The Phoenix Universe

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September 19, 2013

Submitted in partial fulfilment of the requirements for the degree of Master of Science of Imperial College London

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To my parents, Maria and Rafael.

Contents

1 INTRODUCTION

The latest results from the *Planck* satellite have been presented as supporting the simplest inflationary models of the early Universe which can be formulated as single field models with plateau potential. However, these models have severe initial condition problems and lead to eternal inflation. One is implicitly making a number of poorly understood assumptions about measures and the multiverse [24]. Moreover, within the context of inflation, the observationally favored potentials are exponentially unlikely since they are less general, require more tuning, inflate for a smaller range of field values and produce exponentially less inflation than power law potentials.

This is not all the story, recent measurements of the top quark and Higgs mass at the LHC and the absence of physics beyond the standard model suggest that the current symmetry breaking vacuum is metastable, with a modest sized energy barrier of $(10^{12}GeV)^4$ protecting us from decay to a true vacuum. The predicted lifetime of the metastable vacuum is large compared to the time since the big bang, so there is no disagreement with observations. A new problem arises: explaining how the Universe managed to get trapped in the false vacuum whose barriers are tiny compared to the Planck density. If the Higgs field lies outside the barrier, its negative potential will tend to cancel the positive energy density of the inflaton preventing it from occurring. On the other hand, considering the case in which the Higgs started in its false vacuum, inflaton would produce de Sitter fluctuations that tend to kick out the Higgs from this minimum unless the inflation potential is plateau like with sufficiently low plateau (see [25]). Those are the same potentials that have the initial conditions and multiverse problems. The search for alternatives to inflation remains as open as ever.

Two main alternatives are the periodic cyclic models and the variable speed of light (VSL) cosmologies. In this work we will investigate the former. Georges Lemaitre introduced the term *phoenix universe* to describe an oscillatory cosmology in the 1920's. This model was ruled out by observations because it required supercritical mass density and couldn't explain dark energy. Jean-Luc Lehners, Paul J. Steinhardt, Neil Turok and others, have proposed a new cyclic theory which avoids these problems: the Universe undergoes an accelerated expansion prior the big crunch, diluting the entropy and black hole density and producing a void cosmos that will reborn from its ashes.

We will proceed by introducing the historical problems as to why cyclic universes have not been fully considered until recently and present a detailed description of the background evolution of the new kind of cyclic theories. Finally we will consider perturbations on the background previously described and we will show how they are in accordance with the latest released *Planck* data.

2 HISTORICAL REVIEW

2.1 Singularity problem

The idea of the cyclic model of the Universe dates back to the early 1920's, when Friedmann, Lemaitre, Robertson and Walker developed their famous set of equations. Starting with the assumption of homogeneity and isotropy of space (which is correct for distances larger than 100Mpc) we are left with

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right] \tag{1}$$

as the line element for the geometry of the Universe. Where, as usual, $a(t)$ is the scale factor, $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$ in spherical coordinates and K is the curvature parameter that takes the value +1 for positive curvature, 0 for the flat case and -1 for negative curvature. Therefore, from the value of K, we have just three possibilities, either we are in an open (hyperbolic), closed (spherical) or flat Universe.

By substituting the metric defined by the previous line element in the Einstein field equation (first published by Einstein in 1915)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \; , \tag{2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda$, and taking the perfect fluid approximation for the right hand side so that the stress tensor takes the form $T^{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu}$, with U^μ being the four velocity of a fluid, so that $T^\nu_\mu = diag(-\rho, p, p, p)$ one can obtain the so called Friedmann equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{K}{a^2} \; , \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \; , \tag{4}$$

where H is the Hubble parameter and G is Newton's constant. We have defined ρ, p and Λ as the fluid density, fluid pressure and cosmological constant respectively and the dot as derivatives with respect to proper time t. We can rewrite them in conformal time $\tau = \int a^{-1}(t)dt$ as

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2 - \frac{K}{a^2} \tag{5}$$

$$\frac{a''}{a} = \frac{4\pi G}{3}(\rho - 3p)a^2 - K \; , \tag{6}$$

where it is supposed $\Lambda = 0$. For this derivation we have used the fact that $\dot{a} = \frac{a'}{a}$ and

$\ddot{a} = \frac{1}{a}\left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]$.

At that time, every possible evolution that our Universe could take had to be solution of these equations. So the question that arises is if a well defined cyclical solution exists. As an example, let us suppose that we are in a closed ($K = 1$) matter or radiation dominated universes. We are going to make use of the equation of state $p = \omega\rho$ and the continuity equation $\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + p) = 0$ so that the energy density of matter ($\omega = 0$) scales as $\rho_m = \rho_0 a^{-3}$ and the one of radiation ($\omega = 1/3$) as $\rho_\gamma = \rho_0 a^{-4}$.

Matter: in the case of dust, we have an equation of state $\omega = 0$, so zero pressure. The acceleration equation (equation (6)) then reads:

$$a'' = \frac{4\pi G}{3}\rho_0 - a \tag{7}$$

where ρ_0 is the energy density of the dust at some arbitrary time and we have used $\rho_m = \rho_0 a^{-3}$. It is easy to check that

$$a(\tau) = \frac{4\pi G}{3}\rho_0(1 - \cos\tau) \tag{8}$$

is a solution.

Radiation: For radiation $\omega = 1/3$ and the acceleration equation(6) becomes

$$a'' = -a \tag{9}$$

in which case

$$a(\tau) = C\sin\tau \tag{10}$$

is a solution with C an arbitrary normalization.

We see that, in both equation (8) and equation (10), we get a periodic solution as $\tau \to \tau + 2\pi$. This would look as a good candidate for our cyclic solution but we face a big problem here. In fact, it is the same problem that came up in 1916 (a little more than a month after the publication of Einstein's theory of general relativity) with the Schwartzschild solution of the field equations for a black hole. The metric is ill defined for some value of the parameters. In the cosmological case, for $a(t) = 0$ (c.f. $r = 0$ in black hole solution), the metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{pmatrix} \tag{11}$$

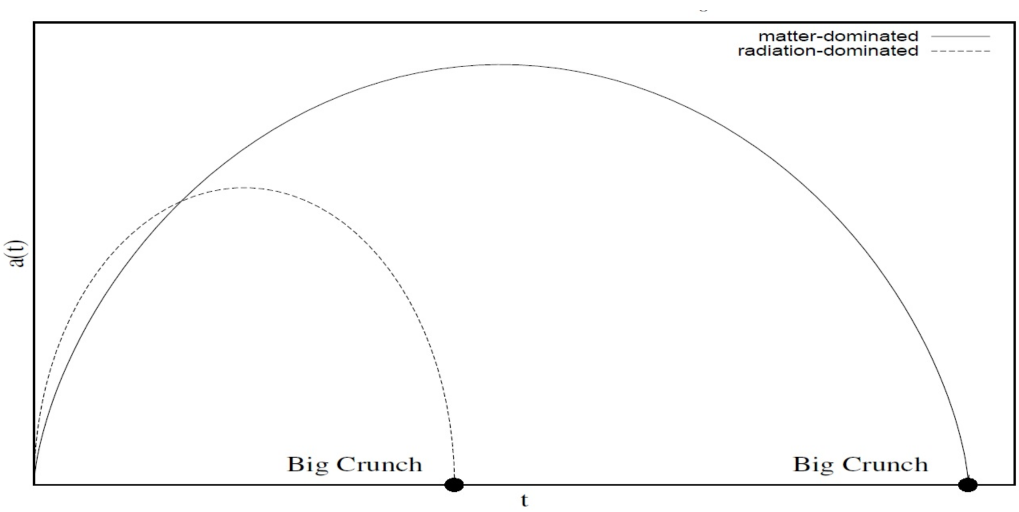

Figure 1: evolution of the scale factor $a(t)$ for the closed Friedmann Universe.

is not invertible i.e. is singular. Hence we have no information about the Riemann (curvature) tensor as it diverges. In the early years of general relativity there was a lot of confusion about the nature of the singularities and it was not clear if this was an actual physical or a coordinate singularity. In the 1960's, the singularity theorems by Hawking and Penrose showed that a big crunch necessarily leads to a cosmic singularity where general relativity becomes invalid. Without a theory to replace general relativity in hand, considerations of whether time and space could exist before the big bang were simple speculation. "Big Bang" became the origin of space and time. However, there is nothing in those theorems that suggest that cyclic behavior is forbidden in an improved theory of gravity, such as string and M theories. That's why the step from big crunch to big bang has to be well described and has to be smooth in any cyclic model. Measurements of the critical mass have shown that the universe is really close to be flat, meaning that we should find an oscillatory solution for a flat geometry rather than a close one.

2.2 Entropy problem

Let's face now other problem suggested by Richard Tolman, reference [2], in the 1930's that would become important in the development of the model. Historically, cyclic cosmologies have been consider attractive because they avoid the issue of initial conditions. The problem is that a study of the thermodynamics of them shows that entropy generated in one cycle would add to the entropy created in the next. Therefore, the duration and maximal size of the Universe increase from bounce to bounce. Extrapolating backwards, the duration of the cycle tends to zero in finite time and the problem of initial conditions remains. This is what we are going to show now.

In order to do this we are going to change our notation slightly, so that it is similar to the one used in the reference. Just for this section we will write the line element of the homogeneous and isotropic Universe as

$$ds^2 = dt^2 - \frac{e^{g(t)}}{[1 + r^2/4R_0^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \tag{12}$$

where R_0 is a constant that can be positive, negative or infinite (plays the role of the curvature parameter K). The time dependence of the spatial part in this line element, instead of being in $a(t)$ as before, is in $e^{g(t)}$, and we will have to take extra care as the metric of the tangent space at any point is $g_{\mu\nu} = diag(1, -1, -1, -1)$, as opposed to the previous $g_{\mu\nu} = diag(-1, 1, 1, 1)$. A detailed calculation on how to compute this metric is done in Appendix 1.

For the purpose of showing the issue, we need first to find the analogue of the second law of thermodynamics $\delta S \geq \frac{\delta Q}{T}$ in general relativity. Starting from this same equation in Galilean coordinates,

$$\frac{\partial}{\partial x^\mu}\left(\phi_0 \frac{dx^\mu}{ds}\right)\delta x \delta y \delta z \delta t \geq \frac{\delta Q_0}{T_0} \tag{13}$$

where ϕ_0 is the proper density of entropy at the point and time of interest as measured by a local observer at rest in the thermodynamic fluid or working substance, the quantities dx_μ/ds are the components of the macroscopic "velocity" of the fluid at that point with respect to the coordinates in use, δQ_0 is the proper heat as measured by a local observer which flows at the proper temperature T_0 into the element of fluid and during the time denoted by $\delta x \delta y \delta z \delta t$, and the signs of equality and inequality refer respectively to the cases of reversible and irreversible processes.

So as to promote this expression to a totally covariant form we can guess for the relativistic second law to be

$$\frac{\partial}{\partial x^\mu}\left(\phi_0 \frac{dx^\mu}{ds}\sqrt{-g}\right)\delta x \delta y \delta z \delta t \geq \frac{\delta Q_0}{T_0} \tag{14}$$

that can be rewritten as

$$\left(\phi_0 \frac{dx^\mu}{ds}\right)_\mu \sqrt{-g}\delta x \delta y \delta z \delta t \geq \frac{\delta Q_0}{T_0} \tag{15}$$

where we have used the tensor density identity $(A^\mu)_\mu\sqrt{-g} = \frac{\partial}{\partial x^\mu}(A^\mu\sqrt{-g})$. Hence, this expression looks quite encouraging as it evidently satisfies the principle of covariance owing to its character as a tensor expression of rank zero, $(\phi_0 dx^\mu/ds)_\mu$ being a scalar since it is the contracted covariant derivative of a vector, $\sqrt{-g}\delta x \delta y \delta z \delta t$ being also a scalar since it is the magnitude of a four dimensional volume expressed in natural measure, and finally $\delta Q_0/T_0$ also being a scalar since it obviously does not depend on the particular coordinates in use. This expression will also satisfy the principle of equivalence since in the tangent space of any point of interest, it will reduce to the special relativity law (13), the contracted covariant derivative $(\phi_0 dx^\mu/ds)_\mu$ being replaced by the ordinary divergence, and the quantity $\sqrt{-g}$ being one.

Once we have a generalized second law we can apply it to the case of study. Using coordinates corresponding to the line element in the form (12) we can take

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0 \qquad \frac{dt}{ds} = 1 \tag{16}$$

due to the comoving character of the coordinates. Consequently

$$\frac{d}{dt}\left(\phi_0 \frac{r^2 \sin\theta e^{\frac{3}{2}g(t)}}{[1 + r^2/4R_0^2]^3}\delta r \delta\theta \delta\phi\right) \geq \frac{\delta Q_0}{T_0} \tag{17}$$

and noting that the proper volume of any element of the fluid would be given by

$$\delta v_0 = \frac{r^2 \sin\theta e^{\frac{3}{2}g(t)}}{[1 + r^2/4R_0^2]^3}\delta r \delta\theta \delta\phi \tag{18}$$

we end up with

$$\frac{d}{dt}(\phi_0 \delta v_0) \geq \frac{\delta Q_0}{T_0}. \tag{19}$$

Now, taking into account the relativistic first law of thermodynamics

$$\frac{\partial(T_\mu^\nu \sqrt{-g})}{\partial x^\nu} - \frac{1}{2}(T^{\alpha\beta}\sqrt{-g})\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = 0 \tag{20}$$

one can show (see Appendix 2) that the energy density and the pressure will obey the following relation,

$$\frac{d}{dt}(\rho \delta v_0) + p\frac{d}{dt}(\delta v_0) = 0 \tag{21}$$

which tells that the proper energy of each element of the fluid would change in accordance with the ordinary equation for the adiabatic expansion or compression. Hence

$$\delta Q_0 = 0, \tag{22}$$

owing to the adiabatic character of the changes. The final form of the relativistic second law takes the form

$$\frac{d}{dt}(\phi_0 \delta v_0) \geq 0 \tag{23}$$

which basically tell us that the entropy of any element of the fluid would ultimately have to increase without limit as the irreversible expansion and contractions continued. It could also remain constant being a reversible process, but all the observations seem to predict otherwise.

Unfortunately, the classical thermodynamics has accustomed us to the idea of a maximum upper value for the possible entropy of an isolated system. To investigate this point it is evident that we may take the proper entropy as depending on the state in accordance with the classical equation

$$d(\phi_0 \delta v_0) = \frac{1}{T_0} d(\rho \delta v_0) + \frac{p_0}{T_0} d(\delta v_0) + \frac{\partial(\phi_0 \delta v_0)}{\partial n_1} dn_1 + ... + \frac{\partial(\phi_0 \delta v_0)}{\partial n_n} dn_n, \tag{24}$$

where the proper energy of the element $\rho \delta v_0$, its proper volume δv_0, and the number of mols $n_1 ... n_n$ of its different chemical constituents are taken as the independent variables which determine its state. In applying this equation to the continued increase in entropy we note in accordance to (21) that the immediate cause which leads to entropy increase cannot be due to the first two terms on the right hand side, since their sum will always be equal to zero. Hence the internal mechanism by which the entropy increase must be due to the presence of the remaining terms on the right hand side corresponding to the irreversible adjustment of composition in the direction of equilibrium.

At first sight it might seem that such an adjustment of concentrations could provide only a limited increase in entropy, since the classical thermodynamics has made us familiar with the existence of a maximum possible entropy for a system having a given energy and volume. The present case differs from the classical case of an isolated system, since the proper energy of every element of fluid does not have to remain constant. This would decrease with time during expansion and increase during contraction. Hence if the pressure tends to be greater during a compression than during the previous expansion the fluid can return to its original volume with increased energy and therefore also with increased entropy. Thus, although the internal mechanism of entropy increase would always be due to the adjustment in concentrations, the possibility for continued entropy increase would have to be due in the long run to an increase in the proper energy of the elements of fluid in the model.

We must now examine the effects of such an increase on the character of later and later cycles. By the computation of the $(0,0)$ component of Einstein's field equations (2), using the stress-energy tensor of the perfect fluid described at the beginning of last section and the metric of the line element (12) we get for the energy density

$$8\pi\rho e^{\frac{3}{2}g} = \frac{3}{R_0^2} e^{\frac{1}{2}g} + \frac{3}{4} e^{\frac{3}{2}g} \dot{g}^2, \tag{25}$$

as an expression which is proportional to the proper energy of any selected element of the fluid with proper volume (18). According to this we see that the volume of any element of the fluid will return to an earlier value when $g(t)$ also returns. Hence the energy of the element at a later return can be greater only in case the square of the velocity $\dot{g}^2$ has a greater value.

Since the value of the energy density at the point of maximum expansion would be given by

$$8\pi\rho = \frac{3}{R_0^2}e^{-g} \tag{26}$$

as $\dot{g} = 0$. The value of g at the maximum will increase without limit so we see that the energy density at this point will get smaller and smaller for later cycles. We may deduce too that the model would spend a greater and greater proportion of its period in a condition of lower density than the observed, for example, at the present in the actual Universe.

Thus, we have proven that the entropy can rise without limit and that in that case, assuming the time spend by the Universe in each cycle is the same, it would be the most of the time in a state of low energy density. We started with just the problem of how to get the initial conditions, and we end up with two: the original one and the fact that for enough amount of cycles, we will have to introduce some fine tunning to explain the high energy density of our Universe today as it would have been more likely to be living in a much less energetic environment.

Those two problems together with the Big Bang being a singularity in Einstein equations are issues the cyclic model proposed here will have to deal with.

In the 1990's tolman's cyclic model based on a closed Universe was ruled out as observations showed that the matter density is significantly less than the critical density and that the scale factor is accelerating. Curiously, the same observations that eliminate Tolman's model would be the key to the kind of cyclic cosmology studied here.

3 OUTLINE OF THE CYCLIC UNIVERSE

In this proposal, the Universe is flat, rather than closed. Instead of having a transition between expansion and contraction due to spatial curvature, it is caused by the introduction of a negative potential energy. Each cycle begins with a "big bang", then there is an expansion phase that includes a period of scalar field, radiation, matter, and quintessence domination. The quintessence would be just the same scalar field with a different value of its potential energy. The acceleration ends, and it is followed by a period of decelerating expansion and then contraction ending up in a "big crunch".

During the accelerated expansion phase, the Universe approaches an almost vacuum state, restoring nearly identical local conditions as existed in the previous cycle. Globally, the entropy would grow from cycle to cycle as Tolman suggested. However, the entropy density, which is all a local observer would be able to measure, has perfect cyclic behavior with entropy density being created at each bounce and being diluted to negligible levels before next one. At the transition from big crunch to big bang matter and radiation are created, restoring the Universe to the high density required for a new big bang phase.

There are two points of view one can take for the understanding of this theory. On one hand we can approach it via the effective 4d theory of the observers "inside" the Universe. On the other hand we could describe the model via the point of view of two 4d boundary branes living in a 5d (or more, depending on the compactification you wish to use) bulk i.e. the point of view of some observer who lives "outside" the Universe. We will mainly use the 4d effective theory for the computations but with continuous references to the 5d one as several of this cyclic model features have a more natural explanation in the later. Let's first get a general picture of how it works in the brane language.

The cyclic model rests heavily in the basic physical notion that the collision between two brane worlds approaching one another along an extra dimension would have generated a hot big bang. This is the so called "ekpyrotic Universe" developed in the early 2000's, see reference [12]. Two four dimensional boundary branes collide, the extra dimension dissapears momentarily and the branes then bounce apart. The ekpyrotic scenario introduces four important concepts: branes approaching one another corresponds to contraction in effective 4 dimensional description, contraction produces a blue shift effect that converts gravitational energy into brane kinetic energy, collision converts some fraction of brane kinetic energy into matter and radiation that can fuel the big bang and the collision and bouncing apart of boundary branes corresponds to the transition from big crunch to big bang.

A key element introduced by the cyclic scenario as opposed to the ekpyrotic one is the assumption of an interbrane potential being the same before and after the collision. After the branes bounce and fly apart, the interbrane potential causes them to draw together and collide again. At distances corresponding to the present day separation between branes, the inter-brane potential energy density should be positive and correspond to the currently observed dark energy. As the brane distance decreases, the interbrane potential becomes negative, the branes approach one another and the scale factor as seen from conventional

Einstein description changes from expansion to contraction. When the branes collide and bounce, matter and radiation are produced and there is a second reversal transforming contraction to expansion so a new cycle begins. The picture goes as follows,

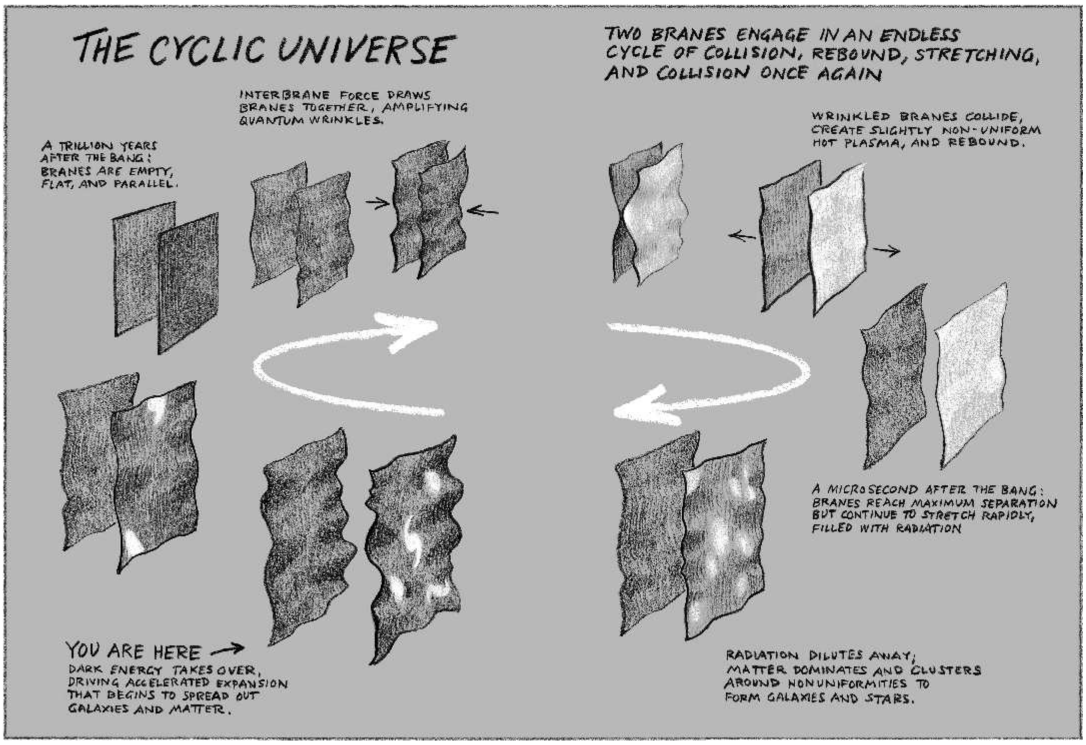

Figure 2: Schematic illustration of the colliding brane picture of the cyclic theory. [Figure taken from the book "Endless Universe" by P.J. Steinhardt and N. Turok].

The central element in the cyclic scenario is a four dimensional scalar field ϕ, parameterizing the interbrane distance or equivalently, the size of the fifth dimension. The branes separation goes to zero as $\phi \to -\infty$ and the maximum brane separation is constrained to some finite value ϕ_{max}. For the most part of our discussion, as already mentioned, we will be framed within the four dimensional effective theory of gravity and matter coupled to the scalar field ϕ. As we will see, the field ϕ will play a crucial role in regularizing the Einstein-frame singularity. Matter and radiation on the brane couple to the scale factor a times a function $\beta(\phi)$ with exponential behavior as $\phi \to -\infty$, such that the product is finite at the brane collision, even though $a = 0$ and $\phi = -\infty$ there.

3.1 Effective Potential

In order to have the phases previously described we would need a potential for the scalar field $V(\phi)$, proposed in reference [6], with the key features:

- The potential tends to zero rapidly as $\phi \to -\infty$.
- The potential is negative for intermediate ϕ as we need an ekpyrotic phase.

- With increasing ϕ the potential gets to a plateau with positive height V_0 given by the present vacuum energy. This positive energy density is essential as we need a phase of accelerated expansion to recover the near vacuum state before the next bounce. However, it is not essential that the positive plateau extends for arbitrarily high values of ϕ since the cyclic solution only explores a finite range of ϕ greater than zero.

An explicit expression would be given by

$$V(\phi) = V_0 \left(1 - e^{-c\phi}\right) F(\phi) \tag{27}$$

where we note that the potential reaches zero when ϕ does. The function of $F(\phi)$ is to turn off the potential rapidly as ϕ goes bellow ϕ_{min}, it should also approach one for $\phi > \phi_{min}$. For example, $F(\phi)$ may be proportional to e^{-1/g_s^2} or e^{-1/g_s} where $g_s \propto e^{\gamma\phi}$ for $\gamma > 0$. This γ would be the string coupling constant. So we get a potential like

$$V_1(\phi) \propto V_0 \left(1 - e^{-c\phi}\right) e^{-1/g_s^2} \tag{28}$$

as well as

$$V_2(\phi) \propto V_0 \left(1 - e^{-c\phi}\right) e^{-1/g_s}. \tag{29}$$

The shape of this potential is shown in Figure 3. In this figure the maximum hight of the potential is approximately given by the constant V_0 i.e. by the value of the vacuum energy observed in todays Universe, of the order of 10^{-120} in Planck units.

3.2 Tour Through One Cycle

Let's now have a rough overview of a whole cycle so that we can afterwards study each stage in much more detail. For our purposes we will be using Figure 3.

Stage 1 represents the present epoch. Nowadays, at a value of $H_0 = (15\ \textit{billion years})^{-1}$, we are at the time when the scalar field is acting as a form of quintessence, the dark energy domination state, in which its potential energy has begun to dominate over matter and radiation. Once the field has reached its maximum and turned back, because the slope at the plateau is so small, ϕ rolls very slowly in the negative direction. When we talk about the acceleration being slow we mean slow compared to inflationary expansion, roughly doubling in size every $H_0^{-1} = 15\ \textit{billion years}$. We would need an acceleration that lasts about trillion years or more (we'll see an easy constraint to satisfy) for the entropy and black hole densities to become negligibly small.

For the sake of simplicity we are going to consider here a simplified version of the equations of motion of the whole theory that we will introduce in future sections (also considered in reference [9]). These are simply obtained by using the Friedmann equations (3) and (4) in a flat cosmological constant free Universe, given by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{30}$$

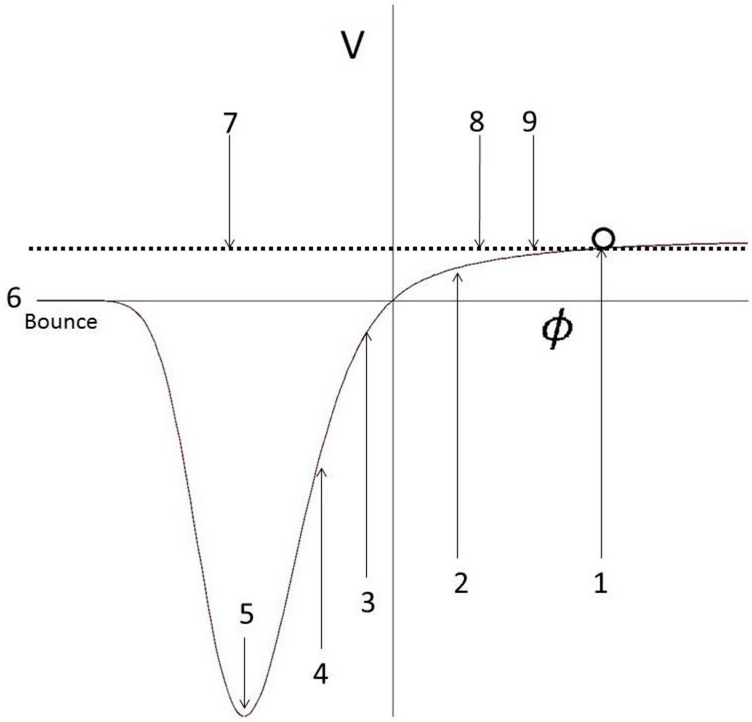

Figure 3: Shape of the potential for the values $V_0 = 1$, $c = 3$ and $\gamma = 0.7$. Observations show that the actual values are $V_0 = 10^{-120}$ in Planck units and $c \approx 10$ but those values wouldn't have allowed us to distinguish all the features of the graph.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ . \tag{31}$$

Recall that the stress-energy tensor for a scalar field is given by the variation of the action defined as the integral over the lagrangian density $\mathcal{L}_\phi = -1/2\partial_\mu\phi\partial^\mu\phi - V(\phi)$,

$$T^\alpha_\beta = g^{\alpha\nu}\partial_\beta\phi\partial_\nu\phi - g^\alpha_\beta\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right] . \tag{32}$$

For a homogeneous field in the rest frame the spatial derivatives vanish and we can identify $T^0_0 = -\rho$ and $T^i_i = p$ to get

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{33}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{34}$$

Hence, for the case of an scalar field, equation (30) and equation (31) become

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{35}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V(\phi)\right). \tag{36}$$

We will generally work in units such that $8\pi G = 1$ except where otherwise noted. When the potential approaches 0, Stage 2, accelerated expansion stops and the scalar kinetic energy becomes comparable to the potential energy. The Universe continues to expand and the kinetic energy decreases as the potential drops below zero. This is the point at which a scale invariant spectrum of fluctuations beyond our current Hubble horizon (large scales) begins to develop in what we will see that could be described as a scale solution with nearly exponential behavior of the potential.

At Stage 3 the potential is sufficiently negative that total energy of the scalar field approaches zero. According to equation (35), $H = 0$ and the Universe would be momentarily static. Going to equation (36), one realizes that $\ddot{a} < 0$ so contraction begins. Perturbations are still evolving.

Once we enter into Stage 4 (almost a second before the big crunch) fluctuations around the current Hubble horizon scale are generated. As the field rolls towards $-\infty$ the scale factor contracts and the kinetic energy increases. In terms of energy conservation, gravitational energy is transformed into scalar field kinetic energy.

Therefore, at Stage 5, the field past the minimum and faces $-\infty$ with kinetic energy becoming increasingly dominant. In fact, this kinetic energy diverges as a approaches 0. This is easy to check as $\ddot{a}/a \to -\infty$ if $a \to 0$ (recall we are decelerating so $\ddot{a} < 0$), so from (36) we see that $\left(\dot{\phi}^2 - V(\phi)\right) \to \infty$, but from Figure 3 we know $V(\phi) \to 0$ so $\dot{\phi}^2$ increases and, in the long term, tends to ∞.

Stage 6 corresponds to the bounce where matter and radiation are generated as the Universe is previously in an state of nearly vacuum. This creation of matter and radiation is possible thanks to the decrease of brane kinetic energy.

Stage 7 happens right after the Big Bang, the acceleration of the Universe has reversed and we encounter a period of scalar field energy density dominated epoch. In order to show that, we are going to make use of the continuity equation

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{37}$$

coming from the time-time component of the energy-momentum conservation $T^{\mu}_{\nu;\nu} = 0$ where

; refers to covariant derivative. Taking the equation of state for a perfect fluid $p = \omega\rho$,

$$\frac{d\rho}{dt} = -3\rho\frac{\dot{a}}{a}\left(1+\omega\right) \tag{38}$$

and getting rid of the time dependence

$$\frac{d\rho}{\rho} = -3\left(1+\omega\right)\frac{da}{a}. \tag{39}$$

For the scalar field case,

$$\omega = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{40}$$

but as $V(\phi) \approx 0$ for the first phase of this Stage, we can say $\omega \approx 1$. This implies that, solving equation (39), $\rho_\phi \propto 1/a^6$. For matter and radiation ω is equal to 0 and 1/3 respectively, so their energy densities scale as $\rho_M \propto 1/a^3$ and $\rho_R \propto 1/a^4$. Hence, for this early stage, the scalar field density clearly dominates over the others. Hence, the motion is almost exactly the time reverse of the contraction between Stage 5 and the big crunch.

However, as the field rolls uphill, radiation becomes important (as it scales like $\rho_R \propto 1/a^4$), breaking the time reversal symmetry. Then we are entering into Stage 8, where radiation dominates and the motion of ϕ is rapidly damped. One expects all these stages (from 2 to 7) to happen really fast compared to the last stages (from 8 to 2) because of the high values of the scalar kinetic energy at the formers. The damping continue during the matter dominated phase (as it scales like $\rho_M \propto 1/a^3$), which begins thousands of years later and undergoes standard cosmology evolution for the next 15 billion years.

At stage 9 the scalar potential energy begins to dominate and cosmic acceleration begins as easily shown by looking at the first Friedmann equation (35), where the kinetic energy is negligible with respect to potential one and the potential energy is approximately equal to the constant V_0,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}V_0 \tag{41}$$

with solution

$$a = a_i e^{\sqrt{\frac{1}{3}V_0}(t-t_i)} \tag{42}$$

which shows an obvious exponential behavior. Eventually, the scalar field rolls back across $\phi = 0$, the energy density becomes zero and cosmic contraction begins so the cycle starts again.

The evolution of the scale factor a is summarized in figure 4.

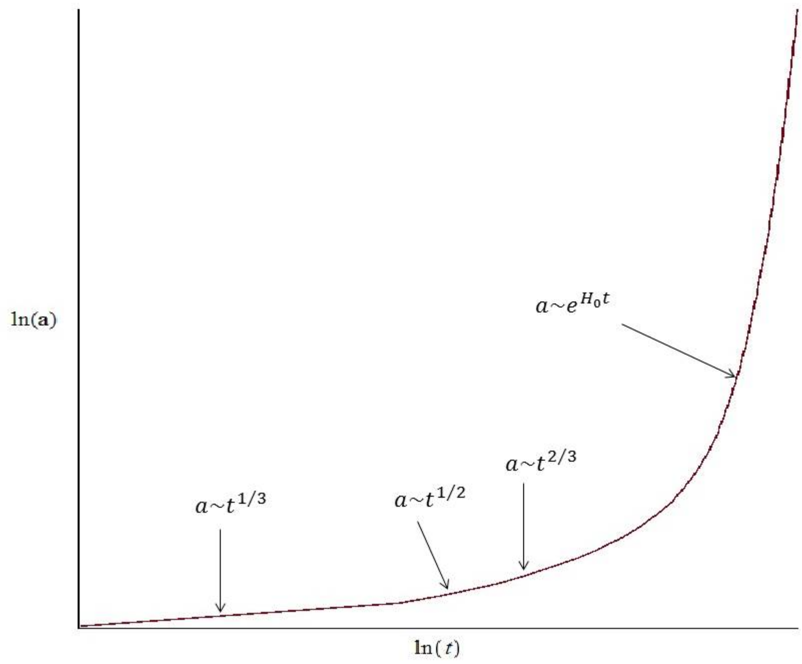

Figure 4: Behavior of a with respect to t for the expansion phase in the cyclic Universe.

4 BRANES POINT OF VIEW

We could start the study of the complete theory by just giving the lagrangian we start with and studying the 4d effective theory it leads to, but there is a natural way to introduce the components that will build it up. The main problem we are trying to solve is, as already mentioned, the divergences appearing at $a = 0$. Naively we could ask ourselves, what if as a approaches zero the energy densities of matter and radiation instead of scaling as $1/a^3$ and $1/a^4$ respectively, would scale as $1/\left(\beta(\phi)a\right)^3$ and $1/\left(\beta(\phi)a\right)^4$, where $\beta(\phi)$ is a function of ϕ which scales as $1/a$ when a tends to zero? Then the densities of matter and radiation would be finite at $a = 0$. It is pretty clear that we would be coupling gravity to this scalar field so, in order to recover Einstein's theory we should be cautious with the explicit form of $\beta(\phi)$.

We will be focusing in the time when a is close to zero i.e. small separation between the branes. This scaling density with $a\beta(\phi)$ can be understood rather simply. But first we shall introduce the Kaluza Klein line element for five dimensional space with a compactified "fifth" dimension.

Kaluza Klein line element: In order to understand how to build a theory of gravity with a compactificatied fifth dimension we better start with the simplest example of the scalar field and then reproduce the same steps in the gravity case.

Consider a 5d free massless scalar field $\phi\left(x^M\right)$ with $M=0,...,4$ and an action

$$S_{5d,\phi}=\int d^5x\left(-\frac{1}{2}\partial_M\phi\partial^M\phi\right). \tag{43}$$

We consider the theory on four dimensional Minkowski space times a circle (where we compactified the fifth variable), $M_4\times S^1$. This is described by letting the coordinate $y\equiv x^4$ have periodicity $y=y+2\pi R$, so

$$\phi\left(x^\mu,y\right)=\phi\left(x^\mu,y+2\pi R\right) \tag{44}$$

and one can expand the y dependence in Fourier modes on the circle

$$\phi\left(x^\mu,y\right)=\sum_{k\in Z}\phi_k\left(x^\mu\right)e^{iky/R}. \tag{45}$$

Substituting into (43), and integrating over y we obtain

$$S_{4d,\phi}=(2\pi R)\int d^4x\left(-\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0\right)-(2\pi R)\sum_{k=1}^{\infty}\int d^4x\left(\partial_\mu\phi_k\partial^\mu\phi_{-k}+\frac{k^2}{R^2}\phi_k\phi_{-k}\right). \tag{46}$$

This describes a 4d theory with a massless scalar ϕ_0 and an infinite tower of massive scalars, known as Kaluza-Klein resonances, labeled by the momentum k, and with masses

$$m_k^2=\frac{k^2}{R^2}. \tag{47}$$

Now, let's follow the same procedure for gravity, i.e. the graviton. We can do so because both are bosonic fields, the way of treating fermionic would be different. Recall the Einstein Hilbert action for 4d gravity

$$S_{4d}=\frac{1}{2k_4^2}\int d^4x\sqrt{-g}R_{4d}\ , \tag{48}$$

where $g=\det(g_{\mu\nu})$, $1/k_4^2=M_p^2/8\pi$, M_p is the Plank mass and R_{4d} is the 4d Ricci scalar of curvature. Promote this action to 5d

$$S_{5d}=\frac{M_5^3}{2}\int d^5x\sqrt{-G}R_{5d}\ , \tag{49}$$

where $G=\det(G_{MN})$, with $M,N=0,...,4$, and R_{5d} is the 5d scalar of curvature. Consider compactifying $y\equiv x^4$ into an S^1, and Fourier expand the metric in exactly the same way as before,

$$G_{MN}\left(x^\mu,y\right)=\sum_{k\in Z}G^k_{MN}\left(x^\mu\right)e^{iky/R}\ . \tag{50}$$

The 4d theory contains a set of massless particles and an infinite tower of massive graviton modes. The massless states turn out to be a 4d graviton $g_{\mu\nu}$, a vector boson A_μ and a scalar σ. We are not going to prove it can be done in a straightforward way from the point of view of representation theory by starting with a 5d representation of the graviton and reducing it to a 4d one. The relations of these modes with the zero mode metric is given by

$$G^0_{MN} = e^{\sigma/3}\begin{pmatrix} g_{\mu\nu} + e^{-\sigma}A_\mu A_\nu & e^{-\sigma}A_\mu(x) \\ e^{-\sigma}A_\mu(x) & e^{-\sigma} \end{pmatrix}. \tag{51}$$

The vector boson arises from the 5d metric mixed component $A_\mu \sim G_{\mu 4}$, and inherit the gauge invariance $A_\mu \to A_\mu - \partial_\mu\lambda$ from the freedom of local parametrization of S^1, $x^4 \to x^4 + \lambda(x^\mu)$. The scalar σ, known as the radion, is a scalar field with vanishing potential, whose vacuum expectation value is arbitrary and parametrizes a microscopic parameter, in this case the S^1 radius via $e^{-\sigma/3} \sim G_{44}$. A better and deeper discussion of this topic can be found in reference [1].

Therefore, the only undone remaining step is choosing a gauge and applying this 5d metric to our case. For simplicity, we choose the gauge that makes $A_\mu = 0$, i.e. $\partial_\mu\lambda = A_\mu$ and we identified the scalar field σ with ϕ as

$$\sigma \equiv -3\sqrt{\frac{2}{3}}\phi. \tag{52}$$

Then, the 5d metric becomes simply

$$G^0_{MN} = e^{-\sqrt{2/3}\phi}\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & e^{3\sqrt{2/3}\phi} \end{pmatrix} \tag{53}$$

and the line element can be written as

$$ds_5^2 = e^{-\sqrt{\frac{2}{3}}\phi}ds_4^2 + e^{-2\sqrt{\frac{2}{3}}\phi}dy^2, \tag{54}$$

where ds_4^2 is the four dimensional line element and y is the fifth spatial coordinate which runs from zero to L.

With this line element one can now write the four dimensional line element in conformal time coordinates $\tau = \int a^{-1}(t)dt$, as $ds_4^2 = a^2(-d\tau^2 + d\vec{x}^2)$. Since from the first Friedmann equation (5) we have $(a'/a)^2 = \frac{1}{6}(\phi')^2$, is clear that $a \propto e^{\phi/\sqrt{6}}$ in the big crunch. Hence a three dimensional comoving volume element $d^3xa^3e^{-\sqrt{\frac{3}{2}}\phi}$ remains finite as a tends to zero. Therefore the density of massive particles $\rho_M \propto 1/\left(\beta(\phi)a\right)^3$ tends to a constant.

What about radiation? The classical argument for its divergence at $a = 0$ states as follows. Consider a set of massive particles in a spacetime with metric $a^2\eta_{\mu\nu}$ which velocities u^μ satisfy the affine parametrization condition $u^\mu u^\nu g_{\mu\nu} = -1$. Hence, if they are comoving i.e. $\vec{u} = 0$, we have $u^0 = a^{-1}$. Then, as photons moving in such a spacetime have constant four momentum $p^\mu = E(1, \vec{n})$ with $\vec{n}^2 = 1$, their energy as seen from the comoving particles would be $-u^\mu p^\nu g_{\mu\nu} = E/a$, which diverges as a tends to zero. However, in the new context, the metric the particles are coupled to, is $e^{-\sqrt{\frac{3}{2}}\phi}a^2\eta_{\mu\mu}$. Consequently, $u^0 = a^{-1}e^{\frac{1}{6}\phi}$ and as proven before $a \propto e^{\phi/\sqrt{6}}$, so the energy of the detected photons is finite as $a \to 0$. The scalar field approaching $-\infty$ cancels the gravitational blueshift.

All this discussion makes us understand why are we going to introduce a term of the form $\beta(\phi)\rho_{R,M}$ in the lagrangian of the full theory.

5 FULL THEORY

Once we have a rough overview of what are the ingredients of the model we are going to move to the study of the full theory in a deeper way (following the track of reference [4]). In order to do so, it will be wise to divide the theory in its multiple phases.

For the first time in this report we present the full action S of the cyclic model (first introduced in reference [6])

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}\mathcal{R} - \frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - V(\phi) + \beta^4(\phi)(\rho_M + \rho_R)\right) , \tag{55}$$

where g is the determinant of the metric $g_{\mu\nu}$, G is Newton's constant and $\mathcal{R}$ is the Ricci scalar. This is an action of a scalar field coupled to gravity and to the fluids ρ_R and ρ_M. For a homogeneous, flat, foliated Universe, we can write in general the line element in terms of the conformal time variable τ

$$ds^2 = a^2(\tau)(-N^2 d\tau^2 + d\vec{x}^2) \tag{56}$$

where a is the Robertson-Walker scale factor as usual and N is the lapse function. It is worthy to stop a while to understand what the lapse function is so that we understand the reason of having it here.

Lapse function: It is a function strongly related with the unit normal vector of a foliated space. In the language of differential geometry, if we foliate the space time in hyperplanes orthogonal to the gradient of t, the lapse function is the proportionality factor between the gradient 1-form dt and the 1-form $\vec{m}$ associated to the normal vector to the hypersurfaces m

$$\vec{m} = -N dt \tag{57}$$

or, in components,

$$m_\alpha = -N\nabla_\alpha t. \tag{58}$$

Then

$$N \equiv \left(-\vec{\nabla}t \cdot \vec{\nabla}t\right)^{-\frac{1}{2}} . \tag{59}$$

The minus sign is chosen so that the vector m is future directed if t is increasing towards the future. Note that the value of N ensures that m is a unit vector i.e. $m \cdot m \equiv g(m, m) = -1$. Note also that $N > 0$. Σ_t would be spacelike iff m is timelike.

The link to our metric is that, if one computes the elapse proper time between the two points p and $p\prime$ the relation obtained is $\delta\tau = N\delta t$.

As it happens with many other concepts in geometry, the best way of understanding an object is having a picture in mind. In this case Figure 5 is the main picture to have a clear idea of what is going on.

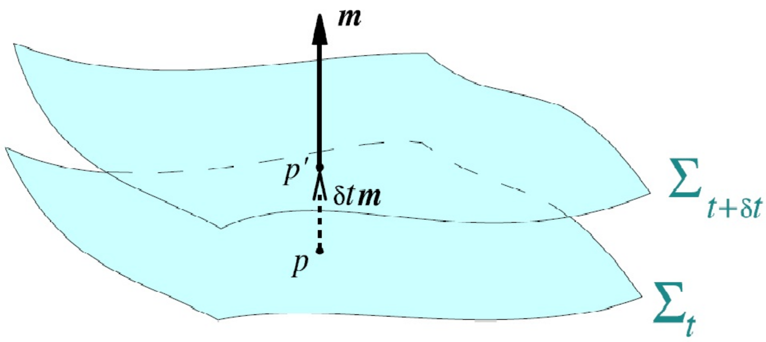

Figure 5: The point $p\prime$ is deduced from $p \in \Sigma_t$ by the displacement δtm belongs to $\Sigma_{t+\delta t}$, i.e. the hypersurface Σ_t is transformed to $\Sigma_{t+\delta t}$ by the vector field δtm. [Figure taken from the notes in "3+1 Formalism and Bases of Numerical Relativity" by Eric Gourgoulhon [5]]

Then, by computing the Ricci scalar $\mathcal{R}$

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = \frac{1}{a^2(\tau)} \left(-\frac{1}{N^2} R_{00} + R_{ii} \right) \tag{60}$$

where $R_{\mu\nu}$ is the Ricci tensor given by

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{61}$$

and the Christoffel symbol which describes the connections between the coordinates is

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2} \left(\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right) \tag{62}$$

we can rewrite the action S as

$$S = \int d^3x d\tau \left[N^{-1} \left(-3a'^2 + \frac{1}{2} a^2 \phi'^2 \right) - N \left((a\beta)^4 (\rho_R + \rho_M) + a^4 V(\phi) \right) \right]. \tag{63}$$

The only unusual term is the coupling of the fluids $\rho_{M,R}$. Recall that for a homogeneous isotropic fluid, the equation of state $p(\rho)$ defines the functional dependence of ρ on the scale factor a, from equation (39), $d\ln\rho/d\ln a = -3(1+\omega)$ with $\omega = p/\rho$.

Assuming that these fluids live on one of the branes they don't couple to the Einstein scale factor a but to the conformaly related scale factor $a\beta(\phi)$ which may be different for the two branes. However, as mentioned above, for large negative values of ϕ one has to have the standard Kaluza Klein result $\beta \approx e^{-\phi/\sqrt{6}}$. This behavior ensures that $a\beta$ is finite at the bounce so the matter and radiation are.

Here we have been foliating space and time without concern, but it is worth remembering that this is not a trivial step. There is still a lot of discussion about the topic. In order to illustrate this, consider the de-Sitter cosmological horizon, which, as an event horizon, can only be understood as a purely 4d phenomenon i.e. it happens in space *and* time. Trying to avoid it would be like trying to avoid tomorrow 10 a.m. On the other hand, with a foliated space-time, avoiding one of these objects looks quite plausible. Thus, we have missed some information in this respect. We are not going to worry much about this as we will use the equations of motion, which turn out to be the same for both actions, (55) and (63).

Therefore, there are two ways of getting the equation of motion, one would be simply varying equation (55) with respect to all its variables (see Appendix 3), the other would be varying (63) with respect to a, N and ϕ, after which N may be set equal to unity. In any way, the equations of motion expressed in terms of proper time t are

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \beta^4\rho_R + \beta^4\rho_M\right) \tag{64}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V + \beta^4\rho_R + \frac{1}{2}\beta^4\rho_M\right) \tag{65}$$

where a dot is a proper time derivative. Note that for the case $\beta = 0$ or $\rho_M = \rho_R = 0$ we recover the equations (35) and (36). The above equations are supplemented (see Appendix 3) by the dynamical equation for evolution of ϕ

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi} - \beta_{,\phi}\beta^3\rho_M \tag{66}$$

and the continuity equation

$$(a\beta)\frac{d\rho_i}{d(a\beta)} = a\frac{\partial\rho_i}{\partial a} + \beta\frac{\partial\rho_i}{\partial\beta} = a\frac{\partial\rho_i}{\partial a} + \beta\frac{\partial\rho_i}{\partial\phi}\frac{\partial\phi}{\partial\beta} = a\frac{\partial\rho_i}{\partial a} + \frac{\beta}{\beta_{,\phi}}\frac{\partial\rho_i}{\partial\phi} = -3(\rho_i + p_i) \tag{67}$$

where p_i is the pressure of the fluid component with energy density ρ_i and the subscript ", ϕ" indicates $d/d\phi$. This last expression can be easily shown taking the continuity equation (37) already seen and promoting it to the full theory $a \to a\beta$. We could have expected the radiation term not to appear in the ϕ-equation as $\rho_R \propto 1/(a\beta)^4$ and so it is just a constant times N in the action, contributing to the Friedmann constraint but not the dynamical equations of motion.

Imposing $\beta(\phi)$ sufficiently flat near the current value of ϕ, these models have modest effects in the late Universe and the standard cosmology is recovered. This is also the basic idea that appears in the so-called chamaleon fields, reference [3], in which there is also the appearance of a fifth force due to the coupling of the energy densities to the scalar field, but whose effect on measurements in the Universe at this time is negligible. Of course, having a measurable fifth force would have been a problem, as the couplings of the Standard Model would have run differently and, for example, we should have had different charge for the electron. However, this fifth force, produces potentially measurable effects, as it will produce violations on the equivalence principle.

5.1 Bounce

With $a \to 0$, the scalar field tends to $-\infty$ so the potential becomes negligible. The scalar field kinetic energy dominates because it scales as a^{-6} whereas radiation and matter scale as a^{-4} and a^{-3} respectively, ignoring β. This, from the Friedmann constraint leads to $\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-6}$ so $\frac{da}{dt} \propto a^{-2}$ and, integrating, $-t \propto a^3$ and so the scale factor a begins to scale as $(-t)^{\frac{1}{3}}$ where the minus appears because we are approaching the big crunch. For the reasons given before $\left(\frac{\dot{a}}{a}\right)^2 \propto \dot{\phi}^2$, what means that $\phi \propto \ln a$, so the scalar field diverges with the logarithm of the scale factor and, consequently, logarithmically with time. Apparently, we have not overcome the problem of the singularity completely as the energy density and Ricci scalar diverges with $(-t)^{-2}$.

Luckily it is only appearances. The singular variables, a and ϕ, can be replace by the non singular variables:

$$a_0 = 2a\cosh((\phi - \phi_\infty)/\sqrt{6}) \tag{68}$$

$$a_1 = -2a\sinh((\phi - \phi_\infty)/\sqrt{6}). \tag{69}$$

where ϕ_∞ is just a constant. From equation (63), in terms of the old variables one has a kinetic line element $-3da^2 + \frac{1}{2}a^2 d\phi^2$, which is clearly singular for $a = 0$. Now, take the change of variables (68) and (69), then,

$$da_0 = 2\left(da\cosh((\phi - \phi_\infty)/\sqrt{6}) + \frac{a}{\sqrt{6}}\sinh((\phi - \phi_\infty)/\sqrt{6})d\phi\right) \tag{70}$$

$$da_1 = 2\left(da\sinh((\phi - \phi_\infty)/\sqrt{6}) + \frac{a}{\sqrt{6}}\cosh((\phi - \phi_\infty)/\sqrt{6})d\phi\right) \tag{71}$$

and as

$$\begin{aligned}
\frac{3}{4}\left(-da_0^2 + da_1^2\right) &= \frac{3}{4}\Bigg[-\left((2\left(da\cosh((\phi - \phi_\infty)/\sqrt{6}) + \frac{a}{\sqrt{6}}\sinh((\phi - \phi_\infty)/\sqrt{6})d\phi\right)\right)^2 + \\
&\qquad \left(2\left(da\sinh((\phi - \phi_\infty)/\sqrt{6}) + \frac{a}{\sqrt{6}}\cosh((\phi - \phi_\infty)/\sqrt{6})d\phi\right)\right)^2\Bigg] \\
&= 3\Bigg[da^2\left(\sinh^2((\phi - \phi_\infty)/\sqrt{6}) - \cosh^2((\phi - \phi_\infty)/\sqrt{6})\right) + \\
&\qquad a^2\frac{1}{6}d\phi^2\left(\cosh^2((\phi - \phi_\infty)/\sqrt{6}) - \sinh^2((\phi - \phi_\infty)/\sqrt{6})\right)\Bigg] \\
&= -3da^2 + \frac{1}{2}a^2 d\phi^2
\end{aligned}$$

we have shown that the line element under the previous reparametrization is $\frac{3}{4}\left(-da_0^2 + da_1^2\right)$, which is perfectly regular at every value. Note that the Einstein frame scale factor a is now given by

$$a = \frac{1}{2}\sqrt{4a^2\cosh^2((\phi-\phi_\infty)/\sqrt{6}) - 4a^2\sinh^2((\phi-\phi_\infty)/\sqrt{6})} = \frac{1}{2}\sqrt{a_0^2 - a_1^2} \tag{72}$$

and that it would vanish in the "light cone" $a_0 = a_1$. Both a_0 and a_1 are "scale factors" as they transform like a under rescaling space-time coordinates (easy to check from their explicit form). We can interpret them as the scale factors of the positive and negative tension branes, so that the coupling of matter to the scalar field is $\beta_0 = 2\cosh((\phi-\phi_\infty)/\sqrt{6})$ for, let's say, brane 0, and $\beta_1 = -2\sinh((\phi-\phi_\infty)/\sqrt{6})$ for, let's say, brane 1.

This is a remarkable result. The new variables a_0 and a_1 define a non singular metric that passes smoothly through the bounce, so the problem with the Einstein Frame parameters a and ϕ, is simply that they are ill defined for certain values of a. No need to say that if the metric $g_{\mu\nu}$ is not singular in one of the coordinate systems, this means it is not singular at all (c.f. Schwartzschild coordinates for the event horizon of a Black Hole). Thus, we overpass one of the main problems introduced in the Historical Review section. Of course we still have to find solutions that match from Big Crunch to Big Bang but this is already a key point of the model.

Now, let's examine the a_0 and a_1 behavior so we have a picture of the trajectories of the cyclic solution in the $a_0 - a_1$ plane. We can have the information required in a compact form by taking a differential equation for $a_0'^2 - a_1'^2$. Based on the same computation from the previous expression for the metric it is easy to check that

$$a_0'^2 - a_1'^2 = -\frac{4}{3}\left(-3a'^2 + \frac{1}{2}a^2\phi'^2\right).$$

Substituting in the expression for a' given by the first equation of motion of the full theory (64)

$$\left(\frac{a'}{a^2}\right)^2 = \frac{1}{3}\left(\frac{1}{2}\phi'^2 a^{-2} + V + \beta^4\rho\right) \tag{73}$$

we get

$$a_0'^2 - a_1'^2 = -\frac{4}{3}\left[-a^4\left(\frac{1}{2}\phi'^2 a^{-2} + V + \beta^4\rho\right) + \frac{1}{2}a^2\phi'^2\right] = \frac{4}{3}\left((a\beta)^4\rho + a^4 V\right).$$

Using relation (72), $a^4 = \frac{1}{16}(a_0^2 - a_1^2)^2$, we end up with

$$a_0'^2 - a_1'^2 = \frac{4}{3}\left((a\beta)^4\rho + \frac{1}{16}(a_0^2 - a_1^2)^2 V(\phi)\right). \tag{74}$$

If the energy density on the right hand side is positive, we say that the trajectory is time-like as $a_0'^2 - a_1'^2 > 0$. If the right hand side is zero $a_0'^2 - a_1'^2 = 0$, like when the potential vanish and we are in an empty Universe $\rho = 0$, then the trajectory is light-like. If the right hand side is negative $a_0'^2 - a_1'^2 < 0$, the trajectory is space-like.

The evolution on the plane (a_0, a_1) is given in Figure 6. This graph tells us about how the scale factor of one of the branes runs as view from the other. The first shocking feature is the continuous expansion of both scale factors except for a small region near the bounce where a_0 decreases slightly. This means that the scale factor in our brane as seen from the other will be expanding forever. Then, how can we match our knowledge in the Einstein's frame about an expansion followed by a contraction i.e. why does the Einstein scale factor a come back to the same value? The answer is that local observers measure physical quantities such as the Hubble constant or the deceleration parameter. Those involve ratios of the scale factor, and its derivatives so a possible normalization of the scale factor cancels out. Hence, to local observers, each cycle appears to be identical to the one before.

This picture solves the entropy problem in an elegant way. The entropy of each brane increases from cycle to cycle, however, the local entropy density decrease due to the late acceleration of the scale factor. The entropy can grow without limit thanks to the infinite extent of the branes. Our universe is not an isolated system.

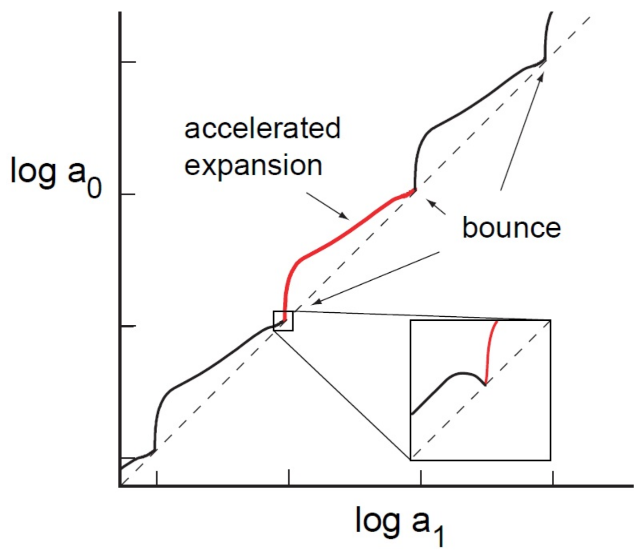

Figure 6: Schematic plot of the a_0−a_1 plane showing a sequence of expansion and contraction. The dashed line represents the "light cone" $a_0 = a_1$ corresponding to a bounce (a=0)[Figure taken from the paper "Cosmic evolution in a Cyclic Universe" by P.J. Steinhardt and N. Turok].

We are now moving to the issue of solving the equations of motion immediately before and after the bounce. Brane scale factors a_0 and a_1 provide the natural setting for this discussion, since neither vanishes at the bounce. We will still be expressing the most of the relevant expressions in singular variables, the reason for this being that those are the ones we can measure.

5.1.1 Incoming solution

Before the bounce, there is little radiation present $\rho \approx 0$, since it has been exponentially diluted in the preceding accelerating phase, and we can approximate $V(\phi) \approx 0$, as the value of the scalar field tends to minus infinity. The Friedmann constraint from the first equation of motion (64) reads $(a'/a)^2 = \frac{1}{6}\phi'^2$ and the scalar field equation becomes

$$\begin{aligned}\ddot{\phi} + 3H\dot{\phi} = 0 \quad &\Longrightarrow -a^{-3}a' + a^{-2}\phi'' + 3a^{-3}a'\phi' = 0 \\ &\Longrightarrow -aa'\phi' + a^2\phi'' + 3aa'\phi' = 0 \\ &\Longrightarrow (a^2\phi')' = 0\end{aligned}$$

where we have used the identity $\frac{d^2\phi}{dt^2} = \frac{d}{ad\tau}\left(\frac{d\phi}{ad\tau}\right) = \frac{1}{a}\left(-\frac{a'}{a^2}\frac{d\phi}{d\tau} + \frac{1}{a}\frac{d^2\phi}{d\tau}\right)$ in the first step. As usual, dots represents derivatives with respect to proper time t and primes with respect to conformal time τ related by $dt = ad\tau$. Therefore, we need to find a solution to the two previous differential equations. Our ansatz for a general solution is

$$\phi = \sqrt{\frac{3}{2}} \ln\left(AH_5(in)\tau\right), \tag{75}$$

$$a = Ae^{\phi/\sqrt{6}} = A\sqrt{AH_5(in)\tau}, \tag{76}$$

$$a_0 = A\left(\lambda + \lambda^{-1}AH_5(in)\tau\right), \tag{77}$$

$$a_1 = A\left(\lambda - \lambda^{-1}AH_5(in)\tau\right), \tag{78}$$

where $\lambda \equiv e^{\phi_\infty/\sqrt{6}}$, A is an integration constant and we will study $H_5(in)$ later on, but for now let us take it approximately as a constant. We choose $\tau = 0$ to be the time when a vanishes so that $\tau < 0$ before collision. Note that in the solution of a we have used the relation we got from the brane point of view and we have substitute the previous ϕ solution. It is clear that expressions (75) and (76) are solutions of $(a'/a)^2 = \frac{1}{6}\phi'^2$ because

$$\left(\frac{a'}{a}\right)^2 = \left(\frac{A\sqrt{AH_5(in)}}{2A\sqrt{AH_5(in)\tau}}\right)^2 = \frac{1}{4\tau^2} = \frac{1}{6}\frac{3}{2}\left(\frac{AH_5(in)}{AH_5(in)\tau}\right)^2 = \frac{1}{6}\phi'^2$$

and of $(a^2\phi')' = 0$ because

$$\left(a^2\phi'\right)' = \left(\sqrt{\frac{3}{2}}\frac{A^3H_5(in)\tau}{AH_5(in)\tau}\right)' = (constant)' = 0\ .$$

Solutions (77) and (78) also make perfectly sense as

$$a_0' - a_1' = \left(\lambda^{-1}A^2H_5(in)\right)^2 - \left(-\lambda^{-1}A^2H_5(in)\right)^2 = 0$$

and, from equation (74), plugging in $\rho \approx 0$ and $V(\phi) \approx 0$ we get the same result, that is, (a_0, a_1) are time-like at the bounce. The Hubble constants as define in terms of the brane scale factors are a_0'/a_0^2 and a_0'/a_0^2 which at $\tau = 0$ are

$$\frac{a_0'}{a_0^2} = \frac{\lambda^{-1}A^2H_5(in)}{A^2\left(\lambda + \lambda^{-1}AH - 5(in)\tau\right)^2} = \lambda^{-3}H_5(in) \tag{79}$$

$$\frac{a_1'}{a_1^2} = \frac{-\lambda^{-1}A^2H_5(in)}{A^2\left(\lambda - \lambda^{-1}AH - 5(in)\tau\right)^2} = -\lambda^{-3}H_5(in) \tag{80}$$

respectively. One can rewrite the proper time as a function of the conformal

$$t = \int a d\tau = A\sqrt{AH_5(in)}\int \tau^{1/2}d\tau = \frac{2}{3}\sqrt{A^3H_5(in)}\tau^{3/2}$$

so

$$\tau = \left(\frac{3}{2}\frac{t}{\sqrt{A^3H_5(in)}}\right)^{2/3}$$

and, substituting in the explicit form of the scalar field which ultimately give us the most of information at this point,

$$\phi = \sqrt{\frac{3}{2}}\ln\left(AH_5(in)\left(\frac{3}{2}\frac{t}{\sqrt{A^3H_5(in)}}\right)^{2/3}\right) = \sqrt{\frac{2}{3}}\ln\left(\frac{3}{2}H_5(in)t\right) \tag{81}$$

The constant $H_5(in)$ has to be negative so that the solution is valid. It has a natural physical interpretation as a measure of the contraction rate of the extra dimension. From equation (54),

$$H_5 \equiv \frac{dL_5}{Ldt_5} \equiv \frac{d(e^{\sqrt{\frac{2}{3}}\phi})}{dt_5} = \sqrt{\frac{2}{3}}\dot{\phi}e^{\sqrt{\frac{3}{2}}\phi} \tag{82}$$

where $L_5 \equiv Le^{\sqrt{\frac{2}{3}}\phi}$ is the proper length of the extra dimension, L is a parameter with dimensions of length, and t_5 is the proper time in the five dimensional metric,

$$dt_5 \equiv ae^{-\sqrt{\frac{1}{6}}\phi}d\tau = e^{-\sqrt{\frac{1}{6}}\phi}dt.$$

As the extra dimension shrinks to zero, H_5 tends to a constant. Actually, this can be straightforwardly shown if we take into account that $\phi \to -\infty$ implies $\frac{1}{2}\dot{\phi}^2 \propto a^{-6}$, because $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $V(\phi) \to 0$ while $\rho_\phi \propto a^{-6}$ (already proven). Then, $\dot{\phi} \propto a^{-3}$, using $a \propto e^{\frac{1}{\sqrt{6}}\phi}$, we get $\dot{\phi} \propto e^{-\sqrt{\frac{3}{2}}\phi}$ and, from (82), we check that H_5 tends to a constant $H_5(in)$ because its ϕ dependence cancels out.

5.1.2 Outgoing solution

Immediately after the bounce, given that the extra dimension shrinks adiabatically and that we can ignore backreaction from particle production, we expect $H_5(out) = -H_5(in)$. The kinetic energy of the scalar field scales as a^{-6} and radiation scales as a^{-4}, so the former dominates at small a. It is convenient to rescale a so that it is unity at scalar kinetic energy-radiation equality, t_r, and denote the Hubble constant H_r. Then we redefine $\widetilde{a} \equiv a/a_r$ where a_r can be computed in the usual way by equating $\Omega_\phi(a_r) = \Omega_R(a_r)$. Scaling to the present $\Omega_\phi(a_0)\left(\frac{a_0}{a_r}\right)^6 = \Omega_R(a_0)\left(\frac{a_0}{a_r}\right)^4$ we get the result $a_r = \sqrt{\frac{\Omega_\phi(a_0)}{\Omega_R(a_0)}}a_0$. Here Ω_i is the usual normalized energy density of the i'th fluid. We will be calling $\widetilde{a}$, a.

The Friedmann constraint reads

$$a'^2 = \frac{1}{2}H_r^2\left(1 + a^{-2}\right) \tag{83}$$

with solution

$$\phi = \sqrt{\frac{3}{2}}\ln\left(\frac{2^{\frac{5}{3}}\tau H_5^{\frac{2}{3}}(out)H_r^{\frac{1}{3}}}{H_r\tau + 2^{\frac{3}{2}}}\right), \tag{84}$$

$$a = \sqrt{\frac{1}{2}H_r^2\tau^2 + \sqrt{2}H_r\tau}, \tag{85}$$

$$a_0 = a\left(\lambda^{-1}e^{\phi/\sqrt{6}} + \lambda e^{-\phi/\sqrt{6}}\right) = A\left(\lambda(1 + \frac{H_r\tau}{2^{\frac{3}{2}}}) + \lambda^{-1}2^{\frac{1}{6}}H_r^{\frac{1}{3}}H_5^{\frac{2}{3}}(out)\tau\right), \tag{86}$$

$$a_1 = a\left(-\lambda^{-1}e^{\phi/\sqrt{6}} + \lambda e^{-\phi/\sqrt{6}}\right) = A\left(\lambda(1 + \frac{H_r\tau}{2^{\frac{3}{2}}}) - \lambda^{-1}2^{\frac{1}{6}}H_r^{\frac{1}{3}}H_5^{\frac{2}{3}}(out)\tau\right), \tag{87}$$

Here the constant $A = 2^{\frac{1}{6}}(H_r/H_5(out))^{\frac{1}{3}}$ has been defined so that we match a_0 and a_1 to the incoming solution. We can check (85)

$$\begin{aligned} a'^2 &= \frac{1}{4}H_r\frac{\left(H_r\tau + \sqrt{2}\right)^2}{\frac{1}{2}H_r\tau^2 + \sqrt{2}\tau} = \frac{1}{2}H_r\frac{\frac{1}{2}H_r^2\tau^2 + \sqrt{2}H_r\tau + 1}{\frac{1}{2}H_r\tau^2 + \sqrt{2}\tau} \\ &= \frac{1}{2}H_r^2\left(1 + \frac{1}{\frac{1}{2}H_r^2\tau^2 + \sqrt{2}H_r\tau}\right) = \frac{1}{2}H_r^2\left(1 + a^{-2}\right) \end{aligned}$$

so the condition is satisfied. The Hubble constant for the positive brane at $\tau = 0$ is

$$\frac{a_0'}{a_0^2} = \frac{A(\lambda H_r 2^{-3/2} + \lambda^{-1} 2^{1/6} H_r^{1/3} H_5^{2/3})}{A^2 \left(\lambda(1 + 2^{-3/2} H_r \tau) + \lambda^{-1} 2^{1/6} H_r^{1/3} \tau\right)^2} = \lambda^{-3} H_5 + \lambda^{-1} H_r^{2/3} 2^{-5/3} H_5^{1/3}. \tag{88}$$

Similarly for the negative brane

$$\frac{a_0'}{a_0^2} = -\lambda^{-3} H_5 + \lambda^{-1} H_r^{2/3} 2^{-5/3} H_5^{1/3}. \tag{89}$$

There is a limiting case when $a_0' = 0$ i.e. $\lambda 2^{-2/3} H_r + \lambda^{-1} 2^{1/6} H_r^{1/3} H_5^{2/3} = 0 \implies H_r = \lambda^{-3} 2^{5/2} H_5$. For $H_r < \lambda^{-3} 2^{5/2} H_5$, the case of relatively little radiation production (can be checked from Friedmann equation (64) as ρ_R is the only "free" parameter under these conditions), immediately after collision a_0 is expanding but a_1 is contracting. On the other hand, for $H_r > \lambda^{-3} 2^{5/2} H_5$, both scale factors expand after collision. We shall focus here on the former case, in which we are close to the adiabatic limit.

If no scalar potential were present we would have a constant solution in the long term

$$\phi_C = \phi(\tau \to \infty) = \sqrt{\frac{3}{2}} \ln\left(\frac{2^{5/3} H_5^{2/3}(out) H_r^{1/3}}{H_r}\right) = \sqrt{\frac{2}{3}} \ln\left(2^{\frac{5}{2}} \frac{H_5(out)}{H_r}\right) \tag{90}$$

However, the presence of the potential $V(\phi)$ alters the expression for the final resting value of the scalar field. As ϕ crosses the potential well traveling in the positive direction, H_5 is renormalized to a different value $\hat{H}_5(out)$. This last value of ϕ provides a reasonable estimate if we use the corrected $\hat{H}_5(out)$, which will be computed in the next section. Note that the ϕ dependence can be simply understood. While the Universe is kinetic energy dominated, a grows as $t^{1/3}$ and ϕ increases logarithmically

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \dot{\phi}^2 \implies \frac{t^{-2/3}}{t^{1/3}} \propto \dot{\phi} \implies \phi \propto \ln(t).$$

However, when the Universe becomes radiation dominated and a grows as $t^{1/2}$, Hubble damping increases and ϕ converges to the previous finite limit.

5.2 Potential Well

Consider the motion of ϕ back and forth across the potential well. The main aim is to make sure of the asymmetry in the behavior before and after the bounce necessary for the cyclic solution to work. Over most of this region the potential may be approximated by $V(\phi) \equiv -V_0 e^{-c\phi}$. For this pure exponential, there is a scaling solution, as shown in reference [8],

$$a(t) = |t|^p, \tag{91}$$

$$V = -V_0 e^{-c\phi} = -\frac{p(1-3p)}{t^2}, \tag{92}$$

where

$$p = \frac{2}{c^2}, \tag{93}$$

which is an expanding or contracting solution depending on whether t is positive or negative. It is useful to notice that relation (92) means

$$\phi = \frac{1}{c}\ln\left(\frac{t^2 V_0}{p(1-3p)}\right) \tag{94}$$

therefore,

$$\dot{\phi} = \frac{2}{ct} .$$

Then, ϕ varies logarithmically with $|t|$, what is reasonable as it matches the solution in the previous section. We are not going to give a rigorous proof of this solution here but we will show that it is consistent with equation of motion introduced earlier. By this time $\beta(\phi)$ is negligible and so we should be solving $\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$, then, on one side

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{p^2}{t^2}$$

and on the other

$$\frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = \frac{1}{3}\left(\frac{2}{c^2t^2} - \frac{p(1-3p)}{t^2}\right) = \frac{1}{3t^2}(p - p + 3p^2) = \frac{p^2}{t^2}$$

so we can say it satisfies the condition.

5.2.1 Incoming solution

At the end of the expanding phase, there is a period of accelerated expansion which makes the Universe empty, homogeneous and flat (and so the brane) followed by ϕ rolling down the potential into the well. The previous solution is accurately followed until ϕ encounters the potential minimum.

Consider the behavior of ϕ under small shifts in the contracting phase. In the background scalar field equation and the Friedmann equation, we set $\phi = \phi_B + \delta\phi$ and $H = H_B + \delta H$, where ϕ_B and H_B are the background quantities. Then equation (66) becomes

$$\ddot{\phi}_B + \delta\ddot{\phi} + 3(H_B + \delta H)(\dot{\phi}_B + \delta\dot{\phi}) + (V_{,\phi} + \delta V_{,\phi}) = 0 .$$

Using (66) again for the background variables and limiting to first, linear, order

$$\delta\ddot{\phi} + 3H_B\delta\dot{\phi} + 3\dot{\phi}_B\delta H + \delta V_{,\phi} = 0 \tag{95}$$

and substituting the scale solution we get

$$\delta\ddot{\phi} + \frac{3p}{t}\delta\dot{\phi} + 3\frac{2}{ct}\delta H + \delta V_{,\phi} = 0\ .$$

Having

$$\delta V_{,\phi} = \frac{d}{d\phi}\left(cV_0 e^{-c\phi}\delta\phi\right) = -c^2 V_0 e^{-c\phi}\delta\phi = -c^2\frac{p(1-3p)}{t^2}\delta\phi$$

and $H = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right)^{1/2}$ so

$$\delta H = \frac{\dot{\phi}\delta\dot{\phi} + \delta V}{\sqrt{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right)^{1/2}},$$

we plug in again the scale solution

$$\delta H = \frac{\frac{2}{ct}\delta\dot{\phi} + \frac{cp(1-3p)}{t^2}\delta\phi}{\sqrt{3}\left(\frac{2}{c^2t^2} - \frac{p(1-3p}{t^2}\right)^{1/2}} = \frac{tc}{\sqrt{3}(2 - c^2p(1-3p))^{1/2}}\left(\frac{2}{ct}\delta\dot{\phi} + c\frac{p(1-3p)}{t^2}\delta\phi\right).$$

Hence, replacing in equation (95)

$$\delta\ddot{\phi} + \left(3p + \frac{3}{\sqrt{3}}\sqrt{\frac{4}{c^2(2 - c^2p(1-3p))}}\right)\frac{1}{t}\delta\dot{\phi} + \left(\frac{3}{\sqrt{3}}\sqrt{\frac{c^2p^2(1-3p)^2}{2 - c^2p(1-3p)}} - c^2p(1-3p)\right)\frac{1}{t^2}\delta\phi = 0$$

and making use of relation (93) we finally end up with

$$\delta\ddot{\phi} + \frac{1+3p}{t}\delta\dot{\phi} - \frac{1-3p}{t^2}\delta\phi = 0. \tag{96}$$

This result is remarkably similar to the one we would obtain if we consider a newtonian perturbation theory in an expanding background, in which we have $\ddot{\delta} + 2H\dot{\delta} + (c_s^2k^2a^{-2} - 4\pi G\rho_0)\delta = 0$ and, for a flat, matter dominated Universe where $a \propto t^{2/3}$, $H = 2/3t$ and $4\pi\rho_0 = 2/3t^2$ we get

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

but here δ being defined like $\delta\rho/\rho_0$. This happens because in both cases we have scaling solutions. Hence, some results can be link to each other. Solutions to (96) are $\delta\phi \approx t^{-1}$

and $\delta\phi \approx t^{1-3p}$ for $p \ll 1$. In the contracting phase, the former solution grows as t tends to zero. However, this solution is simply an infinitesimal shift in the time to the big crunch $\delta\phi \approx t^{-1} \propto \dot{\phi}$. Such a shift provides a solution to Einstein-scalar equations because they are time translation invariant (conserve the energy as another effect of this), but it is physically irrelevant since it can be removed by a redefinition of the zero of time. The second solution is a physical perturbation mode and it decays as t tends to zero. Therefore, we find that the background solution is an attractor in this phase.

Consider now the incoming and outgoing velocity parametrized by $H_5(in)$ and $H_5(out)$. We can calculate the value of the incoming velocity within the scaling solution by treating the prefactor $F(\phi)$ of the potential (3) as a Heaviside function

$$F(\phi) = \begin{cases} 1 & \phi > \phi_{min} \\ 0 & \phi < \phi_{min} \end{cases}$$

where ϕ_{min} is the value of ϕ at the minimum of the potential. We compute the velocity of the field as it approaches ϕ_{min} and use energy conservation at the jump in V to infer the velocity afterwards. From the right $\frac{1}{2}\dot{\phi} + V = 2/c^2t^2 - p(1-3p)/t^2 = 3p^2/t^2$ where we made use of relation (93). This must equal the total energy $\frac{1}{2}\dot{\phi}^2$ evaluated just to the left of ϕ_{min}. Then as $V_{min} = p(1-3p)/t_{min}^2 \implies t_{min} = \sqrt{p(1-3p)/V_{min}}$, we find that $\dot{\phi} = \sqrt{6}p/t = \sqrt{6pV_{min}/(1-3p)}$ at the minimum so, according to definition (82)

$$H_5(in) \approx -\frac{\sqrt{8}|V_{min}|^{\frac{1}{2}} e^{\sqrt{\frac{3}{2}}\phi_{min}}}{c\sqrt{1-6c^{-2}}} \,. \tag{97}$$

This solution can be matched with the outgoing expanding solution with

$$H_5(out) = -(1+\chi)H_5(in), \tag{98}$$

that is positive by construction and where χ is a small parameter arising from the inelasticity of the collision. To obtain cyclic behavior we need χ to be positive i.e. the outgoing velocity to exceed the incoming one. There are two main effects within this theory that can cause χ to be positive. Either χ is generically greater than zero if more radiation is generated on the negative tension brane than on the positive at collision, or χ gets a positive contribution from the coupling of $\beta(\phi)$ to the matter created by the branes at the collision. We shall simply assume a given small positive χ.

5.2.2 Outgoing solution

The outgoing solution is very close to the time reverse of the incoming since χ is very small. Consequently, we can keep using the scale solution but with positive t. However, the contribution of χ becomes increasingly significant with time. For χ greater than zero, H_5 remains positive and ϕ overshoots the potential well. We can analyze this overshoot by treating χ as a perturbation and using the solutions $\delta\phi \approx t^{-1}$ and $\delta\phi \approx t^{1-3p}$ got earlier. The latter grows in the expanding phase as opposed to the contracting phase. One can compute the perturbation in δH_5 in this growing mode by matching at ϕ_{min} as before finding $\delta H_5 = 12\chi H_5^B/c^2$ where H_5 is the background value at the minimum. Beyond this point, δH_5 grows with the exponential inside H_5^B, i.e. with $e^{\sqrt{\frac{3}{2}}\phi}$ and using the scaling solution (94) we check $\delta H_5 \propto e^{\sqrt{\frac{3}{2}}\phi} \propto e^{\ln t^{\sqrt{6}/c}} = t^{\sqrt{6}/c}$ whereas in the background scaling solution H_5 decays with $e^{(\sqrt{\frac{3}{2}}-c/2)\phi}$.

When the perturbation is of the order of the background value $\delta H_5 \approx H_5^B$, the trajectory no longer obeys the scaling solution and the potential becomes irrelevant. The departure from the scaling trajectory occurs at the value

$$\phi_{dep} = \phi_{min} + \frac{2}{c} \ln \frac{c^2}{12\chi} \ . \tag{99}$$

As ϕ passes beyond ϕ_{dep} the kinetic energy overwhelms the negative potential and passes onto the plateau V_0 with H_5 nearly constant

$$\hat{H}_5(out) \approx \chi \left(\frac{c^2}{12\chi}\right)^{\sqrt{6}/c} H_5(in) \ , \tag{100}$$

until the radiation, matter, and vacuum energy become significant and H_5 is damped away to zero.

5.3 Radiation, matter and quintessence

Once the scalar field surpass the potential well, it gets onto the plateau V_0 where the Universe goes through the well known stages of radiation and matter domination to end up in the quintessence or dark energy domination. It is fully known from standard cosmology what happens with the evolution of the scale factor and the densities of energy, matter and radiation. Accordingly, we will pay more attention to variables such as ϕ and H_5.

As already discuss, while overcoming the potential well, $H_5(out)$ is nearly canceled. Hubble damping begins and the value of scalar field tends to the value in equation (90)

$$\phi_C = \sqrt{\frac{2}{3}} \ln \left(2^{\frac{5}{2}} \frac{\hat{H}_5(out)}{H_r} \right) \tag{101}$$

The dependence is quite clear, increasing $\hat{H}_5(out)$ pushes ϕ further, likewise lowering H_r delays radiation domination, allowing the logarithmic increase in ϕ to continue longer in the kinetic energy dominated phase.

With the kinetic energy redshifting, the gently sloping potential gradually becomes important, finally reversing the scalar field motion. The solution of the scalar field equations is, after expanding solution (84) for large τ, converting to proper time $t = \int a(\tau)d\tau$ and matching,

$$\dot{\phi} \approx \frac{\sqrt{3}H_r}{a^3(t)} - a^{-3}\int_0^t dta^3 V_{,\phi} \; . \tag{102}$$

We define $a(t)$ to be unity at kinetic-radiation equality density. During the matter and radiation eras, the first term scales as t^{-2} and $t^{-3/2}$ respectively. For a slowly varying field, $V_{,\phi}$ is nearly constant, and the potential gradient term scales linearly with t, so eventually dominates.

As the field has turned around and starts to roll back towards the potential well, the second term in equation (102) dominates. We require to be a substantial epoch of vacuum energy domination before the next big crunch as we need to dilute the density of radiation, matter, and black holes. The number of e-foldings is given by

$$N_e \equiv \ln\left(\frac{a(t_{final})}{a(t_{initial})}\right) = \int_{t_i}^{t_f} H dt$$

that, in terms of the potential, is

$$N_e = \int_{t_i}^{t_f} H\frac{dt}{d\phi}dt = \int_{t_i}^{t_f} \frac{H}{\dot{\phi}}d\phi = \int_{\phi_i}^{\phi_f} \frac{V}{V_{,\phi}}d\phi.$$

In our case, for the potential of the model

$$N_e = \int_{\phi_i}^{\phi_f} \frac{V}{V_{,\phi}}d\phi \approx \frac{e^{c\phi_c}}{c^2}. \tag{103}$$

Let us put a concrete example, say that we want the number of baryons per Hubble radius to be diluted below unity, then, we set $e^{3N_e} \geq 10^{80}$ or $N_e \geq 60$. This would be easily satisfied if ϕ_c is of order one in Plank units.

We can safely say that ϕ leaves the slow roll regime when $e^{-c\phi}$ exceeds $3/c^2$. At this point, V_0 in the potential becomes irrelevant and we get the, now familiar, scale solution for ϕ until we reach the potential minimum. At this point, density perturbations are generated via the ekpyrotic mechanism. The Einstein factor a is still expanding. The Universe finally enters contraction when the density of the scalar field reaches zero at a negative value of the potential energy.

Contraction phase is well described by the scaling solution, in which, as a reminder, $a \approx (-t)^p$ and $\dot{\phi} = 2/ct = \sqrt{2p}/t$ with $t < 0$. Then, taking back the non-singular variables we see that from equation (69)

$$\dot{a}_1 = -2\left(\dot{a}\sinh\left(\frac{1}{\sqrt{6}}(\phi - \phi_\infty)\right) + \frac{1}{\sqrt{6}}\dot{\phi}a\cosh\left(\frac{1}{\sqrt{6}}(\phi - \phi_\infty)\right)\right),$$

and substituting the scale solution

$$\dot{a}_1 = -2\left(p\frac{(-t)^p}{-t}\sinh\left(\frac{1}{\sqrt{6}}(\phi-\phi_\infty)\right)+\frac{1}{\sqrt{6}}\frac{\sqrt{2p}}{t}(-t)^p\cosh\left(\frac{1}{\sqrt{6}}(\phi-\phi_\infty)\right)\right).$$

Therefore, using again equations (69) and (68), we end up with a differential equation for the velocity of the brane's scale factor of the form

$$\dot{a}_1 = \frac{1}{t}\left(pa_1 - \sqrt{\frac{p}{3}}a_0\right) \ , \tag{104}$$

which is greater than zero for $p < 1/3$. Thus, even when a is undergoing contraction, the effect of the motion of ϕ is enough to make a_1 expand throughout this phase. Hence, matter residing on this brane would see continuous expansion all the way to the big crunch. The same argument but in the expanding phase shows that a_1 actually undergoes a small amount of contraction for a brief period of time.

5.4 Duration of each epoch

We now study how rapidly ϕ travels before and after the bounce. The time spent to the left of the potential well is basically the same in the ingoing and outgoing stages as long as $\chi \ll 1$, that is,

$$|t_{min}| \approx \frac{c}{3\sqrt{2|V_{min}|}} \ . \tag{105}$$

For the outgoing solution, right before the radiation domination i.e. in the epoch where we have left the scaling solution, we can integrate the definition of H_5 to get the time dependence with respect to ϕ

$$t(\phi) = \int\frac{\phi}{\dot{\phi}} = \sqrt{\frac{2}{3}}\int\frac{e^{\sqrt{\frac{3}{2}}\phi}}{H_5(\phi)}d\phi \approx \frac{2}{3}\frac{e^{\sqrt{\frac{3}{2}}\phi}}{\hat{H}_5(out)} \ . \tag{106}$$

The formula for the time before the big crunch in the scaling solution for large values of c looks

$$t(\phi) = -\sqrt{\frac{2}{|V_{min}|}}\frac{e^{\frac{c}{2}(\phi-\phi_{min})}}{c} = -\frac{6e^{\frac{c}{2}(\phi-\phi_{min})}}{c^2}|t_{min}| \ , \tag{107}$$

where the exponential factor makes the time to the big crunch longer than the time from the big bang. Physically, this effect is due to the increase in the brane velocity H_5 after the bounce which implies a positive value of χ.

When does the scalar field turn around? This, of course, depends on the explicit form of the potential. It can happen either in the radiation, matter or quintessence epochs. We can roughly make the computation by taking, $V_0 \approx t_0^{-2}$, where t_0 is the present age of the universe, around 10^5 times the matter domination length t_m. For example, if ϕ_{max} is reached in the radiation era, from equation (102),

$$\frac{t_{max}}{t_m} \approx 10^4 \left(\frac{t_r}{t_m}\right)^{\frac{1}{5}} \left(\frac{V}{V_{,\phi}}(\phi_C)\right)^{\frac{2}{5}} < 1 \; . \tag{108}$$

For a turn around in the matter era we require,

$$3 \times 10^{-4} \leq \left(\frac{t_r}{t_m}\right)^{\frac{1}{6}} \left(\frac{V}{V_{,\phi}}(\phi_C)\right)^{\frac{1}{3}} \leq 30 \; . \tag{109}$$

Finally, if the field runs to very large ϕ_C , so that $V_{,\phi}/V(\phi_C) \approx ce^{-c\phi_C}$ is exponentially small, ϕ turns around in the dark energy dominated era. This maximum value of ϕ_C can be computed by taking (103) for large c and in the limit $t_r \gg \chi^{-1} t_{min}$ (remember that the subscript r refers to the time of scalar field-radiation equality) as

$$\phi_C - \phi_{min} \approx \sqrt{\frac{2}{3}} \ln\left(\chi \frac{t_r}{t_{min}}\right) \; . \tag{110}$$

6 PERTURBATIONS

In this section we will discuss the generation of energy density, or scalar, perturbations. At this point, it is well known that the production of those perturbations happens in the ekpyrotic phase, hence, we will direct our efforts to the perturbative study in this scenario. Our final aim is to show that, as mentioned in reference [13], the cyclic models are not under such a severe pressure due to *Planck 2013* data. That is why the computations for now on won't be done in so much detail as in previous sections.

Although the outcome will be similar to the density fluctuation spectrum for inflation, the mechanism is very different. In the inflationary picture, the physical mechanism for generating fluctuations lies on the existence of a strong gravitational background in which de Sitter fluctuations excite the inflaton and all other fields with masses smaller than the Hubble parameter H during inflation. Nevertheless, in the ekpyrotic picture, the mechanism is non-gravitational and only excites scale-invariant fluctuations in fields with steep potentials.

6.1 Scalar Field Perturbations

Let's first consider the simple example of an scalar field in the ekpyrotic phase in the absence of gravity,

$$S = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 + V_0 e^{-c\phi} \right) . \tag{111}$$

The scale invariance is the result of three main features. First, the action is clearly classically scale invariant as shifting $\phi \to \phi + \epsilon$ and a rescaling of the coordinates $x^\mu \to x^\mu e^{c\epsilon/2}$ just rescales the action by $e^{c\epsilon}$, hence, is a symmetry. Second, by re-scaling $\phi \to \phi/c$ and redefining V_0, the constant c can be absorbed into Planck's constant $\hbar \to \hbar/c^2$ in the expression $iS/\hbar$ governing quantum theory. Finally, note that ϕ has dimensions of mass in 4d.

Taking into account that we come from an accelerating diluting phase, sensible initial conditions for the background are homogeneity and zero energy density. The solution for the scalar field is determined by the scaling symmetry (94) and the dependence of the perturbation is given by $\delta\phi \propto t^{-1}$. On long wavelengths, for frozen modes causally disconnected i.e. $|kt| \ll 1$, we expect perturbations to follow this dependence. Explicitly, set $\phi = \phi_b(t) + \delta\phi(t, \vec{x})$ so the field equation becomes

$$\delta\ddot{\phi} = -V_{,\phi\phi}\delta\phi + \nabla^2\delta\phi . \tag{112}$$

Using $V_{,\phi\phi} = c^2 V = -c^2\dot{\phi}_b^2/2 = -2/t^2$ and expanding $\delta\phi(t, \vec{x}) = \sum_k (a_k \chi_k(t) e^{ikx} + h.c.)$, where a_k is the annihilation operator and $\chi_k(t)$ the positive frequency modes we get the dependence

$$\ddot{\chi}_k = \frac{2}{t^2}\chi_k - k^2\chi_k . \tag{113}$$

Computing the variance of the fluctuation for (112)

$$\langle \delta\phi^2 \rangle = \hbar \int \frac{k^2 dk}{4\pi^2} \frac{1}{k^3 t^2} \propto \hbar t^{-2} \ . \tag{114}$$

Note that it is invariant under under $\phi \to c\phi$ and $\hbar \to c^2\hbar$, and that the constant of proportionality is dimensionless. Therefore, $\delta\phi$ must have a scale invariant spectrum and $n_s - 1 = 0$.

6.1.1 First generalization

Let's try to make this condition of scale invariance a little bit less restrictive by redefining the ekpyrotic potential as

$$V = -V_0 e^{-\int c(\phi) d\phi} \ , \tag{115}$$

where we set $c(\phi)$ to be a slowly varying function so that we assure the nearly scale invariance. The background solution obeys

$$-t\sqrt{2V_0} = \int e^{\frac{1}{2}\int c(\phi)d\phi} d\phi \ . \tag{116}$$

We can expand in derivatives of c as $e^{1/2\int c(\phi)d\phi} = \frac{2}{c}\frac{d}{d\phi} e^{1/2\int c(\phi)d\phi}$ and integrate twice by parts to have

$$\int e^{\int c/2} = \frac{2}{c} e^{\int c/2} \left(1 + 2\frac{c_{,\phi}}{c^2}\right) - \int 4 \left(\frac{c_{,\phi}}{c^3}\right)_{,\phi} e^{\int c/2} \ . \tag{117}$$

Hence, making use of the fact $V_{,\phi\phi} = (c^2 - c_{,\phi})V$ and combining (116) and (117) we end up with

$$V_{,\phi\phi} = -\frac{2}{t^2}\left(1 + 3\frac{c_{,\phi}}{c^2}\right) + \mathcal{O}(c_{,\phi\phi}) \tag{118}$$

The correction $3c_{,\phi}/c^2$ can be treated as a constant to first approximation, so we are able to compute the spectral index as follows. The positive frequency solution (113) is promoted by changing $2 \to 2(1 + 3(c_{,\phi}/c^2))$, which give us the Hankel function $H_\nu^{(2)}(-kt)$, with $\nu = \frac{3}{2}(1 + \frac{4}{3}c_{,\phi}/c^2)$. Taking the small argument expansion of the Hankel function

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^2 e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2} - \nu)_m \Gamma(\nu + m + \frac{1}{2})}{m\Gamma(\nu + \frac{1}{2})(2iz)^m} + \mathcal{O}(z^{-p})\right] \tag{119}$$

the term k^{-3} in (114) becomes instead $k^{-3(1+(4/3)c_{,\phi}/c^2)}$. Therefore, to leading order in derivatives of c, the spectral index is given by

$$n_s - 1 = -4\frac{c_{,\phi}}{c^2} \ . \tag{120}$$

This index is the non gravitational contribution to the fluctuation spectrum.

6.1.2 Second generalization

We now generalize the potential to two decoupled fields with an approximately invariant spectrum. There is no good reason to do so yet, as we have been working with just one, but this is going to become a crucial point, as we will see later, in the transformation from entropic to curvature perturbations. The generalization is straightforward,

$$V_{tot} = -V_1 e^{-\int c_1(\phi_1) d\phi_1} - V_2 e^{-\int c_2(\phi_2) d\phi_2} \tag{121}$$

where V_1 and V_2 are positive constants. The only conditions that we impose are that both fields have to diverge simultaneously to minus infinity with slow varying c_i. As before, we will have a nearly scale invariant fluctuations. In particular, we can define the "entropic" perturbation as

$$\delta s \equiv \frac{\dot{\phi}_1 \delta\phi_2 - \dot{\phi}_2 \delta\phi_1}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}} \tag{122}$$

The entropy perturbation satisfies the relation proven in reference [14]

$$\ddot{\delta s} + \left(k^2 + V_{ss} + 3\dot{\theta}^2\right)\delta s = 0 \tag{123}$$

where

$$V_{ss} = \frac{\dot{\phi}_2^2 V_{,\phi_1\phi_1} - 2\dot{\phi}_1\dot{\phi}_2 V_{,\phi_1\phi_2} + \dot{\phi}_1^2 V_{,\phi_2\phi_2}}{\dot{\phi}_1^2 + \dot{\phi}_2^2} \tag{124}$$

$$\dot{\theta} = \frac{\dot{\phi}_2 V_{,\phi_1} - \dot{\phi}_1 V_{,\phi_2}}{\dot{\phi}_1^2 + \dot{\phi}_2^2} \tag{125}$$

For the simplest case we get $\dot{\theta} = 0$ which background's scalar field trajectory is a straight line. The relation is then exactly the one governing single field fluctuations since we can write $\dot{\phi}_2 = \gamma\dot{\phi}_1 \implies V_{tot} = V(\phi_1) + \gamma^2 V(\phi_2/\gamma)$ where γ is a proportionality constant, we have integrated (125), and we can call $\phi_1 = \phi$. Hence, the power spectrum of δs is scale invariant. The results from the previous section (114) are recovered.

6.1.3 Including gravity

Including gravity into our general ekpyrotic action in full units,

$$S = \int d^4\sqrt{-g}\left(\frac{1}{16\pi G}\mathcal{R} - \frac{1}{2}\sum_{i=1}^{N}(\partial\phi_i)^2 - \sum_{i=1}^{N} V_i(\phi_i)\right) , \tag{126}$$

where all the potentials have the exponential shape $V_i(\phi_i) = -V_i e^{-c_i \phi_i}$. The general scaling solution becomes

$$a = (-t)^p \,, \quad \phi_i = \frac{2}{c_i} \ln(-A_i t) \,, \quad V_i = \frac{2A_i^2}{c_i^2} \,, \quad p = \sum_i \frac{2}{c_i^2} \,. \tag{127}$$

The entropy perturbation equation (123) in flat space and for just two fields is replaced, see reference [14], by

$$\ddot{\delta s} + 3H\dot{\delta s} + \left(\frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2 \right) \delta s = \frac{4k^2 \dot{\theta}}{a^2 \sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}} \Phi \tag{128}$$

where Φ is the Newtonian potential. Focusing again in the case $\dot{\theta} = 0$ so that the Newtonian potential cannot be a source of fluctuations we can solve the equations in a simple way. Because of the scaling symmetry of the background we have again $\dot{\phi}_2 = \gamma \dot{\phi}_1$.

It is convenient to define a rescaled entropy field in terms of conformal time $\delta S \equiv a(\tau) \delta s$ so that (128) becomes

$$\delta S'' + \left(k^2 - \frac{a''}{a} + a^2 V_{,\phi\phi} \right) \delta S = 0 \,. \tag{129}$$

The crucial term governing the spectrum of the perturbations is then

$$\tau^2 \left(\frac{a''}{a} - a^2 V_{,\phi\phi} \right) \,. \tag{130}$$

When this quantity is close to 2, one recovers the scale invariant solution. Let's define now

$$\epsilon \equiv \frac{3}{2}(1+\omega) \equiv \frac{\dot{\phi}_1^2 + \dot{\phi}_2^2}{2H^2} = \frac{(1+\gamma^2)\dot{\phi}^2}{2H^2} = \frac{c^2}{2(1+\gamma^2)} \tag{131}$$

We study (130) as an expansion in inverse powers of ϵ and its derivatives with respect to $N = \ln(a/a_f)$, where note that N decreases as we go downhill. The first term is obtained by differentiating the relation $\dot{H} = -(1/2) \sum_i \dot{\phi}_i^2$,

$$\frac{a''}{a} = 2H^2 a^2 \left(1 - \frac{1}{2} \epsilon \right) \tag{132}$$

The second term is found by is found differentiating (131) twice with respect to time and using the background solutions and the definition of N,

$$a^2 V_{,\phi\phi} = -a^2 H^2 \left(2\epsilon^2 - 6\epsilon - \frac{5}{2} \epsilon_{,N} \right) + \mathcal{O}(\epsilon^0) \tag{133}$$

We just have to express $\mathcal{H} \equiv (a'/a) = aH$ in terms of τ. From equation (132) we obtain

$$\mathcal{H}' = \mathcal{H}^2(1-\epsilon) \ , \tag{134}$$

which integrates to

$$\mathcal{H}^{-1} = \int_0^\tau d\tau_0(\epsilon - 1) \ . \tag{135}$$

Inserting unity $1 = d\tau'/d\tau'$ in the integral and integrating by parts we have

$$\mathcal{H}^{-1} = \epsilon\tau \left(1 - \frac{1}{\epsilon} - (\epsilon\tau)^{-1} \int_0^\tau \epsilon' \tau_0 d\tau_0 \right) \ . \tag{136}$$

Iterating

$$-(\epsilon\tau)^{-1} \int_0^\tau \epsilon' \tau_0 d\tau_0 = \frac{\epsilon'\tau}{\epsilon} - (\epsilon\tau)^{-1} \int_0^\tau \frac{d}{d\tau_0}(\epsilon'\tau_0) d\tau_0 \ . \tag{137}$$

Now, given the fact that $\epsilon' = \mathcal{H}\epsilon_{,N}$, and that to leading order $1/\epsilon$, $\mathcal{H}$ can be replaced by its value in the scaling solution, $\mathcal{H}\tau = \epsilon^{-1}$, we can rewrite the second term on the right hand side as

$$(\epsilon\tau)^{-1} \int_0^\tau \frac{d}{d\tau_0}(\epsilon'\tau_0) d\tau_0 = (\epsilon\tau)^{-1} \int_0^\tau \frac{1}{\epsilon} \left(\frac{\epsilon_{,N}}{\epsilon}\right)_{,N} d\tau_0 \ , \tag{138}$$

which shows that this term is of order $1/\epsilon^2$ and can thus be neglected. Putting all these pieces together

$$\mathcal{H}^{-1} = \int_0^\tau d\tau_0(\epsilon - 1) \approx \epsilon\tau \left(1 - \frac{1}{\epsilon} - \frac{\epsilon_{,N}}{\epsilon^2}\right) \ , \tag{139}$$

and substituting in the crucial term entering the entropy perturbation equation,

$$\tau^2 \left(\frac{a''}{a} - a^2 V_{,\phi\phi}\right) = 2\left(1 - \frac{3}{2\epsilon} + \frac{3}{4}\frac{\epsilon_{,N}}{\epsilon^2}\right) \ . \tag{140}$$

Using the same argument to get the spectral tilt as the one for equation (120) we get

$$n_s - 1 = \frac{2}{\epsilon} - \frac{\epsilon_{,N}}{\epsilon^2} \ . \tag{141}$$

The first term on the right hand side is a gravitational contribution that tends to blueshift the spectrum while the second tends to redshift this.

6.2 Conversion of entropic to curvature perturbations

In general, there are two independent modes, curvature perturbations and time delay fluctuations. In inflation, the curvature perturbation on comoving hypersurfaces is the growing mode. In the contracting phase, however, the roles are reversed, the curvature fluctuations shrink to zero, and the time delay modes grow. This does not, by itself, create fluctuations in temperature and density. A mechanism is needed to go from time delay modes to curvature ones before or at the bounce.

A theorem by Bardeen (1980) shows that the curvature fluctuation amplitude is conserved for modes outside the horizon. If this were true for the cyclic model, decaying curvature fluctuations before the bounce would imply negligible curvature fluctuations after it. Then, the cyclic model would be inconsistent with observations. It can be shown that (see reference [15]) if the cyclic picture could be reduced to a 4d effective theory with a single scalar field (representing the radion), no method of conversion before the bounce is known.

However, the 4d effective picture is just an approximation describing a few of the degrees of freedom of the 5d colliding brane picture. The branes define a precise hypersurface for the bounce, the time slice in which each point on one brane is in contact with a point in the other (see reference[16]). Since ϕ is the modulus field that determines the bounce between the branes, one could imagine that this corresponds to a surface with uniform $\delta\phi = -\infty$. However, the scalar field measures the distance between branes only in the case where they are static. If the branes are moving, there are corrections to the distance relation due to excitation of the bulk modes that depends on the brane speed H_5 and the bulk curvature scale.

The colliding brane picture cannot, then, reduce to a single scalar field degree of freedom plus gravity. If it did, we would be able to choose a gauge where the scalar field is uniform and there is no curvature perturbations. On the other hand, analyzing the problem in 5d, one can show that it is possible to choose a gauge where one degree of freedom is spatially uniform, but not both. This means that there are always some local quantities that indicate non uniformity, and so there is a mixing of growing and decaying modes near the bounce resulting in a nearly scale invariant spectrum of curvature fluctuations.

The same effect can be obtained in a 4d effective theory by introducing two scalar fields with scale invariant fluctuations in a contracting ekpyrotic phase. This is the reason why we introduced a multiple scalar field study of perturbations in the previous subsection. One of the fields produce time delay modes, whereas the other results in a second scale invariant entropic mode that can be transformed into curvature mode right before the big crunch.

This entropic mechanism shows that we can produce a scale invariant spectrum of perturbations in a contracting phase and that an extra dimension and branes are not required to have an ekpyrotic or cyclic model. This is a remarkable result, the brane picture is still easier to understand certain concepts and it still fits perfectly into the model, but is not necessary for getting the results expected (see reference [17]). We put an example here of this perturbation transformation.

Suppose there are two scalar fields, ϕ_1 and ϕ_2, living on the half plane $-\infty < \phi_1 < \infty$, $-\infty < \phi_2 < 0$. The scalar field bounce off a boundary in moduli space creating the curvature perturbation. Following the steps of reference [14] the curvature perturbation $\mathcal{R}$, using the linearized Einstein-scalar field equations, is given by

$$\dot{\mathcal{R}} = -\frac{H}{\dot{H}}\left(g_{ij}\frac{D^2\phi^i}{Dt^2}s^j - \frac{k^2}{a^2}\Psi\right) \tag{142}$$

where $g_{ij}(\phi)$ is the Kahler metric that becomes δ_{ij} for the flat space, D^2/Dt^2 is the usual geodesic operator that reduces to an ordinary derivative in the flat case, and the $N-1$ entropy perturbations

$$s^i = \delta\phi^i - -\dot{\phi}^i\frac{g_{jk}\dot{\phi}^j\delta\phi^k}{g_{lm}\dot{\phi}^l\dot{\phi}^m} \tag{143}$$

are the components of $\delta\phi^j$ orthogonal to the background trajectory. We can rewrite in our case

$$s^1 = \frac{-\dot{\phi}_2\delta s}{\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}}\ , \qquad\qquad s^2 = \frac{\dot{\phi}_1\delta s}{\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}}\ , \tag{144}$$

so the curvature perturbations are generated by entropy perturbations. Integrating (142), for details see reference [18],

$$\langle\mathcal{R}^2\rangle = \hbar\frac{c^2|V_{min}|}{3\pi^2}\frac{\gamma^2}{(1+\gamma^2)^2}(1+\ln(t_{end}/t_{ref}))^2\int\frac{dk}{k} \equiv \int\frac{dk}{k}\Delta^2_{\mathcal{R}}(k)\ , \tag{145}$$

where t_{end} is the time at the end of the ekpyrotic phase and t_{ref} is the time at which the conversion occurs, in this example, $t_{ref} > t_{end}$. Observations on the current Hubble horizon indicate $\Delta^2_{\mathcal{R}}(k) \approx 2.2\times10^{-9}$ and so (forgetting the logarithm) $c|V_{min}| \approx 10^{-3}$ in Planck units $M_{pl} = 1$.

6.3 Spectral index and non gaussianity

When the entropic perturbations are suddenly converted into curvature perturbations, the latter inherit the spectral tilt given in (141). As a first approach we can compute the tilt by re-expressing (141) in terms of the number of e-folds N_e, where $dN_e = (\epsilon-1)N$ and $\epsilon \gg 1$. Then,

$$n_s - 1 = \frac{2}{\epsilon} - \frac{d\ln\epsilon}{dN_e}\ . \tag{146}$$

If we estimate $\epsilon \approx N_e^\alpha$, the previous equation changes to

$$n_s - 1 \approx \frac{2}{N_e^\alpha} - \frac{\alpha}{N_e} \tag{147}$$

so the sign is sensitive to α. The parameter α is roughly equal to one for nearly exponential potentials so the spectral tilt will be slightly blue $n_s \approx 1+1/N \approx 1.02$. However, if $\alpha > 0.14$, the spectral tilt is red. The potential deviates from the exponential regime when is close to the minimum though, and we can have, for example, values of $\alpha \approx 2 \implies n_s \approx 0.97$. These examples represent the range of values that can be achieved by entropically-induced curvature perturbations, roughly $0.97 < n_s < 1.02$.

A second way of analyzing the spectral index is by taking the explicit potential from the second generalization section and applying it directly onto (141). Using the relation $\dot{\phi}_2 = \gamma\dot{\phi}_1$:

$$n_s - 1 = \frac{4(1+\gamma^2)}{c^2 M_{Pl}^2} - \frac{4c_{,\phi}}{c^2} , \tag{148}$$

where we have restored the factors of Planck mass so that we clearly see that the first term in the right hand side is a gravitational term. For pure exponentials, $c_{,\phi} = 0$, and taking plausible values as $c = 20$ and $\gamma = 1/2$ we get $n_s \approx 1.01$. Taking into account the deviation of the potential from pure exponential, one can use as an example $c \propto \phi^\lambda$ and $\int c(\phi)d\phi \approx 125$. Then, as $n_s - 1 = -0.03\lambda/(1+\lambda)$, we will have a range of $0.97 < n_s < 1$ for $0 < \lambda < \infty$ that agree with the model independent computation we had.

In order to compare the cyclic spectral tilt with the inflationary one, it is useful to express the former in terms of the parameters

$$\bar{\epsilon} \equiv \left(\frac{V}{V_{,\phi}}\right)^2 = \frac{1}{c^2} = \frac{1}{2(1+\gamma^2)\epsilon} , \qquad \bar{\eta} \equiv \left(\frac{V}{V_{,\phi}}\right)_{,\phi} . \tag{149}$$

Then, the spectral index reads

$$n_s - 1 = \frac{4(1+\gamma^2)}{M_{Pl}^2}\,\bar{\epsilon} - 4\bar{\eta} . \tag{150}$$

Finally, for inflation,

$$n_s - 1 = -6\epsilon + 2\eta \tag{151}$$

where $\epsilon \equiv (1/2)(M_{Pl}V_{,\phi}/V)^2$ and $\eta \equiv M_{Pl}^2 V_{,\phi\phi}/V$ are the slow-roll parameters. Both contributions have a gravitational origin. Thus, the range of spectral tilts for the simplest inflationary and ekpyrotic models are slightly different. The ekpyrotic is closer to $n_s = 1$, but there is also a big overlap, specially if we consider more general potentials.

Concerning non-gaussianity, we are going to see that it is reasonable to expect a higher values in the cyclic model rather than inflationary ones. A general density fluctuation spectrum can be characterized by the n-point correlation functions

$$\langle \rho(x_1)\rho(x_2)...\rho(x_{n-1})\rho(x_n)\rangle \ . \tag{152}$$

The discussion above about the scale invariance and tilt referred to the two point function. If the spectrum is gaussian, all n-point fluctuations for odd n are zero while for even n you can expand them in terms of two-point functions. Both inflation and ekpyrotic models predict a predominantly gaussian spectrum but also small non-gaussian contributions. These can be detected by measuring three-point functions i.e. the bispectrum. A deviation from zero would be a sign of non- gaussianity.

In both theories the non gaussianity is due to non-linear evolution of scalar fields that varies from point to point i.e. local non-gaussianity. It can be expressed as a correction to the leading linear gaussian curvature perturbation, $\mathcal{R}_L$. Following the convention in reference [19] $\mathcal{R} = \mathcal{R}_L - \frac{3}{5} f_{NL}\mathcal{R}_L^2$. The parameter f_{NL} is nearly scale invariant and can be both positive and negative, it is also a measure of non-gaussianity. A positive sign corresponds to negative skewness in the CMB (more cold spots) and positive skewness in the matter distribution.

There is a simple intuitive way of understanding the generically several order of magnitudes difference between inflationary/ekpyrotic models in the f_{NL} prediction. The difference in the equation of state during the period in which density perturbations are generated is the key. The density perturbations spectra is coming from scalar fields developing nearly scale invariant fluctuations evolving along an effective potential $V(\phi)$. However, the potential is nearly constant during an inflationary phase in order to obtain $\omega \approx -1$ or, equivalently, $\epsilon_{inf} \equiv \frac{3}{2}(1+\omega) \ll 1$. By contrast, the potential should be exponentially steep and negative to obtain $\epsilon_{ekp} \gg 1$ in the ekpyrotic phase. The non-gaussian amplitude for inflation depends on the deviation of the potential from perfect flatness. Nevertheless, a steep potential means that the scalar field in the ekpyrotic model necessarily have nonlinear self-interactions making the non-gaussianity rise. Predictions from cyclic models show f_{NL} of order $\mathcal{O}(100)$ while the value given by inflation is $f_{NL} \leq 0.1$ (see reference [20]).

6.4 ***Planck 2013*** data: the controversy

The data recently released by the *Plank* satellite team is extraordinarily precise. It shows a nearly scale invariant, adiabatic and gaussian perturbations. These findings undoubtedly suggest a predominance of the simplest inflationary models. However, the particular models favored by the data are exponentially unlikely to have produced our universe compared to more generic power-law potentials and only produce inflation in cases where the universe is already surprisingly smooth. That's why it is worth to reconsider cyclic models and to check whether the data fits in them. The low non-gaussianity seems to be a big issue but we will show that it is still possible to understand this within the context of the cyclic theories (see reference [21]).

The problem with the apparent discrepancies come from considering a particular form of scalar field potential (exponential), a particular way of generating curvature perturbations (entropic) and a particular choice of conversion from entropic to curvature perturbations. Furthermore, the *Planck 2013* collaboration reports that the same data fits the cyclic predictions for a range of parameters by shifting the conversion to the kinetic phase occurring after the smoothing phase and still keeping to the exponential form.

Starting with the scalar field $V = -V_1 e^{c_1\phi_1} - V_2 e^{c_2\phi_2}$, and then performing a rotation in field space into the ekpyrotic direction σ (defined to point tangentially to the background trajectory, with $\dot{\sigma} = (\dot{\phi}_1^2 + \dot{\phi}_2^2)^{1/2}$) and the transverse direction s, we get

$$V = -V_0 e^{\sqrt{2\epsilon}\sigma}\left[1 + \kappa_2 \epsilon s^2 + \frac{\kappa_3}{3!}\epsilon^{3/2}s^3 + \frac{\kappa_4}{4!}\epsilon^2 s^4 + ...\right] \tag{153}$$

where V_0 is just a constant, $\epsilon = \frac{3}{2}(1+\omega)$ as usual and, in this case,

$$\kappa_2 = 1\,, \qquad \kappa_3 = 2\sqrt{2}\frac{c_1^2 - c_2^2}{|c_1 c_2|}\,, \qquad \kappa_4 = 4\frac{c_1^6 + c_2^6}{c_1^2 c_2^2 (c_1^2 + c_2^2)}\,, \tag{154}$$

with $1/\epsilon = 2/c_1^2 + 2/c_2^2$. The ekpyrotic phase ends when the steeply falling potential reaches the minimum at V_{ek-end}, a short time before the bounce. The entropy perturbations obey the equation of motion

$$\ddot{\delta s} + 3H\dot{\delta s} + V_{,ss} = 0 \tag{155}$$

that can be solved in terms of Hankel functions, and the boundary condition stating that in the far past the solution should approximate the Minkowski vacuum. One gets $\delta s = \frac{\sqrt{-\pi t}}{2} H_\nu^{(1)}(-kt)$ where $\nu = \left(\frac{1}{4} + 2\kappa_2 - \frac{2}{\epsilon}\kappa_2 - \frac{1}{\epsilon} + \frac{3}{2}\frac{\epsilon_{,N_e}}{\epsilon}\kappa_2\right)^{1/2}$. Here we have neglected terms of order $1/\epsilon^2$. At late times $(-kt) \to 0$, so we obtain $\delta s \approx \frac{1}{\sqrt{2}(-t)k^\nu}$, implying that at the end of the ekpyrotic phase

$$\delta s(t_{ek-end}) \approx \frac{|\epsilon V_{ek-end}|^{1/2}}{\sqrt{2}k^\nu}\,. \tag{156}$$

Hence, the spectral index is given by

$$n_s - 1 = 3 - 2\nu \approx \frac{4}{3}(1-\kappa_2) + \frac{2}{\epsilon} - \frac{\epsilon_{,N_e}}{\epsilon}\,, \tag{157}$$

where $\kappa_2 \approx 1$ so that we recover our previous result. The measured *Planck* value for the tilt is $n_s = 0.9603 \pm 0.0073$, see reference [11]. This number is well recovered for the natural values $\kappa_2 = 1.06$ and $\epsilon = 50$. Recall that, for the cyclic model ϵ is typically of the order of N_e, so $N_e \approx 50$.

As already mentioned, for cyclic models, one intuitively expects significant departures from gaussianity compared to inflation because the potential is steeper and, thus, self interactions are guaranteed. However, this doesn't mean that every non-gaussian estimator has to be large. Non-gaussian corrections are local, and they can be analyze by studying the classical equations of motion to higher orders in perturbation theory. We expand the curvature perturbation as we did in the previous section but to a higher order,

$$\mathcal{R} = \mathcal{R}_L + \frac{3}{5} f_{NL} \mathcal{R}_L^2 + \frac{9}{25} g_{NL} \mathcal{R}_L^3 \ . \tag{158}$$

The parameters f_{NL} and g_{NL} describe the deviations from gaussianity of the bispectrum and trispectrum respectively. In order to compute these parameters we need to know the entropy perturbations to third order. From reference [22],

$$\delta s = \delta s_L + \frac{\kappa_3 \sqrt{\epsilon}}{8} \delta s_L^2 + \epsilon \left(\frac{\kappa_4}{60} + \frac{\kappa_3^2}{80} - \frac{2}{5} \right) \delta s_L^3 \ , \tag{159}$$

where terms of order $1/\epsilon$ are neglected. To see how the entropy perturbations becomes curvature fluctuations, it is convenient to use the formula given in reference [22],

$$\dot{\mathcal{R}} = \frac{2 H_b \delta V}{\dot{\sigma}_b^2 - 2\delta V} \tag{160}$$

where the subscript b denotes background quantities and $\delta V \equiv V(t, x^i) - V_b(t)$. Solving the above formula numerically (reference [20]) for a smooth conversion lasting on the order of Hubble time, we have approximately

$$f_{NL} = \pm 5 + \frac{3}{2} \kappa_3 \sqrt{\epsilon} \tag{161}$$

$$g_{NL} = \left(-40 + \frac{5}{3} \kappa_4 + \frac{5}{4} \kappa_3^2 \right) \epsilon \ , \tag{162}$$

where the sign in front of 5 depends on the details of the conversion process. Together with the spectral tilt, these two expressions summarize the key predictions of the scalar density perturbations to compare with any experimental data. In addition, cyclic models predict no observable tensor modes on large scales (see reference [23]) as the gravitational wave background is negligible.

There have been introduced three dimensionless model independent parameters κ_i, so the expressions differ from what we saw before. However, the essential result remains, the model predicts more non-gaussianity than simple inflationary models. We see that in the formula for f_{NL} both ± 5 and $\frac{3}{2}\kappa_3\sqrt{\epsilon}$ are considerably grater than one so we expect this quantity to be bigger than the inflationary one. The *Planck* results bound this parameter to $f_{NL} = 2.7 \pm 5.8$ at the 1σ level. This is easily satisfy for $-0.76 < \kappa_3 < 0.33$ or $0.18 < \kappa_3 < 1.27$ with $\epsilon = 50$ as suggested before. These allow ranges of κ_3 at the 1σ level of $\mathcal{O}(1)$. One can check that, using the potential (153), for $\kappa_2 \approx 1$ and c_1 of the same order of c_2, we may expect $\kappa_3 \in [-1, 1]$. Then, the potential is roughly symmetric and $\kappa_4 \approx 4$. This agrees with the data, what means that cyclic models are not under such a "severe pressure".

No fine tuning is needed for the value of f_{NL} to fit in the new released data, not even if it is zero, suiting the simplest inflationary models. On the other hand, if it turns to be greater than one, it would be inconsistent with inflation but not with cyclic behavior.

Focusing now in the trispectrum g_{NL} we see that the strong bounds on the bispectrum forces κ_3 not to be much greater than unity, so $\kappa_4 \approx 4$ and, noting (162), we end up with

$$g_{NL} \approx -35\epsilon \, . \tag{163}$$

Consequently, as ϵ's value is about 50 we encounter the condition $g_{NL} \leq -1700$, which is a key test for the simple cyclic models described here.

7 CONCLUSIONS

We started this study of the cyclic model of the Universe by proposing three problems cyclic models have encountered throughout the history. By now, those three problems have been fully solved. Firstly, the issue of the Big Bang being a singularity no longer exists thanks to the introduction of a coupling between the energy densities of matter and radiation with the scalar field(s). This was explicitly shown by the change of variables (68) and (69). The coupling is set in the same way as the chameleon fields so that it is negligible at this moment of time. Moreover, we are changing the laws of gravity, effectively breaking the equivalence principle, leading to potentially measurable effects. Secondly, we have managed to make sure that the entropy can increase unlimited via the introduction of the 5d picture. As the scale factors of the branes increase more and more along every cycle (see figure (6)), the entropy is able to grow even if the total entropy within our Hubble horizon remains unchanged or decrease. Finally, and related to the entropy problem, the cyclic Universe starts each of the cycles in the same conditions as the one before. This means that we can extend the cyclic behavior to minus infinity in time, removing in the way any possible worries about initial conditions.

Note this last result solves one of the main philosophical uncertainties cosmologists likely have. Our view of physics as a science that answers the question of *how nature works* is based on the statement (or proof from other basic postulates) of physical laws. Since Newtonian dynamics, those laws are a set of mathematical expressions that, no matter what initial conditions you plug in, give you the evolution of the system (in the case of quantum mechanics a set of possible evolutions). Let's consider as an example the motion of a stone when one throws it from the surface of the earth. If you know the laws of parabolic motion, you will be able to describe the stone trajectory for any initial value of the velocity. Analogously, if we want to find the laws to understand the Universe, we should be able to substitute any set of initial conditions to explain its evolution. But that is impossible as we can only see one Universe, and so we would never be sure whether our guess is right or not. The physical laws can explain everything we measure, but we will still be doubtful about having the right choice because we cannot compare them with anything else. Inflation is an attractor so the information of initial conditions dilutes in time. We simply haven't got enough data. In our model, as the Universe has no initial conditions, we don't find this insecurity any more.

Some other questions arises from what we have seen. For example, *why is the Universe homogeneous, isotropic and flat*? This is a rather simple one. The Universe is made homogeneous and isotropic during the period of the preceding cycle when quintessence dominates and there is an undergoing cosmic acceleration. This ensures that the branes are flat and parallel as they begin to approach. From the point of view of a 4d observer, any deviation from flatness in the contracting epoch is diminished by more than it grows during the subsequent expanding phase. Then, we can see that dark energy has an essential role both by reducing the entropy and black hole density of the previous cycle, and triggering the turnaround from an expanding to a contracting phase.

Let's focus on the question of *how big is the Universe*? From the 4d point of view, the

Universe oscillates between periods of expansion to periods of contraction. However, from the brane point of view, the universe is infinite in the sense that branes have infinite extent. That is why it is possible for the total entropy to increase from cycle to cycle, and, at the same time, have entropy density become nearly zero prior to each bounce.

Then, if the entropy from the point of view of a 4d observer doesn't always increase, *what determines the arrow of time*? Indeed, for a local observer there is no clear means of defining so. From the global perspective, though, one of the boundary branes is forever expanding in the "forward" direction. That would be a way of defining it.

It has not been mentioned in the dissertation but the question of *what is dark matter*? is simply answered. As the branes are living in a 5d space and there is a coupling between the radion (scalar field) and gravity, the only force allowed to travel through the fifth dimension is gravity. Hence, the gravitational effect of the matter in one of the branes would be observed on the other as *dark matter*.

The nature of the cosmological constant has changed drastically, from trying to explain why the vacuum energy of the ground state is zero, to explaining why the current potential energy is so small. Then, *why is the cosmological constant so small*? We don't know. The value depends on both the shape of the potential and the precise transfer of energy and momentum at the bounce. The problem is no longer tuning a vacuum energy, and so new approaches become possible. A good candidate is the old proposal of introducing a mechanism which causes the cosmological constant to relax to smaller values (see [26]). Starting out large, it naturally decreases but its downward drift slows dramatically as it becomes small. In the case it slips below zero, gravitational collapse follows swiftly. The result is that, for a vast majority of the time and throughout almost all of space, the cosmological constant is tiny and positive. Attempts to incorporate this idea into big bang models have failed because relaxation process takes longer than fourteen billion years. There is plenty of time in a cyclic universe instead.

There are also implications for the dark energy equation of state. We are living in a period in which $\omega = (\frac{1}{2}\dot{\phi}^2 - V)/(\frac{1}{2}\dot{\phi}^2 + V)$ is close to -1, since the kinetic energy is becoming negligible. The tendency is, after the turnaround, to increase to values even higher than 1. Take as an example $V \propto -e^{-c\phi}$ (ekpyrosis) with $c \gg 1$, which implies $\omega = (c^2 - 3)/3 \gg 1$. This have potentially measurable effects.

The point already encountered in the previous section about being able to describe all the theory as an effective 4d theory is quite remarkable. Due to the entropic mechanism we don't need any feature from 5d brane picture to compute the right features of the curvature fluctuations. Therefore, by proposing the cyclic model we wouldn't be assuming more ingredients than inflationary processes.

Finally we have seen that all data fit within the cyclic scenario. There is still an open discussion about the non-gaussianity parameter measured by the *Planck* satellite. However, as shown before, this parameter is highly model dependent, so one can illustrate that it is yet acceptable. Nevertheless, this value of the bispectrum implies a range of values for the trispectrum that can be the key to the confirmation of this theory. Another testable key result which we didn't pay much attention to, is the fact that tensor modes have to be negligible, so, if we were able to detect gravitational waves, their existence would disprove the cyclic model.

To sum up, although inflation is widely accepted, this cyclic model must be taken into account when trying to explain the behavior of the Universe and appears as a strong alternative to the current standard cosmology, solving some of the problems it suffers from but adding new ones. It is and will be a trendy topic for discussion in the near future for cosmologists.

Appendices

A Appendix 1

Because of our assumption of spatial isotropy, it is evident that we may require our coordinate system to be such that the line element will exhibit spherical symmetry around the origin of coordinates. As an starting point, we can express the line element in comoving coordinates in the most general possible form exhibiting spatial spherical symmetry.

$$ds^2 = -e^{\lambda}dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + e^{\nu}dt^2 + 2adrdt$$

where λ, μ and ν are funtions of r and t. We can substitute a new time-like variable t' defined by the equation

$$dt' = \eta(adr + e^{\nu}dt) ,$$

where η is an integrating factor which makes the right hand side a perfect differential. Then, we shall have

$$e^{\nu}dt^2 + 2adrdt = \frac{dt'^2}{\eta^2 e^{\nu}} - \frac{a^2}{e^{\nu}}dr^2 .$$

So that substituting in the line element, and dropping primes, we get the simpler form

$$ds^2 = -e^{\lambda}dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + e^{\nu}dt^2 ,$$

To proceed further in the simplification, we consider the components of gravitational acceleration for a free test particle. These would be determined by the equations for a geodesic

$$\frac{d^2x^{\sigma}}{ds^2} + \Gamma^{\sigma}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0 ,$$

and for the case of a particle at rest with respect to r, θ and ϕ this would give us

$$\frac{d^2r}{ds^2} = -\Gamma^1_{44}\left(\frac{dt}{ds}\right)^2 \quad , \quad \frac{d^2\theta}{ds^2} = -\Gamma^2_{44}\left(\frac{dt}{ds}\right)^2 \quad and \quad \frac{d^2\phi}{ds^2} = -\Gamma^3_{44}\left(\frac{dt}{ds}\right)^2 .$$

Since this test particle is spatially at rest with respect to our system of comoving coordinates, it is also at rest with respect to a local observer moving with the matter in the neighborhood. However, in accordance with spatial isotropy, such a observer must obtain measures independent of direction. Hence, previous spatial accelerations must be zero so as the Christoffel symbols related. Computing them with the previous line element, one can check that this implies

$$\frac{\partial\nu}{\partial r} = \frac{\partial\nu}{\partial\theta} = \frac{\partial\nu}{\partial\phi} = 0$$

This shows that ν is just a function of time and allow us to introduce a new time variable

$$t' = \int e^{\frac{1}{2}\nu} dt \ .$$

We then obtain an expression for the line element of the form

$$ds^2 = -e^{\lambda} dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2 \ .$$

By our hypothesis of isotropy we can write too the useful relation

$$\frac{\partial \lambda}{\partial t} = \frac{\partial \mu}{\partial t}$$

that give us the possibility of a further simplification in the line element by the substitution

$$\frac{dr'}{r'} = e^{\frac{1}{2}(\lambda - \mu)} \frac{dr}{r}$$

so that

$$ds^2 = -e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2 \ .$$

Another consequence of isotropy on the function μ is that

$$\frac{\partial^2 \mu}{\partial r \partial t} = 0 \ .$$

Therefore μ must be taken as the sum of a function of r and t

$$\mu(r,t) = f(r) + g(t) \ .$$

Using Rieman's tensor $R_{\mu\nu}$ one can compute the value of the energy tensor $T_{\mu\nu}$ via Einstein's field equations leading to

$$T_1^1 = -e^{-\mu}\left(\frac{f'^2}{4} + \frac{f'}{r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda$$

$$T_2^2 = T_3^3 = -e^{-\mu}\left(\frac{f''}{2} + \frac{f'}{2r}\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda$$

where primes indicate derivatives with respect to r and dots with respect to t. Again because of the isotropy we have the symmetry in the three spatial directions $T_1^1 = T_2^2 = T_3^3$. Therefore, we obtain the relation

$$\frac{d^2 f}{dr^2} - \frac{1}{2}\left(\frac{df}{dr}\right)^2 - \frac{1}{r}\frac{df}{dr} = 0$$

which solution takes the form

$$e^{f(r)} = \frac{1/c_2^2}{(1 - c_1 r^2/4c_2)^2}$$

where c_1 and c_2 are the first and second integration constants respectively. At last, returning to the previous expression for the line element, absorbing the constant factor $1/c_2^2$ and redefining $-c_1/c_2 = 1/R_0^2$, we can write the line element in the final form

$$ds^2 = dt^2 - \frac{e^{g(t)}}{[1 + r^2/4R_0^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2)$$

which matches with relation (12).

B Appendix 2

From the first law of thermodynamics in the covariant form given by equation (20) and taking into account that $T_1^1 = T_2^2 = T_3^3 = p$ and $T_0^0 = \rho$ for a perfect fluid we can compute the $\mu = 0$ component obtaining

$$\frac{\partial}{\partial t}(\rho\sqrt{-g}) + \frac{1}{2}p_0\sqrt{-g}\left(g^{11}\frac{\partial g_{11}}{\partial t} + g^{22}\frac{\partial g_{22}}{\partial t} + g^{33}\frac{\partial g_{33}}{\partial t}\right) = 0 \ .$$

By inserting the expressions for the metric $g_{\mu\nu}$ defined by the line element in (12), this reduces to the result

$$\frac{\partial}{\partial t}\left(\rho\frac{r^2\sin\theta e^{\frac{3}{2}g(t)}}{[1 + r^2/4R_0^2]^3}\right) + p\frac{\partial}{\partial t}\left(\frac{r^2\sin\theta e^{\frac{3}{2}g(t)}}{[1 + r^2/4R_0^2]^3}\right) = 0 \ .$$

Using the expression for the comoving volume given in expression (18)

$$\delta v_0 = \frac{r^2\sin\theta e^{\frac{3}{2}g(t)}}{[1 + r^2/4R_0^2]^3}\delta r\delta\theta\delta\phi$$

we end up with

$$\frac{d}{dt}(\rho\delta v_0) + p\frac{d}{dt}(\delta v_0) = 0$$

being this the result wanted as it matches perfectly with relation (21). This result is thermodynamically important since it shows that there will be no heat flow into or out of the elements of fluid composing the model. Also shows that the proper energy of each element of fluid as measured by a local observer would change with the proper volume of it.

C Appendix 3

Starting with the action of the full theory

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}\mathcal{R} - \frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - V(\phi) + \beta^4(\phi)(\rho_M + \rho_R)\right) \ ,$$

where $\beta(\phi)$ is the coupling between ϕ and ρ causing the energy densities to remain finite at the big bang, we compute the equations of motion for a metric with line element $ds^2 = -dt^2 + a^2 d\vec{x}^2$. Defining $k = 8\pi G$ we can rewrite the action as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2k}\mathcal{R} + \mathcal{L}_M\right) \ ,$$

where $\mathcal{L}_M = -\frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - V(\phi) + \beta^4(\phi)(\rho_M + \rho_R)$. Then, the e.o.m. would be given by

$$
\begin{aligned}
0 = \delta S &= \int \left[\frac{1}{2k}\frac{\delta(\sqrt{-g}\mathcal{R})}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x \\
&= \int \left[\frac{1}{2k}\left(\frac{\delta \mathcal{R}}{\delta g^{\mu\nu}} + \frac{\mathcal{R}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \right) + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} d^4x \, .
\end{aligned}
$$

Hence

$$
\frac{\delta \mathcal{R}}{\delta g^{\mu\nu}} + \frac{\mathcal{R}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -2k\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \, . \tag{164}
$$

Right Hand Side: The right hand side of (164) is proportional to the stress-energy tensor

$$
T_{\mu\nu} \equiv -2\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M
$$

where we have used that

$$
\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\delta g_{\mu\nu}\right) = -\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right) \, .
$$

The second step was achieved by the use of Jacobi's identity $\delta g = \delta \det(g_{\mu\nu}) = g g^{\mu\nu}\delta g_{\mu\nu}$. Thus $\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}$. And using the explicit form of $\mathcal{L}_M$

$$
T^\alpha_\beta = g^{\alpha\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^\beta} - g^\alpha_\beta\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi) - \beta^4(\phi)(\rho_M + \rho_R)\right] \, .
$$

For the homogeneous part of the field only the time derivatives are relevant, so

$$
T^\alpha_\beta = -g^\alpha_0 g^0_\beta\left(\frac{d\phi}{dt}\right)^2 + g^\alpha_\beta\left[\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi) + \beta^4(\phi)(\rho_M + \rho_R)\right]
$$

For the perfect fluid in this theory we will have

$$
\begin{aligned}
\hat{\rho} = T^0_0 &= \frac{1}{2}\dot{\phi}^2 + V(\phi) - \beta^4(\phi)(\rho_M + \rho_R) \, , \\
\hat{p} = T^i_i &= \frac{1}{2}\dot{\phi}^2 - V(\phi) - \beta^4(\phi)(\rho_M + \rho_R) \, ,
\end{aligned} \tag{165}
$$

note that in the second case we are not summing over the index i, it is just the (i,i) component.

Left Hand Side: For the left hand side of (164), we have to compute the variation of the Ricci scalar. First, consider the Riemann tensor

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$
$$\Downarrow$$
$$\delta R^{\rho}_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^\rho_{\nu\sigma} - \partial_\nu \delta\Gamma^\rho_{\mu\sigma} + \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma} \ .$$

Now, since $\delta\Gamma^\rho_{\nu\mu}$ is the difference between two connections, it is a tensor too. Therefore, calculating its covariant derivative,

$$\nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\nu\lambda}\delta\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\mu\nu}\delta\Gamma^\rho_{\nu\sigma}$$

it becomes clear that we can rewrite $\delta R^\rho_{\sigma\mu\nu}$ as the difference of the terms

$$\delta R^\rho_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\sigma}) \ .$$

Contracting indices we get the variation over the Ricci tensor

$$\delta R_{\mu\nu} = \delta R^\rho_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\mu}) \ .$$

and the variation over the Ricci scalar $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$ then reads

$$\delta\mathcal{R} = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\rho_{\rho\mu}) \ ,$$

where we have used the compatibility of the metric in the second step i.e. $\nabla_\sigma g^{\mu\nu} = 0$. Note that the last term times $\sqrt{-g}$ becomes a total derivative as $\sqrt{-g}\nabla_\mu A^\mu = \partial_\mu(\sqrt{-g}A^\mu)$. Hence, by Stokes theorem, this only yields to a boundary term when integrated. Thus, when the variation $\delta g^{\mu\nu}$ vanishes at infinity, this term does not contribute to the action and we obtain

$$\frac{\delta\mathcal{R}}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{166}$$

Placing together both results (165) and (166) in equation (164) we get, as expected, a promotion of Einstein's equation as we have mixed terms in the stress energy tensor,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Finally, making use of the FRW metric, it is obtained

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\hat{\rho}$$

for the $(0,0)$ component, and

$$2\dot{a}^2 + a\ddot{a} - 3a^2\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] = 8\pi G a^2 \hat{p}$$

for the (i,i) components. Substituting the first one into the second and expanding the values of $r\hat{h}o$ and $\hat{p}$ one gets the equations of motion given in section 5,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \beta^4\rho_R + \beta^4\rho_M\right)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V + \beta^4\rho_R + \frac{1}{2}\beta^4\rho_M\right)$$

Now, taking the time derivative of the first Friedmann equation of the full theory and replacing the acceleration equation, we end up with the dynamical equation for ϕ

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi} - \beta_{,\phi}\beta^3\rho_M$$

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