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Division Algebras, Magic Squares and Supersymmetry

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Abstract

Of the four exclusive normed division algebras, only the real and complex numbers prevail in both mathematics and physics. The noncommutative quaternions and the nonassociative octonions have found limited physical applications. In mathematics, division algebras unify both classical and exceptional Lie algebras with the exceptional ones appearing in a table known as the magic square generated by tensor products of division algebras. This work reviews the normed division algebras and the magic square as well as necessary preliminaries for its construction. Space-time transformations, pure super Yang-Mills theories in space-time dimensions $D = 3$, 4, 6, 10, dimensional reduction and truncation of supersymmetry are also described here by the four division algebras. Supergravity theories, seen as tensor products of super Yang-Mills theories, are described as tensor products of division algebras leading to the identification of a magic square of supergravity theories with their U-duality groups as the magic square entries, providing applications of all division algebras to physics and suggesting division algebraic underpinnings of supersymmetry. Other curious uses of octonions are also mentioned.

Contents

1 Preface

This dissertation is the outcome of studies undertaken in the months June 2013 to September 2013 under Professor Michael Duff and his students at Imperial College London. The subject matter is division algebras and their relations to, first, symmetry, and, second, to supersymmetry.

Ongoing research by this group, which I have been given the opportunity to get a firsthand account of, seeks to establish the division algebras as the underpinnings of theories of supersymmetry. This year, the group published two articles on the subject where super Yang-Mills theories and supergravity theories were treated using division algebras. [1][2] Earlier work used octonions to relate black holes and quantum information theory. [3] It is the work of this year that will be described here.

This is an introductory text that assumes no prior knowledge of the subject. It is developed beginning with division algebras, through symmetry and finally to supersymmetry. The structure of the text is as follows. After the outline of the subject matter, basic definitions, which may be skipped by some readers, are provided. The division algebras are thereupon defined and constructed using the Cayley-Dickson procedure. The unusual division algebras, the quaternions and the octonions, are then elaborated on, their properties described. The split algebras, not division algebras but related, are also introduced.

Integral division algebra elements are defined and used to describe root lattices of interesting Lie algebras. The Cayley-Dickson procedure is also applied here. Using only root lattices described by integral elements, hints of a magic square are seen. Some preliminaries of the actual magic square constructions are then introduced, the Jordan algebras and other derived algebras. Three constructions of the magic square are described. A table of maximal compact subalgebras corresponding to the magic square is also provided.

Space-time transformations, spinors and vectors in the critical dimensions $D = 3, 4, 6, 10$ are then formulated using division algebra elements. Dimensional reduction from $D = 10$ to $D = 6, 4, 3$ is described. Super Yang-Mills theory is further elaborated on, associating also a division algebra with the supersymmetry of the theories. Theories of supergravity are then constructed as tensor products of two super Yang-Mills theories. A magic square of supergravity theories is found. A possible generalisation of the magic square is discussed. The work is concluded and an outlook is provided.

1.1 Acknowledgements

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2 A pressing need

The octonions are infamous for the nonassociativity of their multiplication. Though their multiplication is nonassociative, the state of things is not so bad. The octonions have associative subalgebras as well as identities relating the order of multiplication of four elements. Irrespective of whether the state of things is ameliorated by their having associative subalgebras and identities to deal with nonassociativity, and before introducing anything concrete at all, I feel a pressing need to dispel the aversion and defeatist attitudes to nonassociative algebras by showing that, with or without the reader's noticing, they are already being used. Some readers may find surprising that there already are at least four nonassociative prevailing operations that enter mathematics and physics on different levels.

1. Subtraction is nonassociative since $a-(b-c)=a-b+c \neq (a-b)-c$, and division is so, too.

2. The vector product is nonassociative since $(a\times b)\times c \neq a\times(b\times c)$. For instance, $(\hat{x}\times\hat{x})\times\hat{y}=0$ whilst $\hat{x}\times(\hat{x}\times\hat{y})=\hat{x}\times\hat{z}=-\hat{y}$.

3. The Lie product is nonassociative since $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$ means that $[a,[b,c]]=[[a,b],c]-[c,[a,b]]\neq[[a,b],c]$ in general.

Nonassociative algebras *can* be interesting. Sometimes, only a product of two elements is needed and so the nonassociativity, which takes at least three elements to be noticed, is never relevant, as is the case of the vector product and its use in electrodynamics or motion of rigid bodies.

3 Introduction

There are only four normed division algebras. These are the real numbers $\mathbb{R}$ of dimension one, the complex numbers $\mathbb{C}$ of dimension two, the quaternions $\mathbb{H}$ of dimension four and the octonions $\mathbb{O}$ of dimension eight. The real numbers satisfy $N(ab) = N(a)N(b)$ where $N(a) = \sqrt{a^2}$ is the norm of the real number a. Such property holds for the complex numbers as well since $N^2(ab) = (ab)(ab)^* = aba^*b^*$ which after reordering is $(aa^*)(bb^*) = N^2(a)N^2(b)$ and this also holds for the quaternions and the octonions. The *division* qualifier comes from the fact that $ab = 0$ if and only if $a = 0$ or $b = 0$ so there is a notion of division and $a^{-1} = a^*/N(a)$ can be identified as the multiplicative (left and right) inverse of a. A *normed division* algebra is one for which $N(ab) = N(a)N(b)$ and when $ab = 0$ it follows that either $a = 0$ or $b = 0$. Hurwitz's celebrated theorem [4] states that there are no other normed division algebras.

A complex number can also be regarded as a pair of real numbers with a particular rule for multiplication of pairs. Likewise, a quaternion can be regarded as a pair of complex numbers and an octonion as a pair of quaternions. This successive construction of division algebras is known as the Cayley-Dickson procedure.

Needless to say, the real numbers and the complex numbers thrive in mathematics as well as physics. The next division algebra after the complex numbers, the noncommutative quaternion algebra, can be constructed by introducing further two new imaginary elements j and k that together with i satisfy their squaring to negative unity $i^2 = j^2 = k^2 = -1$, cyclicity $ij = k$,

$jk = i$, $ki = j$ and their noncommutativity where $ij = -ji$, $ik = -ki$ and $jk = -kj$.

A general quaternion is written $q = a_0 + a_1 i + a_2 j + a_3 k$ and its norm is $N(q) = a_0^2 + a_1^2 + a_2^2 + a_3^2$ which is preserved by the group $SO(4)$. An imaginary quaternion is one spanned by imaginary elements alone such that the quaternion has no real (or scalar) part $\mathrm{Re}(q) = a_0 = 0$. The three components of the imaginary quaternion (the vector part), a_1, a_2 and a_3, can be identified with the x, y and z components, respectively, of a vector in Euclidean space $\mathbb{R}^3$. A rotation in three dimensions, fully specified by an axis of rotation $p = p_1 i + p_2 j + p_3 k$ and an angle of rotation α about this axis, is accomplished by a two-sided multiplication $q' = rqr^*$ where $r = \cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2})p$ and r^* is a conjugation of r, generalising the complex conjugation by changing the sign of all imaginary units. The scalar and vector product as well nabla, div and curl, used in vector analysis, arise naturally from the quaternion product. The quaternion product ab of two imaginary quaternions a and b is $ab = -a \cdot b + a \times b$ where the scalar product is the real (scalar) part and the vector product is the imaginary (vector) part of the new quaternion. Nabla was first introduced as $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ and div and curl are the scalar and vector parts, respectively, of the quaternion product of nabla and any imaginary quaternion. These operations were adopted in vector analysis that superseded the quaternions.

Never mainstream, they have since only resurfaced as curiosities or failed attempts. As a curiosity, the Pauli algebra which is isomorphic to the Clifford algebra of $\mathbb{R}^3$ is also isomorphic to the complex quaternions $\mathbb{C} \times \mathbb{H}$, and Dirac theory is 2×2 over the Pauli algebra. [5][6] Clifford algebras generalise the quaternions in higher dimensions but abandon the division algebra property. Parallel to establishing the foundations of quantum mechanics, isospin was an unsolved problem and it was attempted explained within the framework of quaternionic quantum mechanics. The group $SU(2)$ is the automorphism group of the quaternions and arises therefore naturally for quaternions. Exact isospin can be accounted for in quaternionic quantum mechanics but isospin is now understood as a broken symmetry, to some extent rendering isospin a nonproblem. [7]

The octonions are nonassociative, and since their multiplication is nonassociative, they cannot be represented by matrices! The subalgebra generated by one octonion, powers of an octonion, is associative, however. The subalgebra generated by any two octonions is also associative. This makes octonions power-associative and alternative, respectively. The general octonion a can be represented by eight real numbers, the real part a_0 and seven other components $a_1, \ldots, a_7$ associated with seven imaginary elements $e_1, \ldots, e_7$ such that $a = a_0 + a_1 e_1 + \ldots + a_7 e_7$, and, sometimes, a unit e_0 is associated with a_0 but it is not strictly necessary. As with the quaternions, all imaginary elements square to negative unity $e_a^2 = -1$. The norm of an octonion is $N(a) = a_0^2 + a_1^2 + \ldots + a_7^2$ preserved by the group $SO(8)$. The octonion algebra has structure constants as any other algebra. These are (excluding e_0) $e_a e_b = C^a{}_{bc} e_c$ where the content of $C^a{}_{bc}$ is best illustrated by the Fano plane, introduced later. Including e_0, the multiplication is $e_a e_b = \Gamma^a{}_{bc} e_c$, and, if the first element is conjugated, $e_a^* e_b = \bar{\Gamma}^a{}_{bc} e_c$. These structure constants, Γ and $\bar{\Gamma}$, can be used to construct the generators of $SO(8)$ in its spinor and conjugate spinor representations, and, as will be seen, generators of $SO(7)$ in its spinor and conjugate spinor representations are obtained from the above by simply restricting e_a to be an imaginary unit.

The physical applications of octonions have been scarce and scattered. Their role in physics is unknown and they are far from mainstream. The octonions made an appearance in the Jordan program of quantum mechanics. In establishing the foundations of quantum mechanics, alternatives to the associative Hilbert space formulation were explored. The Jordan program proposed to discover a new algebraic setting for quantum mechanics where operations on observables, represented by Hermitian matrices, were also observables in principle. [8] In the Hilbert space formulation, the composition (matrix multiplication) of observables is not an observable unless the observables commute and the adjoint operator (complex conjugate transpose) is just the identity map on observables, hence trivial. The Hilbert space formulation thus has superfluous operations. The only observable operations on Hermitian matrices are, in fact, powers, scalar multiplication and addition. Elements of the Jordan algebra, the alternative algebraic setting for quantum mechanics, are Hermitian matrices and the product is the Jordan product. For two Hermitian matrices A and B, the Jordan product is $A \circ B = \frac{1}{2}(AB + BA)$ which, unlike the matrix product, is commutative. Further imposing an axiom, the Jordan identity, implies the associativity (unambiguity) of all powers. Unlike the matrix algebra, the Jordan algebra is a nonassociative algebra. The complete classification of Jordan algebras contains amongst other matrix structures Hermitian quaternionic $n \times n$ matrices but also 2×2 and 3×3 Hermitian octonionic matrices. The Hermitian $n \times n$ quaternionic matrices and 2×2 octonionic matrices are *special*. They can be embedded in an associative algebra. The Jordan algebra of octonionic 3×3 matrices is *exceptional*. It cannot be embedded in an associative algebra and is hence unreachable by the associative matrix algebra Hilbert space formulation. Deemed too small to contain quantum mechanics, unique but only 27-dimensional, and too isolated to generalise to the infinite-dimensional case, in which there were later found to be no exceptional Jordan algebras, the program was abandoned.

Octonions reappeared in another failed attempt to describe the exact $SU(3)$ symmetry, quark structure and confinement. [7][9] This is no far-fetched setup. The automorphism group of the octonions is the exceptional group G_2 of which $SU(3)$ is a subgroup. Other work, less known, claims to fit the Standard Model into tensor products of division algebras $\mathbb{T} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. [5]

Applications of octonions and Jordan algebras to supersymmetry and string theory came later. A one-to-one correspondence between the critical dimensions of supersymmetric theories and the division algebras was discovered. [10] Later, simple super Yang-Mills theories in dimensions $D =$ 3, 4, 6, 10 were related to the division algebras [11] and elaborated on [12][13]. Superstrings and super Yang-Mills theories exist in two dimensions higher than the dimensions of the division algebras and 2-brane theories in three dimensions higher. In $D = 3, 4, 6, 10$, the number of on-shell degrees of freedom of vectors and spinors match the dimensions of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively, and, hence, single division algebra elements can be used to describe vectors and spinors on-shell. The isomorphisms $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$, $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$, $\mathfrak{sl}(2,\mathbb{H}) \cong \mathfrak{so}(5,1)$ and $\mathfrak{sl}(2,\mathbb{O}) \cong \mathfrak{so}(9,1)$ relate space-time transformations in $D = 3, 4, 6, 10$ to 2×2 matrices over the division algebras. In the dimensions of the division algebras, the vector and spinor representations are also equal.

It turns out that a vanishing quantity necessary for the supersymmetry of both super Yang-Mills and superstring theory relies on the alternativity of the algebra. [12] The only quantum mechanically consistent superstring and supermembrane theories are related to the octonions, suggesting a link between quantization of supersymmetric extended objects and nonassociativity. [7]

In mathematics, numerous works deal with octonions. There is no scarcity of applications here. Relevant to the research presented here are the mathematical applications of division algebras to constructions of classical and exceptional simple Lie algebras. Division algebras organise the classical simple Lie algebras. [14][15] There are three infinite families of classical Lie algebras: $\mathfrak{so}(n)$, $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$. These are constructed in an easy manner. $\mathfrak{so}(n)$ is just the set of $n \times n$ matrices over the *real numbers* $x \in \mathbb{R}[n]$ that satisfy $x^* = -x$ and are traceless $\mathrm{tr}(x) = 0$. $\mathfrak{su}(n)$ is defined similarly but with the $n \times n$ matrices over the *complex numbers* $x \in \mathbb{C}[n]$ instead. The algebra $\mathfrak{sp}(n)$ consists of $n \times n$ matrices over the *quaternions* with $x^* = -x$. There are only six exceptional simple Lie algebras. Are these then related to octonions? It turns out that division algebras also organise the rare exceptional Lie algebras but the simple construction above does not generalise to the octonionic case. It is more subtle since matrices over the octonions do not automatically satisfy the Jacobi identity due to their nonassociativity. [16] As previously mentioned, the automorphism group of octonions is G_2 so for at least one Lie algebra, octonions are involved. [7] In fact, octonions are implicated for the remaining cases $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$, too. The construction that relates the exceptional Lie algebras to octonions is exactly the magic square construction, often displayed as a 4×4 table whose cells are Lie algebras. The construction accepts two division algebras $\mathbb{K}_1$ and $\mathbb{K}_2$, turns $\mathbb{K}_2$ into a Jordan algebra $H_3(\mathbb{K}_2)$ of Hermitian 3×3 matrices over $\mathbb{K}_2$ first, and returns a simple Lie algebra $L_3(\mathbb{K}_1, H_3(\mathbb{K}_2))$. Details are omitted here but tensor products $\mathbb{K}_1 \otimes H_3(\mathbb{K}_2)$ and subalgebra related to $\mathbb{K}_1$ and $H_3(\mathbb{K}_2)$ enter the construction. [15] The magic of the magic square is that it is symmetric even if the division algebras $\mathbb{K}_1$ and $\mathbb{K}_2$ are, a priori, not treated equally. Later constructions of the magic square are manifestly symmetric leaving out $H_3(\mathbb{K}_2)$ and using only $\mathbb{K}_2$, considering tensor products $\mathbb{K}_1 \otimes \mathbb{K}_2$ but modifying the inner workings of L_3 instead. The rows and columns of the magic square correspond to $\mathbb{K}_1$ and $\mathbb{K}_2$ that both range from $\mathbb{R}$ to $\mathbb{O}$. The last row (column) relates $\mathbb{R} \otimes \mathbb{O}$ to $\mathfrak{f}_4$, $\mathbb{C} \otimes \mathbb{O}$ to $\mathfrak{e}_6$, $\mathbb{H} \otimes \mathbb{O}$ to $\mathfrak{e}_7$ and, finally, $\mathbb{O} \otimes \mathbb{O}$ to $\mathfrak{e}_8$.

The dimensional reduction of 11-dimensional supergravity acquires non-trivial symmetries known as U-duality groups due to toroidal compactification. [17] The series of exceptional groups E_6, E_7 and E_8 appear in reductions of $D = 11$ to $D = 5$, 4, 3, respectively. Other slots of the Tits-Rosenfeld-Freudenthal magic square, arise from reducing $D = 9$ and $D = 8$ supergravity with only one slot of the magic square absent. This was the first instance of exceptional groups arising as symmetries in physics without being put in by hand, suggesting deep connections between extended theories of supergravity and exceptional groups. [18]

One way to view the content (multiplets) of a theory of supergravity is as a tensor product of the content (multiplets) of two theories of super Yang-Mills. Given the former identification of super Yang-Mills theories in the critical dimensions with corresponding division algebras, a tensor product of division algebra elements, each representing a super Yang-Mills field, can therefore be identified with a supergravity field. Bearing this in mind and recalling the above construction of the magic square consisting in tensor products of division algebras and the discovery of exceptional U-duality groups in lower-dimensional theories of supergravity, this suggests that the Lie algebras of the magic square construction can be identified with the symmetries of the supergravity theories obtained as a tensor products of super Yang-Mills theories. In fact, this identification was found to be true in $D = 3$. [2]

The dimensional reduction of vectors and spinors can also be described by elements of division algebras. Spinors in a higher dimension reduce to pairs of spinors in a lower dimension, reversing the Cayley-Dickson procedure. For instance, a spinor in $D = 10$, described by an octonion, reduces to two spinors, a pair of quaternions, in $D = 6$. As a result of the compactification, the former space-time symmetries of just one object become internal symmetries transforming two objects into one another.

Incidentally, *integral* elements over the four division algebras can be defined. The set of integral quaternions and integral octonions fashion the root lattices of $SO(8)$ and E_8, respectively. [19] Other root lattices of Lie algebras can be fashioned in a somewhat unified way ultimately beginning from the simple roots of $SU(2)$ and the weights of its vector representation. A hint of a magic square also appears here.

4 Preliminaries

Definitions of composition algebras, division algebras, polarisation and automorphisms are reviewed. The reader familiar with these concepts may skip this section and proceed to the next where the division algebras are constructed using the Cayley-Dickson procedure. The split algebras will also be constructed using the same produce modulo some signs.

4.1 Nomenclature

A composition algebra $\mathbb{K}$ over $\mathbb{R}$ with a nondegenerate quadratic form N and an associated bilinear form $\langle x, y\rangle$ is one for which $N^2(xy) = N^2(x)N^2(y)$ holds.

A division algebra is one for which $xy = 0$ implies that $x = 0$ or $y = 0$ which is true for composition algebras if the form is positive-definite. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only positive-definite composition algebras and hence division algebras.

Polarisation or linearisation obtains an bilinear form from the form N. The inner product $\langle x, y\rangle = \frac{1}{2}(N^2(x + y) - N^2(x) - N^2(y))$.

An automorphism of an algebra is a one-to-one mapping of the algebra onto itself with all operations of the algebra preserved. The set of automorphisms forms a group. Often, it is the case that automorphisms are formed by having elements of the algebra act on itself and at the same time preserve the algebraic structure. One example is conjugation. Such automorphisms are called inner automorphisms. These are to be distinguished from outer automorphisms where extrinsic elements may act on the algebra. Examples of outer automorphisms are reflections and symmetries of Dynkin diagrams such as the triality of the $SO(8)$ Dynkin diagram. The action of the outer automorphism on the roots of the algebra is to permute the three elements but permutations are not themselves elements of $SO(8)$.

5 Constructing the normed division algebras

Although there are other composition algebras, Hurwitz's aforementioned theorem states that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only four normed division algebras. The real numbers of dimension one are real, that is to say, the action of conjugation acts trivially on them. They are also associative and commutative. The complex numbers of dimension two lose the property of trivial conjugation but inherit all the other properties. The quaternions of dimension four inherit all the properties of the complex numbers but lose commutativity and the octonions of dimension eight, which otherwise inherit the properties of the quaternions, lose associativity.

There is a procedure known as the Cayley-Dickson construction that generates a sequence of algebras that double in dimension and makes manifest why the quaternions are noncommutative and the octonions nonassociative. The first four algebras in this sequence are exactly the normed division algebras. The reader is already familiar with the first step of the procedure and the main features of its generalisation but the trivial conjugation and commutativity of the real numbers mask the details.

A complex number can be regarded as a pair of real numbers $z = (a, b)$ which is most commonly written $z = a + ib$ with the imaginary unit i introduced. The multiplication of two complex numbers $z_1 = a+ib$ and $z_2 = c+id$ is just $z_3 = z_1 z_2 = (a,b)(c,d) = (a+ib)(c+id) = (ac-bd)+i(ad+bc) = (ac-bd, ad+bc)$. If (a,b) were a pair of complex numbers or a pair of quaternions the correct quaternion or octonion multiplication, respectively, would be $(a,b)(c,d) = (ac - db^*, cb + a^*d)$ with conjugation defined as $(a,b)^* = (a^*, -b)$. The norm squared is $zz^* = (a,b)(a,b)^* = (aa^* + bb^*, 0) = aa^* + bb^*$ and the inner product is $\langle (a,b),(c,d)\rangle = \frac{1}{2}((a,b)^*(c,d) + (c,d)^*(a,b)) = (\frac{1}{2}(a^*c + c^*a + bd^* + db^*), 0) = \frac{1}{2}(a^*c + c^*a + bd^* + db^*)$. Imaginary elements will greatly simplify these ghastly expressions at the cost of results dependent on choice of basis.

A pair of octonions (a,b) is called a sedenion which is of dimension 16. Sedenions are not division algebras. They have zero divisors. [14] They are not alternative but the power of a sedenion is still a well-defined notion.

5.1 The quaternions

The above construction of the quaternions as a pair of complex numbers may look obscure, but by introducing imaginary elements, the familiar rules of the multiplication of imaginary units will arise. Consider two complex numbers $(x_0, x_1) = x_0 + x_1 i$ and $(x_3, x_2) = x_3 + x_2 i$ where i is the imaginary unit of the complex numbers that satisfies $i^2 = -1$. A quaternion q is then a pair of complex numbers $((x_0,x_1),(x_3,x_2))$ which can also be represented by introducing a new imaginary unit k such that $q = (x_0,x_1) + k(x_3,x_2) = (x_0 + x_1 i) + k(x_3 + ix_2) = x_0 + x_1 i + kix_2 + kx_3$. It can be left at this, having just two imaginary units i and k but then the element ki would then have to be used throughout which is instead named $j = ki$ such that $q = x_0 + x_1 i + x_2 j + k x_3$. Conjugation is represented by $((x_0, x_1), (x_3, x_2))^* = ((x_0, x_1)^*, -(x_3, x_2)) = x_0 - x_1 i - x_2 j - x_3 k$ and the norm is $((x_0, x_1), (x_3, x_2))((x_0, x_1), (x_3, x_2))^* = (x_0, x_1)(x_0, x_1)^* + (x_3, x_2)(x_3, x_2)^* = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

The imaginary element k is represented as $((0,0),(1,0))$ and its square is $k^2 = ((0,0),(1,0))((0,0),(1,0)) = ((0,0)(0,0) - (1,0)(1,0)^*, (0,0)(1,0) + (0,0)^*(1,0)) = (-(1,0),(0,0)) = -1$. By the same procedure, it can be seen that $ij = k$ and $ji = -k$. The full multiplication table can be seen in Figure 1. This algebra is clearly not commutative. The quaternionic

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Figure 1: The multiplication of imaginary quaternionic elements.

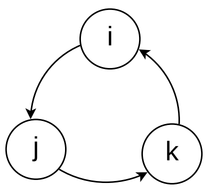

Figure 2: The simple cyclic multiplication of the quaternionic elements.

multiplication can be summarised with the Levi-Civita symbol: $e_a e_b = -\delta_{ab} + \epsilon_{abc} e_c$. The group that preserves the norm for real numbers is $\mathcal{Z}_2$. For complex numbers it is $SO(2) \cong U(1)$ and given that the norm of a quaternion is $N(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$, the group that preserves the norm is $SO(4)$.

In general for division algebras, and not just for quaternions, the inner product defined earlier as $\langle (a,b),(c,d) \rangle = \frac{1}{2}((a,b)^*(c,d) + (c,d)^*(a,b)) = \frac{1}{2}(a^*c + c^*a + bd^* + db^*)$ can also be seen as: $\langle x, y \rangle = \mathrm{Re}(xy^*) = \mathrm{Re}(x^*y) = \frac{1}{2}\mathrm{Re}(xy^* + x^*y)$ and collective conjugation conjugates each element but re-orders the multiplication: $(xy)^* = y^*x^*$.

Since quaternions do not commute, the commutator may be a convenient quantity to consider: $[x, y] = xy - yx$. If any argument is conjugated, the commutator changes sign. For quaternions $[e_a, e_b] = 2\epsilon_{abc}e_c$.

Rotations are two-sided operations. The axis of rotation and the angle of rotation is set by a vector b and α, respectively. Another vector $c = \cos(\frac{\alpha}{2}) + \sin(\frac{\alpha}{2})b$ is formed and the rotated vector is $v' = cvc^*$. Rotation is reversed by conjugation of the vector c.

The left L_c and right multiplication R_{c^*} determined by one element c is isomorphic to $SU(2)$ which $SO(3)$ is isomorphic to. By making the elements of left and right multiplication distinct this becomes an $SU(2) \times SU(2) \cong SO(4)$ transformation. [20]

The elements i, j and k are not the only that square to -1. As long as it is an imaginary quaternion and it has unit norm then it squares to -1. There are also complex subalgebras in the quaternion algebra. They are spanned by a real part and a unit imaginary quaternion such that $\mathbb{C} = \{a + bm\}$ where m is an imaginary unit quaternion. In analogy to complex numbers, $e^{m\alpha} = \cos(\alpha) + m\sin(\alpha)$ and any quaternion q, or octonion if m is a unit octonion, can be written $q = re^{m\alpha}$ where $r = |q|$. [21]

The automorphism group of the quaternions, the set of transformations that preserve the quaternionic multiplication, is $SU(2)$.

5.2 The octonions

The octonions can now be constructed as a pair of quaternions and a similar procedure to the one in the preceding section can be followed to obtain their multiplication rules. By introducing imaginary elements, their multiplication can be made easy, though dependent on basis. Since the procedure is analogous to the one for quaternions, only the results will be given here. Consider two quaternions, p and p'. The octonion k will then be a pair of those such that $k = p + e_7 p'$, and by expressing the quaternions in terms of their three imaginary elements, labelled for convenience, e_1, e_2 and e_4, $p = p_1 + e_1 p_1 + e_2 p_2 + e_4 p_4$ and $p' = p_7 + p_3 e_1 + p_6 e_2 + p_5 e_4$, the octonion k is written $k = (p_0 + p_1 e_1 + p_2 e_2 + p_4 e_4) + e_7(p_7 + p_3 e_1 + p_6 e_2 + p_5 e_4)$. The elements $e_7 e_1$, $e_7 e_2$ and $e_7 e_4$ could be used throughout or new elements could be introduced for convenience $e_3 = e_7 e_1$, $e_6 = e_7 e_2$ and $e_5 = e_7 e_4$. All elements square to -1 and the full multiplication table can be seen in Figure 3. From the multiplication, The index cycling rule (modulo 7) can be read off,

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Figure 3: The multiplication of octonionic imaginary units.

$e_i e_{i+1} = e_{i+3}$, as well as the index doubling identity (modulo 7) that says that $e_i e_j = e_k$ implies $e_{2i} e_{2j} = e_{2k}$. Noncommutativity $e_1 e_2 = e_4 \neq e_2 e_1$ and nonassociativity can be seen. Nonassociativity takes three so $(e_2 e_3) e_7 = -1$ whilst $e_2(e_3 e_7) = e_2 e_1 = -e_4 \neq (e_2 e_3) e_7 = e_4$. However, there are also quaternionic subalgebras in the octonion algebra for which there is associativity. There are also complex subalgebras, for which there is commutativity. The multiplication of octonionic imaginary units can be illustrated by the

Fano plane, which also happens to be the smallest finite projective plane. [14] It has been drawn in Figure 4. The seven labelled points correspond to the

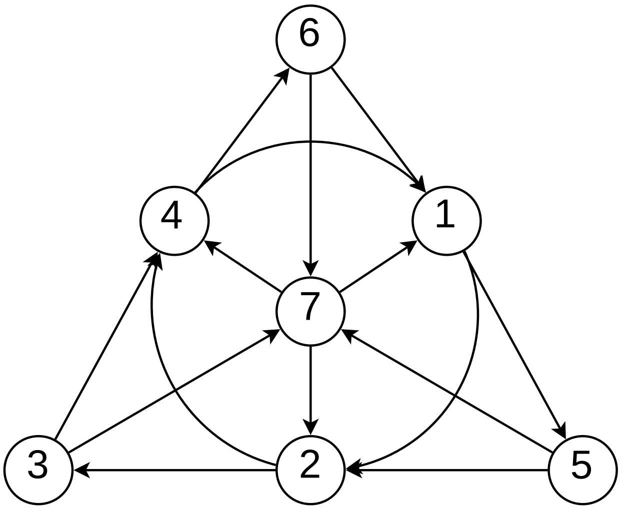

Figure 4: The Fano plane. Multiplication of imaginary units.

seven imaginary elements. There are seven lines each connecting three points. They close as circles $e_1 \to e_2 \to e_4 \to e_1$, $e_2 \to e_3 \to e_5 \to e_2$ and so forth. The circle on the picture also counts as a line. On the drawing, it is the most truthful as all the other lines in fact close like the circle. There are arrows on the lines to denote the direction of multiplication. Going against the arrow gives the negative result. The lines (circles) are the quaternionic subalgebras. This section began with only e_1, e_2, e_4 and one more element e_7. It is now easy to see that this is enough. The other elements can be reached by multiplication of e_1, e_2 and e_4 with e_7. Conjugation changes the sign of all the imaginary elements and the norm is the sum of the squared components. The group that preserves the octonionic norm is $SO(8)$. The transformations that preserve the octonionic multiplication is isomorphic to the exceptional group G_2. $SU(3)$ is the maximal subgroup of G_2. It is the automorphism group involving six octonionic units. [9]

The octonions are power-associative which means that the subalgebra of any one element is associative. This is easily explained. Without loss of generality, it can be assumed that the octonion sits in a complex subalgebra. The octonions are alternative which means that the subalgebra formed by any two elements is associative. Since there are only two octonions, they can be assumed without loss of generality to sit in a quaternionic subalgebra which explains alternativity. Furthermore, the octonions satisfy the Moufang laws involving four octonions which say that $(xyx)z = x[y(xz)]$, $z(xyx) = [(zx)y]x$, and $(xy)(zx) = x(yz)x$. [4]

If e_0 is included as the basis of the real part, then the octonionic multiplication rule can be written as $e_i e_j = (\delta_{i0}\delta_{jk} + \delta_{0j}\delta_{ik} - \delta_{ij}\delta_{0k} + C_{ijk})e_k$ where C_{ijk} is totally antisymmetric, zero if any index is zero, and equal to $+1$ when ijk coincide with the lines of the Fano plane $124, 235, \ldots, 713$.

Since the algebra is nonassociative, the associator is nontrivial $[e_i, e_j, e_k] = 2Q_{ijkl}e_l$ where Q_{ijkl} is equal to 1 on the quadrangles of the Fano plane which are 3567, 4671, $5712, \ldots$, 2456, equal to -1 for odd permutations of these numbers and zero otherwise. Actually, $Q_{ijkl} = -\frac{1}{3!}C_{abc}\epsilon_{abcijkl}$. [1] Quadrangles can be seen as those points left over after removing a particular line. The associator itself is imaginary. Permuting the elements in the associator has the effect of changing the overall sign. Conjugating an element in the associator also changes the overall sign.

5.3 Split algebras

The Cayley-Dickson procedure can be used to construct other algebras known as the split algebras $\tilde{\mathbb{C}}, \tilde{\mathbb{H}}$ and $\tilde{\mathbb{O}}$ which are also composition algebras. It involves changing signs in the Cayley-Dickson construction. [22] The first step to get other 2-dimensional hypercomplex numbers (algebras) is to introduce a real number μ in the Cayley-Dickson procedure and let multiplication in the new algebra be $(a,b)(c,d) = (ac - \mu d^* b, da - bc^*)$. As before, an element (a,b) in the new algebra may be represented as $z = a + eb$ and it is readily found that $e^2 = -\mu$. This follows from starting with $e = (0,1)$ and so $e^2 = (0,1)(0,1) = (-\mu, 0) = -\mu$. The squared norm of a $N(z) = zz^* = z^*z = a^2 + \mu b^2$ and so three algebras distinct algebras can arise:

1. the complex numbers, $\mathbb{C}$ when $\mu = 1$ with norm $N(z) = a^2 + b^2$,
2. the split complex numbers $\mathbb{C}(-1)$ with $\mu = -1$ and norm $N(z) = a^2 - b^2$,
3. the dual numbers $\mathbb{C}(0)$ with $\mu = 0$ and norm $N(z) = a^2$.

Of these, only the complex numbers form a division algebra.

Continuing by now considering a pair of complex numbers z_1 and z_2 such that $z = (z_1, z_2)$ represented as $z_1 + ez_2$ where, as before, $e^2 = -\mu$. Again, there are three nonisomorphic algebras:

1. the quaternions $\mathbb{H} = \mathbb{H}(1,1)$ with $\mu = 1$ norm $N(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$,
2. the split quaternions $\tilde{\mathbb{H}} = \mathbb{H}(1,-1)$ with $\mu = -1$, norm $N(x) = x_0^2 + x_1^2 - x_2^2 - x_3^2$ and imaginary elements $i^2 = -1$ but $k^2 = j^2 = 1$,

3. the semiquaternions $\mathbb{H}(1,0)$ with $\mu = 0$, norm $N(x) = x_0^2 + x_1^2$ and imaginary elements $i^2 = -1$ and $j^2 = k^2 = 0$.

Starting with a pair of quaternions, analogous considerations generate the octonions $\mathbb{O}(1,1,1)$ ($\mu = 1$), the split octonions $\tilde{\mathbb{O}} = \mathbb{O}(1,1,-1)$ and the semioctonions $\mathbb{O}(1,1,0)$. The split octonions have three imaginary elements that square to -1 and four that square to 1. Their norm is $N(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2$.

In general, $\mathbb{H}(\mu_1, \mu_2)$ and $\mathbb{O}(\mu_1, \mu_2, \mu_3)$ could also have been considered and the forms $N(x) = x_0^2 + \mu_1 x_1^2 + \mu_2 x_2^2 + \mu_1\mu_2 x_3^3$ and $N(x) = x_0^2 + \mu_1 x_1^2 + \mu_2 x_2^2 + \mu_3 x_4^2 + \mu_1\mu_3 x_5^2 + \mu_2\mu_3 x_6^2 + \mu_1\mu_2\mu_3 x_7^2$ would have been found.

Incidentally, there is a reason for their being called split algebras. [15] The equation defining the square of some of the imaginary elements can be *split* into two factors: $i^2 = 1 \Rightarrow i^2 - 1 = 0 \Rightarrow (i+1)(i-1) = 0$. It is here easy to see that this is not a division algebra: There are zero divisors since two nonzero elements $i+1$ and $i-1$, not zero themselves, multiply to give zero.

6 The integral octonions

Any element x of a division algebra $\mathbb{A}$ satisfies the rank equation: $x^2 - (x + x^*)x + xx^* = 0$. [19] The factor in the second term $(x + x^*)$ is twice the real part and the last term is the norm squared xx^*. If the elements of a set A obey the rank equation with integer norm, if the double scalar part is integer, and if

1. A is closed under multiplication and subtraction,

2. A contains the identity,

3. A is not a subset of a larger set that also satisfies 1 and 2,

then the elements of A are called *integral* elements. There is a set of 240 (integral) octonions that satisfies these properties. This corresponds to the scaled root lattice of E_8. There is also a set of 24 quaternions that satisfies these properties. This is the scaled root lattice of $SO(8)$. The roots here are normalised to unity as opposed to the standard $\sqrt{2}$ norm in the Cartan-Killing classification.

Weyl reflections applied to the simple roots generate the remaining. Consider two roots r_a and r_b. The reflection of r_a with respect to the hyperplane that has r_b as normal vector generates a new root $r_{ab} = -r_b r_a^* r_b$. [7]

In the following, for convenience, the permutations $e_4 \leftrightarrow e_3$, $e_6 \leftrightarrow e_5$ will be made that differ from the rest of the text.

6.1 The root lattice of $SO(8)$

The root lattice of $SO(8)$ which has a quaternionic description is given by the set $A_0 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\}$, which give the required 24 roots. The root lattice of $SO(8)$ could have been made from $SU(2)^4$. Tensor products are just weights added in the Cayley-Dickson fashion. The root lattice of $SU(2)$ is simply $\{\pm 1\}$ and the weights of the spinor representation are $\{\pm\frac{1}{2}\}$. The adjoint representation of $SO(8)$ decomposes to $SU(2)^4$ as $\mathbf{28} = (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$ from which one can easily obtain the roots of $SO(8)$. The roots $(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$ are made of the roots (adjoint) of one $SU(2)$ and zeros $(((\pm 1, 0), 0), 0) = \pm 1$. The roots of $(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$ are $(((0,\pm 1),0),0) = \pm e_1$ and so $(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1})$ corresponds to $(((0,0),\pm 1),0) = \pm e_2$ and $(((0,0),0),\pm 1) = \pm e_3$. Finally, $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) = (((\pm\frac{1}{2},\pm\frac{1}{2}),\pm\frac{1}{2}),\pm\frac{1}{2}) = \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$.

6.2 The root lattice of E_8

The root lattice of E_8 consists of the set ± 1, $\pm e_i$, $\frac{1}{2}(\pm e_j \pm e_k \pm e_l \pm e_m)$ and $\frac{1}{2}(\pm 1 \pm e_n \pm e_p \pm e_q)$ where $i = 1,\ldots,7$, $jklm = 1246$, 1257, 1345, 1367, 2356, 2347, 4567 and $npq = 123$, 147, 165, 245, 267, 346, 357. Incidentally, the subset of imaginary roots form the E_7 lattice. Given that there is an arbitrariness in the choice of imaginary elements, it is to be expected that this is not a unique set of roots and, indeed, that is the case. There are seven other octonionic integer sets of E_8 roots. [19]

By considering the decomposition of the adjoint representation of E_8 under $SO(8)\times SO(8)$, this list of E_8 roots can easily be generated. The decomposition is $\mathbf{248} = (\mathbf{28},\mathbf{1}) + (\mathbf{1},\mathbf{28}) + (\mathbf{8}_v,\mathbf{8}_v) + (\mathbf{8}_s,\mathbf{8}_c) + (\mathbf{8}_c,\mathbf{8}_s)$. The weights of the vector representation $\mathbf{8}_v$ are $A_1 = \{\frac{1}{2}(\pm 1 \pm e_1), \frac{1}{2}(\pm e_2 \pm e_3)\}$ which can be obtained by knowing that $\mathbf{8}_v = (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$. The weights of the spinor representation are $\mathbf{8}_s = (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$ are $A_3 = \{\frac{1}{2}(\pm 1 \pm e_3), \frac{1}{2}(\pm e_1 \pm e_2)\}$ and those of the conjugate spinor representation $\mathbf{8}_c = (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$ are $A_2 = \{\frac{1}{2}(\pm 1 \pm e_2), \frac{1}{2}(\pm e_3 \pm e_1)\}$.

Incidentally, description makes the triality of $SO(8)$ manifest. Clearly, permutations of the three imaginary elements e_1, e_2, e_3 will map the representations $A_1 \to A_2 \to A_3$ into one another. Looking at the decomposition, the root lattice of E_8 is $(A_0,0) + (0,A_0) + (A_1,A_1) + (A_3,A_2) + (A_2,A_3)$.

The maximal compact subalgebra of E_8 is $SO(16)$ and its root lattice consists of the set ± 1, e_i, $\frac{1}{2}(\pm e_j \pm e_k \pm e_l \pm e_m)$ and $\frac{1}{2}(\pm 1 \pm e_n \pm e_p \pm e_q)$ where, now, $i = 1, \ldots, 7$, $jklm = 2356, 2347, 4567$ and $npq = 123, 147, 165$. Curiously, these are quaternionic subalgebras and their quadrangles!

These roots may become important for a division algebra formulation of the supergravity Lagrangians. Root vectors already appear in Lagrangian descriptions of supergravity theories where they are used to describe the cosets that scalars of compactified theories parameterise. [17] The $\mathcal{N} = 16$, $D = 3$ supergravity theory has scalars that parameterise the 128-dimensional $E_8/SO(16)$ coset. Here, roots of both E_8 and $SO(16)$ have been given.

6.3 A first magic square

Interesting root lattices arise by pairing roots of the algebras $SU(3)$, $Sp(3)$ and F_4 with one another in the Cayley-Dickson fashion. [19]

The F_4 root lattice turns out to be just the sum of the weights of the vector, spinor and conjugate spinor representations of $SO(8)$ and the roots of $SO(8)$: $\mathbf{48} = \mathbf{24} + \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c$.

The $SU(3)$ root lattice is $\{\pm\frac{1}{2}(1+e_1), \pm\frac{1}{2}(e_2-e_1), \pm\frac{1}{2}(1+e_2)\}$ generated by the simple roots $\{\frac{1}{2}(1+e_1), \frac{1}{2}(e_2-e_1)\}$.

The $Sp(3)$ roots are $\{\pm 1, \pm e_1, \pm e_3, \pm(\pm 1 \pm e_1), \frac{1}{2}(\pm e_1 \pm e_3), \frac{1}{2}(\pm 1 \pm e_3)\}$ generated by the simple roots $\{\frac{1}{2}(-1+e_1), \frac{1}{2}(1-e_3), e_3\}$.

The result of matching roots amongst the Lie algebras above can be summarised in a 4×4 table with the three aforementioned Lie algebras in the margins. Short roots are matched with short roots and long roots are matched with zeros. The result is shown in Figure 6. Save the empty upper left corner, this is curiously the magic square, albeit without much explanatory power. It is presented here as a curiosity.

In a subtle way, this construction is probably establishing the simple fact that $(\mathbb{K}_1 \otimes \mathbb{R}) \otimes (\mathbb{R} \otimes \mathbb{K}_2) = \mathbb{K}_1 \otimes \mathbb{K}_2$ since the Cayley-Dickson procedure, which was used for matching, itself is related to tensor products and the algebras in the margin come from tensor products where one division algebra is $\mathbb{R}$. This is just a guess.

	$SU(3)$	$Sp(3)$	F_4
$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$Sp(3)$	$SU(6)$	$SO(12)$	E_7
F_4	E_6	E_7	E_8

Figure 5: The magic square. Paired roots of algebras in the margin produce root lattices of curious groups.

Ramond notes that the Lie algebras of the magic square are interrelated, allowing embeddings to be read off the magic square. [20] Division algebras also finds applications in establishing other embeddings, not necessarily related to the magic square. [16]

It has been shown how integral elements of division algebras can be used to describe root lattices of interesting Lie algebras. It is curious that the set of integral octonionic elements coincides with the scaled root lattice of E_8 and the set of integral quaternionic elements coincides with the scaled root lattice of $SO(8)$. Triality was manifest in the quaternionic description of the $SO(8)$ roots and weights.

7 The Jordan Algebras

The remnants of the failed Jordan program, the Jordan algebras, first introduced as an alternative formulation of quantum mechanics centred on observables, was picked up in mathematics and used in the first construction of the magic square.

The Jordan algebra is a real vector space with a commutative bilinear product $x \circ y = y \circ x$ which satisfies the Jordan identity $(x^2 \circ) \circ x = x^2 \circ (y \circ x)$. The Jordan identity implies that Jordan algebras satisfy power-associativity.

A classification of Jordan algebras exists. Simple finite-dimensional formally real[1] Jordan algebras are isomorphic to one of five types of Jordan algebras. [14][8] Four of the five types, one per division algebra, are $n \times n$ Hermitian matrix algebras $\mathfrak{h}_n(\mathbb{A})$ over a division algebra $\mathbb{A}$ with the Jordan (anticommutator) product $a \circ b = \frac{1}{2}(ab + ba)$ where ab and ba are matrix products. When the division algebra is $\mathbb{O}$, there is a further constraint to $n \leq 3$. The case $n = 3$ is an exceptional Jordan algebra which means that it cannot be realised as a subalgebra over some real associative algebra with multiplication given by $x \circ y = \frac{1}{2}(xy + yx)$. It is also the only exceptional Jordan algebra. The (infinitesimal) transformations that preserve the Jordan product form the exceptional Lie algebra $\mathfrak{f}_4$!

The last type of Jordan algebra is the spin group that lives in $\mathbb{R}^n \oplus \mathbb{R}$. An element in this space is represented by the pair $(\mathbf{x}, t)$ with Jordan product $\circ$ represented by $(\mathbf{x}, t) \circ (\mathbf{x}', t') = (t\mathbf{x}' + t'\mathbf{x}, \mathbf{x} \cdot \mathbf{x}' + tt')$. All spin groups can be realised as a certain subspace of Hermitian $2^n \times 2^n$ matrices.

[1]Sum of elements squared is zero only when the elements are individually zero: $x_1^2 + \ldots + x_n^2 = 0 \Rightarrow x_1 = \ldots = x_n = 0$.

The algebras $\mathfrak{h}_1(\mathbb{O})$ and $\mathfrak{h}_2(\mathbb{O})$ are not exceptional. They are special which means that they can be realised as a subalgebra of an associative algebra algebra over the real numbers. The Jordan program sought an algebra that could not be made from a real associative algebra where the Jordan product $\circ$ was implemented as $\frac{1}{2}(ab + ba)$. Only $\mathfrak{h}_3(\mathbb{O})$ is exceptional but it was too small and too isolated for it to be useful in formulating quantum mechanics. There was a hope that exceptional Jordan algebras would arise in the infinite-dimensional case. For instance, as a matrix algebra the Heisenberg algebra can only be infinite-dimensional, since, by taking the trace of the commutation relations: $\mathrm{Tr}([q,p]) = 0 \neq i\hbar\mathrm{Tr}(I)$. [8] No infinite-dimensional exceptional Jordan algebras exist.

Later works have disputed that the 27-dimensional exceptional Jordan algebra is too small. [23][7] The space of all quantum states of the exceptional Jordan algebra is the coset $F_4/SO(9)$ which is also called the Moufang projective plane. $SO(9)$ here acts as a stability group leaving the quantum state invariant, analogous to phase transformations in quantum mechanics. It has been speculated, that this $SO(9)$, also the little group of $SO(11)$, refers to the Poincare transformations that leave a quantum state, a particle with a particular momentum in an 11-dimensional theory of space-time, invariant. Curiously, a combined F_4 transformation and an $SO(9)$ translation leave behind $SU(3) \times SU(2) \times U(1)$ as the stability group. [7]

This could also have relations to bosonic strings. F_4 has a curious embedding inside $SO(26)$ as $\mathbf{26} = \mathbf{26}$. [23] Unfortunately, nothing concrete has yet emerged from these reflections.

8 Preparations

Current constructions of magic squares patch together algebras and subsets of algebras in tensor products and direct products to make the Lie algebras of the magic square. These subsets and subalgebras will now be introduced. Only the absolute minimum is introduced. Proofs are omitted. The definitions as well as the constructions of the magic square are taken from a review of the constructions. [15]

Consider an algebra $\mathcal{A}$. The left and right multiplication maps take an element y of this algebra $\mathcal{A}$ and left or right multiply it by another element x such that $L_x(y) = xy$ and $R_x(y) = yx$.

The derivation algebra Der$\mathcal{A}$ of an algebra $\mathcal{A}$ is a Lie algebra with elements $D \in \text{Der}\mathcal{A}$ which are maps that act on elements x, y of the original algebra $\mathcal{A}$ in such a way that $D(xy) = D(x)y + xD(y)$ where the bracket is the commutator. The derivations are essentially the infinitesimal analogues of automorphisms. For alternative algebras, which include the division algebras, and for Jordan algebras, derivation algebras can be constructed from left and right multiplication maps. Take two elements of the alternative algebra x, y and to each pair can be associated a $D_{x,y}$ such that $D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y]$ so its action on an element z is $D_{x,y}(z) = [[x, y], z] - 3[x, y, z]$ where the last term is an associator. The real numbers and complex numbers have trivial derivation algebras. This also follows from the above action of the derivation. The real numbers and the complex numbers are commutative and associative so the derivation algebra is trivial since the commutator and associator vanish.

Another very important algebra is the triality algebra. It is a triple of linear maps A, B and C that map elements of the algebra back to the algebra itself $\mathcal{A} \to \mathcal{A}$. It is defined by the relation $A(xy) = (Bx)y + x(Cy)$ for all $x, y \in \mathcal{A}$. For composition algebras, this triple (A, B, C) must furthermore be a subset of $(\mathfrak{so}(\mathbb{K}), \mathfrak{so}(\mathbb{K}), \mathfrak{so}(\mathbb{K}))$ also written $3\mathfrak{so}(\mathbb{K}) = \mathfrak{so}(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K})$ where $\dot{+}$ denotes a direct product. $\mathfrak{so}(\mathbb{K})$ is the norm-preserving algebra of the composition algebra $\mathbb{K}$. It is empty for $\mathbb{R}$, $\mathfrak{u}(1)$ for $\mathbb{C}$, $\mathfrak{so}(4)$ for $\mathbb{H}$ and $\mathfrak{so}(8)$ for $\mathbb{O}$. The triality algebra of $\mathbb{O}$ is $\mathfrak{so}(8)$. For $\mathbb{H}$, it is $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and for $\mathbb{C}$ it is $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$.

9 The magic square constructions

Three magic square constructions are presented here: the Tits-Freudenthal construction, the Vinberg construction and the triality construction. All constructions have as input two composition algebras and return a Lie algebra. The three constructions are isomorphic to one another. If the two composition algebras are division algebras, the result is a Lie algebra from the magic square. The last two constructions are manifestly symmetric in their treatment of the two composition algebras $\mathbb{K}_1$ and $\mathbb{K}_2$.

9.1 The Tits-Freudenthal construction

The magic square is shown in Figure 6. It is, remarkably, symmetric. Choosing $\mathbb{K}_1 = \mathbb{O}$ and $\mathbb{K}_2 = \mathbb{O}$ gives E_8.

This was first obtained by considering two composition algebra $\mathbb{K}_1$ and $\mathbb{K}_2$. One composition algebra is used to form a Jordan algebra $\mathbb{J} = H_3(\mathbb{K}_2)$.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Figure 6: The famous magic square.

An inner product can be defined on this Jordan algebra given by $\langle X, Y\rangle = \frac{1}{2}\mathrm{Tr}(X \circ Y)$ where $\circ$ is the Jordan product $X \circ Y = XY + YX$, the anticommutator.

T is a Lie algebra and it is defined as $T(\mathbb{K}_1, \mathbb{J}) = \mathrm{Der}\mathbb{K}_1 \dot{+} \mathrm{Der}\mathbb{J} \dot{+} \mathbb{K}_1' \otimes \mathbb{J}'$. Had $\mathbb{J}$ just been a composition algebra like $\mathbb{K}_1$ then the definition would have been manifestly symmetric.

The prime ($'$) here means orthogonal to identity. When $\mathbb{K}$ is a composition algebra the prime means an imaginary element of the division algebra, hence orthogonal to the identity. $\mathbb{J}'$ are those elements of the Jordan algebra that are orthogonal to the identity of $\mathbb{J}$. Since there is an inner product, there is also here a notion of orthogonality.

It is useful to define a product that only takes place on $\mathbb{J}'$ as $A * B = A \circ B - \frac{4}{n}\langle A, B\rangle I$ where I is identity and n is the matrix row and column dimension of the Jordan algebra.

The prescription $\mathrm{Der}\mathbb{K} \dot{+} \mathrm{Der}\mathbb{J} \dot{+} \mathbb{K}' \otimes \mathbb{J}'$ should be understood as follows. There is the Lie subalgebra $\mathrm{Der}\mathbb{K} \oplus \mathrm{Der}\mathbb{J}$ and it can act on $\mathbb{K}' \otimes \mathbb{J}'$ just the usual way since $\mathrm{Der}\mathbb{K}$ has already got a defined action on $\mathbb{K}'$ and $\mathrm{Der}\mathbb{J}$, too, already has a defined action on $\mathbb{J}'$. The only undefined product is what it means for $a \otimes A \in \mathbb{K}' \otimes \mathbb{J}'$ to act on $b \otimes \mathbb{B} \in \mathbb{K}' \otimes \mathbb{J}'$. This is given by $[a \otimes A, b \otimes B] = \frac{1}{n}\langle A, B\rangle D_{a,b} - \langle a, b\rangle [L_A, L_B] + \frac{1}{2}[a, b] \otimes (A * B)$ where $\langle a, b\rangle$ is the inner product of the composition algebra and $\langle A, B\rangle$ is the inner product of the Jordan algebra. The first term containing $D_{a,b}$ is an element of Der$\mathbb{K}$. The second term containing $[L_A, L_B]$ is an element of Der$\mathbb{J}$ and the last term $\frac{1}{2}[a,b] \otimes (A * B)$ is an element of $\mathbb{K}' \otimes \mathbb{J}'$.

The split algebras can be used instead of the division algebras in $\mathbb{K}_1$ and valid Lie algebras are obtained. The algebras are of the same kind except they differ in numbers of noncompact and compact generators. They are shown in Figure 7.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	$F_4(52)$
$\breve{\mathbb{C}}$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$E_6(26)$
$\breve{\mathbb{H}}$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{sp}(6,\mathbb{H})$	$E_7(25)$
$\breve{\mathbb{O}}$	$F_4(-4)$	$E_6(-2)$	$E_7(5)$	$E_8(24)$

Figure 7: The split magic square.

The number in the parenthesis denotes the difference between the number of noncompact generators and compact generators of the algebra. The appearance of noncompact generators can be traced back to the Cayley-Dickson procedure where the signature of the forms were changed.

9.2 The Vinberg construction

A clearly symmetric construction of the magic square is the Vinberg construction which takes in two composition algebras $\mathbb{K}_1$ and $\mathbb{K}_2$ and forms the tensor product $\mathbb{K}_1 \otimes \mathbb{K}_2$. Multiplication is defined very simply as $(u_1 \otimes v_1)(u_2 \otimes v_2) = u_1 v_1 \otimes u_2 v_2$.

The vector space $V_3(\mathbb{K}_1, \mathbb{K}_2) = A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2) \dot{+} \mathrm{Der}\mathbb{K}_1 \dot{+} \mathrm{Der}\mathbb{K}_2$ is clearly symmetric with respect to the two composition algebras and it is a Lie algebra once the Lie product is defined as follows. The vector space $\mathrm{Der}\mathbb{K}_1 \oplus \mathrm{Der}\mathbb{K}_2$ is a Lie subalgebra. $A'_n(\mathbb{K}_1 \otimes \mathbb{K}_2)$ stands for $n \times n$ traceless ($'$) and antisymmetric matrices over $\mathbb{K}_1 \otimes \mathbb{K}_2$.

Given two elements $D \in \mathrm{Der}\mathbb{K}_1 \oplus \mathrm{Der}\mathbb{K}_2$ and $A \in A'_3(\mathbb{K}_1 \otimes \mathbb{K}_2)$ in the vector space, the action of D on A is such that $[D, A]$, written $D(A)$, is the *matrix* on which the derivation algebras have acted element by element where $\mathrm{Der}\mathbb{K}_1$, naturally, acts on $\mathbb{K}_1$ of $\mathbb{K}_1 \otimes \mathbb{K}_2$ and $\mathrm{Der}\mathbb{K}_2$ on $\mathbb{K}_2$ of $\mathbb{K}_1 \otimes \mathbb{K}_2$ alone.

For two elements $A, B \in A'_3(\mathbb{K}_1 \otimes \mathbb{K}_2)$, the Lie product is $[A, B] = (AB - BA)' + \frac{1}{3} \sum_{ij} D_{a_{ij}, b_{ji}}$ where $D_{p \otimes q, u \otimes v} = \langle p, u \rangle D_{q,v} + \langle q, v \rangle D_{p,u}$. The first term is the traceless part of an ordinary matrix commutator. The second term is an element of the direct sum of derivations $\mathrm{Der}\mathbb{K}_1 \oplus \mathrm{Der}\mathbb{K}_2$ where a_{ij} and b_{ji} are matrix elements and there is a sum over the indices.

9.3 The triality construction

The triality construction makes the Lie algebras as $L_3(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Tri}\mathbb{K}_1 \oplus \mathrm{Tri}\mathbb{K}_2 \dot{+} 3\mathbb{K}_1 \otimes \mathbb{K}_2$ where the all-important Lie products will now be defined. The significance of $3\mathbb{K}_1 \otimes \mathbb{K}_2$ is a direct sum of tensor products such that $3\mathbb{K}_1 \otimes \mathbb{K}_2$ is made up of elements $(x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2)$ where subscripts denote which composition algebra the elements belong to. This direct product of tensor product elements is then split into three terms themselves elements like $(x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2)$ such that $F_1(x_1 \otimes x_2) + F_2(y_1 \otimes y_2) + F_3(z_1 \otimes z_2) = (x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2)$.

Let T_1 be an element of the first triality algebra, itself a triplet, $(T_{\alpha 1}, \bar{T}_{\alpha 2}, \bar{T}_{\alpha_3})$ and T_2 of the second triality algebra, also a triplet. The bars denote that the element which the triality acts upon is conjugated before triality acting on it and the result of the action is then again conjugated. The action of T_1 can only be on parts that have $\mathbb{K}_1$ so $[T_1, F_i(x_1 \otimes x_2)] = F_i(T_{1i}x_1 \otimes x_2)$. Here, the i in the subscript of T_{1i} determines which element of the first triality triplet acts.

Given two elements of the composition algebra, x and y, an element of the triality algebra $T_{x,y} = (4S_{x,y}, R_y R_{\bar{x}} - R_x R_{\bar{y}}, L_y L_{\bar{x}} - L_x L_{\bar{y}})$ can be associated where and the action of $S_{x,y}(z) = \langle x, z\rangle y - \langle y, z\rangle x$.

A map θ takes the triality triplet (A, B, C) and acts with combined conjugation on the first and third element after cyclically permuting the first to be the last: $(\bar{B}, C, \bar{A})$. The Lie product of two F_i and F_j is an F_k (where ijk is a cyclic permutation of 123) $[F_i(x_1 \otimes x_2), F_j(y_1 \otimes y_2)] = F_k(\bar{y}_1\bar{x}_1 \otimes \bar{y}_2\bar{x}_2)$, and when $F_i(x_1 \otimes x_2)$ acts on $F_i(y_1 \otimes y_2)$ then the result is $\langle x_2, y_2\rangle \theta^{1-i} T_{x_1,y_1} + \langle x_1, y_1\rangle \theta^{1-i} T_{x_2,y_2}$ which is an element of $\mathrm{Tri}\mathbb{K}_1 \oplus \mathrm{Tri}\mathbb{K}_2$. If $i = 1$ then $\theta^{1-i} = \theta^0 = \mathrm{id}$, the identity map, and if $i = 2$ the inverse map θ^{-1} acts. For θ^{-2} the inverse map acts twice, as the notation suggests. This is the content of the triality construction.

9.4 Maximal compact subalgebras

The maximal compact subalgebras of Lie algebras in the magic square are shown in Figure 8. They can also be constructed by prescriptions based on division algebras. [2][15]

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3)$	$\mathfrak{u}(3)$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$
$\mathbb{C}$	$\mathfrak{so}(3)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(6)$	$\mathfrak{sq}(4)$
$\mathbb{H}$	$\mathfrak{u}(3)$	$\mathfrak{so}(6)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(6)$	$\mathfrak{su}(8)$
$\mathbb{O}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$	$\mathfrak{sq}(4)$	$\mathfrak{su}(8)$	$\mathfrak{so}(16)$

Figure 8: The maximal compact subalgebras.

10 Division algebras in supersymmetry

The dimensions of the division algebras 1, 2, 4 and 8 are also the dimensions of the little groups of $D = 3, 4, 6, 10$ theories. Pure super Yang-Mills theories, which have a gauge field A and a gaugino λ but no scalar fields ϕ, only exist in these dimensions. Superstrings also only exist in these dimension. Depending on the dimension, the on-shell degrees of freedom of the super Yang-Mills theory can therefore be described by either one octonion, one quaternion, one complex number or one real number. This holds not only for the vector but also for the spinor so that each is described by an element of the same division algebra $\mathbb{A}$. The work presented in the following sections can be found in the recent publications of Duff et al. [1][2]

10.1 Spinors and vectors

The isomorphism $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ states that space-time with the vector x^μ and Lorentz transformations $M^\mu{}_\nu$ can equally well be described by Hermitian

2×2 matrices and their transformations such that

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \leftrightarrow x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad x'^\mu = M^\mu{}_\nu x^\nu \leftrightarrow x' = AxA^\dagger.$$

In the new language, the Minkowski norm becomes the determinant and to preserve the Minkowski norm, the Lorentz transformations represented by the matrix A must be unimodular. The matrix x in which the space-time vector is encoded, is partitioned into the Pauli matrices σ_1, σ_2, σ_3 and the identity matrix I such that $x = x^0 I + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$. This familiar case is repeated because an unconventional extension follows.

The claim is now that just as $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$, and, in fact, $\mathfrak{so}(1,2) \cong \mathfrak{sl}(2,\mathbb{R})$, it also holds that $\mathfrak{so}(1,5) \cong \mathfrak{sl}(2,\mathbb{H})$ and $\mathfrak{so}(1,9) \cong \mathfrak{sl}(2,\mathbb{O})$. The last statement, in particular, will need further elaboration, since, for instance, $AxA^\dagger$ is a product of three elements and possibly ambiguous for the nonassociative octonions. So considering the $D = n + 2$-dimensional theory means looking at $\mathfrak{so}(1, n+1) \cong \mathfrak{sl}(2,\mathbb{A})$ where n is the dimension of the corresponding division algebra $\mathbb{A}$.

The appeal of the division algebra approach to describing $\mathfrak{so}(1, n+1)$ is that the matrices stay 2×2 and the 2×2 matrix elements stay simple. In $D = 4$ there is $\sigma_\mu = \bar{\sigma}^\mu = (-I, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma}_\mu = \sigma^\mu = (I, \sigma_1, \sigma_2, \sigma_3)$ where the middle two matrices σ_1, σ_2 will now be reconsidered. They are the special case of the collection $\sigma_\mu = \bar{\sigma}^\mu = (-I, \sigma_{a+1}, \sigma_{n+1})$ where

$$\sigma_{a+1} = \begin{pmatrix} 0 & e_a^* \\ e_a & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\} \text{ and } \sigma_{n+1} = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where, above, $\mathbb{A} = \mathbb{C}$ and so $n = 2$, $e_0 = 1$ and $e_1 = i$. This means that for $\mathbb{A} = \mathbb{H}$, two additional matrices are added where σ_3 and σ_4 that look like σ_2 but have units j or k instead where σ_2 has i. These satisfy $\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} I$ and $\bar{\sigma}^\mu \sigma^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} I$ and can therefore be used to generate space-time transformations of spinors and conjugate spinors.

A generalisation of the spinor transformation $\delta\Psi = \frac{1}{4}\lambda^{\mu\nu}\sigma_{\mu\nu}\Psi = \frac{1}{4}\lambda^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu\Psi$ and that of the conjugate spinor transformation $\delta\Phi = \frac{1}{4}\lambda^{\mu\nu}\bar{\sigma}_{\mu\nu}\Phi = \frac{1}{4}\lambda^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu\Phi$ is now due. A priori, it cannot generalise directly since, in the octonionic case, this will be a multiplication of more than two elements and is hence generally ill-defined for octonions unless order of multiplication is provided.

If the generalisation of this expression is simple it could be that it is $(\sigma_{[\mu}\sigma_{\nu]})\Psi$ or $\sigma_{[\mu}(\sigma_{\nu]}\Psi)$. By counting alone, it can be seen that $(\sigma_{[\mu}\sigma_{\nu]})$ cannot add up to the $10 \cdot 9/2 = 45$ generators of $SO(1,9)$. The latter can be shown to be the right ones by checking the commutation relations, considering the action of the purported generators on an arbitrary octonion and finding that the commutation relations are those of the Lorentz algebra. The generators are the octonionic operators

$\hat{\sigma}_{\mu\nu}(\cdot) = \frac{1}{2}[\sigma_\mu(\bar{\sigma}_\nu\cdot) - \sigma_\nu(\bar{\sigma}_\mu\cdot)]$ and $\hat{\bar{\sigma}}_{\mu\nu}(\cdot) = \frac{1}{2}[\bar{\sigma}_\mu(\sigma_\nu\cdot) - \bar{\sigma}_\nu(\sigma_\mu\cdot)]$ where $(\cdot)$ is a slot for an octonion. The Hermitian matrix X associated with the vector X^μ can be constructed as $X^\mu\bar{\sigma}_\mu$ and $\bar{X} = X^\mu\sigma_\mu = X - (\text{Tr}X)I$.

For $\mathfrak{sl}(2,\mathbb{C})$, the transformation of X was $\delta X = \frac{1}{4}\lambda^{\mu\nu}(\sigma_{\mu\nu}X - X\bar{\sigma}_{\mu\nu})$. The right transformation in $\mathfrak{sl}(2,\mathbb{A})$ turns out to be $\delta X = \frac{1}{4}\lambda^{\mu\nu}(\sigma_\mu(\bar{\sigma}_\nu X) - X(\bar{\sigma}_\mu\sigma_\nu))$. Since vectors can be constructed from spinors, this result can also be derived from the spinor and conjugate spinor transformations alone.

Now, the little groups will be considered. As noted above, the on-shell degrees of freedom of the spinor, conjugate spinor and vector representations of $SO(n)$, will each be represented by just one division algebra element. Restriction to $SO(n)$ transformations are made by setting the first and the last element of the antisymmetric $\lambda^{0\mu} = \lambda^{(n+1)\mu} = 0$. It is convenient because the transformations then only involve the generalised σ_2 matrices and so

$$\delta\Psi = \frac{1}{4}\lambda^{\mu\nu}\hat{\sigma}_{\mu\nu}\Psi = \frac{1}{4}\theta^{ab}\begin{pmatrix} e_a^*(e_b\Psi^1) \\ e_a(e_b^*\Psi^2) \end{pmatrix}$$

and the transformation of the spinor and the conjugate spinor can be read off directly as $\delta\Psi^1 = \delta\psi = \frac{1}{4}\theta^{ab}e_a^*(e_b\psi)$ and $\delta\Psi^2 = \delta\chi = \frac{1}{4}\theta^{ab}e_a(e_b^*\phi)$ where ψ and χ were introduced in place of Ψ^1 and Ψ^2, respectively. For the vector, the situation is analogous. The transformation is

$$\delta X = \frac{1}{4}\theta^{ab}\begin{pmatrix} 0 & e_a^*(e_b x^*) - x^*(e_a e_b^*) \\ e_a(e_b^* x) - x(e_a^* e_b) & 0 \end{pmatrix}$$

such that $\delta x = \frac{1}{4}\theta^{ab}[e_a(e_b^* x) - x(e_a^* e_b)]$ where x are the components of X discounting the first and last component. This is frugal. Transformations are formed by simply multiplying unit imaginary elements.

Recall that the multiplication of two division algebra elements is $e_a e_b =$

$\Gamma^a{}_{bc}e_c$ and multiplication by a conjugated unit element e_a^* is $e_a^* e_b = \bar{\Gamma}^a{}_{bc}e_c$ where $\Gamma^a{}_{bc} = (\delta_{a0}\delta_{bc}+\delta_{b0}\delta_{ac}-\delta_{ab}\delta_{0c}+C_{abc})$ and $\bar{\Gamma}^a{}_{bc} = (\delta_{a0}\delta_{bc}-\delta_{b0}\delta_{ac}+\delta_{ab}\delta_{0c}-C_{abc})$. A multiplication of a division algebra element x by e_a results in $e_a x = x_b e_a e_b = x_b \Gamma^a{}_{bc}e_c = e_c \bar{\Gamma}^a{}_{cb}x_b$ which is a multiplication of the components of x by a matrix. It was here used that $\Gamma^a{}_{bc} = \bar{\Gamma}^a{}_{cb}$.

The structure constants satisfy $\Gamma^a\bar{\Gamma}^b + \Gamma^b\bar{\Gamma}^a = 2\delta^{ab}I$ and $\bar{\Gamma}^a\Gamma^b + \bar{\Gamma}^b\Gamma^a = 2\delta^{ab}I$ which imply that $\Sigma^{ab} = \frac{1}{2}\Gamma^{[a}\bar{\Gamma}^{b]}$ and $\bar{\Sigma}^{ab} = \frac{1}{2}\bar{\Gamma}^{[a}\Gamma^{b]}$ are generators of $SO(n)$ in the spinor and conjugate spinor representations, respectively.

Likewise, an expansion of the vector $x = x_a e_a$ in the expression for δx yields $\delta x = e_a \theta^{ab} x_b$ from which can be deduced that $J_{[ij]kl} = \delta_{ki}\delta_{jl} - \delta_{kj}\delta_{il}$.

The $SO(n-1)$ generators in the spinor and vector representations are easily obtained. Simply disregard left multiplication by e_0 such that there is only $e_i e_a$ where a is allowed to range from 0 to 7 but i only from 1 to 7. This makes $\Gamma^i{}_{ab}$ antisymmetric in the lower indices and they satisfy the Clifford algebra of $SO(n-1)$ such that $\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = -2\delta^{ij}I$ so the generators are $\Sigma^{[ij]} = \frac{1}{2}\Gamma^{ij} = \frac{1}{2}\Gamma^{[i}\Gamma^{j]}$ and the spinors, conjugate spinors and vectors transform as before except with the transformations consisting only of multiplications by imaginary elements.

10.2 Dimensional reduction

The process of dimensional reduction can also be described using division algebras. In the critical dimensions $D = 10,\ 6,\ 4,\ 3$ the dimensional reduction of spinors assumes a simple form on-shell. The decomposition of a spinor is seen as a case of Cayley-Dickson construction. In $D = 10$ the spinor, written as an octonion, decomposes into two spinors expressed as two quaternions in $D = 6$, four spinors expressed as four complex numbers in $D = 4$ or eight real numbers in $D = 3$.

The on-shell vector of $D = 10$, also written as an octonion, decomposes into a vector of $D = 6$ expressed as a quaternion and four remaining components identified as scalars. The imaginary elements associated with the scalar components disclose the existence of an internal symmetry. Upon compactification, the former space-time symmetry manifests itself as an internal symmetry.

Reducing the supersymmetry of the theory amounts to removing vectors and spinors. These vectors and spinors are associated with imaginary division algebra units and so it will turn out that reduction of supersymmetry amounts to a consistent deletion of points in the Fano plane.

The $\mathcal{N} = 1$ pure super Yang-Mills theory in $D = 10$ has two fields, the vector x and the superpartner spinor ψ that on-shell transform as representations of $SO(8)$. There is no internal symmetry here. All symmetry is space-time symmetry. The vector and spinor are parameterised as octonions such that $x_{\mathbb{O}} = x_a e_a$ and $\psi_{\mathbb{O}} = \psi_a e_a$, respectively, which transform as described before.

10.2.1 Reduction to six dimensions

In $D = 6$ and $\mathcal{N} = 2$, the little group of the space-time symmetry is $SO(4)$ and there will be an internal symmetry that is also $SO(4)$. Both can be described as $SU(2) \times SU(2) \cong SO(4)$ but $SU(2)$ is also isomorphic to $Sp(1)$ which can be realised as a multiplication by imaginary quaternions. Recall that the family $\mathfrak{sp}(n)$ is generated by anti-Hermitian matrices of quaternions. For $n = 1$, there is just one element in the matrix, an imaginary quaternion. In total, the embedding is $SO(8)_{\text{st}} \supset (Sp(1) \times Sp(1))_{\text{st}} \times (Sp(1) \times Sp(1))_{\text{int}}$ where the subscript st denotes space-time symmetry and the subscript int denotes internal symmetry. Vectors decompose as $\mathbf{8}_v = (\mathbf{2}, \mathbf{2}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ whilst spinors decompose as $\mathbf{8}_s = (\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$. This will now be realised with elements of division algebras. The vector $x_{\mathbb{O}}$ is grouped and renamed as

$$\begin{aligned}
x_{\mathbb{O}} &= x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 \\
&= (x_0 + x_4e_4 + x_5e_5 + x_7e_7) + x_1e_1 + x_2e_2 + x_3e_3 + x_6e_6 \\
&= (x_0 + x_4e_4 + x_5e_5 + x_7e_7) + \phi_1e_1 + \phi_2e_2 + \phi_3e_3 + \phi_6e_6 \\
&= x_{\mathbb{H}} + \phi_{\mathbb{H}^c}.
\end{aligned}$$

where the subscript $\mathbb{H}$ refers to the quaternion subalgebra e_0, e_4, e_5, e_7 and $\mathbb{H}^c$ is the associated quadrangle 6123. The spinor $\psi_{\mathbb{O}}$ is grouped and renamed as

$$\begin{aligned}
\psi_{\mathbb{O}} &= \psi_0 + \psi_1e_1 + \psi_2e_2 + \psi_3e_3 + \psi_4e_4 + \psi_5e_5 + \psi_6e_6 + \psi_7e_7 \\
&= (\psi_0 + \psi_4e_4 + \psi_5e_5 + \psi_7e_7) + e_3(\psi_3 + \psi_6e_4 + \psi_2e_5 + \psi_1e_7) \\
&= (\psi_0 + \psi_4e_4 + \psi_5e_5 + \psi_7e_7) + e_3(\chi_3 + \chi_6e_4 + \chi_2e_5 + \chi_1e_7) \\
&= \psi_{\mathbb{H}} + e_3\chi_{\mathbb{H}}
\end{aligned}$$

which is an undoing (Cayley-Dickson) of the octonion into the quaternion subalgebra e_0, e_4, e_5, e_7. The associated quadrangle is 6123. These refer to the compactified directions. Given that there has been a partitioning, it is useful to consider what set multiplied elements of lines and quadrangles belong to. Demarcate elements on the line with a hat (ˆ) and elements on the quadrangle with a check (ˇ). When referring to imaginary elements alone subscripts or superscripts i and j will be used. Two units on the line multiplied together give units on the line $e_{\hat{a}}e_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}e_{\hat{c}}$. They sit in the quaternion subalgebra. A unit on the quadrangle multiplied by a unit on the line gives units on the quadrangle $e_{\check{a}}e_{\hat{b}} = \Gamma^{\check{a}}{}_{\hat{b}\check{c}}e_{\check{c}}$. Two units on the quadrangle multiplied give units on the line $e_{\check{a}}e_{\check{b}} = \Gamma^{\check{a}}{}_{\check{b}\hat{c}}e_{\hat{c}}$. In short, lines map objects into themselves and quadrangles map into opposite objects.

Given that e_3 has been singled out it is also useful to consider its products with the other elements. Unambiguous left multiplication of $-e_3$ on both sides of $e_3e_{\hat{a}} = \Gamma^3{}_{\hat{a}\check{b}}e_{\check{b}}$ gives $e_{\hat{a}} = -e_3\Gamma^3{}_{\hat{a}\check{b}}e_{\check{b}}$, and, similarly, for the quadrangle elements, $e_{\check{a}} = -e_3\Gamma^3{}_{\check{a}\hat{b}}e_{\hat{b}}$. It is in both cases unambiguous with no parentheses needed because only two imaginary elements are involved.

Now, the transformations of the spinor and vector will be considered. The transformation parameters θ^{ab} are split as $\theta^{\hat{a}\hat{b}}$, $\theta^{\check{a}\check{b}}$ and $\theta^{\hat{a}\check{b}}$ but with $\theta^{\hat{a}\check{b}} = 0$. The transformation of the spinor $\psi_{\mathbb{O}}$ proceeds as follows. The starting point is the $D = 10$ space-time transformation which is split into lines and quadrangles. The transformation of the spinor is

$$\begin{aligned}\delta\psi_{\mathbb{O}} &= \frac{1}{4}\theta^{ab}e_a^*(e_b\psi_{\mathbb{O}})\\ &= \frac{1}{4}\theta^{\hat{a}\hat{b}}e_{\hat{a}}^*(e_{\hat{b}}\psi_{\mathbb{O}}) + \frac{1}{4}\theta^{\check{a}\check{b}}e_{\check{a}}^*(e_{\check{b}}\psi_{\mathbb{O}})\\ &= \frac{1}{4}\theta^{\hat{a}\hat{b}}e_{\hat{a}}^*(e_{\hat{b}}(\psi_{\hat{c}}e_{\hat{c}} + e_3\chi_{\hat{c}}e_{\hat{c}})) + \frac{1}{4}\theta^{\check{a}\check{b}}e_{\check{a}}^*(e_{\check{b}}(\psi_{\hat{c}}e_{\hat{c}} + e_3\chi_{\hat{c}}e_{\hat{c}}))\end{aligned}$$

where the split into internal and space-time parameters was made in going from second line to third line. Notice that the first term containing $e^*_{\hat{a}}(e_{\hat{b}}(\psi_{\hat{c}}e_{\hat{c}})$ has space-time indices only so the multiplication is amongst elements of the associative quaternion subalgebra and therefore the parentheses can be moved to group the unit elements $(e^*_{\hat{a}}e_{\hat{b}})\psi_{\hat{c}}e_{\hat{c}}$, abbreviated as $u_{\text{st}}\psi_{\mathbb{H}}$. An important identity that is useful here is that $e_{\hat{b}}(e_3e_c) = e_3(e^*_{\hat{b}}e_c)$. This can be used to manipulate the second term to pull e_3 out to the left such that it becomes $e_3(u^{\text{other}}_{\text{st}}\chi_{\mathbb{H}})$ where $u^{\text{other}}_{\text{st}}$ refers to some product of unit elements associated with space-time. The third term is manipulated into $\theta^{\check{a}\check{b}}e^*_{\check{a}}(e_{\check{b}}e_{\hat{c}}) = \theta^{\check{a}\check{b}}e_{\hat{c}}(e^*_{\check{a}}e_{\check{b}})$, which can be checked to hold. It is a reordering that frees the space-time unit and displaces the internal indices and multiplies them with one another, abbreviated as $\psi_{\mathbb{H}}u_{\text{int}}$. The last manipulation, the manipulation of $\frac{1}{4}\theta^{\check{a}\check{b}}e^*_{\check{a}}(e_3\chi_{\hat{c}}e_{\hat{c}})$ is the most nontrivial. The resort is to write the product in terms of structure constants and reorder them by using the fact that they obey the Clifford algebra and by also using the aforementioned identity involving Γ^3 to eventually separate and order e_3 to be on the far left. The net effect of these manipulations is $e_3(\chi_{\mathbb{H}}u^{\text{other}}_{\text{int}})$.

The transformations should of the form $\delta\psi_{\mathbb{O}} = \delta\psi_{\mathbb{H}} + e_3\delta\chi_{\mathbb{H}}$ which explains why it was important to manipulate the expressions to get e_3 separated and on the far left. The transformations are $\delta\psi_{\mathbb{H}} = u_{\text{st}}\psi_{\mathbb{H}} + \psi_{\mathbb{H}}u_{\text{int}}$ and $\delta\chi_{\mathbb{H}} = u^{\text{other}}_{\text{st}}\chi_{\mathbb{H}} + \chi_{\mathbb{H}}u^{\text{other}}_{\text{int}}$ where u_{st} and u_{int} turn out to be imaginary unit quaternions and $u^{\text{other}}_{\text{st}}$ and $u^{\text{other}}_{\text{int}}$ turn out to be other imaginary unit quaternions.

This establishes that ψ and χ transform as representations of $(Sp(1) \times Sp(1))_{\text{st}} \times (Sp(1) \times Sp(1))_{\text{int}}$. As stated earlier, the left or right action of imaginary quaternions is isomorphic to $Sp(1)$. The action of the four $Sp(1)$ in $(Sp(1)\times Sp(1))_{\text{st}}\times(Sp(1)\times Sp(1))_{\text{int}}$ correspond to u_{st}, $u_{\text{st}}^{\text{other}}$, u_{int} and $u_{\text{int}}^{\text{other}}$, respectively. The infinitesimal transformation of an object that transforms as a tensor product is a sum of terms where each term corresponds to a factor of the tensor product and is a multiplication of the object with that factor alone. The conclusion is that ψ transforms under the first and third $Sp(1)$ and χ transforms under the second and fourth $Sp(1)$ as anticipated.

The transformation of the vector proceeds in a similar way and starts with the full space-time transformation $\delta x_{\mathbb{O}} = \frac{1}{4}\theta^{ab}(e_a(e_b^* x_{\mathbb{O}}) - x_{\mathbb{O}}(e_a^* e_b))$ which after the split of the vector and the transformation parameters into internal and space-time parts $x_{\hat{a}}e_{\hat{a}} + \phi_{\check{a}}e_{\check{a}}$ is also

$$\begin{aligned}
\delta x &= \frac{1}{4}\theta^{\hat{a}\hat{b}}(e_{\hat{a}}(e_{\hat{b}}^* a_{\mathbb{O}}) - a_{\mathbb{O}}(e_{\hat{a}}^* e_{\hat{b}})) + \frac{1}{4}\theta^{\check{a}\check{b}}(e_{\check{a}}(e_{\check{b}}^* a_{\mathbb{O}}) - a_{\mathbb{O}}(e_{\check{a}}^* e_{\check{b}})) \\
&= \frac{1}{4}\theta^{\hat{a}\hat{b}}(e_{\hat{a}}(e_{\hat{b}}^*(x_{\hat{c}}e_{\hat{c}} + \phi_{\check{c}}e_{\check{c}}))) - (x_{\hat{c}}e_{\hat{c}} + \phi_{\check{c}}e_{\check{c}})(e_{\hat{a}}^* e_{\hat{b}})) \\
&\quad + \frac{1}{4}\theta^{\check{a}\check{b}}(e_{\check{a}}(e_{\check{b}}^*(x_{\hat{c}}e_{\hat{c}} + \phi_{\check{c}}e_{\check{c}}))) - (x_{\hat{c}}e_{\hat{c}} + \phi_{\check{c}}e_{\check{c}})(e_{\check{a}}^* e_{\check{b}})) \\
&= (u_{\text{st}}^{\text{other}} x_{\hat{c}}e_{\hat{c}} - x_{\hat{c}}e_{\hat{c}}u_{\text{int}}) + e_3(u_{\text{int}}^{\text{other}}\phi_{\hat{c}}e_{\hat{c}} - \phi_{\hat{c}}e_{\hat{c}}u_{\text{int}}) = \delta x_{\mathbb{H}} + e_3\delta\phi_{\mathbb{H}}
\end{aligned}$$

where in the last line the same identifications of unit imaginary quaternions were made, but as can be seen, the imaginary elements are differently arranged. They are arranged in such a way that, unlike the spinor, only imaginary unit quaternions associated with space-time transformations transform the vector and only imaginary unit quaternions associated with the internal transformations act on the scalars. As anticipated, the vector only transforms under the first and second of $(Sp(1) \times Sp(1))_{\text{st}} \times (Sp(1) \times Sp(1))_{\text{int}}$

corresponding to $u_{\rm st}$ and $u_{\rm st}^{\rm other}$ whilst the scalars only transform under the third and fourth $Sp(1)$ corresponding to $u_{\rm int}$ and $u_{\rm int}^{\rm other}$.

10.2.2 Reduction to four dimensions

The more intricate decomposition is the reduction from $D = 10$ and $\mathcal{N} = 1$ to $D = 4$ and $\mathcal{N} = 4$. The $SO(8)_{\rm st}$ space-time symmetry becomes an $SO(2)_{\rm st}$ space-time symmetry and an $SO(6)_{\rm int}$ internal symmetry. The original $D = 10$ vector is decomposed into a $D = 4$ vector and six scalars. The original $D = 10$ spinor is decomposed into four spinors. $SO(6)_{\rm int}$ is also isomorphic to $SU(4)_{\rm int}$ and, as is well-known, $SO(2)_{\rm st}$ is isomorphic to $U(1)_{\rm st}$.

The decomposition of the vector representation is to the $U(1)_{\rm st}$ neutral second rank antisymmetric representation of $SU(4)_{\rm int}$ and two singlets, one singlet with charge one and the other with charge minus one: $\mathbf{8}_v = \mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}$. The spinor decomposes into the fundamental and antifundamental representation of $SU(4)_{\rm int}$ with opposite charges of one half: $\mathbf{8}_s = \mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}$.

The starting point is again the octonion vector $x_{\mathbb{O}}$ which is split into a subalgebra $x_{\mathbb{O}} = x_{\mathbb{C}} + \phi_{\mathbb{C}^c} = x_{\underline{a}} e_{\underline{a}} + \phi_{\bar{b}} e_{\bar{b}} = (x_0 + x_3 e_3) + \phi_1 e_1 + \phi_2 e_2 + \phi_4 e_4 + \phi_5 e_5 + \phi_6 e_6 + \phi_7 e_7$ where, now, $\underline{a} = 0,\ 3$ and $\bar{b} = 1,\ 2,\ 4,\ 5,\ 6,\ 7$. The vector is represented by a complex number and six scalars associated with the remaining six imaginary elements. The original octonion spinor is split into four spinors, each represented by a complex number. The original octonion spinor effectively becomes a quaternion over the complex numbers:

$\psi_{\mathbb{O}} = (\psi_{\mathbb{C}})_{\bar{a}} e_{\bar{a}} = (\psi_0 + e_3\psi_3) + (\psi_4 + \psi_6 e_3) e_4 + (\psi_5 + \psi_2 e_3) e_5 + (\psi_7 + \psi_1 e_3) e_7 = (\psi_{\mathbb{C}})^{\hat{a}} e_{\hat{a}}$ where $\hat{a} = 0, 4, 5, 7$ is a basis for quaternions, each element now a complex number $\psi_{\mathbb{C}}$.

The effect of multiplying the octonion spinor by an imaginary unit that is internal, $e_{\bar{i}}$, conjugates the complex components of the quaternion and in addition multiplies them by a 4×4 matrix $\Upsilon^{\bar{i}}{}_{\hat{a}\hat{b}}$ whose commutator with a complex conjugated $\Upsilon^{*\bar{j}}{}_{\hat{a}\hat{b}} = \bar{\Upsilon}^{\bar{j}}{}_{\hat{a}\hat{b}}$ turn out to be one of the $6 \cdot 5/2 = 15$ Hermitian and traceless generators of $SU(4)$. There are 6 elements in the set $\bar{i} = 1, 2, 4, 5, 6, 7$ and so there are $6 \cdot 5/2 = 15$ commutators (generators). Since the matrices are furthermore 4×4, this means that this is the fundamental representation of $SU(4)$. The generators in the antifundamental representations are obtained by conjugating these generators.

The transformation parameters θ^{ab} are again divided into $\theta^{\bar{a}\bar{b}}$, $\theta^{\underline{ab}}$ and $\theta^{\bar{a}\underline{b}} = 0$. The spinor transformation is $\delta\psi_{\mathbb{O}} = \frac{1}{4}\theta^{ab}e_a^*(e_b\psi_{\mathbb{O}}) = \frac{1}{2}\theta^{03}e_3(\psi_{\mathbb{C}}^{\hat{a}}e_{\hat{a}}) + \frac{1}{4}\theta^{\bar{i}\bar{j}}e_{\bar{i}}^*(e_{\bar{j}}(\psi_{\mathbb{C}}^{\hat{a}}e_{\hat{a}})) = \frac{1}{2}\theta^{03}e_3\psi_{\mathbb{C}}^{\hat{a}}e_{\hat{a}} - \frac{1}{2}\theta^{\bar{i}\bar{j}}(T^{[\bar{i}\bar{j}]}{}_{\hat{a}\hat{b}}\,\psi_{\mathbb{C}}^{\hat{b}})e_{\hat{a}}$

The spinor transforms by left multiplications of two unit elements, where the last unit multiplied is conjugated. Each multiplication by an imaginary element produces a factor of Υ, one of them conjugated. Since the transformation parameters are antisymmetric $\theta^{\bar{a}\bar{b}}$, this matrix product is antisymmetrised and therefore generates $SU(4)$ in the fundamental representation. For the space-time transformation there is only one parameter θ^{03}. The spinor is multiplied by $\frac{1}{2}e_3$ which is an infinitesimal transformation of $U(1)$ with positive charge $\frac{1}{2}$. In total, the complex spinor $\psi_{\mathbb{C}}^{\hat{a}}$ transforms as $\mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}$ of $SU(4)_{\text{int}} \times U(1)_{\text{st}}$ as anticipated. One term comes from the real part and the other term from the imaginary part of the complex number.

The octonion vector transforms as $\delta a_{\mathbb{O}} = \frac{1}{4}\theta^{ab}(e_a(e_b^*a_{\mathbb{O}}) - a_{\mathbb{O}}(e_a^*e_b)) = \frac{1}{2}\theta^{03}(-e_3a_{\mathbb{C}} - a_{\mathbb{C}}e_3) + \frac{1}{2}\theta^{03}(-e_3\phi_{\mathbb{C}^c} - \phi_{\mathbb{C}^c}e_3) + \frac{1}{4}\theta^{\bar{i}\bar{j}}(e_{\bar{i}}(e_{\bar{j}}a_{\mathbb{C}}) - a_{\mathbb{C}}(e_{\bar{i}}^*e_{\bar{j}})) + \frac{1}{4}\theta^{\bar{i}\bar{j}}(e_{\bar{i}}(e_{\bar{j}}\phi_{\mathbb{C}^c}) - \phi_{\mathbb{C}^c}(e_{\bar{i}}^*e_{\bar{j}})) = -\theta^{03}e_3a_{\mathbb{C}} + \frac{1}{2}\theta^{\bar{i}\bar{j}}e_{\bar{l}}J_{[\bar{i}\bar{j}]\bar{l}\bar{k}}\phi_{\bar{k}}$ which establishes that the vector transforms according to $\mathbf{1}_1 + \mathbf{1}_{-1}$ of $SU(4)_{\rm int} \times U(1)_{\rm st}$ and the scalars according to $\mathbf{6}_0$. Instead of looking for the second rank antisymmetric representation of $SU(4)$ the vector representation of $SO(6)$ was identified as the representation that the scalars transform according to.

Admittedly, some steps have been omitted in this derivation which is more of a sketch of a derivation. The reader has already seen enough manipulations with division algebra elements and a full-fledged derivation of the above would be very detailed and as a result too tedious to read.

10.2.3 Reduction to three dimensions

The simplest reduction is the reduction from $D = 10$ to $D = 3$ since, here, the space-time symmetry is trivial. The decomposition of $SO(8)_{\rm st}$ is into $SO(7)_{\rm int}$. The spinor of $SO(8)_{\rm st}$ becomes the spinor of $SO(7)_{\rm int}$ such that $\mathbf{8}_s = \mathbf{8}$. The vector of $SO(8)_{\rm st}$ becomes a vector and a singlet of $SO(7)_{\rm int}$ such that $\mathbf{8}_v = \mathbf{7} + \mathbf{1}$.

The vector $x_{\mathbb{O}}$ is trivially split as $x_{\mathbb{O}} = x_{\mathbb{R}} + \phi_{\mathbb{R}^c} = x_0 + \phi_i e_i$, the spinor is expressed as $\psi_{\mathbb{O}} = (\psi_{\mathbb{R}})_a e_a = \psi_a e_a$ and the transformation parameters θ^{ab} are split as θ^{0i} and θ^{ij}. As before, $\theta^{0i} = 0$. The transformations are $\delta\psi_{\mathbb{O}} = \frac{1}{4}\theta^{ij} e_i^*(e_j e_a)\psi_a$ and $\delta x_{\mathbb{O}} = \frac{1}{4}\theta^{ij}(e_i(e_j^* e_k \phi_k) - \phi_k e_k(e_i^* e_j)$ but it was already mentioned in a previous section that by restricting the spinor and vector transformations to only involve imaginary unit elements are transformations of spinors and vectors of $SO(n-1)$ are obtained, which here is $SO(7)$.

It can also with a tedious but straightforward computation using the structure constants be shown, directly, that $\delta\psi_a = -\frac{1}{2}\theta^{ij}\Sigma^{[ij]}{}_{ab}\psi_b$ and $\delta\phi_l = \frac{1}{2}\theta^{ij}J_{[ij]lk}\phi_k$ which is a spinor and vector transformation of $SO(7)$, respectively. Though elementary, such computations can easily occupy whole sheets of paper.

10.2.4 Reduction of supersymmetry

A theory in $D = 10$ and $\mathcal{N} = 1$ was the starting point. Each step down doubles the supersymmetry such that $D = 6$ gives $\mathcal{N} = 2$, $D = 4$ gives $\mathcal{N} = 4$ and $D = 3$ gives $\mathcal{N} = 8$. These are the maximal supersymmetries. For each step down, the spinor splits into two spinors of the lower-dimensional theory. This is the Cayley-Dickson procedure in reverse.

The content of nonmaximal supersymmetries can also be obtained using the division algebra formalism. Think about the truncation in terms of the spinors. In order to halve the supersymmetry in $D = 6$, a quaternion subalgebra must be preserved and four imaginary elements must be removed. This corresponds to a deletion of those components associated with the quadrangle 3612 from $\psi_{\mathbb{O}} = (\psi_0 + \psi_4 e_4 + \psi_5 e_5 + \psi_7 e_7) + e_3(\chi_3 + \chi_6 e_4 + \chi_2 e_5 + \chi_1 e_7) \rightarrow \psi_0 + \psi_4 e_4 + \psi_5 e_5 + \psi_7 e_7$. For the vector, $x_{\mathbb{O}} = (x_0 + x_4 e_4 + x_5 e_5 + x_7 e_7) + \phi_1 e_1 + \phi_2 e_2 + \phi_3 e_3 + \phi_6 e_6 \rightarrow x_0 + x_4 e_4 + x_5 e_5 + x_7 e_7$. There is nothing special about this quadrangle. What is important is that it is the quadrangle corresponding to the quaternion algebra that will be preserved. An equal number of bosonic degrees of freedom $\phi_{\breve{a}}$ will also disappear. One is left with a line in the Fano plane and hence a theory defined over the quaternions.

In obtaining a $D = 4$ and $\mathcal{N} = 2$ theory from $D = 4$ and $\mathcal{N} = 4$ theory, two of the four spinors are removed. The quadrangle 5271 may be chosen such that $\psi_{\mathbb{O}} = \psi_{\mathbb{C}}^{\bar{a}} e_{\bar{a}} = (\psi_0 + e_3\psi_3) + (\psi_4 + \psi_6 e_3)e_4 + (\psi_5 + \psi_2 e_3)e_5 + (\psi_7 + \psi_1 e_3)e_7 \rightarrow (\psi_0 + e_3\psi_3) + (\psi_4 + \psi_6 e_3)e_4$ and, correspondingly, for the vector $x_{\mathbb{O}} = (x_0 +$

$x_3e_3) + \phi_1e_1 + \phi_2e_2 + \phi_4e_4 + \phi_5e_5 + \phi_6e_6 + \phi_7e_7 \to (x_0 + x_3e_3) + \phi_4e_4 + \phi_6e_6$. A further reduction is to remove 46 which leaves only one spinor and one vector and gives a $D = 4$ and $\mathcal{N} = 1$ theory.

In the case of $D = 3$, the difference is that all imaginary elements can be deleted. Deleting a quadrangle brings it to $\mathcal{N} = 4$, deleting two imaginary elements to $\mathcal{N} = 2$ and deleting the last imaginary element gives $\mathcal{N} = 1$ which is a rather uninteresting theory.

10.3 Super Yang-Mills

The dimensional reductions and truncations of supersymmetry of an $\mathcal{N} = 1$ and $D = 10$ pure super Yang-Mills theory done in the preceding section suggest that two division algebras $\mathbb{A}_\mathcal{N}$ and $\mathbb{A}_n$ can be used to specify a pure super Yang-Mills theory with extended, possibly nonmaximal, supersymmetry. The on-shell degrees of freedom were described by an element of a division algebra $\mathbb{A}_n$. In $D = 4$, $\mathbb{A}_n$ was $\mathbb{C}$. The vectors and spinors were here represented by complex numbers. The maximal supersymmetry is in this case $\mathcal{N} = 4$ and the spinor, originally an octonion, was rewritten (in the manner of the Cayley-Dickson procedure) as a complex-valued quaternion $\mathbb{H} \otimes \mathbb{C}$. Truncation of supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ amounted to deleting two of the three imaginary elements associated with the quaternion resulting in a complex algebra that can thus be associated with $\mathcal{N} = 2$. This suggests that a division algebra $\mathbb{A}_\mathcal{N}$ also be associated with extended supersymmetry such that $\mathcal{N} = 1, 2, 4, 8$ correspond to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively. An oxidation algebra[2] $\mathbb{A}_{n\mathcal{N}}$ can be associated with the two algebras $\mathbb{A}_n$ and $\mathbb{A}_\mathcal{N}$.

[2]There was no name for it in the article [1] so this is my suggestion.

Before truncation of supersymmetry was considered this oxidation algebra was always $\mathbb{O}$. An overview is given in Table 9. Before introducing trunca-

$(D, \mathcal{N})$	1	2	4	8
10	$\mathbb{OR} \sim \mathbb{O}$			
6	$\mathbb{HR} \sim \mathbb{H}$	$\mathbb{HC} \sim \mathbb{O}$		
4	$\mathbb{CR} \sim \mathbb{C}$	$\mathbb{CC} \sim \mathbb{H}$	$\mathbb{CH} \sim \mathbb{O}$	
3	$\mathbb{RR} \sim \mathbb{R}$	$\mathbb{RC} \sim \mathbb{C}$	$\mathbb{RH} \sim \mathbb{H}$	$\mathbb{RO} \sim \mathbb{O}$

Figure 9: Division algebra formulations of n+2-dimensional super Yang-Mills theories with $\mathcal{N}$ = 1,2,4,8.

tion of supersymmetry, only the theories with maximal supersymmetry were dealt with, corresponding to the theories in the diagonal. Dimensional reduction is a step down the rows and a truncation of supersymmetry is a step to the left.

Before proceeding to theories of supergravity, triality will first be discussed. It was a component in the later manifestly symmetric magic square constructions and it was introduced as a triplet of $\mathfrak{so}(n)$ elements (A, B, C) for which it holds that $A(xy) = (Bx)y + x(Cy)$ for all $x, y \in \mathbb{A}$ but this sounds very abstract with no clear physical picture. One way to think about triality is in the following manner. A spinor and a conjugate spinor can be used to form a vector $x = \chi\psi^* = \chi^b\psi^c\bar{\Gamma}^a{}_{bc}e_a$, or a spinor can be made from a vector and a conjugate spinor $\psi = x^*\chi = x^a\chi^c\Gamma^a{}_{bc}e_b$, or a conjugate spinor can be made from a vector and a spinor $\chi = x^a\psi^c\bar{\Gamma}^a{}_{bc}e_b$. Supposing that all three objects x, χ, ψ^* are then transformed such that $\delta x = A(x)$, $\delta\chi = B(\chi)$ and $\delta\psi^* = C(\psi^*)$, then, still insisting that x is made from ψ and χ, these transformations would have to be related, and their relation is such that A, B, and C satisfy $\delta x = A(x) = \delta(\chi\psi^*) = (\delta\chi)\psi^* + \chi(\delta\psi^*) = (B\chi)\psi^* + \chi(C\psi^*)$

which is exactly the condition that defines triality! These triality algebras exactly match the symmetries of the $g = 0$ Lagrangian of $D = 3$ super Yang-Mills. [1]

Super Yang-Mills theories were identified using two division algebras so the triality definition is undefined for cases when there is more than one division algebra involved. The $D = 3$ case is an extreme case where space-time symmetry is trivial. A generalisation is needed. It turns out that to get the full symmetries of the Lagrangian of any super Yang-Mills theory discussed here it suffices to consider a subset of the oxidation algebra $\mathbb{A}_{n\mathcal{N}}$ restricted such that the transformations of the vector A must respect the space-time symmetry of the vector: $\widetilde{\mathrm{Tri}}(\mathbb{A}_{n\mathcal{N}}, \mathbb{A}_n) = \{(A, B, C) \in \mathfrak{so}(n) | A(xy) = B(x)y + xC(y), A(\mathbb{A}_n) \subseteq \mathbb{A}_{n\mathcal{N}} = \mathbb{A}_n\}$. [1]

10.3.1 Lagrangians

Briefly, attention will now be turned to the super Yang-Mills Lagrangian. It will not be treated in detail but merely presented. Supersymmetry transformations will be introduced, written using division algebra elements.

The $n + 2$-dimensional $\mathcal{N} = 1$ super Yang-Mills theory has a gauge field ${A_\mu}^A$ and a spinor field Ψ^A that transform as the adjoint representation of a gauge group G where $A = 0, \ldots, \dim(G)$. The field strength is ${F_{\mu\nu}}^A = \partial_\mu {A_\nu}^A - \partial_\nu {A_\mu}^A + g {f_{BC}}^A {A_\mu}^B {A_\nu}^C$ and the covariant derivative acts as $D_\mu \Psi^A = \partial_\mu \Psi^A + g {f_{BC}}^A {A_\mu}^B \Psi^C$.

As discussed earlier, the spinor is an element of $\mathfrak{sl}(2, \mathbb{A}_n)$. It is furthermore a Grassmann variable, so the division algebras used here are not defined over the real numbers but rather over Grassmann variables. The Lagrangian needs a kinetic spinor term that is real, invariant under space-time transformations and bilinear. One such product is given by $\mathrm{Re}(i\Psi^\dagger\Phi) = \frac{i}{2}(\Psi^\dagger\Phi - \Phi^\dagger\Psi)$, adapted here as $\mathrm{Re}(i\Psi^\dagger\bar{\sigma}^\mu\partial_\mu\Psi)$. There is no ambiguity in having a multiplication of three division algebra elements here since the associator is purely imaginary and only the real part is considered so any reordering of multiplication order would contribute with an associator whose real part is trivially zero.

The action is

$$S = \int \mathrm{d}^{(n+2)}x \left(-\frac{1}{4}F_{\mu\nu}{}^A F_{\mu\nu}{}^A - \mathrm{Re}\left(i\Psi^{\dagger A}\bar{\sigma}^\mu D_\mu \Psi^A\right)\right). \qquad (10.3.1)$$

The supersymmetry transformations are $\delta A_\mu{}^A = \mathrm{Re}(i\Psi^{\dagger A}\bar{\sigma}_\mu\epsilon)$ and $\delta\Psi^A = \frac{1}{2}\hat{F}\epsilon$ where $\hat{F} = F_{\mu\nu}\hat{\sigma}^{\mu\nu}$. Packaged as a matrix, the supersymmetry transformation of the vector is $\delta\bar{A}^A = \delta A_\mu{}^A\sigma^\mu = i(\Psi^A\epsilon^\dagger - \epsilon\Psi^{\dagger A})$.

The action of the general super Yang-Mills theory associated with an algebra $\mathbb{A}_n$ and an oxidation algebra $\mathbb{A}_{n\mathcal{N}}$ is

$$\begin{aligned} S(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}}) = \int \mathrm{d}x^{(n+2)}\Big(& -\frac{1}{4}F_{\mu\nu}{}^A F_{\mu\nu}{}^A - \frac{1}{2}D_\mu\phi^{*A}D^\mu\phi^A \\ & - \mathrm{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu D_\mu\Psi^A) - g f_{BC}{}^A \mathrm{Re}(i\Psi^{\dagger A}\varepsilon\phi^B\Psi^C) \\ & - \frac{1}{16}g^2 f_{BC}{}^A f_{DE}{}^A(\phi^{*B}\phi^D + \phi^{*D}\phi^B)(\phi^{*C}\phi^E + \phi^{*E}\phi^C)\Big) \end{aligned} \qquad (10.3.2)$$

where ε is a 2×2 matrix like the Pauli matrix σ_2 but without the imaginary elements. Vectors decompose as $A_{\mathbb{A}_{n\mathcal{N}}} = A_{\mathbb{A}_n} + \phi_{\mathbb{A}_n^c}\varepsilon$ where the subscript $\mathbb{A}_n$ refers to the subalgebra of the division algebra corresponding to space-time symmetry and $\mathbb{A}_n^c$ refers to the remaining elements, that is to say, the elements complementary to $\mathbb{A}_n$. The supersymmetry transformations become $\delta\bar{A}^A = i(\Psi^A\epsilon^\dagger - \epsilon\Psi^{\dagger A})_{\mathbb{A}_n}$, $\delta\phi^A = -\frac{i}{2}\mathrm{Tr}\left(\varepsilon\left(\Psi^A\epsilon^\dagger - \epsilon\Psi^{\dagger A}\right)_{\mathbb{A}_n^c}\right)$ and $\delta\Psi^A = \frac{1}{2}\hat{F}^A\epsilon + \frac{1}{2}\sigma^\mu\varepsilon(D_\mu\phi^A\epsilon) + \frac{1}{4}f_{BC}{}^A\phi^C(\phi^B\epsilon)$.

The proof of supersymmetry on-shell turns out to rest on the alternativity of the algebra which is a property shared by all division algebras. [12] Off-shell, closure of the supersymmetry algebra requires addition of auxiliary fields. The spinor has dimension $2n$ and the vector has dimension $n+2-1 = n+1$. In total, $2n-(n+2-1) = n-1$ additional degrees of freedom are needed. The counting suggests that an imaginary division algebra element carrying scalars be added. The usual nondynamical term associated with these scalar fields is added to the action. A careful analysis finds that, unless this field transforms according to the fundamental representation of G_2, the supersymmetry transformations break the Lorentz invariance of the theory. For quaternions the auxiliary fields must transform as the adjoint representation of $Sp(1)$. [1]

To further obtain closure of the supersymmetry algebra in $D = 10$, the supersymmetry parameter, the spinor ϵ, must be restricted to have its first component ϵ_1 have no imaginary part. This restriction is ultimately due to the nonassociativity of the octonions. [1] Associators which are nontrivial for the octonions obstruct the closure. The off-shell closure of the $D = 10$ supersymmetry algebra is intricate but the octonions elucidate its failure and subsequent patching.

10.4 Supergravity and the magic square

In more than one context theories of supergravity arise from (tensor products of) super Yang-Mills theories. The multiplets of $D = 10$ Type IIA and Type IIB supergravity can be obtained from a tensor product of $D = 10$ super Yang-Mills multiplets. [24] The difference between Type IIA and Type IIB supergravity is then whether or not the same spinor representation was used in both the left and right super Yang-Mills multiplets. Type IIA is nonchiral whilst Type IIB is chiral. Before proceeding to construct theories of supergravity from theories of super Yang-Mills using division algebras the construction of Type IIA from two $D = 10$ super Yang-Mills theories will be presented.

Dynkin labels will be used in the following to denote irreducible representations. The irreducible representations of D_4 are denoted $[1, 0, 0, 0]_{D_4}$ (fundamental representation), $[0, 1, 0, 0]$ (adjoint representation), $[0, 0, 1, 0]_{D_4}$ (spinor representation), $[0, 0, 0, 1]_{D_4}$ (conjugate spinor representation) and $[0, 0, 1, 1]_{D_4}$ (third rank antisymmetric representation). The irreducible representations of B_4 are denoted $[1, 0, 0, 0]_{B_4}$ (fundamental representation), $[0, 1, 0, 0]_{B_4}$ (adjoint representation), $[0, 0, 1, 0]_{B_4}$ (third rank antisymmetric representation) and $[0, 0, 0, 1]_{B_4}$ (spinor representation).

The $D = 11$ supergravity which is understood as the infrared limit of M-theory with $SO(9)$ as the little group has a graviton $[2, 0, 0, 0]_{B_4}$, a gravitino $[1, 0, 0, 1]_{B_4}$ and a 3-form $[0, 0, 1, 0]_{B_4}$. Dimensional reduction of the graviton to $D = 10$ where the little group is $SO(8)$ gives $[2, 0, 0, 0]_{D_4} + [1, 0, 0, 0]_{D_4} + [0, 0, 0, 0]_{D_4}$ which is another graviton, a gauge field and a scalar.

The gravitino $[1,0,0,1]_{B_4}$ reduces to two gravitinos with opposite chirality $[1,0,0,1]_{D_4} + [1,0,1,0]_{D_4}$ and two fermions $[0,0,1,0]_{D_4} + [0,0,0,1]_{D_4}$. The 3-form $[0,0,1,0]_{B_4}$ reduces to $[0,0,1,1]_{D_4} + [0,1,0,0]_{D_4}$, a 3-form and a 2-form. The resulting $D = 10$ theory is the Type IIA supergravity with the field content $[2,0,0,0]_{D_4} + [1,0,0,0]_{D_4} + [0,0,0,0]_{D_4} + [1,0,0,1]_{D_4} + [1,0,1,0]_{D_4} + [0,0,1,0]_{D_4} + [0,0,0,1]_{D_4} + [0,0,1,1]_{D_4} + [0,1,0,0]_{D_4}$.

The $D = 10$ super Yang-Mills theory whose little group is $SO(8)$ has a vector $[1,0,0,0]_{D_4}$ and a spinor $[0,0,1,0]_{D_4}$ or a conjugate spinor $[0,0,0,1]_{D_4}$. When performing the tensor product of two super Yang-Mills multiplets, a left one $([1,0,0,0] + [0,0,1,0])$ and a right one $([1,0,0,0] + [0,0,0,1])$, the result is the field content of Type IIA supergravity. There are four tensor products to be computed. The first tensor product $[1,0,0,0][1,0,0,0] = [2,0,0,0] + [0,1,0,0] + [0,0,0,0]$ gives the NS-NS sector. The tensor product $[0,0,1,0][0,0,0,1] = [0,0,1,1] + [1,0,0,0]$ is the R-R sector. The last two parts are $[1,0,0,0][0,0,0,1] = [1,0,0,1] + [0,0,1,0]$ and $[1,0,0,0][0,0,1,0] = [1,0,1,0] + [0,0,0,1]$. Collecting all terms gives the Type IIA supergravity $[2,0,0,0]+[0,1,0,0]+[0,0,0,0]+[0,0,1,1]+[1,0,0,0]+[1,0,0,1]+[0,0,1,0]+[1,0,1,0]+[0,0,0,1]$. The chiral Type IIB theory arises from considering the product $([1,0,0,0] + [0,0,1,0])([1,0,0,0] + [0,0,1,0])$.

Other work has looked at relating supergravity scattering amplitudes to scattering amplitudes of super Yang-Mills theories. [25]

Also, given that the infrared limit of a string theory is a theory of supergravity it is not unreasonable to expect a manifestation of the relations between supergravity and super Yang-Mills in string theory as well. Indeed, that turns out to be the case. Relations exist, the KLT relations, that describe closed string tree amplitudes (gravity) as products of open string (gluon) tree amplitudes. [26]

Obtaining supergravity fields by a tensor product of super Yang-Mills fields will now be done in $D = 3$. The graviton and gravitino have $D \cdot (D - 3)/2 = 0$ degrees of freedom. 2-forms have $(D-2)(D-1)/(2 \cdot 1) = 1$ degree of freedom and are hence dual to scalars. It has already been shown how the super Yang-Mills fields are formulated with division algebras so all that needs to be done is perform the tensor product of two division algebras $\mathbb{A}_L$ and $\mathbb{A}_R$. The spinors are not division algebras over the real numbers but division algebras over Grassmann variables. The $D = 3$ supergravity fields as tensor products of super Yang-Mills fields are listed in Figure 10.

$\mathbb{A}_L/\mathbb{A}_R$	$(A_\mu)_R \in \mathrm{Re}\mathbb{A}_R$	$\phi_R \in \mathrm{Im}\mathbb{A}_R$	$\lambda_R \in \mathbb{A}_R$
$(A_\mu)_L \in \mathrm{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \mathrm{Re}\mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\varphi \in \mathrm{Re}\mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \mathrm{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi_L \in \mathrm{Im}\mathbb{A}_L$	$\varphi \in \mathrm{Im}\mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\varphi \in \mathrm{Im}\mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\chi \in \mathrm{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\lambda_L \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \mathrm{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \mathrm{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

Figure 10: The fields of supergravity as tensor products of super Yang-Mills fields.

The supersymmetry of the resulting supergravity $\mathcal{N}$ is equal to the sum of supersymmetries of the respective super Yang-Mills theories $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$.

The gravitino is grouped such that $\Psi_\mu = \begin{pmatrix} \mathbb{A}_L \otimes \mathrm{Re}A_R \\ \mathrm{Re}\mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix} \cong \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}$ and $g_{\mu\nu} \in \mathbb{R}$. The spinors and scalars are grouped so that $\varphi, \chi = \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}$.

The three-dimensional theory has a trivial space-time symmetry and only internal symmetry. The former space-time symmetry manifests itself as internal symmetry as a result of compactification. As a result of toroidal compactification, hidden symmetries of the theory known as U-dualities appear.

It may be seen as a generalisation of electromagnetic duality. U-dualities also appear in string theory where they are combinations of T-duality and S-duality mixing radii and couplings. [27]

In contrast to dimensional reduction, oxidation goes the other way and attempts to identify the higher-dimensional theory from which a lower-dimensional theory originates. The highest-dimensional theory that a low-dimensional one comes from is called the oxidation endpoint. The U-duality groups depend on this oxidation endpoint. For theories whose oxidation point is the $D = 11$ supergravity, the exceptional groups E_6, E_7 and E_8 arise as U-duality groups in dimensions $D = 5, 4, 3$, respectively. [17] The U-duality groups of theories in $D = 3,\ 4,\ 5$ whose oxidation points are $D = 11,\ 10,\ 9$ supergravity form a subsquare of the magic square! However, one slot of the magic square containing the F_4 does not appear in this scheme of dimensional reduction.

A *whole* magic square of theories of supergravity has been found using the approach outlined in this dissertation. [2] The rows and columns of this magic square, which are associated with the four division algebras, are identified with left and right super Yang-Mills theories where each cell in the magic square, a symmetry, corresponds to the U-duality group of the supergravity theory that arises as a tensor product of the respective super Yang-Mills theories (division algebras).

A magic square construction that treats the left and right division algebras $\mathbb{A}_L$ and $\mathbb{A}_R$ symmetrically has been crucial to this identification. As has been described, the triality construction is a construction of the magic square that is manifestly symmetric in its two parameters. Furthermore, the triality algebras on their own have been associated with full symmetries of super

Yang-Mills theories. Using the triality construction, but modified in some of the commutators using the Weyl unitary trick to get the right number of noncompact and compact generators, $\mathcal{L}_3 = \text{Tri}(\mathbb{A}_L) \oplus \text{Tri}(\mathbb{A}_R) \dot{+} 3(\mathbb{A}_L \otimes \mathbb{A}_R)$ gives the U-duality groups shown in Figure 11.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N}=2$ $f=4$ $G=SL(2,\mathbb{R})$ $H=SO(2)$	$\mathcal{N}=3$ $f=8$ $G=SU(2,1)$ $H=SU(2)\times SO(2)$	$\mathcal{N}=5$ $f=16$ $G=USp(4,2)$ $H=USp(4)\times USp(2)$	$\mathcal{N}=9$ $f=32$ $G=F_{4(-20)}$ $H=SO(9)$
$\mathbb{C}$	$\mathcal{N}=3$ $f=8$ $G=SU(2,1)$ $H=SU(2)\times SO(2)$	$\mathcal{N}=4$ $f=16$ $G=SU(2,1)^2$ $H=SU(2)^2\times SO(2)^2$	$\mathcal{N}=6$ $f=32$ $G=SU(4,2)$ $H=SU(4)\times SU(2)\times SO(2)$	$\mathcal{N}=10$ $f=64$ $G=E_{6(-14)}$ $H=SO(2)\times SO(10)$
$\mathbb{H}$	$\mathcal{N}=5$ $f=16$ $G=USp(4,2)$ $H=USp(4)\times USp(2)$	$\mathcal{N}=6$ $f=32$ $G=SU(4,2)$ $H=SU(4)\times SU(2)\times SO(2)$	$\mathcal{N}=8$ $f=64$ $G=SO(8,4)$ $H=SO(8)\times SO(4)$	$\mathcal{N}=12$ $f=128$ $G=E_{7(-5)}$ $H=SO(12)\times SO(3)$
$\mathbb{O}$	$\mathcal{N}=9$ $f=32$ $G=F_{4(-20)}$ $H=SO(9)$	$\mathcal{N}=10$ $f=64$ $G=E_{6(-14)}$ $H=SO(2)\times SO(10)$	$\mathcal{N}=12$ $f=128$ $G=E_{7(-5)}$ $H=SO(12)\times SO(3)$	$\mathcal{N}=16$ $f=256$ $G=E_{8(8)}$ $H=SO(16)$

Figure 11: Supergravity theories with extended supersymmetry $\mathcal{N}$, f degrees of freedom, U-duality groups G and their maximal compact subgroups H.

The U-duality group G is a global symmetry of the theory and H which is the maximal compact subgroup of G is a local symmetry. The scalars of the theory parameterise cosets G/H. In the special case of $\mathbb{R}\otimes\mathbb{O}$, the scalars are points in a genuine projective plane, the aforementioned Moufang plane $\mathbb{OP}^2$. Save the scalars, the fields transform as representations of H.

The supergravity theories with $\mathcal{N} > 8$ are pure. The supergravity theories with $\mathcal{N} \leq 8$ are coupled to matter such that the squaring to get $\mathcal{N} = 2, 3, 4, 5, 6, 8$ supergravity theories gives supergravity coupled to an additional $k = 1, 1, 2, 1, 2, 4$ matter multiplets, respectively.

This construction is not directly applicable to supergravity theories in $D = 4$, $D = 6$ and $D = 10$ theories. The method works in $D = 3$ because the space-time symmetry is trivial and therefore, effectively, the triality algebras only depend on one division algebra, which is what the triality construction of the magic square requires. A generalisation of the triality algebra $\widetilde{\text{Tri}}(\mathbb{A}_{n\mathcal{N}}, \mathbb{A}_n)$ has been introduced in a previous section to account for the full symmetries of super Yang-Mills theories, but a construction that yields the U-duality groups of the higher-dimensional supergravity theories is still under development. Such a construction will depend on three division algebras. One division algebra will account for the space-time symmetry, one division algebra for the left super Yang-Mills theory and one for the right super Yang-Mills theory. In $D = 4$, the division algebra associated with the supersymmetry of the super Yang-Mills theory is at most the quaternion algebra so the square at this level is a 3×3 square. In $D = 6$, the division algebra is at most the complex algebra and so the supergravity theories that can be made form a 2×2 square. In $D = 10$, the division algebra associated with the supersymmetry is real (trivial) and there is only one theory. This pyramidal collection of supergravity theories parameterised by the three division algebras, has been dubbed a *magic pyramid*.

11 Summary

The division algebras were introduced. They were then constructed in a unified manner in accordance with the Cayley-Dickson procedure. The properties of the quaternions were discussed. The properties of the octonions were then discussed. Split algebras were constructed and listed using a modified Cayley-Dickson procedure. Jordan algebras were briefly introduced and classified. A description of root lattices of interesting Lie algebras were then described using integral division algebra elements. Considering only root lattices, an informal hint of a magic square was noted. Relevant subalgebras of algebras and their maps were then described. These subalgebras were then used to construct the magic square. The Tits-Freudenthal, Vinberg and triality constructions of the magic square were then presented. Finally, recent work of the department was described. After introducing space-time transformations described using division algebra elements symmetries, dimensional reductions and truncations of supersymmetry of super Yang-Mills theories were described using division algebras. Supergravity theories were then constructed as tensor products of super Yang-Mills theories. A magic square of supergravity theories was found. It was discussed how this can be extended to a magic pyramid of supergravity theories.

12 Outlook

A generalisation of the magic square, the magic pyramid, is not yet complete since a generalisation of the magic square triality construction is needed, possibly involving the new triality algebra definition. Lagrangians of supergravity theories described as elements of division algebras and the Lagrangians of dimensionally reduced theories have not yet been developed. These are under development.

It was noted that the exceptional Jordan algebra has a relation to $SO(9)$ (and possibly $SO(26)$ through F_4) which is the stability group of quantum states in octonionic quantum mechanics. The little group of an 11-dimensional theory is also $SO(9)$ and these Poincare transformations leave a quantum state with a particular momentum invariant. Interestingly, the Standard Model was found to also arise as the stability group after a joint F_4 and a unidrectional $SO(9)$ transformation. Although no major results have emerged from these considerations, physically relevant groups do appear. These could be further hints of something significant yet to be discovered in these octonionic structures.

Division algebras failed to explain, first, isospin and, later, quark structure. Maybe they describe an even deeper layer, and maybe that deeper layer is string theory or M-theory.

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