$E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry

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Contents

1 Introduction

1.1 Supergravity

Modern theoretical physics bases its work and logic on the belief that *a priori* acceptable symmetries should be the background frame of every fundamental theory.
This belief has been leading the last sixty years' research to the Standard Model and its components, such as Quantum Chromo-Dynamics, but also towards more fundamental theories such as Supersymmetry, Supergravity and String Theories. On that track, it has been understood that Supergravity is the low-energy limit of M theory. Thus, it seems interesting to analyse eleven-dimensional supergravity[1] in order to look for underlying symmetries which would help to understand such a fundamental theory.

The first modern theory of gravity, General Relativity, is built on the symmetry group of diffeomorphisms. Gravitational effects are understood to be due to local curvature of space-time, which is locally Poincaré covariant. The key to understanding this theory is to give oneself convenient mathematical tools in order to express these fundamental symmetries. In this case, the underlying symmetry group is the group of diffeomorphisms on a four-dimensional manifold (our space-time). Differential geometry allows us to easily write down a metric, connections and a torsion ; one would then be able to set up Lagrangians in a coordinate independent way. It is thus the naturally covariant language one needs to make the fundamental symmetries obvious.

Several kinds of theories can be built up following the same sort of arguments. For instance, a global Supersymmetry is a symmetry relating integer-spin particles − bosons − to half-integer-spin ones − fermions. It is generated by Supersymmetry fermionic operators $\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\}$ which act on our particles states.
The Poincaré algebra, defined as the set of translations $\{P^\mu\}$ and Lorentz

[1] The number of dimensions of our theory - eleven - is due to the necessity to include the graviton (≤ 11) but also the gauge group of the Standard Model (≥ 11).

transformations $\{M^{\mu\nu}\}$ and by the following relationships:

$$[P^\mu, P^\nu] = 0 \tag{1.1a}$$

$$[P^\mu, M^{\rho\sigma}] = -i\,(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho) \tag{1.1b}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,(\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma}) \tag{1.1c}$$

can be extended. Indeed, one builds up the SuperPoincaré algebra by including these new fermionic operators and setting up additional relationships using them:

$$\{Q_\alpha, \bar{Q}^{\dot\beta}\} = 2\,(\sigma^\mu)_\alpha^{\ \dot\beta}\,P^\mu \tag{1.2a}$$

$$[M^{\mu\nu}, Q_\alpha] = -i\,(\sigma^{\mu\nu})_{\alpha\beta}\,Q_\beta \tag{1.2b}$$

$$[M^{\mu\nu}, \bar{Q}^{\dot\beta}] = -i\,(\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}\,\bar{Q}^{\dot\beta} \tag{1.2c}$$

$$\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} \qquad \{P^\mu, Q_\alpha\} = 0 = \{P^\mu, \bar{Q}^{\dot\alpha}\}$$

The resulting theory is a globally supersymmetric version of General Relativity.
Pursuing further this extension, the local version of Supersymmetry can be added via superdiffeomorphisms. Defining a supermanifold as a manifold with bosonic and fermionic[2] coordinates, one can set up local invariance under some graded diffeomorphisms. Supergravity is the theory formed by such an extension of General Relativity.

1.2 Generalised Geometry

Generalised Geometry is a mathematical tool which allows to reformulate our theory in a geometrical picture. Indeed, one can extend the local symmetry group to include both diffeomorphisms and gauge transformations. One would then have a natural language to express a covariant theory with respect to these two kinds of transformations.
Therefore, one defines several generalised mathematical objects, such as a generalised tangent space and generalised vectors, with a differential structure including generalised metric, connections and torsion. The goal is to introduce them in a coherent way that makes obvious the covariance. It is

[2]Fermionic coordinates anticommunte since they are Grassman-valued spinors.

believed that this language will shed light on the underlying symmetries and theories.

The structure group, the group of diffeomorphisms in General Relativity, has to be generalised first in order to include more fundamental symmetries. Since this work is dedicated to the study of Supergravity and to the clarification of String Theory backgrounds, one should look for symmetries already present in such theories, such as dualities.

At first, five ten-dimensional String Theories, in which the number, shape, size and twisting of dimensions beyond the usual $3+1$ ones influence the predicted physics, had been developped and seemed unrelated. In the mid-1990s, a new kind of connections between the distinct theories was found and called dualities. They linked the theories together and unified them into a broader eleven-dimensional one, called M theory.

The T-duality expresses the fact that different geometries of the manifold (and especially of the extra-dimensions) may result in the same physical effects[3]. Another duality relates a strongly coupled theory to a weakly coupled one, the coupling constants being inversely proportional. The combination of these two gives rise to U-duality.

In the following, two groups will be considered for our extension: the T-duality and the U-duality groups, that is $O(d,d)$[4] and $E_{d(d)}$[5], with d the dimension of our space-time manifold. Instead of our usual tangent space TM, we have generalised tangent spaces such as, in $d=4$ dimensions, $TM \oplus T^*M$ in the first case, and $TM \oplus \wedge^2 T^*M$ in the second one. One can also consider the general case where the dimension d can take any value $d \leq 7$. In any case, the extension allows us to look at all the transformations (diffeomorphisms and gauge transformations) in this generalised space, that is on a generalised vector which will include the gauge fields.

In this work, we will aim on $E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry, which is an

[3]A theory built on a manifold with one dimension compactified over a circle can be related by T-duality to another theory on the same kind of space-time: the radius of the circles are inversely proportional.

[4]The indefinite orthogonal group of split signature.

[5]The split form of the simple exceptional Lie group E_d.

extension of the U-duality case[6]. The generalised $E_{d(d)} \times \mathbb{R}^+$ tangent bundle will be studied both through the split frames one can define on this space and the differential structure which is needed for the theory. A particular attention will be given to the metric compatible, torsion-free generalised connections.

In order to introduce all the notions and specificities of Generalised Geometry, the first section will focus on the low-dimension case $d = 4$. The two generalised tangent spaces mentioned before, stemming from both T- and U-duality groups, will be studied with their linear and differential structures, and especially the connections and metric.

1.3 Notations

In the following, we consider a manifold M of dimension $d \leq 7$, with its tangent bundle TM. Its restriction to the usual $d = 4$ Minkowski space-time manifold is chosen to have $(+, -, -, -)$ signature.

Several generalised tangent spaces can be defined and used:

$$E_1 \simeq TM \oplus T^*M \tag{1.3}$$

$$E_2 \simeq TM \oplus \wedge^2 T^*M \tag{1.4}$$

$$E \simeq TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^7 T^*M) \tag{1.5}$$

In each case, we define a generalised vector V:

$$V = v + f \qquad V \in E_1,\ v \in TM,\ f \in T^*M \tag{1.6}$$

$$V = v + \omega \qquad V \in E_2,\ v \in TM,\ \omega \in \wedge^2 T^*M \tag{1.7}$$

$$V = v + \omega + \sigma + \tau \qquad V \in E,\ v \in TM,\ \omega \in \wedge^2 T^*M,\ \sigma \in \wedge^5 T^*M,\ \tau \in (T^*M \otimes \wedge^7 T^*M) \tag{1.8}$$

In terms of indices, the space-time representations are covered by Greek indices such as μ, ν, ρ, $\sigma\ldots$; if one wants to define an index for a bigger representation than the space-time one, Latin indices might be used, such as m, n, p, $q\ldots$

[6] Known as the "trombone symmetry", this $\mathbb{R}^+$ factor is important to specify the isomorphism between the generalised tangent space and a sum of vector and form spaces.

In the case where one defines coordinates x^μ on the manifold, the generalised vector's components are indiced using a capital letter: for $V \in E$, we would have:

$$V^M = (v^\mu;\, \omega_{\mu_1\mu_2};\, \sigma_{\mu_1\mu_2\mu_3\mu_4\mu_5};\, \tau_{\nu,\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7})$$

with μ, ν, $\mu_k \ldots = 1 \ldots d$, and thus $M = 1 \ldots r$ where $r \equiv (d + \frac{d(d-1)}{2} + \ldots)$ is the dimension of the generalised tangent space.
In different kinds of frame, we might prefer using capital letters A, $B, \ldots$ instead of a couple of letters (such as $(\mu_1\mu_2)$, or rather $(m_1 m_2)$ as we will see). This index would then cover the appropriate range of numbers $-$ for instance $A = 1, \ldots 6$ for a four-dimensional two-form field index: $V^{\mu_1\mu_2} \equiv V^A$.

One can generalise the usual partial derivative operator $\partial_\mu \in T^*M$ by defining an operator acting on the dual of the generalised tangent space E_i^*. For the high dimensional version, one simply defines:

$$\partial_M \equiv \begin{cases} \partial_\mu & \text{for } M = \mu \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

with $\partial_M \in E^*$. This is equivalent to an embedding of the action of the usual operator using the inverse of the isomorphism defining E with respect to the direct sum of tensor bundles.

We here define carefully different kinds of actions and contractions. Defining:

$$\begin{aligned} & h \in TM \otimes T^*M \\ & z \in \wedge^p TM, \qquad 1 \le p \le d \\ & \rho \in \wedge^q T^*M \text{ and } \chi \in \wedge^{8-q} T^*M, \quad 1 \le q \le d \end{aligned}$$

we first recall the action of $GL(d, \mathbb{R})$ on usual tensors:

$$(h \cdot v)^\mu = h^\mu{}_\nu \, v^\nu \tag{1.10a}$$

$$(h \cdot \omega)_{\mu_1\mu_2} = -h^\nu{}_{\mu_1}\, \omega_{\nu\mu_2} - h^\nu{}_{\mu_2}\, \omega_{\mu_1\nu} \tag{1.10b}$$

We will also need details about the contraction of high rank tensors:

$$\begin{cases} (z \diamond \rho)_{\mu_1 \ldots \mu_{q-p}} \equiv \left({}^1\!/\!{}_{p!}\right)\, z^{\nu_1 \ldots \nu_p}\, \rho_{\nu_1 \ldots \nu_p \mu_1 \ldots \mu q-p} & \text{if } p \le q \\ (z \diamond \rho)^{\mu_1 \ldots \mu_{p-q}} \equiv \left({}^1\!/\!{}_{q!}\right)\, z^{\nu_1 \ldots \nu_q \mu_1 \ldots \mu p-q}\, \rho_{\nu_1 \ldots \nu_q} & \text{if } p \ge q \end{cases} \tag{1.11a}$$

$$(z \diamond \tau)_{\mu_1 \ldots \mu_{8-p}} \equiv \left({}^1\!/\!{}_{(p-1)!}\right)\, z^{\nu_1 \ldots \nu_p}\, \tau_{\nu_1, \nu_2 \ldots \nu_p \mu_1 \ldots \mu 8-p} \tag{1.11b}$$

$$(j\, \rho \wedge \chi)_{\mu, \mu_1 \ldots \mu_7} \equiv \left({}^{7!}\!/\!{}_{p!\,(7-p)!}\right)\, \rho_{\mu, [\mu_1 \ldots \nu_p}\, \chi_{\mu_{p+1} \ldots \mu_7]} \tag{1.11c}$$

2 Starting with $TM \oplus T^*M$ and $TM \oplus \wedge^2 T^*M$

As mentioned above, Generalised Geometry gives a geometrical picture to our theory which is naturally covariant with respect to diffeomorphisms and form field transformations. It can be developped by introducing two different symmetry groups: T- and U-duality. Equivalently, it can be built on two different structure groups, respectively, $O(d,d)$ and $E_{d(d)}$. In this section, the manifold M is a $d=4$ spin-manifold, on which we consider a patching of a local connection and an open covering $\{U_i\}$.

2.1 Generalised tangent bundle

First, let us give our generalised tangent space a structure, starting with an inner product between two generalised vectors, $V = v + \omega$, $U = u + \sigma$, both in $(E_k)^2$:

$$\begin{aligned} \langle \cdot\,,\,\cdot \rangle : E_k \times E_k &\to \wedge^{k-1} T^*M \\ (V, U) &\longmapsto \langle V,\, U \rangle \equiv \frac{1}{2}\left(i_v \sigma + i_u \omega\right) \end{aligned} \tag{2.1}$$

Let us now build a generalised Lie derivative of $U = u + \sigma$ with respect to $V = v + \omega$:

$$L_V U \equiv [v, u] + (\mathcal{L}_v \sigma - i_u \mathrm{d}\omega) \tag{2.2}$$

Let us also define the Courant bracket as the antisymmetrisation of the generalised Lie bracket:

$$\begin{aligned} [\![\,\cdot\,,\cdot\,]\!] :\; & E_k \times E_k \to E_k \\ (V,\, U) \;\mapsto\; & [\![V, U]\!] \equiv [v, u] + \mathcal{L}_v \sigma - \mathcal{L}_u \omega - \frac{1}{2}(i_u \mathrm{d}\omega - i_v \mathrm{d}\sigma) \end{aligned} \tag{2.3}$$

It can be seen that diffeomorphisms preserve the Courant bracket, as they do with the usual Lie bracket.

Finally, let us look at transformations which preserve the Courant bracket. For B a $(k+1)$-form, one can define the endomorphism[7]:

$$e^B :\, V \;\to\; e^B(V) = v + (\omega + B(v)) \tag{2.4}$$

[7]The choice for the notation of this operator will be clearer in the following.

where $e^B(V) \in E_i$, that is $(\omega + B(v)) \in T^*M$ or $\wedge^2 T^*M$. Choosing for instance $B(v) = (-1)^{k+1}\, i_v B$, one can apply this operator to the Courant bracket and get:

$$[\![e^B(V), e^B(U)]\!] = e^B\, [\![V,\, U]\!] - i_v i_u\, \mathrm{d}B \tag{2.5}$$

This shows that the Courant bracket is preserved by such a transformation if and only if $\mathrm{d}B = 0$.
It can even be demonstrated that for any transformation and to preserve the Courant bracket, it must be a diffeomorphism, a B-field transformation with $B \in \wedge^{k+1}T^*M$ such that $\mathrm{d}B = 0$ – as introduced just before under the notation e^B – or a composition of the two.

2.1.1 T-duality and $O(d,d)$ generalised tangent space

In the case where we choose to extend the symmetry group with T-duality, the generalised tangent space is isomorphic to the sum $TM \oplus T^*M$. The main object is thus a generalised vector $V = v + f$.
Let us define a theory where, together with a metric g, a closed form field $F = \mathrm{d}f$ plays a central role. Diffeomorphisms act on $v \in TM$ preserving its properties, such as the inner product or the Lie bracket one can define on the tangent space. They also preserve the metric's properties since it is a symmetric $(0,2)$-tensor. Similarly, the one-form field transformation $f \to f + g$ under the condition $\mathrm{d}g = 0$ preserves the closed form field F.
One thus needs to extend this tangent space TM with its symmetries and build a generalised transformation which will include the two previously mentioned transformations. Let us write down the inner product between two generalised vectors $(V, U) \in (E_1)^2$ in a matrix language. First, one can write the generalised vector $V = (\, v \quad \omega\,)^{tr}$. Thus, defining:

$$M = \frac{1}{2}\begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \tag{2.6}$$

one has:

$$\langle V,\, U\rangle = V^{tr}\, M\, U \tag{2.7}$$

Since M is a symmetric real matrix, it can be diagonalised using a real orthogonal matrix P. Its eigenvalues are $\{-1,\, 1\}$ with the same multiplicity.

The group of morphisms preserving the inner product is thus isomorphic to $O(d,d)$.

It can be proven that the Lie algebra related to the symmetry group preserving the inner product, denoted by $\mathfrak{so}(E_1)$, can be decomposed as follows:

$$\mathfrak{so}(E_1) = \wedge^2 TM \oplus \mathrm{End}\, TM \oplus \wedge^2 T^* M \tag{2.8}$$

where $\mathrm{End}\, TM = TM \oplus T^*M$. Indeed,

$$\begin{aligned} \mathfrak{so}(E_1) &= \{Q : Q^{tr} M + MQ = 0\} \\ &= \left\{ Q = \begin{bmatrix} A & C \\ B & -A^{tr} \end{bmatrix}, \quad \begin{array}{r} A \in \mathrm{End}\, TM \\ B \in \wedge^2 T^* M \\ C \in \wedge^2 TM \end{array} \right\} \end{aligned} \tag{2.9}$$

By exponentiating $Q_B = \left[\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right]$, one gets the operator for the B-field transformation, which gives (2.4). Hence the notation, slightly simplified, introduced previously.
Similarly, by exponentiating $Q_C = \left[\begin{smallmatrix} 0 & C \\ 0 & 0 \end{smallmatrix}\right]$, the bi-vector C leads to the transformation $e^C(v+\omega) = (v - i_\omega C) + \omega$ (simplifying here again the notation). Finally, the endomorphisms are represented by the diagonal part of Q. This gives the decomposition (2.8).

One can finally look for a generalised metric which would combine our fields, *i.e.* both the metric g and the form field B.
Let us start by defining a splitting of our space E_1 into a maximal subspace on which the inner product is positive definite, denoted by E_1^+, and its orthogonal complement $E_1^- \equiv (E_1^+)^\perp$. The inner product is negative definite on E_1^- and we can then define a metric operator:

$$\begin{aligned} G:\ E_1 = E_1^+ \oplus E_1^- \ &\rightarrow\ E_1 \\ V = V^+ + V^- \ &\mapsto\ GV \equiv \langle V, \cdot \rangle \mid_{E_1^+} - \langle V, \cdot \rangle \mid_{E_1^-} \end{aligned} \tag{2.10}$$

where the equivalence $E_1^* = TM \oplus T^*M = E_1$ has been used.
Let us now define an operator $\psi : TM \to T^*M$ in order to find an explicit splitting. We require ψ to satisfy:

$$\langle v + \psi(v),\, v + \psi(v) \rangle > 0 \qquad \forall\, v \in TM \tag{2.11}$$

One can then simply define the splitting via:

$$E_1^+ \equiv \{v + \psi(v),\, v \in TM\} \tag{2.12}$$

The important point here is that ψ, as an operator acting on vectors and giving one-forms, can be seen as a $(0,2)$-tensor, and can thus be decomposed into a symmetric part and an antisymmetric one. These are the two objects we wanted to merge into a generalised metric: the usual metric g which is a symmetric $(0,2)$-tensor, and the B-field which is a two-form *i.e.* an antisymmetric $(0,2)$-tensor. More precisely, g is a Riemannian metric on TM. This condition is equivalent to the defining condition of ψ (2.11): the $B(v) = -i_v B$ part does not contribute to the value of the inner product, due to the nilpotency of i_v.
Noting that $e^B(v + g(v)) = v + g(v) - i_v B = v + \psi(v)$, one has:

$$E_1^+ = e^B \bar{E}_1^+ \qquad \text{with } \bar{E}_1^+ \equiv \{v + g(v),\, v \in TM\} \tag{2.13}$$

$$E_1^- = e^B \bar{E}_1^- \qquad \text{with } \bar{E}_1^- \equiv \{v - g(v),\, v \in TM\} \tag{2.14}$$

where $\bar{E}_1^{\pm}$ is defined with g instead of ψ, *i.e.* with $B = 0$. In this case, one can note that:

$$\begin{aligned} \bar{G}(2v) &= \bar{G}(V^+ + V^-) = V^+ - V^- = 2g(v) \\ \bar{G}(2g(v)) &= \bar{G}(V^+ - V^-) = V^+ + V^- = 2v \end{aligned}$$

where $\bar{G}$ is the generalised metric when $B = 0$. Hence $\bar{G} = \left[\begin{smallmatrix} 0 & g^{-1} \\ g & 0 \end{smallmatrix}\right]$.
Besides, since $E_1^+ = e^B \bar{E}_1^+$ and $GE_1^+ = E_1^+$, one has $e^{-B} G e^B = \bar{G}$. Finally, we get in this matrix representation:

$$\begin{aligned} G &= \begin{bmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{bmatrix} \begin{bmatrix} 0 & g^{-1} \\ g & 0 \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ -B & \mathbb{1} \end{bmatrix} \\ &= \begin{bmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{bmatrix} \end{aligned} \tag{2.15}$$

Note that the need for B to be a closed form is equivalent to the one for e^B to preserve the Courant bracket.

2.1.2 U-duality and $E_{d(d)}$ generalised tangent space

A three-form field, that we will denote by B in the following, is appearing in Supergravity (especially in four dimensions) as well as in several String Theories. It would thus be interesting to write down a Generalised Geometry language including such a field and the related symmetries. One can build the generalised tangent space $E_2 = TM \oplus \wedge^2 T^*M$ on a $d = 4$-dimensional manifold M and a theory where a metric g and a closed form field $F = \mathrm{d}B$ play a central role.

The first important thing to note is that the inner product (2.1) defined above is no longer bilinear, and is thus no longer preserving the group of transformations. We then need to find another general linear group on our generalised tangent space. Looking at the natural group of transformations on TM, $GL(4, \mathbb{R})$, and at the U-duality symmetry group, $E_{4(4)} \simeq SL(5, \mathbb{R})$, one can choose to use $GL(5, \mathbb{R})$ which include both these groups[8].
Let us first find another representation for our generalised vector that makes more sense regarding the symmetry group. In the following, the Latin indices will cover the $GL(5, \mathbb{R})$ dimensions $m,\, n,\, p \ldots = 1 \ldots 5$, whereas the Greek ones will still denote the manifold dimensions $\mu,\, \mu_i,\, \nu_i \ldots = 1 \ldots 4$.
Our generalised vector is written as V^{mn} ; the vector v, element of TM, as $v^\mu = V^{5\mu}$. Using the isomorphism presented in details in appendix A, one can express our two-form field ω as $V^{\nu_1\nu_2} = \frac{1}{2}\, \epsilon^{\nu_1\nu_2\mu_1\mu_2}\, \omega_{\mu_1\mu_2}$. Finally, we have $V^{mn} = -V^{nm}$, hence a generalised vector $V = v + \omega$ written as a 5×5 matrix.

In this representation, we now need to express our transformations in a coherent way, that is with $V' = Q\, V\, Q^{tr}$ with $Q \in GL(E_2) \simeq GL(5, \mathbb{R})$, or equivalently $V'^{pq} = Q^p{}_m V^{mn} Q_n{}^q$. As for the previous case (E_1), the transformations can be decomposed into several specific cases. Naturally, the first one is the linear transformations represented by a block diagonal matrix:

$$Q_A = \begin{bmatrix} A\, (\det A)^\kappa & 0 \\ 0 & (\det A)^\lambda \end{bmatrix} \tag{2.16}$$

[8]Even if $SL(5, \mathbb{R})$ seems to be enough at first since we could include $GL(4, \mathbb{R})$ in it, we will see later that this extension is necessary.

with[9] $A \in GL(TM)$ and $\kappa,\, \lambda \in \mathbb{R}$. One thus has:

$$V'^{5\mu} = (Q_A)^5{}_\nu\,(Q_A)^\mu{}_\rho\,V^{\nu\rho} = (\det A)^{\kappa+\lambda}\;A^\mu{}_\rho\,V^{5\rho} \tag{2.17}$$

$$V'^{\mu\nu} = (Q_A)^\mu{}_\rho\,(Q_A)^\nu{}_\sigma\,V^{\rho\sigma} = (\det A)^{2\kappa}\;A^\mu{}_\rho\,A^\nu{}_\sigma\,V^{\rho\sigma} \tag{2.18}$$

In order to preserve the symmetries, one needs to fix $\kappa + \lambda = 0$ (hence $V^{5\mu} \in TM$) and $2\kappa = 1$ (hence $V^{\mu\nu} \in (\det TM) \otimes \wedge^2 T^*M$).
Note that these conditions on κ and λ, *i.e.* on $V^{5\mu}$ and $V^{\mu\nu}$ to transform as a vector field and a two-form density field, are too restrictive to be compatible with $Q_A \in SL(5,\,\mathbb{R})$. Indeed, one could not impose $\det(Q_A) = 1$ without restricting A to be in $SL(TM)$ (instead of $GL(TM)$). Hence why we choose $GL(E_2)$ to be $GL(5,\,\mathbb{R})$ as opposed to $SL(5,\,\mathbb{R})$.
Glancing at the way an object such as $P^m = (\,p^\mu \quad p\,)^{tr}$ transforms under this operator, it can be seen that we have:

$$\begin{aligned} p'^\mu &= (\det A)^{1/2}\,A^\mu{}_\nu\,p^\nu \quad &&\Rightarrow\; p^\mu \in (\det TM)^{1/2} \otimes TM \\ p' &= (\det A)^{-1/2}\,p \quad &&\Rightarrow\; p \in (\det TM)^{-1/2} \end{aligned}$$

Therefore $P^m \in (\det TM)^{1/2}\,(TM \otimes (\det TM)^{-1})$.

Let us now look at the two other kinds of transformations, *i.e.* the shear transformations in the $\wedge^2 T^*M$ and TM directions as defined by e^B and e^C in the previous case[10]. First, we define $(Q_B)^{mn} \equiv \left[\begin{smallmatrix} 0 & B^\mu \\ 0 & 0 \end{smallmatrix}\right]$ such that $e^{Q_B} \in GL(5,\,\mathbb{R})$ (in the following denoted by e^B). One has:

$$\begin{aligned} V'^{mn} &= (e^B)^m{}_p\,(e^B)^n{}_q\,V^{pq} \\ &= V^{pq} + \begin{bmatrix} B^\mu v^\nu - B^\nu v^\mu & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.19}$$

Indeed, Q_B does generate a shear transformation in the $\wedge^2 T^*M$ direction by shifting the two-form part of V (the ω part) by an antisymmetric product of the vector part of V (the v part) and the parameter of the transformation (the B-field).

[9] Note that here, the morphism $A \in GL(TM)$ is different from the one $A \in \mathrm{End}\,(TM)$ used previously in (2.9), even if they could be closely related.

[10] Except in the fact that e^B was a shear transformation in the T^*M direction since the generalised tangent space was $E_1 \simeq TM \oplus T^*M$.

Furthermore, we define $(Q_C)^{mn} \equiv \left[\begin{smallmatrix} 0 & 0 \\ C_\nu & 0 \end{smallmatrix}\right]$ such that $e^{Q_C} \in GL(5,\, \mathbb{R})$ (in the following denoted by e^C). Again, one has:

$$\begin{aligned} V'^{mn} &= (e^C)^m_{\ p}\,(e^C)^n_{\ q}\,V^{pq} \\ &= V^{pq} + \begin{bmatrix} 0 & -C_\rho V^{\rho\mu} \\ C_\rho V^{\rho\nu} & 0 \end{bmatrix} \end{aligned} \tag{2.20}$$

Here too, Q_C generates a shear transformation in the TM direction by shifting antisymmetrically the v part by a contraction of the ω part and the parameter of the transformation (the C-field).

Finally, the space of transformations can be decomposed as follows:

$$GL(E_2) = GL(5,\, \mathbb{R}) \simeq GL(4,\, \mathbb{R}) \oplus \wedge^3 T^*M \oplus \wedge^3 TM \tag{2.21}$$

2.2 Differential structure

Since our generalised tangent spaces have been previously provided with a common differential structure, that is a generalised Lie derivative (2.2) and a Courant bracket (2.3), we will now focus on the latest tangent space. Let us then rewrite our Courant bracket in a matrix language as developped for $E_2 \simeq TM \oplus \wedge^2 T^*M$:

$$\begin{aligned} [\![V,\, U]\!] &= [\![V^{5\mu} + V^{\mu_1\mu_2},\, U^{5\mu} + U^{\mu_1\mu_2}]\!] \\ &= [V,\, U]^{5\mu} + \big(\mathcal{L}_V U - \mathcal{L}_U V - \frac{1}{2}\, d(i_U V - i_V U)\big)^{\mu_1\mu_2} \\ &= (V^{5\nu}\partial_\nu U^{5\mu} - U^{5\nu}\partial_\nu V^{5\mu}) + \big(V^{5\nu}\partial_\nu U^{\mu_1\mu_2} - U^{5\nu}\partial_\nu V^{\mu_1\mu_2}\big) \\ &\quad - \frac{3}{2}\big(V^{[\mu_1\mu_2}\partial_\nu U^{5\nu]} - U^{[\mu_1\mu_2}\partial_\nu V^{5\nu]} + V^{[5\nu}\partial_\nu U^{\mu_1\mu_2]} - U^{[5\nu}\partial_\nu V^{\mu_1\mu_2]}\big) \end{aligned}$$

or, in a matrix layout:

$$[\![V,\, U]\!]^{mn} = \begin{pmatrix} \begin{matrix} V^{5\nu}\partial_\nu U^{\mu_1\mu_2} - U^{5\nu}\partial_\nu V^{\mu_1\mu_2} \\ -\frac{3}{2}\big(V^{[\mu_1\mu_2}\partial_\nu U^{5\nu]} - U^{[\mu_1\mu_2}\partial_\nu V^{5\nu]} \\ +V^{[5\nu}\partial_\nu U^{\mu_1\mu_2]} - U^{[5\nu}\partial_\nu V^{\mu_1\mu_2]}\big) \end{matrix} & \begin{matrix} V^{5\nu}\partial_\nu U^{5\mu_1} \\ -U^{5\nu}\partial_\nu V^{5\mu_1} \end{matrix} \\ V^{5\nu}\partial_\nu U^{5\mu_2} - U^{5\nu}\partial_\nu V^{5\mu_2} & 0 \end{pmatrix} \tag{2.22}$$

In order to check the consistency, we need to verify that B-field transformations are still a symmetry of the Courant bracket, *i.e.* that they satisfy:

$$e^B \, [\![V,\, U]\!] = [\![e^B V,\, e^B U]\!] \quad \Leftrightarrow \quad \mathrm{d}B = 0 \tag{2.23}$$

It can firstly be noted that, defining $B^\mu = \epsilon^{\mu\nu\rho\sigma} \, B_{\nu\rho\sigma}$ *i.e.* the dual for the three-form field B, one has $\frac{1}{2}\, \epsilon^{\mu_1\mu_2\nu\rho}\, V^{5\sigma}\, B_{\nu\rho\sigma} = -2B^{[\mu_1}\, V^{\mu_2]5}$. Also, the v part of our generalised vector is invariant under the action of e^B, whereas the ω part is modified into:

$$V'^{\mu_1\mu_2} \equiv V^{\mu_1\mu_2} + \frac{1}{2}\, \epsilon^{\mu_1\mu_2\nu\rho}\, V^{5\sigma}\, B_{\nu\rho\sigma} = V^{\mu_1\mu_2} - 2B^{[\mu_1}\, V^{\mu_2]5} \tag{2.24}$$

One can then compute:

$$\begin{aligned}
&[\![e^B\, V\, (e^B)^{tr},\; e^B\, U\, (e^B)^{tr}]\!]^{mn} \\
&\qquad = \left[\!\!\left[\begin{bmatrix} V'^{\mu_1\mu_2} & -V^{5\mu_1} \\ V^{5\mu_2} & 0 \end{bmatrix},\; \begin{bmatrix} U'^{\mu_1\mu_2} & -U^{5\mu_1} \\ U^{5\mu_2} & 0 \end{bmatrix} \right]\!\!\right] \\
&\qquad = [\![V,\, U]\!]^{mn} + 2 \begin{bmatrix} \begin{matrix} \left(V^{5\nu}\, \partial_\nu U^{5[\mu_2} - U^{5\nu}\, \partial_\nu V^{5[\mu_2}\right) B^{\mu_1]} \\ -\, V^{5[\mu_1} U^{\mu_2]5}\, \partial_\nu B^\nu \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

and identify $\left(i_{[V,U]} B\right)^{\mu_1\mu_2} = 2\left(V^{5\nu}\, \partial_\nu U^{5[\mu_2} - U^{5\nu}\, \partial_\nu V^{5[\mu_2}\right) B^{\mu_1]}$, *i.e.* the first part of the extra term, as well as $(i_V i_U dB)^{\mu_1\mu_2} = V^{5[\mu_1} U^{\mu_2]5}\, \partial_\nu B^\nu$, *i.e.* the second part. Hence the final equality:

$$\begin{aligned}
&[\![e^B\, V\, (e^B)^{tr},\; e^B\, U\, (e^B)^{tr}]\!]^{mn} \\
&\qquad = [\![V,\, U]\!]^{mn} + \left(i_{[V,U]} B\right)^{\mu_1\mu_2} - 2\, (i_V i_U\, \mathrm{d}B)^{\mu_1\mu_2} \\
&\qquad = \left(e^B\, [\![V,\, U]\!]\, (e^B)^{tr}\right)^{mn} \quad \Leftrightarrow \quad \mathrm{d}B = 0
\end{aligned} \tag{2.25}$$

which implies[11] the conservation of the Courant bracket under B-field transformations, under the necessary and sufficient condition that the three-form field B be closed.

[11] This was known but has now been proven in the matrix language previously developped.

2.3 Generalised connections and metric

In order to complete the structure of our generalised tangent space, an affine connection is needed, as well as a metric, both being invariant under $GL(5, \mathbb{R})$.

Let us first introduce an affine connection:

$$\begin{aligned} \nabla : E_2 \times E_2 &\to E_2 \\ (V,\, U) &\mapsto \nabla_V U \end{aligned} \tag{2.26}$$

satisfying the following properties, $\forall\, V, U, T \in (E_2)^3$ and $\forall\, f \in \mathscr{C}^\infty(M)$[12]:

$$\nabla_V\,(U+T) = \nabla_V U + \nabla_V T \tag{2.27a}$$

$$\nabla_{V+U}\, T = \nabla_V T + \nabla_U T \tag{2.27b}$$

$$\nabla_{f\,V}(U) = f\,(\nabla_V U) \tag{2.27c}$$

$$\nabla_V (f\,U) = V[f] + f\,(\nabla_V U) \tag{2.27d}$$

where we define $V[f] = (v+\omega)[f] \equiv v[f]$.

We also define a basis for E_2 with antisymmetric matrices labelled by A and with only -1 and 1 at the position $(m+1,\, m)$ and $(m,\, m+1)$:

$$\left\{ e_A \equiv \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & -1 & \\ & 1 & & \\ 0 & & & 0 \end{bmatrix},\ A = 1, \ldots 10 \right\} \tag{2.28}$$

and connections $\Omega_{A\ B}^{\ C} \in \mathbb{K}$ $-$ $\mathbb{K}$ the space of connection $-$ with[13]:

$$\Omega_{A\ B}^{\ C}\, e_C \equiv \nabla_A\, e_B \tag{2.29}$$

Then, one can work out the properties (2.27) in terms of $\{e_A\}$ and get:

$$\nabla_V U = V^A \left(e_A[U^B] + \Omega_{A\ C}^{\ B} U^C\right) e_B = V^A \left(\partial_A U^B + \Omega_{A\ C}^{\ B} U^C\right) e_B$$

since the form part of $e_A[U^B]$ vanishes. Here, one can recognise and define the usual form of the covariant derivative in terms of partial derivative and

[12] Note that here and in the following, f is no longer an element of T^*M but a continuous function over the manifold M.

[13] We slightly simplify the notation again, assimilating $\nabla_{e_A} \equiv \nabla_A$

connection.

The last point to check is the invariance of these connections under $GL(5,\,\mathbb{R})$ transformations, by defining an action over our tangent space E_2 (in (mn) indices):

$$\begin{aligned} (\ \ ,\ \ ): \qquad & \mathbb{K} \times E_2 \ \to \ E_2 \\ (\Omega_{pq}{}^{mn}{}_{rs},\, V^{rs}) \mapsto (\Omega \cdot V)^{mn}_{pq} & \equiv (\Omega_{pq})^m_r\, V^{rn} - (\Omega_{pq})^n_r\, V^{rm} \end{aligned} \tag{2.30}$$

Note that this definition is coherent with the action of the affine connection ∇_V on $P^m \in (\det TM)^{1/2}\, (TM \otimes (\det TM)^{-1})$.

Finally, let us find a generalised metric on E_2 as we did for E_1. As usual, it must satisfy several key properties, such as being covariant under $GL(5,\,\mathbb{R})$ transformations, and including the fields our theory would need in one object, mathematically coherent.

Our theory contains at least the usual metric g and the three-form field B. Using the isomorphism between $\wedge^3 T^*M$ and $(\det TM) \otimes T^*M$, our first assumption could be that $G \in ((\det TM) \otimes T^*M) \oplus S^2 T^*M$, which seems reasonable apart from the number of degrees of freedom. Indeed, given that our generalised metric G must be symmetric (since g is), it should have ${}^{5\times 6}\!/_{2} = 15$ d.o.f. ; the usual metric contains ${}^{4\times 5}\!/_{2} = 10$ d.o.f. and the three-form field ${}^{4\times 3\times 2}\!/_{3\times 2} = 4$. There is an extra freedom that we will include either as the G^{55} term or as an overall multiplying factor.

Finally, recalling $V \in T^*M \oplus \wedge^2 T^*M$ and that we need to apply the same transformations on our generalised vectors V^{mn} and on the inverse of the generalised metric G^{mn}, we assume $G^{-1} \in S^2TM \oplus TM \oplus (\det TM)$.

Hence:

$$G^{mn} = \begin{bmatrix} G^{\mu_1\mu_2} & G^{\mu_1 5} \\ G^{5\mu_2} & G^{55} \end{bmatrix} = \begin{bmatrix} \tilde{g}^{\mu_1\mu_2} & X^{\mu_1} \\ X^{\mu_2} & \gamma \end{bmatrix} \tag{2.31}$$

with a density scalar field $\gamma \in \det TM$, a vector field $X \in TM$ such that $X^\mu \sim B^\mu = \epsilon^{\mu\nu\rho\sigma} B_{\nu\rho\sigma}$, and an inverse metric $\tilde{g} \in S^2TM$. Note that this metric is not necessarily g, the one on our manifold M, but has the same signature $(1,3)$. This point raises the question of the signature of our generalised metric G: we choose it to have a $(2,3)$ signature $-$ and not $(1,4)$ $-$ in order to be adequatly developped for our physical theories.

Indeed, the fermionic sector of four-dimensional Supergravity is built with the subgroup $Spin(2,3)$ of the Clifford Algebra, leading to such a signature. Moreover, one of the three-form supergravity solutions, that is AdS_4, is naturally embedded in $\mathbb{R}^{2,3}$, confirming the choice for G.

Let us now apply the $GL(5,\,\mathbb{R})$ transformations on the different components of the generalised metric, starting with the linear operator Q_A:

$$X'^{\mu} = A^{\mu}{}_{\nu}\, X^{\nu} \quad \Rightarrow \quad G^{5\mu} \in TM \tag{2.32a}$$

$$\gamma' = (\det A)^{-1}\, \gamma \quad \Rightarrow \quad G^{55} \in (\det TM)^{-1} \tag{2.32b}$$

$$\tilde{g}'^{\mu_1\mu_2} = (\det A)\, A^{\mu_1}_{\ \nu_1} A^{\mu_2}_{\ \nu_2}\, \tilde{g}^{\mu_1\mu_2} \quad \Rightarrow \quad G^{\mu_1\mu_2} \in (\det TM) \otimes S^2TM \tag{2.32c}$$

One can see that this is not coherent. Recalling the determinant of the metric is a density scalar of rank two and writting $\sqrt{|\det\tilde{g}|} \equiv \mathfrak{g}$, we redefine the generalised metric's components. The metric part is a density and should include a γ^{-1} factor: $G^{\mu_1\mu_2} \equiv \mathfrak{g}\,\gamma\,\tilde{g}^{\mu_1\mu_2}$. The vector part needs to include this extra-factor as well: $G^{5\mu_2} \equiv \gamma\, X^{\mu}$. Finally, we adjust the rank of the density scalar: $G^{55} \equiv \mathfrak{g}^{-1}\,\gamma$.
Hence the final form of the generalised inverse metric:

$$\begin{aligned} G^{mn} &= \mathfrak{g}\,\gamma \begin{bmatrix} \tilde{g}^{\mu_1\mu_2} & \mathfrak{g}^{-1}\, X^{\mu_1} \\ \mathfrak{g}^{-1}\, X^{\mu_2} & \mathfrak{g}^{-2} \end{bmatrix} \\ &\in (\det TM) \otimes \Big(S^2TM \oplus (\det TM)^{-1}\, TM \oplus (\det TM)^{-2}\Big) \end{aligned} \tag{2.33}$$

We also define, in the case $B = 0$ and using here the usual manifold metric,

$$\bar{G}^{mn} \equiv \mathfrak{g}\,\gamma \begin{bmatrix} g^{\mu_1\mu_2} & 0 \\ 0 & \mathfrak{g}^{-2} \end{bmatrix} \tag{2.34}$$

which should satisfy $G^{-1} = e^B\, \bar{G}^{-1}\, (e^B)^{tr}$ under a shear transformation in the $\wedge^2 T^*M$ direction – as was developped proviously for V^{mn}. This leads to:

$$G^{mn} = \mathfrak{g}\,\gamma \begin{bmatrix} g^{\mu_1\mu_2} + \mathfrak{g}^{-2}\, B^{\mu_1} B^{\mu_2} & \mathfrak{g}^{-2}\, B^{\mu_1} \\ \mathfrak{g}^{-2}\, B^{\mu_2} & \mathfrak{g}^{-2} \end{bmatrix} \tag{2.35}$$

Given that $B^{\mu} \in (\det TM) \otimes T^*M$, this is coherent with the definition of the generalised metric. One thus has $\tilde{g}^{\mu_1\mu_2} = g^{\mu_1\mu_2} + \mathfrak{g}^{-2}\, B^{\mu_1} B^{\mu_2}$ and $\mathfrak{g}\, X^{\mu} = B^{\mu}$, which are once again coherent with the previous work.

Finally, let us check the consistency with the other shear transformation, generated by Q_C: one should have $G^{-1} = e^C\, \bar{G}^{-1}\, (e^C)^{tr}$. One gets:

$$G^{mn} = \mathfrak{g}\,\gamma \begin{bmatrix} g^{\mu_1\mu_2} & g^{\mu_1\nu}\, C_\nu \\ g^{\mu_2\nu}\, C_\nu & g^{\mu\nu} C_\mu C_\nu + \mathfrak{g}^{-2} \end{bmatrix} \tag{2.36}$$

which is again coherent, since $C_\mu \in (\det TM)^{-1}\, T^*M \simeq \wedge^3 TM$.

Lastly, one wants to construct the generalised metric G_{mn} (not its inverse). Let us start with the easiest case:

$$\bar{G}_{mn} \equiv \mathfrak{g}^{-1}\,\gamma^{-1} \begin{bmatrix} g_{\mu_1\mu_2} & 0 \\ 0 & \mathfrak{g}^2 \end{bmatrix} \tag{2.37}$$

One can then invert one of the relationships between G^{mn} and $\bar{G}^{mn}$, giving $G = (e^{-B})^{tr}\, \bar{G}\, e^{-B}$; therefore:

$$G_{mn} \equiv \mathfrak{g}^{-1}\,\gamma^{-1} \begin{bmatrix} g_{\mu_1\mu_2} & -g_{\mu_1\nu} B^\nu \\ -g_{\mu_2\nu} B^\nu & \mathfrak{g}^2 + B^\mu g_{\mu\nu} B^\nu \end{bmatrix} \tag{2.38}$$

Not only does this transformation give a coherent metric, but it also confirms that $G \in (\det TM)^{-1} \otimes \left(S^2 T^*M \oplus (\det TM)\, T^*M \oplus (\det TM)^2 \right)$.
One can finally check that the other shear transformation gives a coherent result:

$$G_{mn} \equiv \mathfrak{g}^{-1}\,\gamma^{-1} \begin{bmatrix} g_{\mu_1\mu_2} + \mathfrak{g}^2\, C_{\mu_1} C_{\mu_2} & -\mathfrak{g}^2\, C_{\mu_1} \\ -\mathfrak{g}^2\, C_{\mu_2} & \mathfrak{g}^2 \end{bmatrix} \tag{2.39}$$

As a conclusion work on our generalised tangent space E_2, it would be interesting to look at metric-compatible connections, *i.e.* connections which preserve the metric defined above. A torsion-free condition can be added in order to develop the generalisation of the Levi-Civita connection, that is the metric compatible torsion-free connection.
First, we recall that in conventional geometry, one has a unique Levi-Civita connection, since its 64 degrees of freedom (d.o.f.) are constrained by 64 equations: 24 from the torsion-free condition and 4×10 to be metric compatible:

$$\Gamma_{\mu\ \rho}^{\ \nu} \quad \text{such that} \quad T_{\mu\ \rho}^{\ \nu} \equiv \Gamma_{\mu\ \rho}^{\ \nu} - \Gamma_{\rho\ \mu}^{\ \nu} = 0 \quad \text{and} \quad \nabla_\mu g_{\nu\rho} = 0 \tag{2.40}$$

Here, the connections contains 250 d.o.f, since it is a five-dimensional $(1,3)$-tensor with two antisymmetrised indices[14] (that is $5 \times 5 \times \frac{5\times 4}{2}$ d.o.f).

[14] One can write the connections as $\Omega_{pq\ r}^{\ \ m}$ from its action over E_2 defined in (2.30).

The torsion is a generalised $(1,2)$-tensor, or equivalently a map defined as $T : E_2 \times E_2 \rightarrow E_2$ and such that:

$$T(V,\, U) \equiv \nabla_V U - \nabla_U V - [V,\, U] = \Omega_{A\ C}^{\ B} \left(V^A U^C - V^C U^A \right) e_B \qquad (2.41)$$

for any $(V,\, U) \in (E_2)^2$. Although the torsion can be seen as some antisymmetrisation of the connections on the lower indices (A, C), one can note that the symmetric part of the trace of the connections remains[15]. It is a symmetric $(0,2)$-tensor with 15 d.o.f ; the traceless part of the torsion is a $(1,3)$-tensor with three antisymmetrised indices giving $5 \times \frac{5\times4\times3}{2\times3} = 50$ d.o.f. Therefore, the torsion-free condition restricts the connections with 65 equations, $250 - 65 = 185$ d.o.f. remaining.

Similarly, the generalised metric is a symmetric $(0,2)$-tensor, *i.e.* contains 15 d.o.f., and the covariant derivative is antisymmetric, giving 10 d.o.f. Thus, the compatibility condition $DG = 0$ combines 150 equations, leaving $185 - 150 = 35$ d.o.f. for the generalised Levi-Civita connections.

The unicity of such connections is thus lost. This is not a problem for the framework developped above, mainly because a Ricci curvature tensor can still be built up uniquely. Indeed, even if the Riemann curvature $-$ which can be defined as the commutator of the covariant derivative $-$ is no longer a tensor, its contraction is and constrains all the freedom left.

[15]Indeed, when expressed with the (m, n) indices instead of the A ones, the trace of the connections does not vanish.

3 $E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry

From its original version where the tangent space is $E_1 \simeq TM \oplus T^*M$ and the underlying structure relies on $O(d,d)$, Generalised Geometry has been extended to include the symmetries appearing in M theory. The generalised tangent space $E \simeq TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^7 T^*M)$ is relevant for the low dimensional $d \leq 7$ restriction of eleven-dimensional Supergravity − the low-energy limit of M theory. It admits an $E_{d(d)}$ structure, which is completed by a "trombone symmetry" to give the $E_{d(d)} \times \mathbb{R}^+$ structure.

In the following, the manifold M is a d-dimensional spin manifold, restricted to $d \leq 7$. Note that if $d < 7$, one would need to ignore one or several terms in the direct sum of tensor bundles that our generalised tangent space E is isomorphic to. One also defines a patching $\{U_j\}$ on M.

3.1 Generalised tangent bundle

The tangent space E is isomorphic to a sum of tensor bundles. To be more accurate, the space E springs from a series of exact extensions:

$$\begin{aligned} 0 \longrightarrow \wedge^2 T^*M \longrightarrow E'' \longrightarrow TM \longrightarrow 0 \\ 0 \longrightarrow \wedge^5 T^*M \longrightarrow E' \longrightarrow E'' \longrightarrow 0 \\ 0 \longrightarrow T^*M \otimes \wedge^7 T^*M \longrightarrow E \longrightarrow E' \longrightarrow 0 \end{aligned} \tag{3.1}$$

One can see this in a more practical way when looking at the elements of sections of E and the way they are patched. Indeed, for $V_{(j)}$, $V_{(k)}$ elements of a section of E over patches U_j, U_k, one has:

$$\begin{aligned} V_{(j)} &= v_{(j)} + \omega_{(j)} + \sigma_{(j)} + \tau_{(j)} \\ &\in \Gamma\left(TU_j \oplus \wedge^2 T^*U_j \oplus \wedge^5 T^*U_j \oplus (T^*U_j \otimes \wedge^7 T^*U_j)\right) \end{aligned} \tag{3.2}$$

and, locally defining $\Lambda_{(jk)}$ and $\Xi_{(jk)}$ two- and five-forms on the overlap $U_j \cap U_k$, one gets:

$$V_{(j)} = e^{\mathrm{d}\Lambda_{(jk)} + \mathrm{d}\Xi_{(jk)}} V_{(k)} \tag{3.3}$$

$$
\begin{aligned}
i.e. \quad v_{(j)} &= v_{(k)} && \text{(3.4a)}\\
\omega_{(j)} &= \omega_{(k)} + i_{v_{(k)}} \mathrm{d}\Lambda_{(jk)} && \text{(3.4b)}\\
\sigma_{(j)} &= \sigma_{(k)} + \mathrm{d}\Lambda_{(jk)} \wedge \omega_{(k)} \\
&\qquad + i_{v_{(k)}} \mathrm{d}\Xi_{(jk)} + {}^1\!/\!_2\, \mathrm{d}\Lambda_{(jk)} \wedge i_{v_{(k)}} \mathrm{d}\Lambda_{(jk)} && \text{(3.4c)}\\
\tau_{(j)} &= \tau_{(k)} + j\, \mathrm{d}\Lambda_{(jk)} \wedge \sigma_{(k)} - j\, d\Xi_{(jk)} \wedge \omega_{(k)} \\
&\qquad + j\, \mathrm{d}\Lambda_{(jk)} \wedge i_{v_{(k)}} \mathrm{d}\Xi_{(jk)} + {}^1\!/\!_2\, j\, \mathrm{d}\Lambda_{(jk)} \wedge \mathrm{d}\Lambda_{(jk)} \wedge \omega_{(k)} \\
&\qquad + {}^1\!/\!_6\, j\, \mathrm{d}\Lambda_{(jk)} \wedge \mathrm{d}\Lambda_{(jk)} \wedge i_{v_{(k)}} \mathrm{d}\Lambda_{(jk)} && \text{(3.4d)}
\end{aligned}
$$

Note that $v_{(j)}$ is a globally defined vector, whereas $\omega_{(j)}$, $\sigma_{(j)}$, $\tau_{(j)}$ are only locally defined tensors. Note also that the generalised tangent bundle E contains all the topological information for our supergravity background.

This generalised vector bundle structure means that from every point $x \in M$, there is a fibre denoted by E_x. Since we have an $E_{d(d)} \times \mathbb{R}^+$ principal bundle, this fibre forms a representation space of the group $E_{d(d)} \times \mathbb{R}^+$. The bundle is also defined by an action of this group, which acts here on the component spaces T_xM, $\wedge^2 T_x^*M$, $\wedge^5 T_x^*M$ and $T_x^*M \otimes \wedge^7 T_x^*M$ via the $GL(d, \mathbb{R})$ subgroup.
Thus, we note that the exact extension (3.1) defining E is directly linked to the structure group. Indeed, without the extension by an $\mathbb{R}^+$ factor, sections of the vector bundle would not transform as tensors ; they would get an additional power of $(\det T^*M)$.

Moreover, one can define a superstructure, a frame bundle F, for E. We define $\{\hat{E}_A,\, A = 1 \dots r\}$ a basis for the fibre E_x, where r is the dimension of the representation of the $E_{d(d)} \times \mathbb{R}^+$ group, *i.e.* the dimension of the generalised tangent space. One can check the values presented in Table 1 below by computing the dimension of each tensor bundle in the sum (the number of which depends on the dimension) and adding them.
The frame bundle F is formed from all the bases and is therefore a $GL(r, \mathbb{R})$ principle bundle. This means that E is seen as a sub-bundle, the natural $E_{d(d)} \times \mathbb{R}^+$ principle sub-bundle of F which is compatible with the patching defined earlier in (3.3).
Let us define $\{\hat{e}_\mu\}$ a basis for T_xM and thus $\{\hat{e}^\mu\}$ a basis for T_x^*M. One then has $\{\hat{e}^{\mu_1\mu_2}\}$ for $\wedge^2 T_x^*M$, $\{\hat{e}^{\nu_1\dots\nu_5}\}$ for $\wedge^5 T_x^*M$ and finally $\{\hat{e}^{\nu,\rho_1\dots\rho_7}\}$ for

d	# of terms	r
7	4	56
6	3	27
5	3	16
4	2	10

Table 1: Dimension of the generalised tangent space

$\left(T_x^*M \otimes \wedge^7 T_x^*M\right)$. Using these, we can easily construct a basis for E_x just by combining them. Hence the expression of a generalised vector $V \in E_x$:

$$V = V^A\,\hat{E}_A = v^\mu\,\hat{e}_\mu + {}^1\!/_2\,\omega_{\mu_1\mu_2}\,\hat{e}^{\mu_1\mu_2} + {}^1\!/_{5!}\,\sigma_{\nu_1\ldots\nu_5}\,\hat{e}^{\nu_1\ldots\nu_5} + {}^1\!/_{7!}\,\tau_{\nu,\rho_1\ldots\rho_7}\,\hat{e}^{\nu,\rho_1\ldots\rho_7} \tag{3.5}$$

If U_j is endowed with a certain choice of coordinates, the natural basis on T_xM is $\{{}^{\partial}\!/_{\partial x^\mu}\}$, hence $\{\hat{E}_M\} = \{{}^{\partial}\!/_{\partial x^\mu}\} \cup \{\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2}\} \cup \ldots$ Note that through the $GL(d,\mathbb{R})$ subgroup of our $E_{d(d)} \times \mathbb{R}^+$ action, one acts on the bases such as $\{\hat{e}_\mu\}$ and $\{\hat{e}^{\mu_1\mu_2}\}$ in the usual way, separately. A general $E_{d(d)} \times \mathbb{R}^+$ transformation, including the patching (3.3), gives an $E_{d(d)} \times \mathbb{R}^+$ basis $\{\hat{E}_A\}$.

Let us have a closer look at these transformations, that is at our group $E_{d(d)} \times \mathbb{R}^+$ and its subgroup $GL(d,\mathbb{R})$. One has a manifold M defining TM, the usual tangent space, and E, the generalised tangent space. In order to define the Lie algebra of the group $E_{d(d)} \times \mathbb{R}^+$, we need to define a space ; let us call it $\mathfrak{F}$:

$$\mathfrak{F} \equiv \mathbb{R} \oplus (TM \otimes T^*M) \oplus \wedge^3 T^*M \oplus \wedge^6 T^*M \oplus \wedge^3 TM \oplus \wedge^6 TM \tag{3.6a}$$

$$\mathcal{F} = c + h + \alpha + \bar{\alpha} + a + \bar{a} \quad \in \mathfrak{F} \tag{3.6b}$$

and its action on $V = (v + \omega + \sigma + \tau) \in E$:

$$\mathcal{F} \cdot v = c\,v + h \cdot v + a \diamond \omega - \bar{a} \diamond \sigma \tag{3.7a}$$

$$\mathcal{F} \cdot \omega = c\omega + h \cdot \omega + v \diamond \alpha + a \diamond \sigma + \bar{a} \diamond \tau \tag{3.7b}$$

$$\mathcal{F} \cdot \sigma = c\sigma + h \cdot \sigma + v \diamond \bar{\alpha} + \alpha \wedge \omega + a \diamond \tau \tag{3.7c}$$

$$\mathcal{F} \cdot \tau = c\tau + h \cdot \tau + j\,\alpha \wedge \sigma - j\,\bar{\alpha} \wedge \omega \tag{3.7d}$$

For more details on the contractions, see Section 1.3, (1.10) and (1.11).
Starting with the action of $m \in GL(d,\, \mathbb{R})$:

$$(m \cdot v)^\mu = m^\mu{}_\nu \, v^\nu$$
$$(m \cdot \omega)_{\mu_1\mu_2} = (m^{-1})^{\nu_1}{}_{\mu_1} \, (m^{-1})^{\nu_2}{}_{\mu_2} \, \omega_{\nu_1\nu_2}$$

and adding the exponentiated action of α and $\bar{\alpha}$, a and $\bar{a}$:

$$\begin{aligned} e^{\alpha+\bar{\alpha}}\, V = v &+ (\omega + i_v\alpha) + (\sigma + \alpha \wedge \omega + {}^1\!/\!{}_2\, \alpha \wedge i_v\alpha + i_v\bar{\alpha}) \\ &+ (\tau + j\,\alpha \wedge \sigma - j\,\bar{\alpha} \wedge \omega + {}^1\!/\!{}_2\, j\,\alpha \wedge \alpha \wedge \omega \\ &\quad + {}^1\!/\!{}_2\, j\,\alpha \wedge i_v\bar{\alpha} - {}^1\!/\!{}_2\, j\,\bar{\alpha} \wedge i_v\alpha + {}^1\!/\!{}_6\, j\,\alpha \wedge \alpha \wedge i_v\alpha) \end{aligned} \tag{3.8}$$

$$\begin{aligned} e^{a+\bar{a}}\, V = (v &+ a \diamond \omega - \bar{a} \diamond \sigma + {}^1\!/\!{}_2\, a \diamond a \diamond \sigma \\ &+ {}^1\!/\!{}_2\, a \diamond \bar{a} \diamond \tau + {}^1\!/\!{}_2\, \bar{a} \diamond a \diamond \tau + {}^1\!/\!{}_6\, a \diamond a \diamond a \diamond \tau) \\ &+ (\omega + a \diamond \sigma + \bar{a} \diamond \tau + a \diamond a \diamond \sigma) \\ &+ (\sigma + a \diamond \tau) + \tau \end{aligned} \tag{3.9}$$

and the $\mathbb{R}^+$ scaling factor via e^δ, $\delta \in \mathbb{R}$, one finally gets an element of $E_{d(d)} \times \mathbb{R}^+$ which takes the form:

$$M \cdot V \;\equiv\; e^\delta \, e^{a+\bar{a}} \, e^{\alpha+\bar{\alpha}} \, m \cdot V \tag{3.10}$$

Note that the exponential expansion is limited to cubic terms. It is linked to the nilpotency (of rank two) of both the actions of $\alpha + \bar{\alpha}$ and $a + \bar{a}$.

One can also define generalised tensors as an extension of generalised vectors. Indeed, for instance, an element of the section of the dual generalised tangent space $E^* \simeq T^*M \oplus \wedge^2 TM \oplus \wedge^5 TM \oplus (TM \otimes \wedge^7 TM)$ can be expressed in the dual basis $\{\hat{E}^A\}$ as $W = W_A \, \hat{E}^A$.
We also need to define the generalised $E_{d(d)} \times \mathbb{R}^+$ structure bundle $\tilde{F}$:

$$\tilde{F} \equiv \Big\{ \Big(x,\, \{\hat{E}_A\}\Big) \,:\, x \in M,\, \{\hat{E}_A\} \text{ an } E_{d(d)} \times \mathbb{R}^+ \text{ basis } \Big\} \tag{3.11}$$

This sub-bundle of F (the frame bundle for E) is an $E_{d(d)} \times \mathbb{R}^+$ principle bundle in the exact same way as in conventional geometry where the frame bundle is a $GL(d,\, \mathbb{R})$ principle bundle.
The adjoint bundle $\mathrm{ad}\, \tilde{F}$ associated with $\tilde{F}$ is actually the space acting on the generalised tangent space to build our group $E_{d(d)} \times \mathbb{R}^+$:

$$\mathrm{ad}\, \tilde{F} \equiv \mathfrak{F} \tag{3.12}$$

An element of a section can be written $R = R^A{}_B\, \hat{E}_A\, \hat{E}^B$ since $\mathrm{ad}\,\tilde{F} \subset E \otimes E^*$. We also define the projection on the adjoint representation:

$$\otimes_{ad} : E^* \otimes E \to \mathrm{ad}\,\tilde{F}$$

Following the same path, we introduce the sub-bundle of the symmetric product of two generalised cotangent bundles:

$$\begin{aligned} N \simeq T^*M \oplus \wedge^4 T^*M \oplus \big(T^*M \otimes \wedge^6 T^*M\big) \\ \oplus \big(\wedge^3 T^*M \otimes \wedge^7 T^*M\big) \oplus \big(\wedge^6 T^*M \otimes \wedge^7 T^*M\big) \end{aligned} \tag{3.13}$$

Sections can also be written $S = S^{AB}\, \hat{E}_A\, \hat{E}_B$ and the projection defined as $\otimes_N : E \otimes E \to N$.

d	E^*	$\mathrm{ad}\,\tilde{F}$	N
7	56	133 + 1	133
6	27	78 + 1	27
5	16	45 + 1	10
4	10	24 + 1	5

Table 2: Dimension of the generalised tensors

The dimension of such generalised tensors − that is the dimension of the representation they belong to − can be computed by adding the dimensions of each term in the direct sum. Table 2 gives their dimensions with respect to the dimension d of our manifold M. Note that higher rank generalised tensors can be built using the same procedure. For more details, see [4].

3.2 Split frame

So far, we defined a frame bundle F for E and its sub-bundle, the generalised $E_{d(d)} \times \mathbb{R}^+$ structure bundle $\tilde{F}$. We now want to define a special class of $E_{d(d)} \times \mathbb{R}^+$ frames via a splitting of the generalised tangent space E following the isomorphism (1.5).

Let $\{\hat{e}_a\}$ be a generic basis for TM and $\{\hat{e}^a\}$ its dual, that is a basis for T^*M. Let $\Delta \in \mathbb{R}$ be a scalar field, A and $\bar{A}$ be three-form and six-form connections and let us patch them on $U_j \cap U_k$:

$$\begin{aligned} A_{(j)} &= A_{(k)} + \mathrm{d}\Lambda_{(jk)} \\ \bar{A}_{(j)} &= \bar{A}_{(k)} + \mathrm{d}\Xi_{(jk)} - {}^1\!/\!_2\, \mathrm{d}\Lambda_{(jk)} \wedge A_{(k)} \end{aligned} \tag{3.14}$$

We can then define a conformal split frame $\{\hat{E}_A\}$ for E:

$$\begin{aligned} \hat{E}_\mu = e^\Delta \, \big(&\hat{e}_\mu + i_{\hat{e}_\mu} A + i_{\hat{e}_\mu} \bar{A} + {}^1\!/\!_2\, A \wedge i_{\hat{e}_\mu} A \\ &+ jA \wedge i_{\hat{e}_\mu} \bar{A} + {}^1\!/\!_6\, jA \wedge A \wedge i_{\hat{e}_\mu} A \big) \end{aligned} \tag{3.15a}$$

$$\begin{aligned} \hat{E}^{\mu_1\mu_2} = e^\Delta \, \big(&\hat{e}^{\mu_1\mu_2} + A \wedge \hat{e}^{\mu_1\mu_2} - j\bar{A} \wedge \hat{e}^{\mu_1\mu_2} \\ &+ {}^1\!/\!_2\, jA \wedge A \wedge \hat{e}^{\mu_1\mu_2} \big) \end{aligned} \tag{3.15b}$$

$$\hat{E}^{\mu_1 \dots \mu_5} = e^\Delta \, \left(\hat{e}^{\mu_1 \dots \mu_5} + jA \wedge \hat{e}^{\mu_1 \dots \mu_5} \right) \tag{3.15c}$$

$$\hat{E}^{\mu, \nu_1 \dots \nu_7} = e^\Delta \, \hat{e}^{\mu, \nu_1 \dots \nu_7} \tag{3.15d}$$

The term "conformal" refers to the $\mathbb{R}^+$ factor and we would have a split frame in the case $\Delta = 0$.

The isomorphism (1.5) is actually realised via the definition of the connection forms A and $\bar{A}$ since one has, in the conformal split frame:

$$\begin{aligned} V^{(A,\bar{A})} &= e^{-\Delta} \, e^{-A_{(j)} - \bar{A}_{(j)}} \, V_{(j)} \\ &= v^\mu \, \hat{e}_\mu + {}^1\!/\!_2\, \omega_{\mu_1\mu_2} \, \hat{e}^{\mu_1\mu_2} + {}^1\!/\!_{5!}\, \sigma_{\nu_1 \dots \nu_5} \, \hat{e}^{\nu_1 \dots \nu_5} \\ &\qquad + {}^1\!/\!_{7!}\, \tau_{\nu, \rho_1 \dots \rho_7} \, \hat{e}^{\nu, \rho_1 \dots \rho_7} \end{aligned} \tag{3.16}$$

which is an element of the section of

$$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus (T^*M \otimes \wedge^7 T^*M)$$

Note that the way the connection forms are patched in (3.14) implies that $e^{-A_{(j)} - \bar{A}_{(j)}} \, V_{(j)} = e^{-A_{(k)} - \bar{A}_{(k)}} \, V_{(k)}$ since we patched the generalised vectors according to (3.3).

Finally, the class of split frames – which is a sub-bundle of $\tilde{F}$ – can be defined as:

$$P_{\text{split}} \equiv \Big\{ \Big(x, \, \{\hat{E}_A\} \Big) \, : \, x \in M, \, \{\hat{E}_A\} \text{ a split frame } \Big\} \; \subset \tilde{F} \tag{3.17}$$

The exponentiated action of $(\alpha + \bar{\alpha})$ on such frames shifts the connection forms: $A \mapsto A + a$, $\bar{A} \mapsto \bar{A} + \bar{\alpha}$. Thus, transformations such as $M = e^{\alpha + \bar{\alpha}} \, m$ with $m \in GL(d, \mathbb{R})$ are endomorphisms of P_{split}. They transform a split frame into another split frame, in opposition with the other kinds of $E_{d(d)} \times \mathbb{R}^+$ transformations with a $e^{a + \bar{a}}$ term – as defined in (3.10).

3.3 Dorfman derivative and exceptional Courant bracket

As mentioned initially, one of the main goals of this structure is to build a transformation which combines infinitesimal diffeomorphisms and gauge transformations. This is the role of the generalised Lie derivative, commonly called Dorfman derivative, which acts on any generalised tensor. As will be seen below, it also encodes the bosonic symmetries of our theory.

Indeed, let again $V = v + \omega + \sigma + \tau$ be an element of the section of our generalised tangent space E and let us define L_V this operator, acting on $U = u + \xi + \phi + \pi$ a generalised vector:

$$\begin{aligned} L_V U \equiv \mathcal{L}_v u + (\mathcal{L}_v \xi - i_u \mathrm{d}\omega) + (\mathcal{L}_v \phi - i_u \mathrm{d}\sigma - \xi \wedge \mathrm{d}\omega) \\ + (\mathcal{L}_v \pi - j\,\phi \wedge \mathrm{d}\omega - j\,\xi \wedge \mathrm{d}\sigma) \end{aligned} \tag{3.18}$$

Here, U is transformed by the action generated by both the vector part v − infinitesimal diffeomorphism − and the form parts ω and σ − gauge transformations, under the A- and $\bar{A}$-form fields.

In order to extend this Dorfman derivative to a derivative on other $E_{d(d)} \times \mathbb{R}^+$ generalised tensors, we should make the symmetry more obvious, that is rewrite the definition (3.18) in a covariant way.
Using the generalised partial derivative operator ∂_M defined in (1.9) for high dimensional manifolds, and defining the action of the Dorfman derivative on a function as the one of the usual Lie derivative $L_V f = \mathcal{L}_v f$, one can write:

$$L_V U^M = V^N \partial_N U^M - (\partial \otimes_{\mathrm{ad}} V)^M{}_N\, U^N \tag{3.19}$$

with $(\partial \otimes_{\mathrm{ad}} V) = \mathrm{d}v + \mathrm{d}\omega + \mathrm{d}\sigma$ the projection of the generalised partial derivative and the generalised vector onto $\mathrm{ad}\,\tilde{F}$.
Written in this form, one can simply extend the Dorfman derivative action to any kind of generalised tensor by taking the projection map's action on the appropriate $E_{d(d)} \times \mathbb{R}^+$ representation.

Then, the Dorfman derivative can be antisymmetrised, defining an exceptional Courant bracket as:

$$\begin{aligned}
[\![V,\, U]\!] \equiv {}& {}^1\!/\!_2\, (L_V U - L_U V) \\
={}& [v,\, u] + \mathcal{L}_v \xi - \mathcal{L}_u \omega - {}^1\!/\!_2\, \mathrm{d}(i_v \xi - i_u \omega) \\
&+ \mathcal{L}_v \phi - \mathcal{L}_u \sigma - {}^1\!/\!_2\, \mathrm{d}(i_v \phi - i_u \sigma) \\
&\qquad + ({}^1\!/\!_2\, \omega \wedge \mathrm{d}\xi - {}^1\!/\!_2\, \xi \wedge \mathrm{d}\omega) \\
&+ {}^1\!/\!_2\, (\mathcal{L}_v \pi - \mathcal{L}_u \tau) + {}^1\!/\!_2\, (j\, \omega \wedge \mathrm{d}\phi - j\, \phi \wedge \mathrm{d}\omega) \\
&\qquad - {}^1\!/\!_2\, (j\, \xi \wedge \mathrm{d}\sigma - j\, \sigma \wedge \mathrm{d}\xi)
\end{aligned} \tag{3.20}$$

It can be noted that the full automorphism group of the exceptional Courant bracket, that is the group of transformations generated by this bracket, is the local symmetry group of Supergravity: local diffeomorphism and closed three- and six-form connections gauge transformations. It can be written as a semi-direct product: $G_{\text{sugra}} = \text{Diff}(M) \ltimes \Omega^3_{\text{cl}}(M) \ltimes \Omega^6_{\text{cl}}(M)$.

Finally, the Dorfman derivative satisfies the Leibniz identity:

$$L_U\,(L_V T) - L_V\,(L_U T) = L_{[\![U,V]\!]} T = L_{L_U V} T$$

with V, U, T generalised vectors. Thus, E has a "Leibniz algebroid" structure.

3.4 Compatible, torsion-free generalised connections

3.4.1 Connections and torsion

We first need to introduce generalised connections in a way that would be compatible with the $E_{d(d)} \times \mathbb{R}^+$ structure. Let us define Ω an element of a section of E^* indiced by M and taking values in $E_{d(d)} \times \mathbb{R}^+$. Hence the first-order linear differential operator D acting on a generalised vector V:

$$D_M V^A = \partial_M V^A + \Omega_M{}^A{}_B\, V^B \tag{3.21}$$

In such a covariant way, one can naturally extend the action of the differential operator D to any higher rank $E_{d(d)} \times \mathbb{R}^+$ generalised tensor.
One can also build up a generalised connection on a conventional one ∇ acting on conventional tensors. Indeed, we embed the action of ∇ in E^*,

acting now on generalised vectors (and tensors):

$$D^{\nabla}_{M}\, V \equiv \begin{cases} (\nabla_\mu\, v^a)\, \hat{E}_a + {}^1\!/\!{}_2\, (\nabla_\mu\, \omega_{a_1 a_2})\, \hat{E}^{a_1 a_2} & \\ \quad + {}^1\!/\!{}_{5!}\, (\nabla_\mu\, \sigma_{a_1 \ldots a_5})\, \hat{E}^{a_1 \ldots a_5} & \text{for } M = \mu \\ \quad + {}^1\!/\!{}_{7!}\, (\nabla_\mu\, \tau_{a, b_1 \ldots b_7})\, \hat{E}^{a, b_1 \ldots b_7} & \\ 0 & \text{otherwise} \end{cases} \tag{3.22}$$

where $\nabla_\mu\, v^a$, $\nabla_\mu\, \omega_{a_1 a_2} \ldots$ are the usual tensors under the action of a conventional connections.

The definition of the generalised torsion follows directly the one for the generalised connection.
First, one needs to define L^D as the Dorfman derivative in which the partial derivative ∂ has been replaced by the covariant one D defined above. It acts on any generalised tensor with respect to a generalised vector as it did before. Hence the generalised torsion:

$$T(V) \cdot U \equiv \left(L^D_V - L_V \right) U \tag{3.23}$$

T can be seen as a generalised tensor, element of a section of $E^* \otimes \mathrm{ad}\, \tilde{F}$ even if some of the components actually vanish[16]. T can also be looked at as a linear map acting on a generalised vector and leading to the adjoint representation: $T : E \to \mathrm{ad}\, \tilde{F}$. $T(V)$ then acts on any generalised tensor as the Dorfman derivative does.
Using $\{\hat{E}_A\}$ and $\{\hat{E}^A\}$ an $E_{d(d)} \times \mathbb{R}^+$ frame and its dual and recalling $\hat{E}^A \left(\hat{E}_B\right) = \delta^A{}_B$, one can get:

$$T(V) = V^C \left[\Omega_C{}^A{}_B - \Omega_B{}^A{}_C - \hat{E}^A \left(L_{\hat{E}_C} \hat{E}_B \right) \right] \hat{E}_A \otimes_{\mathrm{ad}} \hat{E}^B \tag{3.24}$$

Note that if we use a coordinate frame, the "frame term" $\hat{E}^A \left(L_{\hat{E}_C} \hat{E}_B \right)$ vanishes. We then recognise the usual definition of the torsion, that is the antisymmetric part of the connections: $T(V)^A{}_B = 2\, V^C\, \Omega_{[C}{}^A{}_{B]}$.

Finally, as in conventional geometry one could want to define a generalised curvature in order to complete the picture. However, since the Dorfman derivative is not antisymmetric, which also means that the exceptional

[16] The generalised torsion is an element of a section of $E^* \oplus K$ where $K \in E^* \otimes \mathrm{ad}\, \tilde{F}$ has elements decomposing as $T = T_A{}^B{}_C\, \hat{E}^A \otimes \hat{E}_B \otimes \hat{E}^C$. For more details, see [4].

Courant bracket does not satisfy the Jacobi identity, one cannot obtain a generalised tensor by using the usual definition:

$$R\,(V,\,U)\;T \equiv [D_V\,,\;D_U]\;T - D_{[\![V,U]\!]}\;T \tag{3.25}$$

Although Ricci curvature tensor and scalar could be defined, they would need some extra structure on E.

3.4.2 Metric

In order to continue our study on the connections, and especially to look for compatible torsion-free ones, we need to build up a generalised metric. This additional structure generalises the set of orthonormal frames related by $O(d)$ transformations in conventional geometry. Here, we would have an H_d structure, where H_d is the maximally compact[17] subgroup of $E_{d(d)}$. Thus, the generalised connections D preserving this H_d structure would be considered as compatible, as we will see below.
Note that the double cover of H_d, denoted by $\tilde{H}_d$, is physically more relevant. Indeed, the fermionic sector of Supergravity – through spinor representations – would require such an extension. These groups are exposed in Table 3 with respect to the dimension d of our manifold.

d	$E_{d(d)}$	$\tilde{H}_d$
7	$E_{7(7)}$	$SU(8)$
6	$E_{6(6)}$	$Sp(8)$
5	$E_{5(5)} \simeq Spin(5,5)$	$Spin(5) \times Spin(5)$
4	$E_{4(5)} \simeq SL(5,\,\mathbb{R})$	$Spin(5)$

Table 3: Double cover of the maximally compact subgroup of $E_{d(d)}$

This set of frames forms an $\tilde{H}_d$ principle sub-bundle of $\tilde{F}$, the generalised structure bundle for E, and we denote it by $P\,\subset\tilde{F}$.

We would like to identify explicitely the frames which are in P. First, in any

[17] In the physically relevant case, one might prefer to consider non-compact versions of H_d, that is modify the signature of the metric to get for instance an $SO(6,1)$ subgroup of $GL(7,\,\mathbb{R})$. One would get the same results as here and would be able to discuss consistent dimensional reductions of eleven-dimensional Supergravity with a timelike dimension.

H_d structure[18], the H_d frame can be chosen to be a conformal split frame with no loss of generality. Equivalently, any H_d frame can be transformed into a conformal split frame via the definition of three- and six-form field connections over a patching of M. Thus, one gets (3.15), and any other frame by an H_d transformation defined as:

$$\hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B \left(H^{-1}\right)^B{}_A \quad \text{and} \quad V^A \mapsto V'^A = H^A{}_B\, V^B \tag{3.26}$$

$$\text{with } H \equiv e^{a+\bar{a}}\, e^{\alpha+\bar{\alpha}}\, h \tag{3.27}$$

with $h \in O(d)$ and the same exponentiated actions as in $E_{d(d)} \times \mathbb{R}^+$ transformations, set up in (3.10).

The action of $h \in O(d) \subset H_d$ on an H_d frame keeps it both orthonormal for a conventional metric $-$ since it simply transforms an orthonormal frame $\{\hat{e}_a\}$ for TM into another one $-$ and in its conformal split form. Therefore, the set of H_d conformal split frames forms an $O(d)$ structure on E: $(P \cap P_{\text{split}}) \subset \tilde{F}$.

In an H_d frame, one can easily define a generalised G metric such as:

$$\begin{aligned} G(V, V) &\equiv |v|^2 + |\omega|^2 + |\sigma|^2 + |\tau|^2 \\ &= v_\mu\, v^\mu + 1/2\, \omega_{\mu_1\mu_2}\, \omega^{\mu_1\mu_2} + \dots \\ &= \delta_{\nu\mu}\, v^\nu\, v^\mu + 1/2\, \delta^{\nu_1\mu_1}\, \delta^{\nu_2\mu_2}\, \omega_{\mu_1\mu_2}\, \omega_{\nu_1\nu_2} + \dots \end{aligned} \tag{3.28}$$

with V a generalised vector and $\delta_{\nu\mu}$ the flat frame metric. But this definition has to be independent of the choice of frame and thus is valid in any conformal split frame. The fields determining the coset element and thus the frame, *i.e.* g the conventional metric, A and $\bar{A}$ the three- and six-form field connections and Δ the scaling factor, are entirely defining the generalised metric G.

Finally, it is useful to define a "$\det E^*$" density scalar the same way we had $\sqrt{g}$, an $SO(d)$-invariant $\det T^*M$ density. In conventional geometry, this was possible because of the embedding $SO(d) \subset SL(d, \mathbb{R}) \subset GL(d, \mathbb{R})$; it is also feasible in our Generalised Geometry since we have the parallel

[18] The choice of an H_d structure is equivalent to the choice of an element of the coset $\left(E_{d(d)} \times \mathbb{R}^+\right)/H_d$

sequence $H_d \subset E_{d(d)} \subset E_{d(d)} \times \mathbb{R}^+$. Thus, we define an H_d-invariant density in terms of the conformal split frame:

$$\mathrm{vol}_G \equiv \sqrt{g}\,\left(e^{\Delta}\right)^{9-d} \tag{3.29}$$

Note that this density is also $E_{d(d)}$-invariant and that one can define it as the determinant of the generalised metric G to a suitable power.

3.4.3 Generalisation of the Levi-Civita connection

In conventional $d = 4$ geometry, the Levi-Civita connection is the unique connection which is both torsion-free and metric compatible. Here, we want to extend this notion and look for the constrained generalised connection, with its possible multiplicity.

Let us first look at the compatibility of our generalised connection. To be compatible with the H_d structure $P \subset \tilde{F}$, it has to satisfy $DG = 0$, that is to act only in the H_d principle sub-bundle. Defining ∇ the Levi-Civita connection with respect to the usual metric g, one can lift it to an action on a generalised vector V in an H_d conformal split frame, as in (3.22). Moreover, the Levi-Civita connection ∇ is H_d compatible since it is $O(d)$ compatible, $O(d)$ being a subset of H_d.
Note that even if we formally work out the compatible connections in a restricted set of frames, the resulting form can easily be adapted to any other frame via an H_d transformation $-$ since the connection is H_d covariant.

Though, in this form, D^{∇} is not torsion-free even if ∇ is by definition torsion-free[19]. Indeed, we can calculate the (generalised) torsion of the generalised connection D^{∇} in an H_d conformal split frame:

$$\begin{aligned} T(V) = e^{\Delta}\big(&- i_v \mathrm{d}\Delta + v \otimes \mathrm{d}\Delta - i_v F + \mathrm{d}\Delta \wedge \omega \\ &- i_v \tilde{F} + \omega \wedge F + \mathrm{d}\Delta \wedge \sigma\big) \end{aligned} \tag{3.30}$$

[19]Beware here of the notion of torsion: when we deal with the generalised connections D^{∇}, we do mean that the *generalised* torsion is vanishing, whereas we actually imply *conventionally* torsion-free when we explicitely refer to the usual connections in conventional geometry, such as the Levi-Civita connection ∇.

where F and $\tilde{F}$ are the field strengths of the A and $\tilde{A}$ form field potentials set up in (3.14) and are defined by:

$$\begin{cases} F \equiv \mathrm{d}A_{(j)} \\ \tilde{F} \equiv \mathrm{d}\tilde{A}_{(j)} - {}^1\!/\!_2\, A_{(j)} \wedge F \end{cases} \tag{3.31}$$

In order to get a generalised Levi-Civita connection, *i.e.* a torsion-free compatible one, we need to modify the embedding of our (usual) Levi-Civita connection (3.22), that is the definition of D^{∇}. One can always write for an H_d compatible generalised connection D the relationship:

$$D_M V^A = D^{\nabla}_M V^A + \Sigma_M{}^A{}_B\, V^B \tag{3.32}$$

where Σ is an element of a section of $(E^* \otimes \mathrm{ad}\, P)$, that is a generalised $(0,1)$-tensor (index downstairs M) taking its values in the adjoint of H_d (indices upstairs and downstairs A and B).
By fixing in the appropriate way this Σ to make the torsion of D vanish, one would define a torsion-free compatible connection. If we have a closer look at the decomposition under H_d of the representations appearing in the torsion, we note that they are all contained in the ones defined by Σ. This means that solutions for this problem exist but not uniquely. Indeed, except for the $d = 3$ case, some of the components of Σ are not contained reciprocally in the torsion representations[20], leaving some unconstrained degrees of freedom.

In order to write down the explicit solution for Σ, we contract it with a generalised vector V to get $\Sigma(V) \in \mathrm{ad}\, P$. We can then express it in the basis for the adjoint of H_d:

$$\begin{aligned} \Sigma(V)_{\mu_1\mu_2} \equiv e^{\Delta} \Big(2 \left(\tfrac{7-d}{d-1}\right) v_{[\mu_1} \partial_{\mu_2]} \Delta + {}^1\!/\!_{4!}\, \omega_{\nu_1\nu_2} F^{\nu_1\nu_2}{}_{\mu_1\mu_2} & \\ + {}^1\!/\!_{7!}\, \sigma_{\nu_1\ldots\nu_5} \tilde{F}^{\nu_1\ldots\nu_5}{}_{\mu_1\mu_2} + C(V)_{\mu_1\mu_2} \Big) & \end{aligned} \tag{3.33a}$$

$$\begin{aligned} \Sigma(V)_{\mu_1\mu_2\mu_3} \equiv e^{\Delta} \Big(\tfrac{6}{(d-1)(d-2)} (\mathrm{d}\Delta \wedge \omega)_{\mu_1\mu_2\mu_3} & \\ + {}^1\!/\!_4\, v^{\nu} F_{\nu\mu_1\mu_2\mu_3} + C(V)_{\mu_1\mu_2\mu_3} \Big) & \end{aligned} \tag{3.33b}$$

$$\Sigma(V)_{\mu_1\ldots\mu_6} \equiv e^{\Delta} \left({}^1\!/\!_7\, v^{\nu} \tilde{F}_{\nu\mu_1\ldots\mu_6} + C(V)_{\mu_1\ldots\mu_6} \right) \tag{3.33c}$$

[20] The torsion is actually an element of the section of $K \oplus E^*$, where $K \subset E^* \otimes \mathrm{ad}\, \tilde{F}$ has not been explicitely defined here (again, see [4] for more details). In general, we have $E^* \otimes \mathrm{ad}\, P \simeq (K \oplus E^*) \oplus U$, where sections of U contain the unconstrained part of Σ. In $d = 3$, we simply have $E^* \otimes \mathrm{ad}\, P \simeq K \oplus E^*$.

where C is the unconstrained part of the connection[21].
Finally, we define the Clifford algebra Cliff(d, $\mathbb{R}$) and its gamma matrices γ^μ satisfying $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$. As well, we denote by $\gamma^{\mu_1\cdots\mu_n} \equiv \gamma^{[\mu_1}\ldots\gamma^{\mu_n]}$ the antisymmetric product of n of them. Using this and the embedding of the double cover $\tilde{H}_d$ in this algebra, we can finally write the connection:

$$\begin{aligned} D_\mu = e^\Delta \Big(\nabla_\mu + \tfrac{1}{2}\left(\tfrac{7-d}{d-1}\right)(\partial_\nu\Delta)\,{\gamma_\mu}^\nu - \tfrac{1}{2\cdot 4!}F_{\mu\nu_1\nu_2\nu_3}\gamma^{\nu_1\nu_2\nu_3} \\ - \tfrac{1}{2\cdot 7!}\tilde{F}_{\mu\nu_1\ldots\nu_6}\gamma^{\nu_1\ldots\nu_6} + \not{C}_\mu\Big) \end{aligned} \tag{3.34a}$$

$$\begin{aligned} D^{\mu_1\mu_2} = e^\Delta\Big(\tfrac{2!}{4\cdot 4!}{F^{\mu_1\mu_2}}_{\nu_1\nu_2}\gamma^{\nu_1\nu_2} + \tfrac{3}{(d-1)(d-2)}(\partial_\rho\Delta)\,\gamma^{\mu_1\mu_2\rho} \\ + \not{C}^{\mu_1\mu_2}\Big) \end{aligned} \tag{3.34b}$$

$$D^{\mu_1\ldots\mu_5} = e^\Delta\left(\tfrac{5!}{4\cdot 7!}{\tilde{F}^{\mu_1\ldots\mu_5}}_{\nu_1\nu_2}\gamma^{\nu_1\nu_2} + \not{C}^{\mu_1\ldots\mu_5}\right) \tag{3.34c}$$

$$D^{\mu,\nu_1\ldots\nu_7} = e^\Delta\left(\not{C}^{\mu,\nu_1\ldots\nu_7}\right) \tag{3.34d}$$

where $\not{C}$ is the embedding of the unconstrained part of the connection in Cliff(d, $\mathbb{R}$), that is:

$$\begin{aligned} \not{C}_\alpha &= \tfrac{1}{2}\left(\tfrac{1}{2!}\,C_{\alpha,\mu_1\mu_2}\gamma^{\mu_1\mu_2} - \tfrac{1}{3!}\,C_{\alpha,\mu_1\mu_2\mu_3}\gamma^{\mu_1\mu_2\mu_3} - \tfrac{1}{6!}\,C_{\alpha,\mu_1\ldots\mu_6}\gamma^{\mu_1\ldots\mu_6}\right) \\ \not{C}^{\alpha_1\ldots\alpha_n} &= \tfrac{1}{2}\left(\tfrac{1}{2!}\,{C^{\alpha_1\ldots\alpha_n}}_{\mu_1\mu_2}\gamma^{\mu_1\mu_2} - \tfrac{1}{3!}\,{C^{\alpha_1\ldots\alpha_n}}_{\mu_1\mu_2\mu_3}\gamma^{\mu_1\mu_2\mu_3} - \tfrac{1}{6!}\,{C^{\alpha_1\ldots\alpha_n}}_{\mu_1\ldots\mu_6}\gamma^{\mu_1\ldots\mu_6}\right) \end{aligned}$$

for $n \in \{2, 5, 7\}$.
As wanted, this defines the generalised Levi-Civita connections, *i.e.* H_d compatible torsion-free generalised connections, here expressed in a basis for the adjoint of H_d and embedded in Cliff(d, $\mathbb{R}$).

[21]This element of a section of $E^* \otimes \mathrm{ad}\,P$ actually leaves on a section of U. Indeed, if we define the projection map on the torsion representation $K \oplus E^*$, C is in its kernel.

4 Conclusion

Through the extension of diffeomorphisms to include gauge transformations, Generalised Geometry gives us a powerful tool to express dimensional reductions of M theory in a simpler way.

As has been shown, the bosonic sector of eleven-dimensional Supergravity, reduced to $d \leq 7$ dimensions, can be rewritten in a larger structure which includes all the symmetries. The generalised tangent space can be endowed with a coherent linear structure. By adding U-duality symmetries, the underlying group defines transformations which combine local diffeomorphisms with three- and six-form field gauge transformations.
Moreover, the whole differential structure, which consists mainly in a Dorfman derivative and connections, is covariantly defined. The generalised metric contains the several bosonic fields of our theory. Together with a subset chain $H_d \subset E_{d(d)} \subset E_{d(d)} \times \mathbb{R}^+$, it allows us to generalise the notion of torsion-free compatible connections $-$ even if we lose the unicity.
Finally, it is noticeable that the generalised tangent space is actually included in a much broader structure. Higher rank generalised tensors are sections of larger $E_{d(d)} \times \mathbb{R}^+$ bundles. More importantly, we defined several kinds of frames that build up a principle bundle and sub-bundles structure in a coherent and physically meaningful way.

Moreover, it is remarkable to see how all of this work is perfectly suited to the fermionic sector. Indeed, this development has been done in a spin-manifold and with the double cover of H_d, in order to include spinor representations. In addition, the generalised Levi-Civita connections defined here are expressed in an embedding of H_d the maximally compact subgroup of $E_{d(d)}$ into the corresponding Clifford algebra Cliff$(d, \mathbb{R})$. This defines $\{\gamma^\mu\}$ the gamma matrices necessary for the extension of this structure to the fermionic sector.

This work could be extended further. Indeed, one could first define tools to express the curvature via a generalised Ricci tensor and the associated scalar. This would allow us to write a Supergravity action in comparison with the Einstein Hilbert action of General Relativity. One would therefore be able to derive equations of motion in such a framework. So far, this work has mainly been carried out by A. Coimbra, C. Strickland-Constable and D. Waldram in [4].

However, several important questions remain, amongst which the question of how extra-dimensions are truncated or wrapped is obviously worth examining. Indeed, it may have several consequences on the lower dimensional cases and thus on the physical interpretation of the action and equations of motion. Besides, the timelike dimension has to be considered carefully, even though this work can easily be modified to non-compact structures $-$ for instance using the non-compact subgroup $SU^*(8)$ instead of its compact version $SU(8)$ in the $E_{7(7)}$ case.

Finally, one of the most important issues remaining is the extension of these tools to $d > 7$ dimensional reductions of Supergravity and M theory. Indeed, as mentioned briefly in this work, the case $d = 7$ already shows the tip of a broader problem through the symmetric part of the Dorfman derivative that we cannot express in an $E_{d(d)}$ covariant way. This is linked to the fact that the exceptional Courant bracket does not satisfy the Jacobi identity. In larger dimensions, the Dorfman derivative itself cannot be written covariantly.

Solutions could emerge from the constraints one can identify in such higher dimensional extensions or even in String Theories, and apply to our manifold. These links would probably enlighten the way our lower dimensional cases are embedded into the eleven-dimensional structure. Another track may also lie in the study of various other formulations, whether or not they are built on geometrical considerations.

A The isomorphism $\wedge^p T^*M \simeq (\det TM) \otimes \wedge^{d-p} T^*M$

In order to describe our mathematical objects in a clearer way, and especially to make the $\mathbb{R}^+$ factor more intuitive, one has to look at the isomorphism $\wedge^p T^*M \simeq (\det TM) \otimes \wedge^{d-p} T^*M$.

Before any further analysis, this isomorphism can simply be interpreted by using the Hodge dual in the case of a manifold M allowing a metric g. Indeed, for a p-form $\omega = \frac{1}{p!}\,\omega_{m_1\ldots m_p} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p}$, one has:

$$\star\omega = \frac{\sqrt{|\, g\,|}}{p!(d-p)!}\;\omega_{m_1\ldots m_p}\,\epsilon^{m_1\ldots m_p}{}_{m_{p+1}\ldots m_d}\,\mathrm{d}x^{m_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{m_d} \tag{A.1}$$

One can also use the metric to transform a $(d-p)$-form into an antisymmetric $(d-p, 0)$-tensor (*i.e.* raising the indices) and get the metric dual:

$$(\star\omega)^* = \frac{\sqrt{|\, g\,|}}{p!(d-p)!}\;\omega_{m_1\ldots m_p}\,\epsilon^{m_1\ldots m_p\, m_{p+1}\ldots m_d}\,\frac{\partial}{\partial x^{m_{p+1}}} \wedge \ldots \wedge \frac{\partial}{\partial x^{m_d}} \tag{A.2}$$

This gives the isomorphism we are looking for.

Still, one can set it up without any metric and get the following relationship between $x \in (\det TM) \otimes \wedge^{d-p} T^*M$ and $\omega \in \wedge^p T^*M$:

$$\begin{cases} x^{m_{p+1}\ldots m_d} = \frac{1}{p!}\,\epsilon^{m_1\ldots m_p\, m_{p+1}\ldots m_d}\,\omega_{m_1\ldots m_p} \\ \omega_{m_1\ldots m_p} = \frac{1}{(d-p)!}\,\epsilon_{m_1\ldots m_p\, m_{p+1}\ldots m_d}\,x^{m_{p+1}\ldots m_d} \end{cases} \tag{A.3}$$

This representation allows us to define all the usual operations on p-forms. Indeed, one has the interior product, with $v \in TM$:

$$\begin{aligned} i_v :\ \wedge^p T^*M &\longrightarrow (\det TM) \otimes \wedge^{d-p+1} T^*M \\ \frac{1}{p!}\,\omega_{m_1\ldots m_p} &\mapsto (i_v x)^{n_p\ldots n_d} = \frac{(-1)^{d-1}}{d-p}\,x^{[n_{p+1}\ldots n_d} v^{n_p]} \end{aligned} \tag{A.4}$$

the exterior derivative:

$$\begin{aligned} \mathrm{d} :\ \wedge^p T^*M &\longrightarrow (\det TM) \otimes \wedge^{d-p-1} T^*M \\ \frac{1}{p!}\,\omega_{m_1\ldots m_p} &\mapsto (\mathrm{d}x)^{n_{p+2}\ldots n_d} = (-1)^{d-p+1}\,\partial_{n_{p+1}} x^{[n_{p+1} n_{p+2}\ldots n_d]} \end{aligned} \tag{A.5}$$

the exterior derivative of the interior product:

$$\begin{aligned} \mathrm{d}i_v :\ \wedge^p T^*M &\longrightarrow (\det TM) \otimes \wedge^{d-p} T^*M \\ \frac{1}{p!}\,\omega_{m_1\ldots m_p} &\mapsto (\mathrm{d}i_v x)^{n_{p+1}\ldots n_d} = (d-p+1)\,\partial_{n_p}(v^{[n_p} x^{n_{p+1}\ldots n_d]}) \end{aligned} \tag{A.6}$$

and finally the Lie derivative (with respect to v):

$$\begin{aligned}\mathcal{L}_v : \wedge^p T^*M &\longrightarrow (\det TM) \otimes \wedge^{d-p} T^*M \\ \frac{1}{p!}\,\omega_{m_1 \dots m_p} &\mapsto (\mathcal{L}_v x)^{n_{p+1}\dots n_d} = v^{n_p} \partial_{n_p} (x^{n_{p+1}\dots n_d}) \\ &\qquad\qquad + (d-p+1)\,(\partial_{n_p} v^{[n_p})\, x^{n_{p+1}\dots n_d]}\end{aligned} \tag{A.7}$$

B Eleven-dimensional Supergravity

In order to embed this work into a more physical point of view, let us have a look first at eleven-dimensional Supergravity and then at its lower dimension restrictions.

Eleven-diemensional Supergravity contains three fields: the metric $\mathcal{G}_{\mu\nu}$, a symmetric $(0,2)$-tensor ; the three-form potential $\mathcal{A}_{\mu\nu\rho}$; the gravitino ψ^e_μ, a one-form carrying a spinor index e. Thus, we can define the Ricci tensor $\mathcal{R}_{\mu\nu}$ and the Ricci scalar $\mathcal{R}$, as well as the field strength four-form $\mathcal{H} \equiv \mathrm{d}\mathcal{A}$. The bosonic action is:

$$S_B = \frac{1}{2\kappa^2} \int (\mathrm{vol}_g\, \mathcal{R} - {}^1\!/\!_2\, \mathcal{H} \wedge *\mathcal{H} - {}^1\!/\!_6\, \mathcal{A} \wedge \mathcal{H} \wedge \mathcal{H}) \tag{B.1}$$

which gives the following equations of motion:

$$\mathcal{R}_{\mu\nu} - {}^1\!/\!_{12} \left(\mathcal{H}_{\mu\rho_1\rho_2\rho_3} \mathcal{H}_\nu{}^{\rho_1\rho_2\rho_3} - {}^1\!/\!_{12}\, g_{\mu\nu}\, \mathcal{H}^2 \right) = 0 \tag{B.2a}$$

$$\mathrm{d} * \mathcal{H} + {}^1\!/\!_2\, \mathcal{H} \wedge \mathcal{H} = 0 \tag{B.2b}$$

In order to deal with the fermionic sector, we define Γ^μ the gamma matrices of the relevant Clifford algebra Cliff$(10, 1, \mathbb{R})$. Therefore, under a supersymmetry transformation parametrised by ε, the variation of the gravitino is:

$$\delta\psi_\mu = \nabla_\mu \varepsilon + {}^1\!/\!_{288} \left(\Gamma_\mu{}^{\nu_1\dots\nu_4} - 8\, \delta_\mu{}^{\nu_1}\, \Gamma^{\nu_2\nu_3\nu_4} \right) \mathcal{H}_{\nu_1\dots\nu_4}\, \varepsilon \tag{B.3}$$

Defining M a d-dimensional spin manifold, $d \le 7$, and $\mathbb{R}^{10-d,\,1}$ the Minkowski space-time in $(11-d)$ dimensions, we now look at restrictions of our eleven-dimensional theory in $\mathbb{R}^{10-d,\,1} \times M$. For that purpose, we choose all the fields to be independant of the flat $\mathbb{R}^{10-d,\,1}$ space-time and thus restrict the theory to leave on M. The indices are split between $a, b = 0, 1 \dots (10-d)$

on the flat space-time, and $m, n = 1 \dots d$ on M.
To build up an action for this restricted theory, the fields need to be defined carefully. The internal components of the metric $\mathcal{G}$ and the potential $\mathcal{A}$ give the restricted metric g and potential A. The corresponding field strength is defined as usual as $F = \mathrm{d}A$. The warp factor Δ remains the same.
Finally, a dual six-form potential $\tilde{A}$ can be introduced on M if $d = 7$. Indeed, in that case a seven-form field strength $\tilde{F}$ can be defined as the eleven-dimensional Hodge dual of the four-form field strength F. Thus, one can define $\tilde{F} = \mathrm{d}\tilde{A} - {}^1\!/\!{}_2\, A \wedge F$.
These field strengths satisfy the Bianchi identities:

$$\mathrm{d}F = 0 \tag{B.4a}$$

$$\mathrm{d}\tilde{F} + {}^1\!/\!{}_2\, F \wedge F = 0 \tag{B.4b}$$

and are related to the eleven-dimensional field strength $\mathcal{F}$ via:

$$F_{m_1 \dots m_4} = \mathcal{F}_{m_1 \dots m_4} \tag{B.5a}$$

$$\tilde{F}_{m_1 \dots m_7} = (*\mathcal{F})_{m_1 \dots m_7} \tag{B.5b}$$

Defining again R_{mn} and R the Ricci tensor and scalar restricted on M, one can write down the bosonic action for this restricted theory:

$$S_B^M = \frac{1}{2\kappa^2} \int \sqrt{g}\, e^{(11-d)\Delta} \Big(R + (11-d)(10-d)\,(\partial\Delta)^2 - \tfrac{1}{2\cdot 4!}\, F^2 - \tfrac{1}{2\cdot 7!}\, \tilde{F}^2 \Big) \tag{B.6}$$

Since this action has the same form as the action S_B for the eleven-dimensional theory, it leads to the same kind of equations of motion for the fields:

$$\begin{aligned} R_{mn} - (11-d)\big[\nabla_m \nabla_n \Delta + \big(\partial_m \Delta\big)\big(\partial_n \Delta\big)\big] & \\ - \tfrac{1}{2\cdot 4!}\, \big(4\, F_{mp_1p_2p_3}\, F_n{}^{p_1p_2p_3} - \tfrac{1}{3}\, g_{mn} F^2\big) & \\ - \tfrac{1}{2\cdot 7!}\, \big(7\, \tilde{F}_{mp_1 \dots p_6}\, \tilde{F}_n{}^{p_1 \dots p_6} - \tfrac{2}{3}\, g_{mn} \tilde{F}^2\big) &= 0 \end{aligned} \tag{B.7a}$$

$$\begin{aligned} R - (10-d)\big[2\,\nabla^2\Delta - (11-d)\big(\partial\Delta\big)^2\big] & \\ - \tfrac{1}{2\cdot 4!}\, F^2 - \tfrac{1}{2\cdot 7!}\, \tilde{F}^2 &= 0 \end{aligned} \tag{B.7b}$$

$$\mathrm{d} * \big(e^{(11-d)\Delta} F\big) - e^{(11-d)\Delta}\big(*\,\tilde{F}\big) \wedge F = 0 \tag{B.7c}$$

$$\mathrm{d} * \big(e^{(11-d)\Delta} \tilde{F}\big) = 0 \tag{B.7d}$$

One also has the same kind of supersymmetry variations for the gravitino:

$$\delta\psi_m = \nabla_m\varepsilon + \tfrac{1}{288}\big(\gamma_m{}^{n_1\ldots n_4} - 8\,\delta_m^{\ n_1}\gamma^{n_2n_3n_4}\big)F_{n_1\ldots n_4}\varepsilon - \tfrac{1}{12\cdot 6!}\,\tilde{F}_{mn_1\ldots n_6}\gamma^{n_1\ldots n_6}\varepsilon \tag{B.8a}$$

$$\delta\rho = \gamma^m\nabla_m\varepsilon - \tfrac{1}{4\cdot 4!}\gamma^{m_1\ldots m_4}F_{m_1\ldots m_4}\varepsilon - \tfrac{1}{4\cdot 7!}\tilde{F}_{m_1\ldots m_7}\gamma^{m_1\ldots m_7}\varepsilon + \tfrac{9-d}{2}\big(\gamma^m\partial_m\Delta\big)\varepsilon \tag{B.8b}$$

where ρ has to do with the trace of ψ_m, ε is the parameter of the supersymmetry transformation and $\{\gamma^m\}$ are the gamma matrices defining Cliff$(d,\,\mathbb{R})$.

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