Big Bang is replaced by the Big Bounce where the classical singularity is avoided due to the loop quantum gravity effects. Early universe inflation is automatically predicted and later stopped by loop quantum cosmology. Loop cosmology is a rapidly developing area where current research aims to consider models of increasing complexity to see whether results from simpler models carry over. In particular, interacting matter and inhomogeneities cannot be analysed using current methods.

Another application of loop quantum gravity is the black hole entropy calculation which fixes the value of the Immirzi parameter. More recent developments of loop quantum gravity have been created in the spinfoam formalism which is a path integral formulation of loop quantum gravity. Most importantly, this formalism allows the calculation of n-point functions and the graviton propagator. There is much work to do in the topic of n-point functions because it is a very recently founded topic in loop quantum gravity.

Loop quantum gravity is a theory that is attacting more research currently because of the number of possible investigation lines for researchers. Still, with only about a hundred researchers working on it worldwide it has to answer to criticism from the vast string theory community who are the main competitors in search for a quantum theory of gravity. While loop quantum gravity does not attempt to unify matter and gravity it can incorporate matter easily. The theory makes predictions and stands on a rigorous foundation, which cannot be said of string theory. However, it has not been proved that general relativity is the low-energy limit of loop quantum gravity, nor has a choice been made of the best way to define a Lorentzian version of the theory. Hopefully these questions will be answered by the community in the future.

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