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Cyclic Universe: Cosmic Evolution and Perturbations Analysis

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Contents

Introduction

Why an Alternative to Inflationary Models

The aim of this dissertation is to describe an alternative model to the standard Big Bang picture for what concerns the history of the universe with special emphasis towards its early stages. From where does this idea derive? The first concept that has captivated me and captured my interest is the notion of Big Bang in the inflationary scenario. Indeed its definition says that it represents a physical singularity of space-time in which the energy density and temperature diverge, are infinite. Moreover it is assumed to be the moment in which time has begun to flow, $t = 0$. The first question that raises in the human mind usually after facing such a definition is: "..and what about before?" Well, the cyclic model of the universe addresses this problem in its own way, since as the name suggests, there is not such a moment in which the "universal clock" starts running for the first time, but just a succession of cycles with the starting point coinciding with the end of the previous one. Philosophically speaking, it furnish a more acceptable view for what concerns the time with respect to the inflationary scenario since, to be honest the problem of an initial set remains, but in the cyclic case is due more to the fact that the human mind finds difficult to accept the idea of infinity and subconsciously needs to define a beginning and an end to everything.

More important, physically speaking, this alternative model includes the Big Bang event, redefining it and describing it as the transition phase between two cycles, with the fundamental difference from inflation that in this picture the temperature and the energy density remains finite and do not diverge.

Another important aspect to keep in count is the fact that, even if inflation has proved thanks to observational evidences in the recent past to be a model with a truly strong predictive power, it requires a precise fine tuning of certain cosmological parameters in order to work. There is not a physical reason for this parameters to have such values. Henceforth, the scientific method runs faster and it turns out to be usually much more productive if there are two competing models that try to explain the same physical situation. In addition to this, the presence of an alternative explanation to the same experimental results allows to understand if the owned data are sufficient in order to declare the plausibility of a theory. As a matter of fact, the cyclic model, supposing at least the same level of fine tuning as inflation, satisfies the same observational constraints as the standard picture. Therefore as it will be showed and discussed later, observational data available nowadays are not enough to rule out one of the two models, or at least they leave the discussion still open.

Furthermore, the discovery of Dark Energy and its effects on the evolution of the universe has shaken a little the inflationary picture. In particular, in the standard scenario the cosmic history appeared to be set once the properties of the scalar field causing the initial accelerated expansion where defined, together with its decays product (radiation and matter). Hence the universe should have just undergone a series of different phases of decelerated expansion after inflation, characterized by the component dominating each period. Instead with the apparition of a phase dominated by Dark Energy, since it can be thought as an element with "negative pressure", has revealed that at the present day the universe is in a period of accelerated expansion. Therefore in order to accommodate the observations with the standard scenario, it is necessary to find a physical explanation linking the predicted part of the history of the universe with the Dark Energy dominated phase or the all picture will seem as an assembly of different components.

On the contrary the cyclic model of the universe predicts the existence of this period of accelerated expansion, assigning it also a precise role and a function in order to allow to the model to work. However it must be noticed that the cyclic picture has been developed after the discovery of Dark Energy and hence it is natural and necessary that the model furnishes a coherent explanation for it. Nevertheless it surely represents an interesting solution to an unsolved problem of the inflationary scenario as we will see later.

As a matter of fact, the fascinating feature of the cyclic theory, together with the fact already mentioned that it allows to go infinitely backwards in time, is that it tries to explain and include all the history of the universe in a single frame where the transition from one part of it to another is justified by physical reasons. In particular, effectively speaking, the connections between phases and the evolution of the cycle is determined by the behaviour of one scalar field during time and its potential.

In the second chapter of the dissertation it will be presented a quick summary of the standard inflationary picture. Firstly we will introduce the so called "cosmological puzzle", a set of problems with regards to the early history of the universe and how inflation solves them. Later it will follow a qualitative description of the evolution of the universe according to the standard model.

In the same fashion but entering more deeply into mathematical details, the third chapter introduces the concept of cyclic universe, describing in the first section the ideas that lead to its formulation and that lie behind its effective form. After we will derive the equations that rule the evolution of the universe in the cyclic picture. Consequently as did for inflation, it will described each phases of the cycle, with particular attention to the one responsible for the solution of the cosmological puzzle in this model. We will see that this phase describes a period in which the universe undergoes a contraction, caused by a particular behaviour of the scalar field potential $V(\varphi)$. Moreover in this chapter it will be described the particular role of the Dark Energy and its interpretation in the cyclic universe.

The fourth chapter carries on the parallel between inflation and cyclic theory focusing the attention on the evolution in the two models during the different phases of the main cosmological parameters with respect to the time, like the scale factor a, the scalar field $\varphi(t)$, the Hubble parameter $H(t)$ and the number of e-folds N. At the end the results will be summarized in two separate tables.

Last but not the least, we finally face in the fifth chapter the analysis of perturbations in both models induced by a single scalar field. This is done in order to underline the fact that, as we will see, at the linear order the cyclic scenario reproduces the same results about the spectrum as the ones obtained by a computation in the inflationary frame. Indeed, also the alternative model retrieves a scale invariant spectrum even if there are some important subtleties with the single scalar field formulation which require some kind of expedient like the introduction of a second scalar field in order to explicit the duality between the two models. However this argument will be faced and discussed qualitatively also in the conclusions.

In the last section of the perturbations chapter we introduce a short analysis of the tensor modes perturbations, which turn out to be really important since the two models give different predictions about their spectrum. Indeed for what concerns inflation the spectrum of the gravitational waves produced during the accelerating stage just after the Big Bang is scale invariant, while on the contrary in the cyclic model the spectrum of the same produced during the contracting phase results to be blue-shifted.

In the conclusive chapter we will analyse the results obtained, particularly focusing the attention on the problems and the proposed solutions in the cyclic background, in order to recover the successful predictions of the inflationary picture.

Inflationary Model Recap

2.1 Flatness and Horizon Problems

Before starting the description of the cyclic model of the universe it is useful to review the standard inflationary scenario in order to allow later quick comparisons between the two models. The main reason for which it has been introduced the idea of inflation in cosmology is that it represents an effective solution to the cosmological puzzle composed by the flatness and homogeneity-isotropy problems.

The flatness problem is related to the fact that if today the universe that we observe is close to be flat, the critical energy density observed is $\Omega_0 \sim 1$, then it must have been much closer to be flat in the past. This because writing the Hubble equation in terms of the critical energy density at any time and manipulating it in order to emphasize the relation between spatial curvature and Ω as

$$\begin{aligned} H^2 &= H^2\Omega - \frac{K}{a^2} \\ \Omega_K \equiv \Omega - 1 &= \frac{K}{a^2H^2} \end{aligned} \tag{2.1.1}$$

where the critical energy density contains contributions from matter or radiation, it is possible to see that if $\Omega = 1$ then also the right hand side of the equation will be zero. Therefore the situation will be stationary but if on the contrary $\Omega_K \neq 0$ then the value of Ω will evolve with time. Indeed, considering for example an universe filled with matter or radiation it's easy to find from the standard Friedmann equations that:

$$\begin{aligned} & a_m \sim t^{2/3} \qquad \text{and} \qquad a_\gamma \sim t^{1/2} \\ \Longrightarrow \qquad & \Omega_K \equiv |\Omega - 1| \sim t^{2/3} \quad \text{or} \quad \sim t \end{aligned} \tag{2.1.2}$$

Just to give an idea of the orders of magnitude we are talking about, if we assume that the time of which experimental observations are made is approximately $t \approx 10^{17}$ s, the time at which it has occurred the matter-radiation equivalence is $t_{eq} \approx 10^{11}$ s and that the Planck time at which the initial conditions were set is $t_{pl} \approx 10^{-43}$ s, we obtain an infinitesimal upper bound for Ω_K:

$$\Omega_K(t_{pl}) < 10^{-61} \qquad \text{if} \qquad \Omega_K(t_0) < 0.02\ . \tag{2.1.3}$$

The homogeneity and isotropy problem goes also under the name of horizon problem because it is closely related with the concept of the comoving particle horizon. Indeed, knowing that at recombination the particle horizon was roughly $\eta_* \sim 200$ Mpc and that the today horizon is $\eta_0 \sim 14$ Gpc, it is clear that the surface of last scattering that is observed nowadays spans lots of regions that were casually disconnected at recombination. Nevertheless the CMB spectrum is homogeneous and isotropic to one part in 10^4 even though no casual physical process could have made regions so homogeneous. It can be estimated that the universe seen today is composed of 10^{78} regions that were casually disconnected at the Planck time and yet the distribution of matter was very smooth over this whole region, in other words, assuming that gravity was always attractive and hence was decelerating the expansion, the homogeneity scale was always larger than the causality scale. Moreover it is assumed, with the support of General Relativity, that inhomogeneity cannot be dissolved by expansion.

2.2 The Solution represented by Inflation

An important parameter to study in order to find an explanation to this problem is the *comoving Hubble radius* defined by $1/aH$ and related to the particle horizon by the relation

$$\eta = \int_0^{a_0} \frac{da}{a} \frac{1}{aH} = \int_0^{a_0} d\,\mathrm{ln} a \frac{1}{aH} \tag{2.2.1}$$

that, while the particle horizon tells us whether two points were ever in casual contact, it tell us whether two points are in casual contact at a time/scale factor a.

Since the modes that are experimentally observed today are super-horizon modes entering the horizon, the stratagem that can help us in recovering the causality relation between the data collected is to realize that if the comoving Hubble radius was larger during the setting of the initial conditions and then decreased in size, even if now the comoving Hubble radius is much smaller than in the past, the comoving particle horizon can be still very large:

$$\eta_0 >> \frac{1}{a_0 H_0} \tag{2.2.2}$$

with most of its contribution coming from early times. Then, if the comoving Hubble radius decreases while the comoving particle horizon keeps increasing linearly, regions that at the beginning were causally connected become casually disconnected. Physically this means that wavelengths that were sub-horizon turn into super-horizon modes.

The only way in which it is possible to have the comoving Hubble radius decreasing with time is to have $\dot{a}$ increasing with time and then $\ddot{a} > 0$. From the acceleration Friedmann equation we can understand which features must own the kind of matter dominating this phase of cosmic evolution:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad \Longrightarrow \qquad p < -\frac{\rho}{3} \quad \text{i.e.} \quad w < -\frac{1}{3} \tag{2.2.3}$$

Therefore this kind of matter must violate the strong energy condition $\rho + 3p > 0$, for example a cosmological constant that implies a de Sitter universe, which expands exponentially quickly. However the exact de Sitter solution would spoil the results obtained by the standard decelerating cosmology. Consequently the *inflation*, this stage of accelerated expansion, must last enough to solve the cosmological puzzle but also end enough quickly with a smooth transition to the decelerating phase as it is shown in the graphic in the next page.

Accordingly to this fact, the question that immediately arises now is how long does inflation last? In cosmology instead of using time in order to express the length of a phase of the universe is useful to introduce the concept of e-fold, which corresponds to the interval of proper time necessary for a patch of space to increase its dimension of a factor of e. Doing the approximation that the main contribution to $1/aH$ is given by the radiation period and hence assuming that from the end of inflation until now the universe has been radiation dominated, let's compare the comoving Hubble radius at the two extremes of the interval:

$$\text{since } H \sim \frac{1}{a^2} \text{ and } \frac{a_0 H_0}{a_e H_e} = \frac{a_e}{a_0} \Longrightarrow \frac{1}{a_e H_e} = \frac{a_e}{a_0}\frac{1}{a_0 H_0} = \frac{T_0}{T_{inf}}\frac{1}{a_0 H_0} \tag{2.2.4}$$

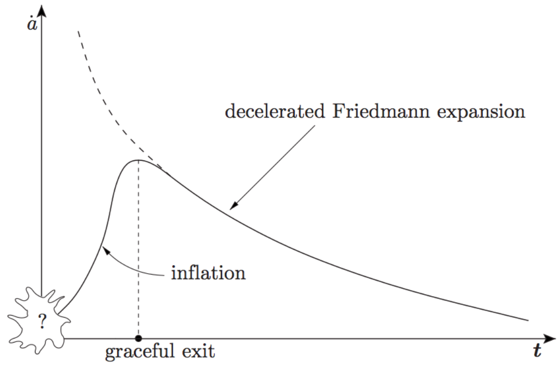

Figure 2.1: $\dot{a}(t)$ with graceful exit from inflationary phase [8]

where a_e is the scale factor at the end of inflation and given that in the radiation era $a \sim 1/T$. T_{inf} is the energy scale associated to the temperature of the universe at the start of the Big Bang picture. With the proper values for the temperatures we obtain the relation

$$\frac{1}{a_e H_e} \approx 10^{-28} \frac{1}{a_0 H_0} \tag{2.2.5}$$

which shows that at the end of inflation the visible universe was 28 orders of magnitude smaller than nowadays.

The minimal requirement to solve the horizon problem is that all the largest scales observed today were sub-horizon at the start of inflation. For this reason the comoving Hubble radius has to decrease by at least 28 orders of magnitude during the inflation. Assuming that, since the Hubble parameter is constant in the de Sitter approximation, H is constant, a must then grow by 28 orders of magnitude during inflation. In the de Sitter universe the scale factor depends exponentially on the time t:

$$a(t_s) = a_e e^{H(t_s - t_e)} \quad t_s < t_e \tag{2.2.6}$$

then the number of e-foldings of the scale factor during the acceleration phase is: $N_i \simeq \ln(10^{28}) \sim 64$.

2.3 Cosmic Evolution

The standard way in order to obtain this superluminal phase of expansion after the Big Bang, interpreted as the beginning of space and time with infinite temperature and energy density, is to introduce a scalar field called the *inflaton* with a proper scalar potential $V(\varphi)$. This potential must satisfy the condition to be approximately flat (slow rolling of the field) for an interval of values of φ in which the kinetic energy is negligible with respect to the potential. As we will see better later for the cyclic universe, this situation implies that since for a scalar field we have:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \qquad \text{and} \qquad p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \tag{2.3.1}$$

the equation of state gives $w \sim -1$ (negative pressure) which makes the inflaton act like a cosmological constant on the "plateu" of the potential, in other words accelerates the expansion.

The height of the plateu determines the rapidity of the acceleration, and it is chosen in such a way that the universe doubles its size every 10^{-35} s.

The inflaton comes since on the plateu, it behaves like a cosmological constant with equation of state $w = -1$. Using the continuity equation derived from the Bianchi Identities:

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}(1+w)\rho \tag{2.3.2}$$

we see that every energy term scales as:

$$\implies \quad \rho(a) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{2.3.3}$$

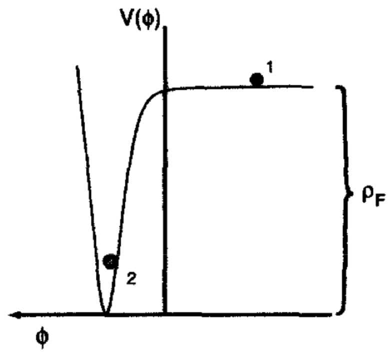

Figure 2.2: $V(\varphi)$ [12]

which justifies the fact that just after the Big Bang the first Friedmann equation describing the emerging universes has the form:

$$H^2 = \frac{1}{3}\left[\frac{\rho_m}{a^3} + \frac{\rho_\gamma}{a^4} + \frac{\sigma^2}{a^6} + ...\rho_I\right] - \frac{k}{a^2} \tag{2.3.4}$$

where ρ_m and ρ_γ are the matter and radiation energy densities at the beginning of inflation ($a = 1$); σ^2 measures the anisotropy and ρ_I is the energy density associated with the inflaton. The dots refer to other possible contributes such as the energy associated with inhomogeneous spatially varying fields and k is the spatial curvature. Immediately we notice that the inflaton's contribute is approximately a-independent. This implies that whilst all other contributes shrink as a increases, ρ_I remains almost constant (φ is on the plateu and therefore $w \approx -1$ fo the inflaton). Therefore this explains why after the Big Bang the universe evolution is determined by the inflaton and depends on its decay products since all others contributions, including the anisotropy and the spatial curvature, are tending to zero as the the universe expands. Hence a complex multivariable initial state which needed a precise fine tuning in order to solve the cosmological puzzle, after a sufficient interval of time of accelerated expansion, is converted into a much more simpler state of the space-time, on which, applying the conditions of being homogeneous, isotropic and spatially flat, is easier and more natural.

The graceful exit previously mentioned is gained through a smooth falling of the inflaton into a potential well at the end of the "plateu". As φ reaches the bottom, it starts oscillating around the minimum and hence decaying into ordinary matter and radiation.

The next phase is then the radiation dominated epoch in which the universe starts to undergo a decelerated expansion, called in this way since the cosmic evolution of the universe is determined by ρ_γ. Indeed from eq. (2.3.3) together with the fact that for the radiation we have $w = -1/3$ it is clear that at the beginning it is the radiation component the one dominating since it scales as $\rho \propto a^{-4}$, while the matter component having equation of state $w = 0$ (pressureless) goes as $\rho \propto a^{-3}$.

The radiation-matter equality occurs when $\rho_\gamma = \rho_m$ when the universe was approximately 50000 years old and represents the starting point of the matter dominated era, which is still a decelerated expanding phase.

Finally, the last turning point, occurring approximately when the universe was 9.65 billions years old, is the matter-Dark Energy equality ($\rho_m = \rho_{de}$) since when the cosmological evolution of the universe has been dominated by this cosmological constant ($w = -1 \rightarrow \rho_{de} = \text{const}$). This phase is different from the other two since it is an accelerating expanding phase of the universe. Moreover from an inflationary cosmology point of view the Dark Energy represents an unpredicted surprise since in the standard model it has no know specific role. Indeed, until its discovery from observational evidences like large scale structures and distant supernovae, it was not supposed to exist, with the belief that the universe would have still continued to expand decelerating. The only way in which it is possible to reconcile the existence of the Dark Energy with the standard inflationary model is to postulate that the inflaton didn't decay just in a combination of matter and radiation but in a mix of matter, radiation and Dark Energy. This requires a fine tuning of the respective proportions for each component in order to save the theory. Instead as we will see later, in the cyclic model the D.E. dominated phase has a specific role in the cosmic evolution. Even it must be noted that the elaboration of the cyclic theory was started after the discovery of D.E. and the same level of fine tuning is necessary, especially in the potential of the equivalent of the inflaton, to make the theory work.

The future in the inflationary model relies on the nature of D.E.:

- if D.E. is a cosmological constant then the accelerated expansion will go on forever and the universe will become colder and colder with the total energy density falling towards zero;
- if D.E. is some sort of quintessence there are different alternatives, indeed it may decay into something else and then put an end to the accelerated expansion.

Cyclic Model of the Universe

3.1 Main Ideas of a Cyclic Universe

The cyclic theory of the universe starts to differ from the standard inflationary models since from the basic concepts, for example, giving a radically new interpretation of the idea of Big Bang. Indeed, while in the inflationary models the Big Bang is seen as the singularity representing the beginning of space and time, in the cyclic model it is nothing more than a transition phase between the end of a cycle of the universe and the starting point of a new one.

Another obvious point is that as the name suggests, the evolution of the universe is cyclic and not "linear" as in the inflationary case.

The solutions to the problems of the homogeneity, isotropy and flatness of the universe and the generation of a nearly scale-invariant spectrum of density fluctuations are furnished in the cyclic model by the idea of a slow contraction phase preceding the Big Crunch, called *ekpyrotic phase*, in which as it will be explained in details later, the equation of state gives $w >> 1$ (where $w = p/\rho$ is the ratio of pressure to energy density). This represents one of the main differences respect the inflationary models because while in the latter what we observe today is a consequence of something happened in a period of accelerated expansion ($\ddot{a} > 0$) just after the Big Bang, in the cyclic universe the experimental observations performed nowadays are the result of what happened in a slow contracting phase before the end of the previous cycle of the universe.

A very useful aspect is the geometrical visualization and dynamical explanation for a cyclic model that can be found in string theory (Horava-Witten and heterotic M-theory) where it is presented the idea of two 3-dimensional colliding *branes* (orbifold planes) separated by a "small" gap in an hidden extra-dimension. All the process can be described for what concerns cosmological aspects by an effective 5d *bulk* space-time picture (3-dimensional branes, extra-dimension and time) even if the M-theory is 10-dimensional, since the other six hidden dimensions do not play an active role. The dynamical idea of the cyclic universe consists then in the periodically slightly inelastic collision of the two branes along the extra spatial dimension which produces each time new matter and new radiation. Our world lies on one of the two branes and the only interaction allowed with the particles present on the other brane is the gravitational one; strong and electroweak interactions are forbidden. In other words, all the matter and forces, except for gravity that can propagate on the whole spacetime, are localized on the 3-dimensional branes.

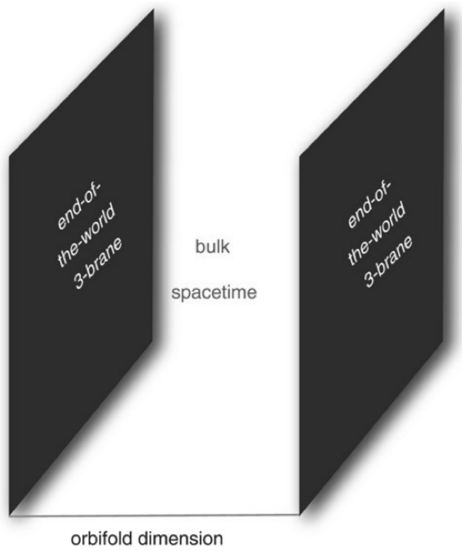

Figure 3.1: the brane-world picture of the universe [7]

Why do the branes collide? The cyclic model postulates the existence of an attractive force between them that in the literature is due to a potential $V(\varphi)$ where φ is a scalar field describing the distance of the two branes along the extra dimension. This potential tends to be flat when the branes are far apart from each other, corresponding to a very weak attraction force. However, as the distance along the extra dimension decreases, the potential as it will be better seen later, decreases almost exponentially, reaching a negative minimum value, in such a way that the attraction force becomes stronger and stronger until it is reached the Big Crunch corresponding to $\varphi \to -\infty$. This also shows that, as said before, in the cyclic model the Big Bang represents a physical event and not a mysterious moment of creation. Indeed, since the collision only occurs in the extra dimension between the branes the singularity in this case means only the momentarily disappearance of that dimension while the branes exist before, during and after the collision. This inter-brane potential energy density is identified in the cyclic model with the Dark Energy, that as observed nowadays represents approximately the 70% of the critical energy density and plays in this model an essential role in restoring the universe to a nearly vacuous state. This aspect is fundamental in order for the cyclic solution to become an attractor that means that the cycling is stable.

Last interesting point in this brief review of the string theory explanation of the cyclic universe is about density perturbations production. Indeed, due to the presence of quantum fluctuations, when the branes are close to collide they are not perfectly flat and smooth everywhere, but instead they are characterised by ripples and wrinkles. This implies that they do not collide everywhere exactly at the same time, but instead there will be zones colliding a little bit earlier and so they will have more time to expand and cool down. On the other side there will be regions colliding slightly later and this means they will be to some extent hotter than the average. Therefore this provide a natural explanation within the model for how temperature fluctuations are produced.

Finally, the introduction of the potential allows to switch to a 4-dimensional effective theory in which all the computations and results concerning the target of this dissertation are much easier to obtain. If it is assumed that the background universe is spatially flat then the resulting metric is

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{3.1.1}$$

where $a(t)^2$ is the scale factor.

3.2 Cosmic Evolution in a Cyclic Universe

The main character of the cosmic evolution in a cyclic model of the universe is, as it was anticipated earlier, the scalar field φ called in the literature as *radion* (the respective one of the *inflaton* in inflationary cosmology) and its potential $V(\varphi)$. It is important to notice that whilst the inflaton governs only the beginning of the cosmic evolution in the standard theory, here the scalar field is the engine that drives the whole scenario. Assuming the radion to be homogeneous, in other words it does not depend on the spatial coordinates $\varphi(t,\vec{x}) \to \varphi(t)$, its Lagrangian density is (adopting from now on units in which $8\pi G = 1$):

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \tag{3.2.1}$$

From the definition of the energy-momentum tensor and from the action for the radion:

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad S = -\int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right] \tag{3.2.2}$$

and varying the action with respect to the inverse metric:

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[\partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + V(\varphi) \right) \right] \delta g^{\mu\nu} \tag{3.2.3}$$

we obtain substituting in (3.2.2) the form of the energy momentum tensor for the radion:

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} g^{\sigma\rho} \partial_\sigma \varphi \partial_\rho \varphi + V(\varphi) \right). \tag{3.2.4}$$

If we consider the universe filled by a perfect fluid we know that the energy-momentum tensor assumes the form $T_{\mu\nu} = \mathrm{diag}\,(-\rho, p, p, p)$ and since the spatial derivatives disappear cause of the homogeneity assumption, the energy density ρ and the pressure p are easily obtained as a function of φ:

$$\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \qquad p_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi) \tag{3.2.5}$$

Then it is now possible to write down the equation of state for the radion field:

$$w = \frac{\frac{1}{2} \dot{\varphi}^2 - V(\varphi)}{\frac{1}{2} \dot{\varphi}^2 + V(\varphi)} \tag{3.2.6}$$

from which it immediate to see that all the cosmological evolution in the cyclic model depends on the form of $V(\varphi)$. One of the easiest form for the potential that can be chosen in order to make the model work and that is motivated my string theory is:

$$V(\varphi) = V_0 \left(1 - e^{-c\varphi} \right) F(\varphi) \tag{3.2.7}$$

where V_0 is the value of today's Dark Energy density, c is positive and typically $c >> 1$. The precise form of $F(\varphi)$ has not cosmological interest as long as it cuts off the steep exponential fall-off of the potential as φ goes from 0 to $-\infty$. As we can see in the next image, in this case the steep decline cuts off near a negative minimum $V_{end} = V(\varphi_{end})$. Indeed, as it will be show few steps later, the value of the potential after having passed V_{end} is no more relevant since in the equation of state will be dominated by the kinetic term. In order to start the description of the cosmic evolution let's consider the full action describing also gravity, radiation and matter together with the scalar field φ:

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2}\mathcal{R} - \frac{1}{2}(\partial\varphi)^2 - V(\varphi) - \xi^4(\varphi)(\rho_m + \rho_\gamma) \right) \tag{3.2.8}$$

where $\mathcal{R}$ is the Ricci scalar, $\xi(\varphi)$ is the coupling constant between φ and the matter ρ_m and radiation ρ_γ energy densities. The reason for the existence of this coupling will be clear later. Then we can now obtain the equation of motion varying the action with respect to the inverse of the metric $\delta g^{\mu\nu}$ so that $\delta S\left(\delta g^{\mu\nu}\right) = 0$ and:

$$\delta S = \frac{1}{2}\int d^4x \sqrt{-g} \left[\mathcal{R}_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\mathcal{R} + \frac{1}{2}\dot{\varphi}^2 - V - \xi^4(\rho_m + \rho_\gamma) \right) \right] \delta g^{\mu\nu}$$

$$\Longrightarrow \quad \mathcal{R}_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\mathcal{R} + \frac{1}{2}\dot{\varphi}^2 - V - \xi^4(\rho_m + \rho_\gamma) \right) = 0. \tag{3.2.9}$$

Computing for example the time-time component of the equation and knowing that for a spatially flat FRW metric we have

$$\mathcal{R} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] \qquad \mathcal{R}_{00} = -3\frac{\ddot{a}}{a} \qquad \mathcal{R}_{ij} = \delta_{ij}\left(\ddot{a}a + 2\dot{a}^2 \right) \tag{3.2.10}$$

it is obtained the first Friedmann equation:

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V + \xi^4\rho_\gamma + \xi^4\rho_m \right) \tag{3.2.11}$$

From the spatial component of the equation follows also the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\varphi}^2 - V + \xi^4\rho_\gamma + \frac{1}{2}\xi^4\rho_m \right) \tag{3.2.12}$$

where $a(t)$ is the scale factor and $H = \dot{a}/a$ is the Hubble parameter and the dot represents a time derivative. The fluid equation of motion can be obtained by the Bianchi's identity:

$$\hat{a}\frac{d\rho_i}{d\hat{a}} = a\frac{\partial\rho_i}{\partial a} + \frac{\xi}{\xi'}\frac{\partial\rho_i}{\partial\varphi_i} = -3(\rho_i + p_i), \qquad i = m, \gamma \quad \text{and} \quad \hat{a} = \xi a. \tag{3.2.13}$$

The equation of motion for the radion can always be obtained through the variation of the action with respect to φ paying attention to a couple of facts:

- $\rho_\gamma \sim (a\xi)^{-4} \longrightarrow \xi^4\rho_\gamma$ does not depend on φ and disappears in the variation;
- since $g_{\mu\nu} = \text{diag}\,[-1, a^2, a^2, a^2]$ then $\sqrt{-g} = a^3$

Then from the action (3.2.8) we have:

$$\delta S = \int d^4x\, a^3 \left(\dot{\varphi}\delta\dot{\varphi} - V_{,\varphi}\,\delta\varphi - 4\,\xi^3\xi_{,\varphi}\,\rho_m\delta\varphi\right) \tag{3.2.14}$$

where integrating by part the first term and remembering that since the variation at the extremes of integration is set $\delta t = 0$ then $\int d^4x\, \frac{d}{dt}\left(\delta\varphi a^3\dot{\varphi}\right) = 0$ and we have:

$$\delta S = \int d^4x \left[-\frac{d}{dt}\left(a^3\dot{\varphi}\right) - V_{,\varphi} - 4\,\xi^3\xi_{,\varphi}\,\rho_m\right]\delta\varphi \tag{3.2.15}$$

and from the immediate equality $\dfrac{d}{dt}\left(a^3\dot{\varphi}\right) = 3a^3H\dot{\varphi} + a^3\ddot{\varphi}$ we obtain

$$\ddot{\varphi} + 3H\dot{\varphi} = -V_{,\varphi} - 4\,\xi^3\xi_{,\varphi}\,\rho_m \tag{3.2.16}$$

the desired equation of motion. Finally in this graphic it is shown the evolution of the potential with respect to the radion field:

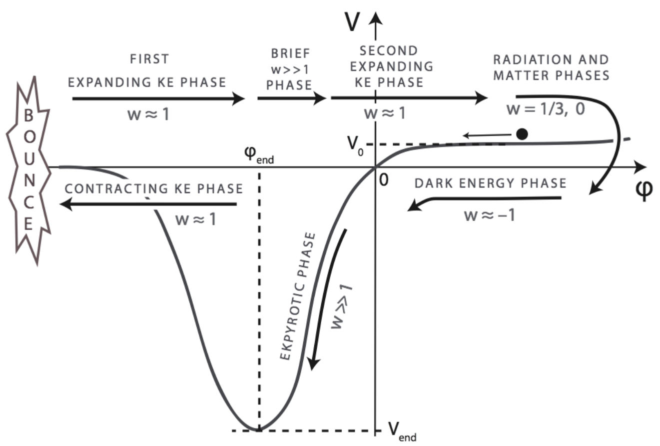

Figure 3.2: $V(\varphi)$ and the stages of one cycle of the universe [9]

3.2.1 Phases of the Cycle: Dark Energy Period

In the next chapter each single phase will be analysed in details but in order to get an idea of what is going on is useful to go through a cycle just using words to grasp the main features of each step.

A good starting point is the nowadays situation that in the figure 3.2 is represented by the filled black circle. At t_0 the universe is in an epoch dominated by Dark Energy. This in the cyclic model is identified with the radion potential that as we can see has a positive and almost constant value, what is called a *plateu*. Due to this particular situation and as consequence of the previous phase (radiation-matter) the kinetic energy term is very small compared to the potential one $V(\varphi) >> \frac{1}{2}\dot{\varphi}^2$ and then from equation (3.2.6) it is immediate to see that $w \sim -1$ and from the acceleration equation in this phase that:

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\varphi}^2 - V(\varphi)\right) \quad \longrightarrow \quad \ddot{a} > 0 \qquad (3.2.17)$$

it is a phase of cosmic acceleration. However it is called a period of *slow* acceleration since the universe needs 15 billions of years to double its size instead of the 10^{-35}s in the inflationary phase.

This period it is estimated to last for trillions of years, since its importance derives from the fact that it allows to restore flatness and homogeneity in the universe emptying it from all the matter and radiation. Actually at the end of this phase the universe has returned to a nearly vacuous state, where there is less than one particle per horizon. This solves the problem of entropy that has afflicted all the previous models of cyclic universe. Indeed in these ones it has been computed that the entropy density rose from one cycle to the next which would have been then longer than the previous, implying that going backwards in time it was impossible to avoid the existence of an initial singularity like in the inflationary models. On the contrary in our model, even if it is still true that the entropy rises from cycle to cycle, the physical entropy density (the entropy per proper volume, measurable inside the observer's horizon) is expanded away each cycle equalizing the length of the expansion and contraction history of on cycle and the next.

3.2.2 Phases of the Cycle: Ekpyrotic Contraction

Due to the very small slope, the field φ rolls down the plateu very slowly until the potential becomes zero at $\varphi = 0$, since when the evolution becomes dominate by the kinetic term which decelerates the expansion. Meanwhile the potential keeps falling below the zero and the $\frac{1}{2}\dot{\varphi}^2$ keeps increasing until the expansion stops ($H = 0$) as we see from the simplified version of the Friedmann equations:

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\varphi}^2 - V(\varphi)\right) \tag{3.2.18}$$

and the universe begins to contract ($\ddot{a} < 0$). Moreover we can see from eq. (3.2.6) that in this phase $w >> 1$. At the beginning, around $V(\varphi) \sim 0$, the contraction is very slow, going on for billions of years.

As it will be accurately described in a specific section later, this is exactly the period in which the nearly scale invariant spectrum of perturbations observed in the next cycle is produced by spatial variations caused by quantum fluctuations. Indeed the spatial variations will imply slightly different collision times for some zones of the brane as written in the end of the chapter's introduction, which turn into energy density and temperature fluctuations after the bounce. To make a comparison, while in the inflationary model the property of scale invariance is given by a having during the acceleration phase H^{-1} nearly costant and the scale factor a expanding much more quickly, in the cyclic model it is obtained by having during the ekpyrotic phase a nearly static and H^{-1} rapidly contracting.

Solutions to Homogeneity, Isotropy and Flatness Problems

It is during the Ekpyrotic Contraction phase that the cosmological puzzle is solved in the cyclic model of the universe. Let's consider again eq. (2.3.4), substituting in this case the energy density of the inflaton ρ_I with the addition of a cosmological constant Λ:

$$H^2 = \frac{1}{3}\left[\frac{\rho_m}{a^3} + \frac{\rho_\gamma}{a^4} + \frac{\sigma^2}{a^6} + ...\Lambda\right] - \frac{k}{a^2} \tag{3.2.19}$$

During a phase in which the universe is contracting, the cosmological constant soon becomes irrelevant with respect to the terms which are scaling as some negative power of the scale factor. Indeed the term that will dominate is the anisotropy one which goes as $\propto a^{-6}$. Analysing more carefully the situation one discovers that the universe does not just become more and more anisotropic, but also grows a large anisotropic spatial curvature and it enters a phase called of *chaotic mixmaster behaviour* as it gets closer to the Big Bounce. The mixmaster behaviour produces severe inhomogeneities, and the universe undergoes stretching and contracting phases in different directions which extremely vary by slightly changing the initial conditions. This represents exactly the contrary of a solution to the cosmological puzzle, an universe that is anisotropic, curved and inhomogeneous as it approaches the Big Crunch.

Whereas, the situation and its evolution drastically changes if instead of an energy density component as the cosmological constant characterized by an equation of state $w \approx -1$ we introduce one with $w >> 1$. This is what happens in the Ekpyrotic phase thanks to the particular behaviour of the radion potential. Indeed the energy term relative to the scalar field goes as $\propto a^{-3(1+w)}$ where $3(1+w) > 6$ and hence it grows faster than the anisotropy term (and all the others). Therefore, as in the inflation case, the longer the contraction phase lasts, the flatter and the more homogeneous and isotropic the universe is as it approaches the Big Bounce. In other words a contracting phase with $w >> 1$ has the same effects in homogenizing, making flat and isotropic the universe as an accelerated expanding phase with $w < -1/3$.

3.2.3 Phases of the Cycle: Big Crunch and Big Bounce

The radion reaches then the bottom of the potential well at V_{end} but given the exponential-steep form of the potential and the blueshift of the kinetic energy of φ due to the previous contraction phase, the minimum is quickly surpassed. This phase is characterized by an equation of state $w \approx 1$ having the $1/2\dot{\varphi}^2$ term that keeps increasing since the the conversion into it of the gravitational energy released by the contraction of the scale factor a. The field then speeds up towards $-\infty$ and reaches it in a finite time.

Here is the point in which the coupling factor $\xi(\varphi)$ plays its crucial role. Indeed as the $a \to 0$ the kinetic term diverges but thanks to the coupling with matter and radiation, at the Crunch and during the Bounce, the energy density and the temperature are finite even if they are very high. This since at the Bounce part of the scalar kinetic energy of the radion is converted into radiation and matter and the universe reheats.

Immediately after the Bounce there is an expanding phase still dominated by kinetic energy since for a small scale factor the the term dominating in the Friedmann equations is the one with the lowest exponent for a. In fact, from continuity equation we have that:

$$\rho_i \propto a^{-3(1+w_i)} \quad \text{and} \quad w_\varphi \approx 1 \qquad \Longrightarrow \qquad \rho_\varphi \propto a^{-6} \tag{3.2.20}$$

that shows how the radion energy density prevails on the other contributes. In the middle of this brief phase there is also a much more shorter $w >> 1$

phase which has no important influence on the cosmic evolution. As a result of the expansion of the universe and of the presence of matter and radiation, the kinetic energy of φ is redshifted and soon the radiation comes to dominate approximately 10^{-25}s after the Big Bang.

3.2.4 Phases of the Cycle: Radiation and Matter Period

The domination of ρ_γ over the other contributes causes the breaking of the time-reversal symmetry, since after the Bounce and until now the motion has been exactly the time reverse of contraction phase driven by the kinetic energy of φ. From the continuity equation we have that $\rho_\gamma \propto a^{-4}$ and then from this phase the evolution is the same as the one described in the inflationary model. In the same way indeed the radiation dominated phase ends when it reached the matter-radiation equality characterised by $\rho_\gamma = \rho_m$ after which begins the matter period.

Furthermore the cosmic evolution in both these phases is a decelerated expansion as in the standard inflationary theory. Finally, the potential of the radion field comes to dominate again, and another cycle starts again with the accelerated expansion phase.

Parameters Evolution Analysis

In this chapter we will analyse in details the evolution of the fundamental parameters describing each phase of the cosmic history of the universe. As did before, it is useful to start with a quick review of them in the inflationary model, in order to set the basis for a comparison and an operative method.

4.1 Inflation Analysis

As it has been mentioned before, in the standard model of cosmology, immediately after the Big Bang there is a period of acceleration caused by the domination of the inflaton scalar field over the other components. Therefore the Friedmann equations are the same of (3.2.18) and in order to obtain the equation of motion for φ is enough to take the time derivative of the first and substitute in the acceleration equation. Finally using the definition of pressure and energy density for a scalar field defined in eq. (3.2.5) we obtain something very similar to what computed in the case of the radion:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0 \tag{4.1.1}$$

The potential described in the introduction and represented in the figure 2.2 describes the behaviour of a slow-rolling scalar field. Indeed the inflaton is directed towards the potential well but the plateu shape of the potential entails that it does it very slowly. Translating this concept in mathematical formalism corresponds with imposing these two conditions:

- $|\ddot{\varphi}| << 3H\dot{\varphi}$, the acceleration of the scalar field is negligible compared to the friction term;

- $\dot{\varphi}^2 << V(\varphi)$, the kinetic energy is much more smaller than the potential.

Applying these two conditions to the equation of motion and to the first Friedmann equation we obtain:

$$3H\dot{\varphi} + V_{,\varphi} \simeq 0 \qquad \text{and} \qquad H^2 \simeq \frac{1}{3}V(\varphi) \tag{4.1.2}$$

which together, knowing that $H = \dfrac{d\ln a}{dt}$, allow us to find:

$$\frac{d\ln a}{dt} = \dot{\varphi}\frac{d\ln a}{d\varphi} \simeq -\frac{V_{,\varphi}}{3H}\frac{d\ln a}{d\varphi} \quad \Longrightarrow \quad V \simeq 3H^2 = -V_{,\varphi}\frac{d\ln a}{d\varphi} \tag{4.1.3}$$

that integrated gives the dependence of the scale factor a during the inflationary phase from the scalar field:

$$\int_{\varphi_i}^{\varphi} -\frac{V}{V_{,\bar{\varphi}}}\,d\bar{\varphi} = \int_{a_i}^{a} d\ln\bar{a} \quad \Longrightarrow \quad a(\varphi) \simeq a_i \exp\left(\int_{\varphi}^{\varphi_i} \frac{V}{V_{,\bar{\varphi}}}\,d\bar{\varphi}\right) \tag{4.1.4}$$

This exponential relation is valid only if both conditions mentioned before are satisfied; this depends on the specific form of the potential, hence it is useful to rewrite them using the relations (4.1.2) as constraints on $V(\varphi)$:

$$\begin{aligned} |\dot{\varphi}^2| << |V| \quad &\Longrightarrow \quad \frac{1}{|V|}\left(\frac{V_{,\varphi}}{3H}\right)^2 << 1 \quad \Longrightarrow \quad \left(\frac{V_{,\varphi}}{V}\right)^2 << 1 \\ |\ddot{\varphi}| << 3H\dot{\varphi} \quad &\Longrightarrow \quad \left|\frac{\dot{\varphi}}{3H}V_{,\varphi\varphi}\right| << 3H\dot{\varphi} \quad \Longrightarrow \quad \left|\frac{V_{,\varphi\varphi}}{V}\right| << 1 \end{aligned} \tag{4.1.5}$$

These are fulfilled for example by a power-law potential of the type $V = \frac{1}{n}\lambda\,\varphi^n$ when $|\varphi| >> 1$. In addition from (4.1.2) we can immediately obtain the time dependence of the scale factor:

$$H \simeq \sqrt{\frac{1}{3}V(\varphi)} \quad \Longrightarrow \quad H = \frac{\dot{a}}{a} = \sqrt{\frac{\lambda}{3n}}\,\varphi^{\frac{n}{2}} \quad \Longrightarrow \quad a(t) = a_0 \exp\left(H_\varphi t\right) \tag{4.1.6}$$

where $H_\varphi = \sqrt{\frac{\lambda}{3n}}\,\varphi^{\frac{n}{2}}$. Therefore as the field decreases (slow-rolls), the H-term falls down and the exponential growth ends.

The time dependence of the inflaton is easy to compute in the case $n = 2$ because in the slow-roll approximation the equation of motion would reduce to:

$$\sqrt{\frac{3\lambda}{2}}\,\dot{\varphi}\varphi + \lambda\varphi = 0 \quad \Longrightarrow \quad \dot{\varphi} = -\sqrt{\frac{2\lambda}{3}} \tag{4.1.7}$$

and then by simply integrating: $\varphi(t) = \varphi_i - \sqrt{\frac{2\lambda}{3}}\,(t - t_i)$, which shows that the value of the radion decreases linearly with the passing of time. If we define t_f as the time at which at the end of inflation the inflaton disappears (decays), the last relation can be rewritten as

$$\varphi(t) = \sqrt{\frac{2\lambda}{3}}\,(t_f - t). \tag{4.1.8}$$

Inverting the relation we see that the inflation lasts for $\Delta t \simeq t_f - t_i \simeq \sqrt{\frac{3}{2\lambda}}\,\varphi_i$ considering $\varphi_i >> \varphi_f$.

The number of e-folds during the inflation, defined by $N_{inf} = \ln{(a_f/a_i)}$ is given by eq. (4.1.4) that in the case of a power-law exponential gives:

$$N_{inf} = \int_{\varphi}^{\varphi_i} \frac{1}{n}\,\bar{\varphi}\,d\bar{\varphi} \simeq \frac{1}{2n}\,\varphi_i^2 \tag{4.1.9}$$

In the introduction it has been presented that, as a minimum requirement for inflation in order to be successful, it must have lasted at least 64 e-folds. Since non-perturbative quantum gravity effects would become relevant if the energy density would reach the Planckian value, let's consider in the case $n = 2$ the scalar field having a mass of $\sim 10^{13}$ GeV for which the maximum value of φ_i in order to remain in the sub-Planckian limit is $\varphi_i \sim 10^6$. Inserting this value in the previous equation we see immediately that inflation could last much and much more than the minimum required.

Finally, the dependence of the Hubble parameter on the time follows immediately from taking the time derivative of the expression of the scale factor in eq. (4.1.6) in the case $n = 2$:

$$H = \frac{\dot{a}}{a} = \frac{d}{dt}(H_\varphi t) = H_\varphi + t\frac{d}{dt}H_\varphi \tag{4.1.10}$$

remembering that $H_\varphi = \dfrac{\lambda}{3}(t_f - t)$ and that $t_f = \sqrt{\dfrac{3}{2\lambda}}\,\varphi_i + t$, we have:

$$H(t) = \sqrt{\frac{\lambda}{6}}\,\varphi_i - \frac{\lambda}{3}\,t \tag{4.1.11}$$

from which we see that in this approximation, while the scale factor in the inflationary phase expands exponentially with time, the Hubble parameter decreases linearly with time and hence the Hubble radius H^{-1} increases much more slower than $a(t)$.

Immediately after the end of inflation starts the radiation dominated period in which the Friedmann equations can be simplified keeping in mind that:

- $p_\gamma = \frac{1}{3}\,\rho_\gamma$;
- we are assuming a spatially flat universe $k = 0$;
- $\rho_\gamma = \rho_0\,(a/a_0)^{-4}$,

and written in conformal time as:

$$a'^2 = \frac{1}{3}\,\rho_0\,a_0^4 \qquad \text{and} \qquad a'' = 0 \tag{4.1.12}$$

hence it follows immediately that $a(\eta) = a_0\,\eta$ and that from the definition of conformal time $dt = a(\eta)d\eta$ integrating we have $t = a_0\,\eta^2/2$, which finally allows us to write the relations with t:

$$a(t) \propto t^{\frac{1}{2}} \qquad \text{and} \qquad H(t) = \frac{1}{2t} \tag{4.1.13}$$

Consequently we can esteem the number of e-folds N_γ occurred during this phase. Assuming that the end of inflation occurred at $t_i \sim 10^{-34}$ s and the radiation-matter equality had been at $t_{eq1} \sim 3.16 \times 10^{11}$ s (50000 years) it is straight forward to compute:

$$N_\gamma = \frac{1}{2}\ln\left(\frac{t_{eq1}}{t_i}\right) \qquad \Longrightarrow \qquad N_\gamma \simeq 52.4 \tag{4.1.14}$$

After the radiation-matter equality the Friedmann equation are dominated by the contribution of ρ_m and then remembering that:

- $\rho_m = \rho_0(a/a_0)^{-3}$;
- $k = 0$;
- $p_m = 0$,

they can be directly written in conformal time as:

$$a'^2 = \frac{1}{3}\,\rho_0\,a_0^3\,a \qquad \text{and} \qquad a'' = \frac{1}{6}\,\rho_0\,a_0^3 \tag{4.1.15}$$

Again it is easy to notice that in this case $a(\eta) \propto \eta^2$ and consequently $t = a_0\,\eta^3$ which allows us to write:

$$a(t) \propto t^{\frac{2}{3}} \qquad \text{and} \qquad H(t) = \frac{2}{3t} \tag{4.1.16}$$

As before we can now compute the number of e-folds N_m occurred during the matter dominated period; considering that it has lasted until
$t_{eq2} \sim 9.65 \times 10^9$ years the relation to use is:

$$N_m = \frac{2}{3} \ln \left(\frac{t_{eq2}}{t_{eq1}} \right) \qquad \Longrightarrow \qquad N_m \simeq 8.1 \tag{4.1.17}$$

Finally, the last known period of the cosmic evolution in the inflationary model is the Dark Energy dominated one. This can be approximated to a de Sitter universe characterised by:

- $\rho_{de} = \text{const}$;
- $k = 0$;
- $p_{de} = -\rho_{de}$, characteristic of a cosmological constant perfect fluid.

Therefore the Friedmann equations become (we can avoid in this case to use the conformal time):

$$H^2 = \frac{1}{3}\rho_{de} \qquad \text{and} \qquad \frac{\ddot{a}}{a} = \frac{1}{3}\rho_{de} \tag{4.1.18}$$

and hence:

$$a(t) \propto \exp(H_{de}t) \qquad \text{and} \qquad H(t) = H_{de} = \text{const} \tag{4.1.19}$$

where $H_{de} = \sqrt{\rho_{de}/3}$. Moreover the number of e-folds N_{de} up to now, considering the current age of the universe equivalent to $t_0 \sim 12.7 \times 10^9$ years, it follows straight forward:

$$N_{de} = \frac{t_0}{t_{eq2}} \qquad \Longrightarrow \qquad N_{de} \simeq 1.43 \tag{4.1.20}$$

4.2 Cyclic Universe

Let's start now the same analysis of the parameters evolution for the different phases in the cyclic model. Before facing the first phase it is useful to remember the form of the potential:

$$V(\varphi) = V_0 \left(1 - e^{-c\varphi}\right) F(\varphi) \quad \text{where} \quad c >> 1, \tag{4.2.1}$$

and the useful equations:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right), \tag{4.2.2}$$

which are the same obtained in the previous chapter but ignoring the coupling term $\xi(\varphi)$ since it becomes relevant only at the moment of the Big Crunch and can be neglected for all the rest of the time.

4.2.1 Dark Energy Period

In this phase the radion is positive ($\varphi > 0$) and the potential is approximately flat, earning the epithet of "plateu" ($V(\varphi) \simeq V_0$). As a consequence of this the kinetic energy of the scalar field is negligible with respect to the potential and therefore the Friedmann equation simplifies to:

$$V_0 >> \frac{1}{2}\dot{\varphi}^2 \quad \Longrightarrow \quad H^2 \simeq \frac{1}{3}V_0, \tag{4.2.3}$$

from which we also see that H as in the de-Sitter universe can be treated as a constant. Moreover the other eq. of (4.2.2) gives:

$$\ddot{\varphi} + 3H\dot{\varphi} \simeq 0. \tag{4.2.4}$$

Now, knowing that $\ddot{\varphi} = \dot{\varphi}\frac{d\dot{\varphi}}{d\varphi}$, the last equation becomes

$$\dot{\varphi}\frac{d\dot{\varphi}}{d\varphi} + 3H\dot{\varphi} = 0. \tag{4.2.5}$$

Since even if it is small on the plateu, the coordinate time derivative of the scalar field is always different from zero, $\dot{\varphi} \neq 0$ and hence we can dividing by it simplifying further the equation:

$$\frac{d\dot{\varphi}}{d\varphi} + 3H \simeq 0 \quad \Longrightarrow \quad \dot{\varphi} \propto 3H\varphi \tag{4.2.6}$$

which integrating gives:

$$\begin{aligned} &\Longrightarrow \quad \int \frac{1}{\bar{\varphi}}\, d\bar{\varphi} = -A3H \int d\bar{t} \\ &\Longrightarrow \quad \ln|\varphi| = -A3Ht + A_1 \qquad A, A_1 = \text{ const.} \\ &\Longrightarrow \quad \varphi(t) = A_2 \exp\left(-3Ht\right) \qquad A_2 = \text{ const.} \end{aligned} \tag{4.2.7}$$

From this result we see that during the D.E. phase the radion decreases exponentially with time going towards $\varphi = 0$. For what concerns the scale factor dependence on time we obtain immediately from the Friedmann equation the classical de-Sitter exponential result:

$$a(t) = a_0 \exp\left(\sqrt{\frac{V_0}{3}}t\right). \tag{4.2.8}$$

In both results we have to keep in mind that the coordinate time is negative and going from $-\infty$ to 0.

In order to compute the number of e-folds in this phase it is necessary to face the problem created by the definition of time in the cyclic picture. Indeed until now it has been assumed that the time is flowing from $-\infty$ towards 0 during one cycle. This does not represents a problem in the other phases but in this one yes since it has been taken for simplicity as the begging of the cycle. Therefore, in order to compute the number of e-folds we have to change the definition of the time-flow which now goes from $t = 0$ corresponding to the previous Big Bounce to some value t during the Dark Energy dominated phase. Hence the number of e-folds results the same as in the inflationary corresponding case:

$$N_{DE} = \frac{t_f}{t_i}. \tag{4.2.9}$$

4.2.2 Ekpyrotic Contraction

Let's consider now the slow contracting ekpyrotic phase in which $\varphi_{min} < \varphi < 0$. In this period the scalar field potential is dominated by the steep falling exponential term and can be approximated as:

$$V(\varphi) \simeq -V_0 \, e^{-c\varphi} \tag{4.2.10}$$

so that the eq. (4.2.2) becomes

$$\begin{aligned} \ddot{\varphi} + 3H\dot{\varphi} + cV_0 e^{-c\varphi} &= 0 \\ H^2 &= \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 - V_0 e^{-c\varphi}\right) \end{aligned} \tag{4.2.11}$$

Now assuming valid the zero-energy universe hypothesis for which the total energy density of the universe is zero, we have that:

$$0 = \frac{1}{2}\dot{\varphi}^2 + V \quad \Longrightarrow \quad \dot{\varphi} = \sqrt{-2V} \quad \Longrightarrow \quad \int d\bar{t} = \int d\bar{\varphi} \frac{1}{\sqrt{2V_0}} e^{\frac{c}{2}\bar{\varphi}} \tag{4.2.12}$$

In order to fix the extremes of integration let's notice that:

- $\bar{t}$ is going from some generic negative value t to $t_{crunch} = 0$;
- $\bar{\varphi}$ is going from some negative value φ to $\varphi_{crunch} \to -\infty$.

Therefore integrating we obtain that:

$$-t = \int_t^0 d\bar{t} = \int_{\varphi}^{-\infty} d\bar{\varphi} \frac{1}{\sqrt{2V_0}} e^{\frac{c}{2}\bar{\varphi}} = \frac{2}{c\sqrt{2V_0}} e^{\frac{c}{2}\varphi}$$

$$e^{\frac{c}{2}\varphi} = -\frac{c\sqrt{2V_0}}{2} t = -\sqrt{\frac{c^2 V_0}{2}} t = -\sqrt{\frac{V_0}{p}} t \quad \text{where} \quad p = \frac{2}{c^2}$$

$$\varphi(t) = \frac{2}{c} \ln\left(\sqrt{\frac{V_0}{p}}(-t)\right) \tag{4.2.13}$$

From the result obtained we see that the radion φ is decreasing logarithmically with time towards $\to -\infty$ which agrees perfectly with what has been stated before.

Consequently we can now compute the dependence on the coordinate time of the scale factor a. From eq. (4.2.13) we immediately obtain that:

$$\Longrightarrow \quad \dot{\varphi} = \frac{2}{c}\frac{1}{t} \quad \text{and} \quad V = -V_0 \exp\left[2\ln\left(\sqrt{\frac{V_0}{p}}(-t)\right)\right] = -\frac{p}{t^2} \tag{4.2.14}$$

which using the second Friedmann equation for a scalar field gives:

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\varphi}^2 - V\right) \quad \Longrightarrow \quad \frac{\ddot{a}}{a} = -\frac{1}{3}\left(2\frac{p}{t^2} + \frac{p}{t^2}\right) = -\frac{p}{t^2} \tag{4.2.15}$$

which integrated twice:

$$\int \frac{1}{a} d\dot{\bar{a}} = -p \int \frac{1}{\bar{t}^2}\, d\bar{t} \quad \Longrightarrow \quad \frac{\dot{a}}{a} = \frac{p}{t} + \text{const}$$
$$\int \frac{d\bar{a}}{\bar{a}} = p \int \frac{d\bar{t}}{\bar{t}} + \text{const} \quad \Longrightarrow \quad a(t) \propto |t|^p \tag{4.2.16}$$

Before describing this result let's compute also the time dependence of the Hubble parameter H:

$$H(t) = \frac{p}{|t|} \tag{4.2.17}$$

These two last equations together shows what it was meant in the beginning when it has been said that during ekpyrotic contraction, the fact that the Hubbe radius $\sim H^{-1}$ is shrinking while the scale factor remains nearly constant allows the production of a scale invariant spectrum. Indeed as we see from eq. (4.2.16), remembering that $p = 2/c$ and that $c >> 1$, given a value a_0 for the scale factor, after an interval of time Δt the new value of a will be $a_0(\Delta t)^p \simeq a_0$ since $(\Delta t)^p \simeq 1$. On the contrary, since the Hubble parameter increases as $|t|^{-1}$, $(t \to 0)$, the Hubble radius is decreasing proportionally to the coordinate time t. This is the mechanism that allows to obtain the same result of inflation at linear order in perturbations.

Finally it is remained only to compute the number of e-folds as a function of time. From the definition it is straightforward to get:

$$N_{ekp} = p \ln\left|\frac{t_f}{t_i}\right| = -p \ln\left|\frac{t_i}{t_f}\right| \tag{4.2.18}$$

where the fact that the time is flowing towards $-\infty$ implies that $|t_i| > |t_f|$. Using this, flipping denominator with the numerator of the logarithm it is possible to bring out a negative sign which explicates the fact that the universe is contracting of $|N_{ekp}|$ e-folds.

Moreover keeping in mind the fact the previous phase of the cycle was the D.E. dominated expanded phase which is estimated to last for trillions of years in order to bring the universe to a nearly vacuum state, since the ekpyrotic phase is instead supposed to be much more shorter, of the order of billions of years, the ratio between t_i and t_f must be very close to the unity. Therefore $N_{ekp} \simeq 0$ and hence we see that being very precise it almost as the universe stops expanding instead of contracting during the ekpyrotic phase. This is not in contradiction with what said before since what matters is the relationship between the growth of the scale factor and the shrinking of the Hubble radius.

4.2.3 Big Crunch and Big Bounce

After the ekpyrotic phase, since the radion has rolled down a steep exponential potential well, its kinetic energy is increased in such a way that the possibility that it remains trapped at the bottom of the well does not exist and on the contrary it passes the minimum V_{min} speeding up towards $\varphi \to -\infty$. Therefore it is a good approximation to assume that this period is dominated by kinetic energy and hence $\frac{1}{2}\dot{\varphi}^2 >> V(\varphi)$. The equations of motion become:

$$\ddot{\varphi} + 3H\dot{\varphi} \simeq 0 \qquad \text{and} \qquad H^2 \simeq \frac{1}{6}\dot{\varphi}^2 \tag{4.2.19}$$

From the second one follows that $H = 1/\sqrt{6}\,\dot{\varphi}$ and then immediately we obtain the relation between the scale factor and the radion:

$$\frac{da}{a} = \frac{1}{\sqrt{6}}d\varphi \quad \Longrightarrow \quad a \propto \exp\left(\frac{\varphi}{\sqrt{6}}\right). \tag{4.2.20}$$

In order to obtain the time-dependence of the scalar field, substituting the second equation of motion into the first and performing a double integration we get:

$$\ddot{\varphi} + \sqrt{\frac{3}{2}}\dot{\varphi}^2 = 0 \qquad \text{labelling} \qquad x = \dot{\varphi}$$

$$\dot{x} = -\sqrt{\frac{3}{2}}x^2 \quad \Longrightarrow \quad \int -\frac{d\bar{x}}{\bar{x}^2} = \sqrt{\frac{3}{2}}\int d\bar{t} \quad \Longrightarrow \quad \frac{1}{x} = \sqrt{\frac{3}{2}} + \text{ const}$$

$$\int d\bar{\varphi} = \sqrt{\frac{2}{3}}\int \frac{d\bar{t}}{\bar{t}} \quad \Longrightarrow \quad \varphi(t) = \sqrt{\frac{2}{3}}\ln|t| + \varphi_0 \tag{4.2.21}$$

Since $|t| \to 0$ this shows that radion is going logarithmically with time towards $-\infty$ which corresponds to the Big Crunch.

Immediately we can compute from the second equation of motion (4.2.19) the time dependence of the Hubble parameter and after the one of the scale factor. Knowing that:

$$\dot{\varphi} = \sqrt{\frac{2}{3}}\frac{1}{|t|} \quad \Longrightarrow \quad H = \frac{1}{3}\frac{1}{|t|} \tag{4.2.22}$$

and then immediately

$$\int \frac{d\bar{a}}{\bar{a}} = \frac{1}{3}\int \frac{d\bar{t}}{|\bar{t}|} \quad \Longrightarrow \quad a(t) \propto |t|^{\frac{1}{3}} \tag{4.2.23}$$

which confirms the fact that this is still a contracting phase of the universe.

The number of e-folds corresponding to this phase as a function of time follows straightforward from the definition:

$$N_{KE} = -\frac{1}{3}\ln\left|\frac{t_i}{t_f}\right| \tag{4.2.24}$$

where once again t_i and t_f have been swapped in order to make explicit that the universe has contracted of $|N_{KE}|$ e-folds during this phase.

The mathematical description of what happens at the Big Crunch and immediately after at the Big Bounce it's beyond the purposes of this dissertation since it requires a string theory background and the formulation of the problem no more in terms of 4D effective theory but in terms of a 5D branes theory. Therefore it can be given just a qualitative description as the one given in the previous chapter. Let's just point out that in the five dimensional picture, the Big Crunch and the Big Bounce are a different kind of singularity with respect to the Big Bang in the standard inflationary scenario. Indeed at the instant $t = 0$ it's only the fifth dimension that disappears momentarily, the dimension describing the distance between the two branes. The others, the three spatial ones relative to the surface of the brane and the temporal one, behave normally.

After the bounce, the situation of the universe is exactly the time-reversal of the one just described. The fact the the time is now flowing from zero towards some positive value implies that all the parameters that were decreasing now increase and vice versa, therefore this kinetic energy dominated phase is a period of expansion for the universe. Therefore rewriting the results just obtained, the parameters behaves like:

$$\varphi(t) = \sqrt{\frac{2}{3}}\ln t + \varphi_0, \qquad H = \frac{1}{3}\frac{1}{t}, \qquad a(t) \propto t^{\frac{1}{3}} \quad \text{and} \quad N_{KE} = \frac{1}{3}\ln\left(\frac{t_f}{t_i}\right) \tag{4.2.25}$$

However, as written previously, this phase is very short since the kinetic energy of the scalar field is soon redshifted due the expansion of the universe and the radiation comes to dominate.

4.2.4 Radiation and Matter Period

During the Big Bounce the energy density of the radion and the temperature do not diverge but remain finite because part of the scalar field energy is converted into matter and radiation. This is the purpose of the coupling function ξ in the action of eq. (3.2.8). However since the the energy density of the radion goes as $\rho_\varphi \propto a^{-6}$ the beginning of the expansion remains dominated by the kinetic energy of the scalar field. Very shortly after the universe enters in a radiation dominated phase which is the same as the one of the inflationary model. The same is true for what concerns the following matter epoch and therefore all the parameters behave in the same way as in the standard picture.

Then the only thing that remains to compute is the behaviour of the radion in these two epochs. Since the first Friedmann equation is dominated in one case by the radiation term and in the other by the matter one, the Hubble parameter can be approximated to be independent of φ. Therefore we just have the second first equation of motion that becomes:

$$\ddot{\varphi} + 3H_i\dot{\varphi} = 0 \quad \text{where} \quad i = \gamma, m \tag{4.2.26}$$

and performing a double integration with $x = \dot{\varphi}$ we have:

$$\int \frac{d\bar{x}}{\bar{x}} = -3H_i \int d\bar{t} \quad \Longrightarrow \quad \dot{\varphi} = \exp\left(-3H_i t\right)$$
$$\int d\bar{\varphi} = \int \exp\left(-3H_i\bar{t}\right)d\bar{t} \quad \Longrightarrow \quad \varphi(t) = \varphi_0 - \frac{1}{3H_i}\exp\left(-3H_i t\right)$$

Substituting the value of the Hubble parameter as a function of t we see that during these two epochs the scalar decreases proportionally with the coordinate time t. Indeed:

$$\varphi_\gamma(t) = \varphi_0 - \frac{2t}{3}e^{-\frac{3}{2}} \quad \text{and} \quad \varphi_m(t) = \varphi_0 - \frac{t}{2}e^{-2}. \tag{4.2.27}$$

Finally, let's see with a simple argumentation the reason for which the cycle should start again reasoning in terms of critical energy density defined as

$$\Omega_i = \frac{\rho_i}{\rho_{cr}} \quad \text{with} \quad \rho_{cr} = \frac{3}{8\pi G}H^2(t) \quad \text{and} \quad i = \varphi, \gamma, m \tag{4.2.28}$$

where the Hubble parameter dependence on time is based on the phase in which it is computed. Therefore lets consider what happens to the critical energy density of the radion Ω_φ during the radiation and matter dominated periods. During the radiation phase, taking the approximation of considering a time t in the middle of it, we have that:

$$\frac{1}{\rho_{cr}} \propto t^2 \quad \Longrightarrow \quad \Omega_\gamma \sim a^{-4}t^2 \sim \text{const} \qquad \text{and} \qquad \Omega_\varphi \sim V(\varphi)t^2 \tag{4.2.29}$$

where it has been considered that since the kinetic energy of the scalar field has been red-shifted cause of the expansion, $\rho_\varphi \sim V(\varphi)$. Therefore the critical energy density of φ keeps increasing and becoming more and more relevant.

The same happens during the matter dominated phase, indeed computing in the same way the dependence of the critical density on the time we get that Ω_m is constant while $\Omega_\varphi \sim V(\varphi)t^2$. Therefore at a certain instant the radion energy density term will become again dominant cause the beginning of a new Dark Energy dominated phase.

In the same fashion we could have proved before that during the kinetic energy dominated expansion phase, the critical energy densities of radiation and matter keep increasing while the critical density of the scalar field remains approximately constant. Indeed we have that:

$$\rho_{cr} \sim H(t)^2 \sim \frac{1}{t^2} \qquad \text{and} \qquad \rho_\varphi \sim \frac{1}{2}\dot{\varphi}^2 \sim \frac{1}{t^2} \quad \Longrightarrow \quad \Omega_\varphi = \text{const} \tag{4.2.30}$$

while

$$\Omega_\gamma \sim t^{\frac{2}{3}} \qquad \text{and} \qquad \Omega_m \sim t \tag{4.2.31}$$

and hence at a certain instant the radiation comes to dominate. However it is important to note that this is a simplified computation since the ρ_{cr} changes during each phase more and more the other contributes become relevant. Nevertheless it furnish an intuitive idea of the reason for which a phase of the cycle follows the other in the described way.

4.3 Recap of the Parameters Evolution

In this section will be summarized all the results obtained for both inflationary and cyclic picture in the analysis of the parameter evolution. Let's begin with inflation:

Inflation Recap

phase	**inflaton** φ	$a(t)$	$H(t)$	e**-folds**
inflation	$\sqrt{\frac{2\lambda}{3}}(t_f - t)$	$a_0 \exp(H_\varphi t)$	$\sqrt{\frac{\lambda}{6}}\varphi_i - \frac{\lambda}{3}t$	$\frac{1}{2n}\varphi_i^2$
ρ_γ dominated	~ 0	$a_0\, t^{\frac{1}{2}}$	$\frac{1}{2t}$	$\frac{1}{2}\ln\left(\frac{t_{eq1}}{t_i}\right)$
ρ_m dominated	~ 0	$a_0\, t^{\frac{2}{3}}$	$\frac{2}{3t}$	$\frac{2}{3}\ln\left(\frac{t_{eq2}}{t_{eq1}}\right)$
ρ_{de} dominated	~ 0	$a_0 \exp(H_{de} t)$	$\sqrt{\frac{\rho_{de}}{3}}$	$\frac{t_0}{t_{eq2}}$

Where we remember that $H_\varphi = \sqrt{\frac{\lambda}{3n}}\varphi^{\frac{n}{2}}$ is not a constant whilst H_{de} it is. This tables regroup the standard results of inflationary cosmology, let's see now the summary of the cyclic scenario analysis.

Cyclic Universe Recap

phase	**radion** φ	$a(t)$	$H(t)$	e**-folds**
D.E. dominated	$\varphi_0 \exp\left(-3Ht\right)$	$a_0 \exp\left(H_{DE}t\right)$	$\sqrt{\frac{V_0}{3}}$	$\frac{t_f}{t_i}$
ekpyrotic p.	$\ln\left[\sqrt{\frac{V_0}{p}}(-t)\right]^{\frac{2}{c}}$	$a_0\lvert t\rvert^p$	$\frac{p}{\lvert t\rvert}$	$-p\ln\left\lvert\frac{t_i}{t_f}\right\rvert$
before the B.C.	$\sqrt{\frac{2}{3}}\ln\lvert t\rvert + \varphi_0$	$a_0\lvert t\rvert^{\frac{1}{3}}$	$\frac{1}{3\lvert t\rvert}$	$-\frac{1}{3}\ln\left\lvert\frac{t_i}{t_f}\right\rvert$
after the B.B.	$\sqrt{\frac{2}{3}}\ln t + \varphi_0$	$a_0\, t^{\frac{1}{3}}$	$\frac{1}{3t}$	$\frac{1}{3}\ln\left(\frac{t_f}{t_i}\right)$
ρ_γ dominated	$\varphi_0 - \frac{2t}{3}\,e^{-\frac{3}{2}}$	$a_0\, t^{\frac{1}{2}}$	$\frac{1}{2t}$	$\frac{1}{2}\ln\left(\frac{t_{eq1}}{t_i}\right)$
ρ_m dominated	$\varphi_0 - \frac{t}{2}\,e^{-2}$	$a_0\, t^{\frac{2}{3}}$	$\frac{2}{3t}$	$\frac{2}{3}\ln\left(\frac{t_{eq2}}{t_{eq1}}\right)$

where a_0 and φ_0 are constants of integration relative to each phase and different between each others.

Perturbations

5.1 Scalar Perturbations in Inflation

One of the greatest merits of inflationary theory is to be able to connect the large-scale structures that we observe nowadays with the micro-physics relative to early times of the universe. Indeed considering the fact that the primordial perturbations are supposed to derive from quantum fluctuations and since the amplitudes of the latter are relevant only on scales close to the Planckian length, it is fundamental the stretching effect which also leaves the amplitudes almost unchanged caused by a stage of cosmic acceleration. Indeed this effect makes these quantum fluctuations substantial on galactic scales.

Moreover the spectrum of inhomogeneities obtained in this way does not depend strictly on the tuning and on the kind of inflationary model chosen. Instead before the introduction of inflation, inhomogeneities were explained by postulating their existence and spectrum just in order to fit observational data. Therefore this shows the predictive power of inflation which can furnish a physical explanation for the CMB anisotropies.

Consequently in this section we will recap how to derive the spectrum caused by small inhomogeneities $\delta\varphi(\eta, \vec{x})$ in the field φ with the approximation of slow-roll inflation. For a scalar field with potential $V(\varphi)$ we have the action:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\,\varphi_{,\nu} - V\right) d^4x \tag{5.1.1}$$

which through the variation of the scalar field:

$$\delta S = \int \left(\sqrt{-g}g^{\mu\nu}\varphi_{,\mu}\,\frac{\partial\delta\varphi}{\partial x^{\nu}} - \sqrt{-g}V_{,\varphi}\,\delta\varphi\right) d^4x \tag{5.1.2}$$

integrating by part and knowing that $\delta\varphi = 0$ when computed at the extremes we get the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\varphi}{\partial x^\nu}\right) + \frac{\partial V}{\partial\varphi} = 0 \tag{5.1.3}$$

In order to obtain the simplest form of the metric for the background it is useful now to choose the well known longitudinal or Newtonian gauge which allows us to put equal to zero two of the four parameters characterizing the scalar perturbations ($E_l = B_l = 0$) leaving only ϕ_l, which is identified as the generalization of the Newtonian potential, and ψ_l related to the curvature perturbation. Nice property of the Newtonian gauge is the fact that the parameters different from zero corresponds to the gauge invariant parameters Φ and Ψ which for definition do not change under coordinate transformation. Therefore the metric $g_{\mu\nu}$ for a scalar perturbation $\delta\varphi(\eta, \vec{x})$ takes the simple form:

$$ds^2 = a^2\left[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)\delta_{ij}dx^i dx^j\right]. \tag{5.1.4}$$

Considering then the inflaton written as $\varphi = \varphi_0(\eta) + \delta\varphi(\eta, \vec{x})$ where $\varphi_0(\eta)$ is the the unperturbed homogeneous part for which the metric $g^0_{\mu\nu}$ is:

$$ds^2 = a^2\left[d\eta^2 - \delta_{ij}dx^i dx^j\right] \tag{5.1.5}$$

it's easy to obtain the Klein-Gordon equation for the homogeneous part:

$$g^{00}\varphi_0'' + \frac{1}{a^4}\varphi_0'\frac{\partial}{\partial\eta}\left(a^4 a^{-2}\right) + \frac{\partial V}{\partial\varphi} = 0 \quad\Longrightarrow\quad \varphi_0'' + 2\varphi_0'\mathcal{H} + a^2\frac{\partial V}{\partial\varphi} = 0 \tag{5.1.6}$$

where $\mathcal{H} = a'/a$ and $\sqrt{-g} = a^4$.

Since computing the Klein-Gordon equation to linear order in metric perturbations and $\delta\varphi$ requires some algebra it will be written here just the final result and the entire computation can be found at the end in the Appendix. The equation is:

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \Delta\delta\varphi + a^2 V_{,\varphi\varphi}\,\delta\varphi - \varphi_0'(\Phi + 3\Psi)' + 2a^2 V_{,\varphi}\,\Phi = 0 \tag{5.1.7}$$

Furthermore this expression has an ulterior simplification reached remembering the property of the Newtonian gauge which says that if the spatial part of the energy-momentum tensor is diagonal, in other words $\delta T^i_j \propto \delta^i_j$ then $\phi_l = \psi_l$ and consequently $\Phi = \Psi$. Indeed, as already seen before, for a scalar field the energy-momentum tensor assumes the form:

$$T^\mu_\nu = g^{\mu\rho}\varphi_{,\rho}\,\varphi_{,\nu} - \left(g^{\gamma\sigma}\varphi_{,\gamma}\,\varphi_{,\sigma} - V(\varphi)\right)\delta^\mu_\nu \tag{5.1.8}$$

and it is immediate to see that using the Newtonian gauge metric (5.1.4) for scalar perturbations, the tensor is diagonal in its spatial part. For this reason we are left with just two variables, Φ and $\delta\varphi$. The second condition needed to solve the system comes from the perturbed Einstein equations:

$$\delta\mathcal{G}^{\mu}_{\nu} = 8\pi G\,\delta T^{\mu}_{\nu} \tag{5.1.9}$$

As showed in Mukhanov cosmology textbook [8], one of the resulting equations that we are going to use is:

$$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi G\, a^2\,\delta T^0_i. \tag{5.1.10}$$

From the expression given for the energy-momentum tensor we see that

$$T^0_i = g^{00}\varphi_{,0}\,\varphi_{,i} \quad\Longrightarrow\quad \delta T^0_i = \frac{1}{a^2}\varphi_0'\,\delta\varphi_{,i} = \frac{1}{a^2}\left(\varphi_0'\,\delta\varphi\right)_{,i} \tag{5.1.11}$$

since $\varphi_0 = \varphi_0(\eta)$ and just considering the linear order in metric perturbations and $\delta\varphi$. Hence (5.1.10) gives:

$$\Phi' + \mathcal{H}\Phi = 4\pi G\,\varphi_0'\,\delta\varphi \tag{5.1.12}$$

The system of equations composed by (5.1.7) and (5.1.12) will be solved in two limiting cases:

- when the perturbations are sub-horizon, their physical wavelength is much smaller then curvature scale $\lambda_{ph} << H^{-1}$;
- for super-horizon perturbations, $\lambda_{ph} >> H^{-1}$.

Remembering the relation (4.1.11) and knowing that inflation last for a very small interval of time, we see that the curvature scale H^{-1} changes infinitesimally during inflation. On the contrary the physical wavelength increases exponentially with time $\lambda_{ph} \sim a/k$ and therefore modes that at beginning were sub-horizon can eventually become super-horizon.

5.1.1 Sub-Horizon Modes

In the case of $\lambda_{ph} << H^{-1}$, condition that can be rewritten as $k\eta >> 1$ since from the definition of conformal time $d\eta = dt/a(t)$ we get that during inflation $\eta \sim (aH)^{-1}$, we are considering distances that are under the curvature scale and therefore it is possible to assume the space-time locally Minkowskian. For this reason the gravitational field does not influence substantially the evolution of the modes and all the terms of Φ or derivatives of it can be neglected. Indeed in equation (5.1.7) the spatial derivative term dominates. Now considering the fact that the conformal time derivative can be expressed in terms of number of e-folds derivative as

$$\frac{d}{d\eta} = \mathcal{H}\frac{d}{dN} \quad \text{where} \quad \mathcal{H} = \frac{a'}{a} \quad \text{and} \quad N = \ln a \tag{5.1.13}$$

Therefore the first term of (5.1.7) can be written as:

$$\begin{aligned}
\delta\varphi'' &= \frac{d}{d\eta}\left(\mathcal{H}\frac{d\,\delta\varphi}{dN}\right) = \mathcal{H}^2\frac{d^2\delta\varphi}{dN^2} + \frac{d\,\mathcal{H}}{d\eta}\frac{d\,\delta\varphi}{dN} \\
&= \mathcal{H}^2\frac{d^2\delta\varphi}{dN^2} + \left[\frac{a''}{a} - \mathcal{H}^2\right]\frac{d\,\delta\varphi}{dN} \\
&= \mathcal{H}^2\frac{d^2\delta\varphi}{dN^2} + \left[\frac{2}{3}a^2V - \mathcal{H}^2\right]\frac{d\,\delta\varphi}{dN} = \mathcal{H}^2\frac{d^2\delta\varphi}{dN^2} + \mathcal{H}^2\frac{d\,\delta\varphi}{dN}
\end{aligned} \tag{5.1.14}$$

Where they have been used both Friedmann equations in slow-roll approximation:

$$\frac{1}{a^2}\mathcal{H}^2 \simeq \frac{1}{3}V \qquad \text{and} \qquad \frac{a''}{a} = \frac{2}{3}a^2V \tag{5.1.15}$$

Finally since also the second term of (5.1.7) can be written as $2\mathcal{H}\delta\varphi' = 2\mathcal{H}^2\,(d\,\delta\varphi/dN)$, if we divide by the a^2 of the fourth term of eq. (5.1.7), all the $\mathcal{H}^2$ become H^2. Therefore dividing again by H^2, since from the Friedmann equation we know that $H^2 \sim V$, we see that the term $\delta\varphi\, V_{,\varphi\varphi}$ vanishes due to the fact that in the slow-roll approximation $|V_{,\varphi\varphi}/V| << 1$.

As a result of this, only the first three terms of eq. (5.1.7) matter and for a plane wave perturbation with comoving wave-number k, the same equation in Fourier space reduces to:

$$\delta\varphi_k'' + 2\mathcal{H}\delta\varphi_k' + k^2\delta\varphi_k \simeq 0 \tag{5.1.16}$$

which introducing the new variable $u_k = a\,\delta\varphi_k$ becomes:

$$u_k'' + \left(k^2 - \frac{a''}{a}\right)u_k = 0. \tag{5.1.17}$$

Now since we are considering modes for which $k >> |\eta^{-1}|$, together with the fact that in inflation $a \sim \eta^{-1}$ and therefore $a''/a \sim \eta^{-2}$, the last term of the previous equation can be ignored and hence the solution is:

$$\delta\varphi_k \simeq \frac{C_k}{a} \exp\left(\pm ik\eta\right) \tag{5.1.18}$$

where C_k is an integration constant depending on the initial conditions. Immediately from this expression is possible to see that the amplitude of the oscillation is not fixed but goes as $\sim a^{-1}$ which means that it decreases exponentially with the passing of the coordinate time t.

5.1.2 Super-Horizon Modes

In order to study the long wavelength modes we will use the slow-roll approximation for an inflation model. Therefore equation (4.1.1) is valid for the homogeneous part of the scalar field φ_0 and hence neglecting the second derivative with respect to the coordinate time t we have the equation:

$$3H\dot{\varphi}_0 + V_{,\varphi} \simeq 0 \tag{5.1.19}$$

Moreover it is necessary now to rewrite the system of equations formed by (5.1.7) and (5.1.12) in terms of the physical time t:

$$\begin{aligned} \delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \Delta\delta\varphi + V_{,\varphi\varphi}\,\delta\varphi - 4\dot{\varphi}_0\dot{\Phi} + 2V_{,\varphi}\,\Phi = 0, \\ \dot{\Phi} + H\Phi = 4\pi\,\dot{\varphi}_0\delta\varphi. \end{aligned} \tag{5.1.20}$$

The next step is to consider the facts that:

- for $\lambda_{ph} >> H^{-1}$ we can ignore the spatial derivative term $\Delta\delta\varphi$;
- in order to find the non-decaying slow-roll solution the terms containing $\delta\ddot{\varphi}$ and $\dot{\Phi}$ can be omitted;
- it is useful to introduce the variable $y \equiv \delta\varphi/V_{,\varphi}$;
- remembering that from the constraints on the slow-roll potential $V(\varphi)$ it derives that $|V_{,\varphi\varphi}/V_{,\varphi}| << 1$;
- during inflation $3H^2 \simeq 8\pi V$.

Henceforth dividing the first of the two equations (5.1.20) by $V_{,\varphi}$ and for the other one multiplying and dividing keeping only the first order in $\delta\varphi$:

$$H\Phi = 4\pi \frac{\delta\varphi}{V_{,\varphi}} \frac{\partial V(\varphi_0 + \delta\varphi)}{\partial\varphi} \dot{\varphi}_0 = 4\pi\, y \frac{\partial V(\varphi_0)}{\partial\varphi_0} \dot{\varphi}_0 = 4\pi y \dot{V}, \tag{5.1.21}$$

the system reduces to:

$$3H\dot{y} + 2\Phi = 0 \quad \text{and} \quad H\Phi = 4\pi y \dot{V}. \tag{5.1.22}$$

Starting with substituting the second equation into the first, the next short algebra steps are:

$$\begin{aligned} 3H\dot{y} + 8\pi y \frac{\dot{V}}{H} &= 0 \\ V\dot{y} + y\dot{V} &= 0 \\ \frac{d(Vy)}{dt} &= 0 \quad \Longrightarrow \quad y = \frac{A}{V} \end{aligned} \tag{5.1.23}$$

and hence for the non-decaying mode the solutions are:

$$\delta\varphi_k = A_k \frac{V_{,\varphi}}{V} \quad \text{and} \quad \Phi_k = 4\pi A_k \frac{\dot{\varphi}_0}{H} \frac{V_{,\varphi}}{V} = -\frac{1}{2} A_k \left(\frac{V_{,\varphi}}{V}\right)^2 . \tag{5.1.24}$$

Consequently we can now plot the evolution of $\delta\varphi_k$ with respect to the scale factor a, underlining the turning points with the labelling of a_k for the scale factor corresponding to the exit from the horizon of the mode with wave-number k and a_f for the scale factor corresponding to the end of inflation.

Then for $a < a_k$ we see the exponentially damped oscillating solution for the sub-horizon modes, while for $a_k < a < a_f$ we see the behaviour of the amplitude for super-horizon modes. The latter particularly, since from (5.1.24) it goes as $\sim V_{,\varphi}/V$, which for example for a power-law exponential means that $\delta\varphi_k \propto \varphi^{-1}$, shows that the amplitude increases slowly once the modes exit the horizon (the value of φ is decreasing with the passing of time). At the end of inflation, the conditions on the potential of the scalar field are no longer valid and in particularly $\sim V_{,\varphi}/V$ becomes of order of unity.

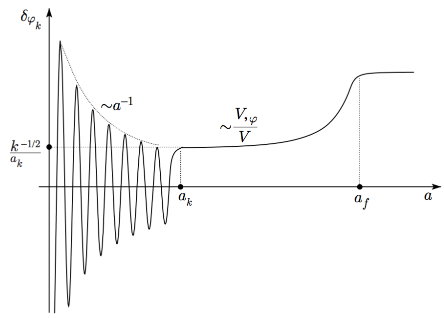

Figure 5.1: behaviour of $\delta\varphi(a)$ [8]

In order to fix the integration constants C_k and A_k we use the fact that as explained in Appendix B, an estimation of the minimal amplitude of quantum fluctuations is $|\delta\varphi_k| \sim k^{\frac{1}{2}}/a$ and hence comparing this relation with the two solution at the moment in which the mode crosses the horizon we get:

$$C_k \sim k^{-\frac{1}{2}} \qquad \text{and} \qquad A_k \sim \frac{k^{-\frac{1}{2}}}{a_k}\left(\frac{V}{V_{,\varphi}}\right)_{k\sim Ha} \tag{5.1.25}$$

Since we are considering perturbations as a Gaussian random process, the spectrum of the inhomogeneities is characterized by the variance of the gravitational potential $\sigma_k^2 = |\Phi_k|^2$. The *power spectrum* $\delta_\Phi^2(k)$ is defined as the dimensionless variance:

$$\delta_\Phi^2(k) \equiv \frac{|\Phi_k|^2 k^3}{2\pi^2} \tag{5.1.26}$$

where $\delta_\Phi(k,t)$ is the typical amplitude of the metric fluctuations. We are interested in computing the power spectrum for modes relative to supercurvature scales in order to reach the target represented by the *spectral index* n_S defined as:

$$n_S - 1 = \frac{d\ln\delta_\Phi^2}{d\ln k}. \tag{5.1.27}$$

Therefore since at the end of inflation the ratio $V/V_{,\varphi} \leftarrow 1$ we have that the solution for Φ_k simplifies and hence:

$$\begin{aligned} \Phi_k &\simeq -\frac{1}{2}A_k \\ \Longrightarrow \delta_\Phi(k,t_f) &\sim A_k k^{\frac{3}{2}} \sim \left(H\frac{V}{V_{,\varphi}}\right)_{k\sim Ha} \sim \left(\frac{V^{\frac{3}{2}}}{V_{,\varphi}}\right)_{k\sim Ha}. \end{aligned} \tag{5.1.28}$$

where it has been used the fact that at the horizon crossing $k \sim Ha$ and that during inflation from (4.1.2) $H \sim V^{\frac{1}{2}}$. Finally we can compute the spectral index n_S. Before proceeding with the algebra let's notice that since the expression depends on quantities (A_k) that have been estimated when the mode crosses the horizon, from the fact that $k \sim Ha$ and that during inflation H is almost constant, we have the useful relation $d\ln k = d\ln a$. This, together with the relation found in the previous chapter (4.1.3) which gives

$d\ln a = -d\varphi\, V/V_{,\varphi}$, is everything we need in order to find n_S:

$$n_S - 1 = \frac{1}{\delta_\Phi^2}\frac{d\,\delta_\Phi^2}{d\ln k} \simeq \frac{1}{\delta_\Phi^2}\frac{d}{d\ln k}\left(\frac{V^3}{V_{,\varphi}^2}\right)_{k\sim Ha}$$

$$\simeq \frac{1}{\delta_\Phi^2}\left[\left(\frac{d\,V^3}{d\ln k}\right)\frac{1}{V_{,\varphi}^2} + V^3\frac{d}{d\ln k}\frac{1}{V_{,\varphi}^2}\right]_{k\sim Ha}$$

$$\simeq \frac{1}{\delta_\Phi^2}\left(\frac{V}{V_{,\varphi}}\right)^2\left[3\frac{d\,V}{d\ln k} - 2\frac{V}{V_{,\varphi}}\frac{d\,V_{,\varphi}}{d\ln k}\right]_{k\sim Ha}$$

$$\simeq \frac{1}{\delta_\Phi^2}\left[-3V + 2\frac{V^2}{V_{,\varphi}^2}V_{,\varphi\varphi}\right]_{k\sim Ha}$$

$$n_S - 1 \simeq -3\left(\frac{V_{,\varphi}}{V}\right)^2 + 2\frac{V_{,\varphi\varphi}}{V} \tag{5.1.29}$$

and defining the slow-roll parameters of inflation:

$$\bar{\epsilon} = \frac{1}{2}\left(\frac{V_{,\varphi}}{V}\right)^2 \qquad \text{and} \qquad \bar{\eta} = \frac{V_{,\varphi\varphi}}{V} \tag{5.1.30}$$

we recover the classical expression

$$n_S - 1 = -6\bar{\epsilon} + 2\bar{\eta}. \tag{5.1.31}$$

In the case of a power-law potential like the one used until now for our simple model of inflation, $V \propto \varphi^n$, it is immediate to see from (5.1.29) that the spectral index becomes:

$$n_S - 1 \simeq \left(-3n^2 + 2n^2 - 2n\right)\varphi_{k\sim aH}^{-2} \simeq -\frac{n\,(n+2)}{\varphi_{k\sim aH}^2} \tag{5.1.32}$$

where the value of the inflaton is the one corresponding at the time at which the mode with wave-number k exits the horizon. Remembering the definition of e-folds during inflation given in (4.1.9), substituting the initial value of the inflaton φ_i with the value of the field when the mode crosses the horizon $\varphi_{k\sim aH}$ and replacing the arbitrary value φ with the value of the scalar field at the end of inflation $\varphi_f = 0$ we see that the number of e-folds that miss from the end of the acceleration phase is given by:

$$N_{missing} = \int_{\varphi_{end}}^{\varphi_{k\sim Ha}} \frac{1}{n}\,\bar{\varphi}d\bar{\varphi} = \frac{1}{2n}\varphi_{k\sim Ha}^2 \tag{5.1.33}$$

and therefore:

$$n_S - 1 \simeq -\frac{n+2}{2N_m} \tag{5.1.34}$$

For example for a massive scalar field with $n = 2$ and considering galactic scales for which $N \simeq 50$ we have $n_S \simeq 0.96$.

5.2 Scalar Perturbations in the Cyclic Model

In order to start the analysis of the spectrum in the case of a single scalar field (the radion), it is useful to recap some of the results or the informations presented before. Since we are considering the perturbations produced during the ekpyrotic phase, the action for φ can be rewritten without the coupling factor $\xi(\varphi)$ which plays a relevant role only at the moment of the Big Crunch. Therefore we rewrite the action as:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}\mathcal{R} - \frac{1}{2}(\partial\varphi)^2 + V_0 e^{-c\varphi}\right] \tag{5.2.1}$$

from which we can be obtained the well known equations of motion (4.2.2) :

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0$$

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right). \tag{5.2.2}$$

The solution of these ones in the ekpyrotic phase are summarized in the last table of the previous chapter. The background metric is the same of the inflationary case eq. (5.1.4).

Therefore let's consider now the linearized Einstein equations $\delta\mathcal{G}^\mu_\nu = 8\pi G\delta T^\mu_\nu$. In case of a diagonal stress-energy tensor as the one for the radion, in terms of the gauge invariant variable Φ, it is a long but standard computation to obtain the components of the Einstein tensor $\mathcal{G}^\mu_\nu$ and the one of T^μ_ν as:

$$\begin{aligned}
\delta\mathcal{G}^0_0 &= \frac{2}{a^2}\left[-3\mathcal{H}(\mathcal{H}\Phi + \Phi') + \Delta\Phi\right] , \\
\delta\mathcal{G}^0_j &= \frac{2}{a^2}\left[\mathcal{H}\Phi + \Phi'\right]_{,j} , \\
\delta\mathcal{G}^i_j &= -\frac{2}{a^2}\left[\Phi(2\mathcal{H}' + \mathcal{H}^2) + \mathcal{H}\Phi' + \Phi''\right]\delta^i_j , \\
\delta T^0_0 &= \frac{1}{a^2}\left[-\varphi_0'^2\Phi + \varphi_0'\delta\varphi' + a^2 V_{,\varphi}\,\delta\varphi\right] \\
\delta T^0_j &= \frac{1}{a^2}\left[\varphi_0'\delta\varphi\right]_{,j} , \\
\delta T^i_j &= \frac{1}{a^2}\left[\varphi_0'^2\Phi - \varphi_0'\delta\varphi' + a^2 V_{,\varphi}\,\delta\varphi\right]\delta^i_j .
\end{aligned}$$

where it has been used the metric (5.1.4), as in the inflationary case φ_0 is the homogeneous part of the field and finally the stress-energy tensor for the radion is of the form $T^{\mu\nu} =$ diag $[\rho, p, p, p]$ with $\rho_\varphi = 1/2\dot{\varphi}^2 + V$ and $p_\varphi = 1/2\dot{\varphi}^2 - V$. Therefore the Einstein equations becomes:

$$\begin{aligned} -3\mathcal{H}(\mathcal{H}\Phi + \Phi') + \Delta\Phi &= 4\pi G \left(-\varphi_0'^2\Phi + \varphi_0'\delta\varphi' + a^2 V_{,\varphi}\,\delta\varphi\right) , \\ \mathcal{H}\Phi + \Phi' &= 4\pi G \left(\varphi_0'\delta\varphi\right) , \\ \Phi(2\mathcal{H}' + \mathcal{H}^2) + 3\mathcal{H}\Phi' + \Phi'' &= 4\pi G \left(-\varphi_0'^2\Phi + \varphi_0'\delta\varphi' - a^2 V_{,\varphi}\,\delta\varphi\right) . \end{aligned} \tag{5.2.3}$$

Let's write now the non perturbed Einstein equation for the background with metric (5.1.5). The components of the Einstein tensor are easy in this case to compute and they are:

$$\mathcal{G}_0^0 = \frac{3\mathcal{H}^2}{a^2}, \qquad \mathcal{G}_i^0 = 0, \qquad \mathcal{G}_j^i = \frac{1}{a^2}(2\mathcal{H}' + \mathcal{H}^2)\delta_j^i \tag{5.2.4}$$

Hence, keeping in mind the stress-energy tensor for φ we can write the Einstein equation for the background:

$$\begin{aligned} \frac{3}{a^2}\mathcal{H}^2 &= 8\pi G \left(\frac{1}{2a^2}\varphi_0' + V\right) , \\ \frac{1}{a^2}\left(2\mathcal{H}' + \mathcal{H}^2\right) &= 8\pi G \left(-\frac{1}{2a^2}\varphi_0'^2 + V\right) . \end{aligned} \tag{5.2.5}$$

Subtracting the second equation from the first it immediately follows that: $\mathcal{H}^2 - \mathcal{H}' = 4\pi G\varphi_0'^2$. Using this relation it is possible to rewrite the equations (5.2.3) as

$$\begin{aligned} -3\mathcal{H}\Phi' - 2\mathcal{H}^2\Phi - \mathcal{H}'\Phi + \Delta\Phi &= 4\pi G \left(\varphi_0'\delta\varphi' + a^2 V_{,\varphi}\,\delta\varphi\right) , \\ \mathcal{H}\Phi + \Phi' &= 4\pi G \left(\varphi_0'\delta\varphi\right) , \\ \Phi(\mathcal{H}' + 2\mathcal{H}^2) + 3\mathcal{H}\Phi' + \Phi'' &= 4\pi G \left(\varphi_0'\delta\varphi' - a^2 V_{,\varphi}\,\delta\varphi\right) . \end{aligned} \tag{5.2.6}$$

The next step consists in subtracting the first of the three equations from the first obtaining:

$$\Phi'' + 6\mathcal{H}\Phi' - \Delta\Phi + 2\Phi(\mathcal{H}' + 2\mathcal{H}^2) = -8\pi G\, a^2 V_{,\varphi}\,\delta\varphi \tag{5.2.7}$$

Together with this, remembering the equation of motion (5.1.6) for the homogeneous part:

$$\varphi_0'' + 2\mathcal{H}\varphi_0' = -a^2 V_{,\varphi} \tag{5.2.8}$$

we see that multiplying the second equation of (5.2.6) by $2\left[2\mathcal{H} + (\varphi_0''/\varphi_0')\right]$ we get:

$$4\mathcal{H}^2\Phi + 4\mathcal{H}\Phi' + 2\mathcal{H}\Phi\frac{\varphi_0''}{\varphi_0'} + 2\Phi'\frac{\varphi_0''}{\varphi_0'} = -8\pi G\, a^2 V_{,\varphi}\,\delta\varphi\,, \tag{5.2.9}$$

and finally subtracting (5.2.9) from (5.2.7) we obtain the final equation

$$\Phi'' + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\Phi' - \Delta\Phi + \left(2\mathcal{H}' - 2\mathcal{H}\frac{\varphi_0''}{\varphi_0'}\right)\Phi = 0\,. \tag{5.2.10}$$

Rewriting the same equation passing in the Fourier space we get:

$$\Phi'' + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\Phi' + \left(k^2 + 2\mathcal{H}' - 2\mathcal{H}\frac{\varphi_0''}{\varphi_0'}\right)\Phi = 0 \tag{5.2.11}$$

In order to simply this equation it is necessary to introduce a new parameter z together with a new variable u defined as

$$z^{-1} \equiv \frac{a'}{a^2}\varphi_0'^{-1} = \mathcal{H}\frac{1}{a\varphi_0'} \qquad \text{and} \qquad u \equiv \frac{a}{\varphi_0'}\Phi \tag{5.2.12}$$

which after long computations allow to rewrite the equation for Φ as:

$$u'' + \left(k^2 - \frac{(1/z)''}{(1/z)}\right)u = 0 \tag{5.2.13}$$

In order to understand better the result obtained for the analysis of single scalar field perturbations in the cyclic model it is fundamental to describe the same also using the other gauge invariant variable ζ, the one representing the curvature perturbation on spatial hypersurfaces. ζ is related to the Newtonian potential Φ by the relations:

$$\zeta = \Phi + \frac{1}{\epsilon}\left(\frac{a}{a'}\Phi' + \Phi\right) \quad \text{and} \quad \Phi = -\epsilon\frac{a'}{ak^2}\zeta' \tag{5.2.14}$$

and ζ's equation of motion is:

$$\zeta'' + 2\frac{z'}{z}\zeta' + k^2\zeta = 0. \tag{5.2.15}$$

As for the Φ's equation, it is useful to introduce the new variable $v = z\zeta$, and hence the equation of motion becomes:

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0\,. \tag{5.2.16}$$

In Appendix C it is shown how the two coefficients of u and v in the two equations, $(1/z)''/(1/z)$ and z''/z respectively, depend on the conformal time η. Here we give just the results:

$$\begin{aligned}\frac{(1/z)''}{(1/z)} &= \frac{\epsilon}{(\epsilon-1)^2\eta^2}\\ \frac{z''}{z} &= \frac{2-\epsilon}{(\epsilon-1)^2\eta^2}\end{aligned} \tag{5.2.17}$$

5.2.1 Sub-Horizon Modes

From this kind of dependence we can study, as in the inflationary case, the limit in which the modes are sub-horizon. In the cyclic picture this corresponds to large $|\eta|$ since the conformal time is flowing from $-\infty \to 0$. Therefore in both equations for u and v the k^2 term dominates over the one depending on z. Immediately the form of the two equations conveys the idea of asymptotically oscillatory solutions. Indeed on scales much smaller than H^{-1} the space-time can be considered with very good approximation to be flat (Minkowskian) and hence it is logical to impose that the solutions satisfy the boundary condition of tending to the Minkowski vacuum:

$$u \to \frac{i}{(2k)^{3/2}} e^{-ik\eta} \qquad \text{and} \qquad v \to \frac{1}{\sqrt{2k}} e^{-ik\eta} \tag{5.2.18}$$

with $\eta \to -\infty$.

Following the standard treatment for this equations of motion described in the literature, since they are Bessel equations, their exact solutions are given by:

$$\begin{aligned} u(x) &= x^{1/2}\left[A^{(1)}H_\alpha^{(1)}(x) + A^{(2)}H_\alpha^{(2)}(x)\right] \\ v(x) &= x^{1/2}\left[B^{(1)}H_\beta^{(1)}(x) + B^{(2)}H_\beta^{(2)}(x)\right] \end{aligned} \tag{5.2.19}$$

where it has been introduced the dimensionless time variable $x = k|\eta|$, $H_s^{(1,2)}(x)$ are Henkel functions and $A^{(1,2)}$, $B^{(1,2)}$ are constants. Moreover α and β are functions of z defined as:

$$\begin{aligned} \alpha &\equiv \sqrt{\frac{(1/z)''}{(1/z)}\eta^2 + \frac{1}{4}} = \frac{1}{2}\left|\frac{\epsilon+1}{\epsilon-1}\right| \\ \beta &\equiv \sqrt{\frac{z''}{z}\eta^2 + \frac{1}{4}} = \frac{1}{2}\left|\frac{\epsilon-3}{\epsilon-1}\right| \end{aligned} \tag{5.2.20}$$

where we have used the relations (5.2.17).
In order to study sub-horizon modes then we need to take the limit $x \to \infty$ for which the Henkel functions assume the asymptotic expression:

$$H_s^{(1,2)}(x) \to \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{s\pi}{2} - \frac{pi}{4}\right)\right] \tag{5.2.21}$$

where we need to take the $+$ sign since in the definition of x the conformal time appears as $|\eta|$. From (5.2.20) and from the boundary conditions (5.2.18) we see that $A^{(2)}$ and $B^{(2)}$ must be null. Then the solutions for the Minkowskian vacuum take the forms:

$$u = \frac{\mathcal{P}_1}{2k}\sqrt{\frac{\pi x}{4k}}H_\alpha^{(1)}(x) \qquad \text{and} \qquad v = \mathcal{P}_2\sqrt{\frac{\pi x}{4k}}H_\beta^{(1)}(x)\,. \tag{5.2.22}$$

The phase factor parameters $\mathcal{P}_1$ and $\mathcal{P}_2$ are computed equating the last expression with (5.2.18). For example for what concerns the first one:

$$i = \mathcal{P}_1 \exp\left[i\left(-2\alpha - 1\right)\frac{\pi}{4}\right] \quad \Longrightarrow \quad \mathcal{P}_1 = \exp\left[i\left(2\alpha + 3\right)\frac{\pi}{4}\right] \tag{5.2.23}$$

In the same way it is obtained the second parameter:

$$1 = \mathcal{P}_2 \exp\left[i\left(-2\beta - 1\right)\frac{\pi}{4}\right] \quad \Longrightarrow \quad \mathcal{P}_2 = \exp\left[i\left(2\alpha + 1\right)\frac{\pi}{4}\right] \tag{5.2.24}$$

5.2.2 Super-Horizon Modes

For what concerns super-horizon modes, corresponding to late times $\eta \to 0$, we need then to take the limit $x \to 0$. The asymptotic expression for the Henkel function in this case is:

$$H_s^{(1)}(x) \;\to\; -\frac{1}{\pi}\Gamma(s)\left(\frac{x}{2}\right)^{-s} \tag{5.2.25}$$

where $s > 0$ and $\Gamma(s)$ is the Euler Γ function. Hence the limit for the super-horizon solutions becomes:

$$u = -\frac{\mathcal{P}_1}{2k}\sqrt{\frac{x}{4\pi k}}\Gamma(\alpha)\left(\frac{x}{2}\right)^{-\alpha} \qquad \text{and} \qquad v = -\mathcal{P}_2\sqrt{\frac{x}{4\pi k}}\Gamma(\beta)\left(\frac{x}{2}\right)^{-\beta}\,. \tag{5.2.26}$$

Therefore using the definition for the power spectrum (5.1.26), we have that for the gravitational potential Φ

$$\delta_\Phi^2(k) = \frac{|\Phi_k|^2 k^3}{2\pi^2} = \frac{k^3}{2\pi^2}\frac{|u|^2\varphi_0'^2}{a^2} \;\propto\; x^{1-2\alpha}. \tag{5.2.27}$$

Using also the definition of spectral index introduced before in (5.1.27) we have that:

$$n_\Phi - 1 = 1 - 2\alpha = 1 - \left|\frac{\epsilon + 1}{\epsilon - 1}\right|. \tag{5.2.28}$$

This result for n_Φ shows that, for what concerns the gauge invariant gravitational potential variable Φ, perturbations during the ekpyrotic contraction ($\epsilon >> 1$) result in a scale-invariant spectrum as in the inflationary case

($\epsilon << 1$). If the computation would have been done keeping the parameter ϵ slowly varying with time instead of constant, as it is shown in the literature, the spectral index can be written in terms of the *fast-roll* parameters $\bar{\epsilon}$ and $\bar{\eta}$:

$$n_\Phi - 1 = -4(\bar{\epsilon} + \bar{\eta}) \quad \text{with} \quad \bar{\epsilon} = \left(\frac{V}{V_{,\varphi}}\right)^2 , \quad \bar{\eta} = 1 - \frac{VV_{,\varphi\varphi}}{V_{,\varphi}^2} . \tag{5.2.29}$$

Therefore if we assume an exponential potential as the one used before to analyse the parameter behaviour in the ekpyrotic phase,$V = -V_0 \exp(-c\varphi)$, the spectral index becomes:

$$n_\Phi - 1 = -\frac{4}{c^2} = -2p \quad \Longrightarrow \quad n_\Phi \simeq 1 \tag{5.2.30}$$

since in the cyclic model it is assumed $c^2 >> 1$.

It seems then that the analysis of the perturbations in the single scalar field model for the cyclic universe reaches the same conclusion as inflation. However if one computes the power spectrum with respect to the gauge invariant variable ζ some problems arise as it will be shown shortly. Computing the power spectrum and therefore the spectral index for the variable ζ is important since when it comes to study the evolution of perturbation modes when they exit the horizon, this variable has the fundamental property of remaining constant on super-horizon scales and starts to change again only when it renters the horizon. Instead if one studies the evolution of the modes outside the horizon using the gauge invariant variable Φ he or she must keep in count that it continues evolving also on super-horizon scales.

Let's see what happens using ζ. From the asymptotic v solution the power spectrum for ζ results to be proportional to:

$$P_\zeta(k) = \frac{k^3}{2\pi^2}\frac{|v|^2}{z^2} \propto x^{3-2\beta} . \tag{5.2.31}$$

Immediately it follows that the spectral index n_ζ is:

$$n_\zeta - 1 = 3 - 2\beta = 3 - \left|\frac{\epsilon - 3}{\epsilon - 1}\right| . \tag{5.2.32}$$

This result highlights the fundamental problem of the single scalar field perturbations model in the ekpyrotic phase. Indeed, while for a parameter $\epsilon << 1$ typical of an inflationary model the spectrum of the curvature perturbation would be scale invariant, in the cyclic picture, since $\epsilon >> 1$ the spectrum results to not be scale invariant but instead blue-shifted. Consequently cyclic model prediction of ζ spectrum are in contradiction with the experimental evidences. The solutions proposed to this problem will be discussed in the final chapter of the dissertation.

5.3 Tensor Perturbations: Gravitational Waves

The last section of this chapter will deal with the tensor perturbations. Instead of presenting as usual what happens before in the inflationary scenario and after study the cyclic one, we will write down as final result the spectral index as function of ϵ and after take the limits for the two cases, $\epsilon \to 0$ for inflation and $\epsilon >> 1$ for the ekpyrotic phase.

Analysing tensor perturbations, in other words the gravitational waves in both models, results to be very interesting since it's the only aspect in which inflationary and cyclic picture diverge, predicting different spectral index as we will find out shortly. Let's write down the perturbed metric in the case of tensor perturbations:

$$ds^2 = a^2 \left[d\eta^2 - (\delta_{ij} - 2h_T Y_{ij}^{(2)}) dx^i dx^j \right] \tag{5.3.1}$$

where $Y_{ij}^{(2)}$ is a tensor harmonic and h_T is a gauge invariant variable. Computing the perturbed Einstein equations as in the scalar case, the resulting equation of motion in the Fourier space is:

$$h_T'' + 2\mathcal{H} h_T' + k^2 h_T = 0. \tag{5.3.2}$$

Proceeding as in the single scalar field case, it is useful in order to eliminated the first derivative of h_T with respect to the conformal time to perform the substitution with the new variable $f_T = a h_T$. Substituting in eq. (5.3.2) we get:

$$\frac{f_T''}{a} - 2Hf_T' - H' f_T + 2\mathcal{H}\left(\frac{f_T'}{a} - Hf_T\right) + k^2 \frac{f_T}{a} = 0$$
$$\frac{f_T''}{a} - \frac{a''}{a^2} f_T + 2H\mathcal{H} f_T - 2H\mathcal{H} f_T + k^2 \frac{f_T}{a} = 0$$

$$\Longrightarrow \quad f_T'' + \left(k^2 - \frac{a''}{a}\right) f_T = 0\,. \tag{5.3.3}$$

Always following the computation for the single scalar field, considering the boundary condition in the far past $\eta \to -\infty$, we impose the Minkowski vacuum solution as limit:

$$f_T \to \frac{1}{\sqrt{2k}} e^{-ik\eta} \tag{5.3.4}$$

Now, looking back to the equation for v in the single scalar field case, eq. (5.2.16), ve notice the similarity with the equation for f_T. Moreover, rewriting the solution for the scale factor a and the radion field φ obtained for the ekpyrotic phase in terms of conformal time η, we see that:

$$a(\eta) \propto |\eta|^{\frac{p}{1-p}} \qquad \text{and} \qquad \varphi(\eta) \propto \ln|\eta|. \tag{5.3.5}$$

This, together with the definition of z in eq. (5.2.12), implies that:

$$z(\eta) \propto \eta^{\frac{2p}{1-p}-\frac{p}{1-p}+1-1} = \eta^{\frac{p}{1-p}} \propto a(\eta) \quad \Longrightarrow \quad \frac{a''}{a} = \frac{z''}{z} \tag{5.3.6}$$

which proves that f_T and v obey to the same equation of motion and hence have the same solution:

$$f_T = \mathcal{P}_2\sqrt{\frac{\pi x}{4k}}H_\beta^{(1)}(x). \tag{5.3.7}$$

Henceforth from the definition of the tensor spectral index as $k^3|f_T|^2 \propto k^{n_T}$ in the super-horizon modes limit, using the same expansion for $H_\beta^{(1)}(x)$ as for the v solution, it is straightforward to get:

$$n_T = 3 - 2\beta = 3 - \left|\frac{\epsilon-3}{\epsilon-1}\right|. \tag{5.3.8}$$

Whilst the result predicted for the power spectrum of ζ in the cyclic problem represents a problem for the theory, the tensor spectral index just computed could represent the experimental turning point in order to choose between inflationary and cyclic picture. Indeed for $\epsilon >> 1$ the gravitational waves produced during the ekpyrotic phase own a strongly blue-shifted spectrum, $n_T \approx 2$. On the contrary for $\epsilon << 1$ in th inflationary case, the spectrum results to be scale invariant.

Unfortunately the amplitudes predicted for the gravitational waves produced during the ekpyrotic phase is so small that observing them is far from being possible with the current technology. Nevertheless they still represents for the future the possibility of performing a crucial observational test which depending on the result can reinforce and confirm the reliability and the predictive power of one of the two model while ruling definitely out the other.

Conclusions

The principal interesting aspect described in the second and third chapter that is worth to recall in these final considerations is the fact that the cyclic model in its formulation gives a unitary explanation for the history of the universe, which in a relative simple way solves the cosmological puzzle and at the same time assigns a meaning and a function to the Dark Energy, which instead for the standard Big Bang picture it has been an unexpected discovery source of many unanswered questions. Again it must be underlined that the formulation of the cyclic model has been made later and also in response to the discovery of D.E. and hence it was a necessary prerequisite to be able to furnish a physical explanation to it in order to be considered a possible alternative to the inflationary model.

Let's consider the results summarized in the two tables at the end of the fourth chapter, especially focusing our attention on the behaviour of the Hubble parameter during the different phases in the two models. This in order to compare the evolution of the Hubble horizon defined as H^{-1} with respect to the scale factor a in the two backgrounds.

In the inflationary picture we immediately see that during the accelerating phase after the Big Bang, as mentioned before, H^{-1} increases approximately as $\sim 1/(A-t)$, where A is a constant. Nevertheless the scale factor increases exponentially during inflation and therefore points that were casually connected may fall out of casual contact by the end of inflation.

On the contrary during the radiation and matter dominated epochs, the Hubble horizon increases linearly with time while $a \sim t^b$ with $b < 1$ and hence during these two phases points that were casually disconnected may be in casual contact after a certain interval of time.

In the last period however the situation changes again, indeed the domination of Dark Energy implies an approximately constant Hubble horizon and hence, since a increases exponentially with time even if much more slower than during inflation, points in the space-time tend to fall out of casual contact.

In the cyclic picture things go a little bit in a different way, indeed even if starting from the Dark Energy phase we recognize the same relation between H^{-1} and a seen in the inflationary scenario, when we look at what happens during the ekpyrotic phase something new appears. In particular during the first stage of contraction ruled by the steep falling exponential potential of the radion, H^{-1} decreases linearly with the coordinate time $|t|$, which is now going towards zero, whilst a is going as $|t|^p$, and hence for definition $p << 1$, stays almost constant. Therefore the same result of inflation is obtained but in a different way, the fall out of contact is caused by the shrinking of the Hubble horizon instead of the more quickly increase rate of the scale factor.

Also during the second stage of the contraction ruled by the kinetic energy of the scalar field we see that since $H^{-1} \sim |t|$ while $a \sim |t|^{\frac{1}{3}}$ the space-time points progressively tend to fall out of casual contact.

After the Big Bounce, thanks to the changing direction of the time flow, the situation in reversed in the expansion ruled by the kinetic energy, with the Hubble horizon increasing more quickly of the scale factor. The becoming casually connected of space-time points goes on at different rates also in the two next phases of the cyclic picture, in the same way of the corresponding radiation and matter dominated phases in the standard scenario.

For what concerns the results obtained in the perturbations chapter, it is interesting to analyse the ostensible duality achieved between the spectrum relative to the single scalar field model in the inflationary and cyclic picture. Indeed the main point for the computation in the cyclic background was to retrieve the successful prediction about the energy density perturbation spectrum given by the inflationary model which are well approximated already by the simple case of one scalar field classical model. However even if the spectrum for the gauge invariant potential variable Φ results to be scale invariant also in the cyclic picture, when we have computed the same spectrum for the other gauge invariant variable ζ we got a blue-shifted spectrum. Hence, since the result that is experimentally relevant is the ζ-spectrum because this gauge invariant variable has the fundamental property of remaining constant once the mode exits the horizon, the single scalar field model is not enough to recover and to prove the duality between ekpyrotic contraction and inflation for what concerns the perturbation spectrum.

The first solution to this problem that has been proposed in the literature, is that at the Big Bounce occurs a mixing between Φ and ζ so that at large scale the scale invariant component of the mix due to Φ dominates. However this stratagem is strongly dependent on the dynamic of what happens during the Big Crunch and the Bounce, which is still not clear and varies from model to model of cyclic universe.

Another way to face the problem is to introduce a second scalar field, which in the higher dimension theories represents the volume modulus of the internal manifold. For example in a four dimensional theory with two scalar fields minimally coupled, one can introduce the potential with the following form:

$$V = -V_1 e^{-\int c_1 d\varphi_1} - V_2 e^{-\int c_2 d\varphi_2} \tag{6.0.1}$$

where c_1 and c_2 are respectively functions of φ_1 and φ_2. The solution in this kind of models, comes from the fact that with the introduction of a second scalar field one then can have entropy perturbations. These can source curvature perturbations and hence the scale-invariance of the spectrum must be proved for entropy perturbations. In the literature concerning cyclic universe this computation is widely described and it is shown that with this prescription one proves the duality between the spectrum of scalar perturbations obtained for inflation and for the ekpyrotic phase. The only aspect that we can underline without performing the computation is that introducing a second field increases the degree of fine tuning of the theory. Indeed, since the degree of fine-tuning of the theory largely lies on the particular shape of the radion potential $V(\varphi)$ and for example in the case of the single scalar field the only parameter for the potential is c which has to satisfy $c^2 >> 1$, in the case of two scalar fields we have instead two functions, $c_1(\varphi_1)$ and $c_2(\varphi_2)$, that will have to satisfy certain requirements in order to make the model work.

Moreover it must be pointed out, as said previously, that it is still not completely clear what happens during the Big Crunch and the Big Bounce and how the perturbed modes behaves in that phase. In the literature can be found numerous hypothesis and computations concerning this still open problem.

Finally we have seen the the tensor spectral index n_T assumes different values in the cases of inflationary or cyclic picture. This is due to the fact that during the phases in which gravitational waves are generated in the two models the gravitational background is completely different. Indeed during inflation the Hubble parameter is large and almost constant, which implies strong gravitational fluctuations and hence a nearly scale invariant spectrum. On the contrary, during the ekpyrotic phase the Hubble parameter is much more smaller, of the order of the nowadays parameter H_0 which is exponentially smaller than the value that it assumes during inflation. As a result of this, the average amplitude of the gravitational perturbations produced is very small, since gravity is almost negligible in that phase. Indeed it is computed in the literature that the gravitational waves produced in a cyclic scenario are tens of orders below the measurable limit for a wide range of frequencies. Moreover, since as we have seen the Hubble parameter increases hyperbolically with time during the ekpyrotic phase, the gravitational waves spectrum tends to be blue-shifted.

In conclusion we can state that even if it represents a fascinating alternative to the standard inflationary picture, the cyclic model of the universe presents at least as many issues that need to be answered as the Big Bang scenario. Nevertheless it offers also interesting and elegant solutions to certain questions which are unsolved or solved in a more elaborate way by the standard model, hence are strongly justified the need and effort in order to measure the gravitational waves spectrum, which as it has been said before, represents the only way known so far to rule out one of the two models.

Appendix

7.1 Appendix A

K.-G. equation to linear order in metric perturbations and $\delta\varphi$

Computing the Klein-Gordon equation to linear order in metric perturbations and $\delta\varphi$ requires some algebra and hence before starting let's underline some facts:

- thanks to Newtonian gauge the metric is diagonal so there are just two cases, $\mu = \nu = 0, i$;
- the expansion of the potential is $\dfrac{\partial V(\varphi_0 + \delta\varphi)}{\partial \varphi} = \dfrac{\partial^2 V}{\partial \varphi^2}\delta\varphi + \dfrac{\partial V}{\partial \varphi}$;
- for convenience let's call $A = (1 + 2\Phi)$ and $B = (1 - 2\Psi)$;
- then we have that $\sqrt{-g} = a^4 A^{\frac{1}{2}} B^{\frac{3}{2}}$.

Now we can start with rewriting equation (5.1.3):

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial(\varphi_0 + \delta\varphi)}{\partial x^\nu}\right) + \frac{\partial^2 V}{\partial \varphi^2}\delta\varphi + \frac{\partial V}{\partial \varphi} = 0 \tag{7.1.1}$$

Let's focus on the first term, starting with the easy case of $\mu = \nu = i$:

$$-\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}a^{-2}B^{-1}\frac{\partial(\varphi_0 + \delta\varphi)}{\partial x^i}\right) = -a^{-2}B^{-1}\frac{\partial^2 \delta\varphi}{\partial x^{i\,2}} \tag{7.1.2}$$

where it has been neglected the term relative to φ_0 since it does not give contributes to the linear approximation equation.

Now let's face the term $\mu = \nu = 0$:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\frac{\partial\varphi}{\partial\eta}\right) = \frac{1}{\sqrt{-g}}\left[\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\frac{\partial\varphi_0}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\frac{\partial\delta\varphi}{\partial\eta}\right)\right] \tag{7.1.3}$$

It is more orderly to analyse separately the two terms of the R.H.S. of the previous equation. The first one gives:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\frac{\partial\varphi_0}{\partial\eta}\right) = g^{00}\varphi_0'' + \varphi_0'\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\right) \tag{7.1.4}$$

where expanding for small Φ and Ψ:

$$\begin{aligned}\sqrt{-g}g^{00} &= a^4 A^{\frac{1}{2}} B^{\frac{3}{2}} A^{-1} a^{-2} \\ &= a^2(1+2\Phi)^{-\frac{1}{2}}(1-2\Psi)^{\frac{3}{2}} \simeq a^2(1-\Phi)(1-3\Psi) + \mathrm{O}(\Psi^2, \Phi^2, \Psi\Phi..) \\ &\simeq a^2(1-(\Phi+3\Psi))\end{aligned} \tag{7.1.5}$$

where the first term of the R.H.S. of (7.1.4) gives using the Klein-Gordon equation for the homogeneous part (5.1.6):

$$\begin{aligned}g^{00}\varphi_0'' &= g^{00}(-2\mathcal{H}\varphi_0' - a^2 V_{,\varphi}) \\ &= -2a^{-2}\mathcal{H}\varphi_0' A^{-1} - V_\varphi' A^{-1}\end{aligned} \tag{7.1.6}$$

and the second term of the R.H.S. of (7.1.4) gives:

$$\begin{aligned}\varphi_0'\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\right) &= \frac{1}{\sqrt{-g}}\varphi_0'\left[2aa'(1-(\Phi+3\Psi)) - a^2(\Phi+3\Psi)'\right] \\ &= \varphi_0'\left[2a^{-2}\mathcal{H}A^{-1}\frac{1-\Phi-3\Psi}{A^{-\frac{1}{2}}B^{\frac{3}{2}}} - \frac{a^2}{\sqrt{-g}}(\Phi+3\Psi)'\right] \\ &\simeq \varphi_0'\left[2a^{-2}\mathcal{H}A^{-1} - a^{-2}(\Phi+3\Psi)'\right] + \mathrm{O}(\Psi^2, \Phi^2, \Psi\Phi..)\end{aligned} \tag{7.1.7}$$

Therefore (7.1.4) is equal to:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\frac{\partial\varphi_0}{\partial\eta}\right) = -a^{-2}\varphi_0'(\Phi+3\Psi)' - V_\varphi' A^{-1}. \tag{7.1.8}$$

The last piece remained to analyse is the second term of the R.H.S. of (7.1.3):

$$\begin{aligned}\frac{\partial}{\partial\eta}\left(\sqrt{-g}g^{00}\frac{\partial\delta\varphi}{\partial\eta}\right) &= a^{-2}A^{-1}\delta\varphi'' + \frac{1}{\sqrt{-g}}\delta\varphi'\frac{\partial}{\partial\eta}(\sqrt{-g}g^{00}) \\ &\simeq a^{-2}A^{-1}\delta\varphi'' + \frac{1}{\sqrt{-g}}\delta\varphi'\frac{\partial}{\partial\eta}(a^2) \\ &\simeq a^{-2}A^{-1}\delta\varphi'' + 2a^{-2}\mathcal{H}\delta\varphi' \end{aligned} \tag{7.1.9}$$

where in the second and third step we have discarded many terms due to expansions since we are interested just in the linear order. Finally we can return to (7.1.1) substituting all the terms found in (7.1.2), (7.1.8) and (7.1.9):

$$\begin{aligned}&- a^{-2}B^{-1}\Delta\delta\varphi \,-\, a^{-2}\varphi_0'(\Phi \,+\, 3\Psi)' \,-\, V_\varphi' A^{-1} \,+\, a^{-2}A^{-1}\delta\varphi'' \\ &+\, 2a^{-2}\mathcal{H}\delta\varphi' \,+\, V_{,\varphi\varphi}\,\delta\varphi + V_{,\varphi} \,=\, 0\end{aligned} \tag{7.1.10}$$

The last step consists just in multiplying everything for a^2A and keeping only the linear order in $\delta\varphi$ and in the metric perturbations. As a result of doing so we obtain the final equation:

$$\delta\varphi'' \,+\, 2\mathcal{H}\delta\varphi' \,-\, \Delta\delta\varphi \,+\, a^2V_{,\varphi\varphi}\,\delta\varphi \,-\, \varphi_0'(\Phi \,+\, 3\Psi)' \,+\, 2a^2V_{,\varphi}\,\Phi \,=\, 0. \tag{7.1.11}$$

7.2 Appendix B

Vacuum Quantum Fluctuations Amplitude (Inflation)

In order to determine the behaviour of the integration constants in the perturbations solutions (C_k and A_k) we need to compute the typical amplitude of vacuum quantum fluctuations $\delta\varphi_L$ on physical scales L (volume $V \sim L^3$). Then for a massless scalar field assumed to be homogeneous within V the action can be written as:

$$S \simeq \int \left(\dot{X}^2 + ... \right) dt \tag{7.2.1}$$

where we have introduced:

- $X = \delta\varphi_L L^{\frac{3}{2}}$ is the canonical quantization variable;
- $P = \dot{X} \sim X/L$ is the conjugate momentum.

Therefore using the Heisenberg uncertainty relation satisfied by these two variables $\Delta X \Delta P \sim 1$ (with $\hbar = 1$) we can compute the minimal amplitude of the quantum fluctuations:

$$\delta\varphi_L L^{\frac{3}{2}} \delta\varphi_L L^{\frac{1}{2}} \sim 1 \quad \Longrightarrow \quad \delta\varphi_L \sim L^{-1} \tag{7.2.2}$$

which together with the fact that $\delta\varphi_L \sim |\delta\varphi_k| k^{\frac{3}{2}}$, where $k \sim a/L$ is the comoving wave-number, gives the result:

$$|\delta\varphi_k| \sim \frac{k^{-\frac{1}{2}}}{a}. \tag{7.2.3}$$

and hence the typical amplitude of fluctuations for a given comoving wave-number k is approximately:

$$\delta_\varphi(k) \sim |\delta\varphi_k| k^{\frac{3}{2}} \sim \frac{k}{a_k} \sim H_{k \sim Ha} \tag{7.2.4}$$

where $Ha_k \sim k$ at the moment of horizon crossing.

7.3 Appendix C

η-dependence of $(1/z)''/(1/z)$ and z''/z

The first step consists in finding the relation between the parameter $p = 2/c^2$ and the parameter characterizing the equation of state $\epsilon = 3/2(1+w)$. From the definition of $w = p/\rho$ for a scalar field as the radion we have:

$$w = \frac{\frac{1}{2}\dot{\varphi}^2 - V}{\frac{1}{2}\dot{\varphi}^2 + V} = \frac{\dot{\varphi}^2}{3H^2} - 1 \tag{7.3.1}$$

In chapter 4 we have proved that in the ekpyrotic phase $\dot{\varphi} = \frac{2}{ct}$ and that $H = \frac{p}{|t|}$, hence:

$$w = \frac{2}{3p} - 1 \quad \Longrightarrow \quad p = \frac{1}{\epsilon}. \tag{7.3.2}$$

Now, we have seen also that in the ekpyrotic contraction the scale factor goes as $a \propto |t|^p$. Rewriting now the parameters for the ekpyrotic phase as functions of the conformal time η and the variable ϵ instead of the coordinate time t and p we have that:

$$\begin{aligned} a(t) = |t|^{\frac{1}{\epsilon}} = |\eta a|^{\frac{1}{\epsilon}} \quad &\Longrightarrow \quad a(\eta) = |\eta|^{\frac{1}{\epsilon - 1}} \\ \varphi(t) \propto \sqrt{\frac{2}{\epsilon}}\ln|a\eta| \quad &\Longrightarrow \quad \varphi(\eta) \propto \frac{\sqrt{2\epsilon}}{\epsilon - 1}\ln|\eta| \\ H(t) = \frac{1}{\epsilon|t|} \quad &\Longrightarrow \quad H(\eta) = \frac{1}{\epsilon}\eta^{\frac{\epsilon}{1-\epsilon}} \end{aligned} \tag{7.3.3}$$

Since they will turn out to be necessary later let's compute some conformal time derivative of H and φ:

$$\begin{aligned} H' &= \frac{1}{1-\epsilon}\eta^{\frac{2\epsilon - 1}{1-\epsilon}}\,, \\ H'' &= \frac{2\epsilon - 1}{(1-\epsilon)^2}\eta^{\frac{3\epsilon - 2}{1-\epsilon}}\,, \\ \varphi' &= \frac{\sqrt{2\epsilon}}{\epsilon - 1}\frac{1}{\eta}\,, \\ \varphi'' &= -\frac{\sqrt{2\epsilon}}{\epsilon - 1}\frac{1}{\eta^2}\,, \\ \varphi''' &= 2\frac{\sqrt{2\epsilon}}{\epsilon - 1}\frac{2}{\eta^3}\,. \end{aligned} \tag{7.3.4}$$

Finally we can start with the coefficient $(1/z)''/(1/z)$. Remembering that $(1/z) = H\varphi'^{-1}$ we compute the second derivative:

$$(1/z)' = H'\varphi'^{-1} - H\frac{\varphi''}{\varphi'^2} \quad \Longrightarrow \quad (1/z)'' = \frac{H''}{\varphi'} - 2H'\frac{\varphi''}{\varphi'^2} - \frac{\varphi'''}{\varphi'^2} + 2\frac{\varphi''^2}{\varphi'^3}\,, \tag{7.3.5}$$

and therefore:

$$\frac{(1/z)''}{(1/z)} = \frac{H''}{H} - 2\frac{H'}{H}\frac{\varphi''}{\varphi'} - \frac{\varphi'''}{\varphi'} + 2\left(\frac{\varphi''}{\varphi'}\right). \tag{7.3.6}$$

Using the derivatives just computed one finds:

$$\begin{aligned}\frac{(1/z)''}{(1/z)} &= \eta^{-2}\left[\frac{2\epsilon^2-\epsilon}{(1-\epsilon)^2} + \frac{2\epsilon}{(1-\epsilon)} - 2 + 2\right]\\ \Longrightarrow \quad &= \frac{\epsilon}{(1-\epsilon)^2\eta^2}.\end{aligned} \tag{7.3.7}$$

Let's pass then to the second coefficient; given $z = H^{-1}\varphi'$ the second derivative immediately follows:

$$z' = H^{-1}\varphi'' - \frac{H'}{H^2}\varphi' \quad \Longrightarrow \quad z'' = 2\frac{H'^2}{H^3}\varphi' - \frac{H''}{H^2}\varphi' - 2\frac{H'}{H^2}\varphi'' + \frac{\varphi'''}{H} \tag{7.3.8}$$

and hence:

$$\frac{z''}{z} = 2\left(\frac{H'}{H}\right)^2 - \frac{H''}{H} - 2\frac{H'}{H}\frac{\varphi''}{\varphi'} + \frac{\varphi'''}{\varphi'}. \tag{7.3.9}$$

Substituting the derivatives:

$$\begin{aligned}\frac{z''}{z} &= \eta^{-2}\left[\frac{2\epsilon^2}{(1-\epsilon)^2} + \frac{\epsilon-2\epsilon^2}{(1-\epsilon)^2} + \frac{2\epsilon-2\epsilon^2}{(1-\epsilon)^2} + \frac{2+2\epsilon^2-4\epsilon}{(1-\epsilon)^2}\right]\\ \Longrightarrow \quad &= \frac{2-\epsilon}{(1-\epsilon)^2\eta^2}.\end{aligned} \tag{7.3.10}$$

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