Non-Geometric Flux and Double Field Theory

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Abstract

Double field theory is formulated in a T-duality invariant way. This theory has unified the metric and two-form into the generalized metric, which is $O(D, D)$ tensor as well as the dilaton and the determinant of the metric into the $O(D, D)$ singlet. When the Scherk-Schwarz compactifition is considered, the fluxes in the the effective theory are turned on. In the case of 3-torus, the H-flux, f-flux, Q-flux and R-flux are linked by T-duality transformation. The H-flux and f-flux are known as the geometric flux, while, Q-flux and R-flux are referred to the non-geometric flux. In this dissertation, the geometric meaning of Q-flux and R-flux are discussed and they are the connection of winding derivative and the field-strength of bivector in dual theory respectively.

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Contents

1. Introduction

In the standard model of particle physics, three forces in nature, such as electromagnetic, weak interaction, and strong interaction can be explained in the quantum field theory. On the other hand, gravity can be described in terms of the geometry of the spacetime known as the general relativity. At some energy scale, one believes that gravity could be unified with the other three forces. However, due to the nonrenormalizable property of gravity, it is difficult to combine gravity into quantum theory. String theory is a alternative theory that might shine a way to quantum gravity because it contains graviton in the spectrum. Moreover, the concept of point particles is broken down, and replaced by the extended objects called strings, when the length scale is closed to the Planck length.

Due to their different configuration from particles, strings admit more symmetries than those found in particle theory. The striking symmetry, which we will discuss in this dissertation, is known as T-duality [1, 2]. When one of the dimension where strings propagate is compactified into a circle of circumference smaller than the string length, strings can warp along this compact direction. Number of times that strings curl along the circle give rise to the winding mode w. The mass spectrum of closed string state with one circular direction is given by [3]

$$M^2 = \big(N + \bar{N} - 2\big) + p^2 \frac{l_s^2}{R^2} + w^2 \frac{R^2}{l_s^2}, \tag{1.1}$$

where N and $\bar{N}$ are number operators for right and left-movers respectively, p is a momentum mode along the circle, w is the winding mode and l_s is a string length. This closed string state also satisfies the level-matching condition

$$N - \bar{N} = pw. \tag{1.2}$$

If the momentum mode p is exchanged with the winding mode w as well as the quantity R/l_s becomes l_s/R, the mass spectrum (1.1) and the level-matching condition (1.2) are still invariant. It implies that in the string point of view, strings cannot distinguish between propagating along the circle with radius R or $1/R$. This duality is known as T-duality which links the small space with the large space. In general, when n dimensions are toroidal compactified, T-duality is generalized into T-duality group $O(n, n, \mathbb{Z})$.

In [1], T-duality is realized as a symmetry of string field theory. In string field theory on the torus, the winding modes are treated on an equal footing as the momentum modes and this gives rise to coordinates that is dual to winding modes. Although the full closed string field theory on torus is so complicated and cannot be studied in more detail, the massless sector has been developed and it is known as double field theory (DFT). In [4], double field theory is a T-duality invariant theory and consists of massless fields in D dimension such as the metric g_{ij}, the Kalb-Ramond two-form b_{ij} and the dilaton ϕ. These spectrums are referred to the supergravity limit of string theory. The reason that it is called the double field theory is because the coordinates in the compact directions are double. When the theory is formulated on the product manifold such as $\mathbb{R}^{d-1,1} \times T^n$ fields depend on the coordinates of $\mathbb{R}^{d-1,1} \times T^{2n}$. This T^{2n} consists of the original torus T^n and another torus T^n corresponding to winding coordinates.

As we mention before, T-duality symmetry originating from toroidal compactification is given by $O(n, n, \mathbb{Z})$. In [5,6], it is useful to double the coordinates in the d-dimensional non-compact directions and it leads to the continuous $O(d, d)$ symmetry in these direction. Moreover, if the DFT is formulated on background $\mathbb{R}^{2D}$, there is a continuous $O(D, D)$ symmetry. Compactification on n-torus breaks the $O(D, D)$ symmetry into $O(d, d) \times O(n, n, \mathbb{Z})$. If the coordinates in non compact directions are restricted to the original set of coordinates, $O(d, d)$ symmetry group is then broken further to $O(1, d-1)$ Lorentz group. Therefore, DFT has included the Lorentz group as well as T-duality group within its framework.

In order to have T-duality invariant theory, fields in DFT can be rearranged into the $O(D, D)$ representation. For instance, the metric g_{ij} and the two-form b_{ij} have been unified into $O(D, D)$ tensor $\mathcal{H}_{MN}$ known as the generalized metric, where M, N are $O(D, D)$ indices. The dilation ϕ and the determinant of the metric g have been combined into an $O(D, D)$ singlet d, such that $e^{-2d} = \sqrt{g}e^{-2\phi}$. Additionally, the generalized metric arises when the first quantisation is considered in the level of string world sheet [2]. Not only the fields, but also the coordinates and dual coordinates are also represented into the generalized coordinates $X^M = (\tilde{x}_i, x^i)$. In DFT, there is another metric which is referred to the $O(D, D)$ invariant metric η_{MN}. The role of this metric η_{MN} is for raising and lowering the $O(D, D)$ indices.

Furthermore, fields in DFT should satisfy the constraint arising from the level-matching condition $L_0 - \bar{L}_0 = 0$. In terms of field representation, it implies that fields A are annihilated by $\partial_i \tilde{\partial}^i(A) = 0$. This constraint is know as the weak constraint. Additionally, the generalized diffeomorphisms is also considered in order to construct the invariant action. The gauge transformation in DFT is given by the generalized Lie derivative generated by a vector field ξ^i and a one-form $\tilde{\xi}_i$. These gauge parameters can be rearranged in to $O(D, D)$ vector representation such that $\xi^M = (\tilde{\xi}_i, \xi^i)$. in the limit when the theory is independent of dual coordinates, the gauge transformation has reduced into ordinary diffeomorphisms and two-form gauge transformation [3–5,7–9]. For the closure of the generalized Lie derivative, the constraint that is stronger than the weak constraint is required [10] and known as the strong constraint. When the strong constraint is imposed, field as well as products of fields are vanished by the condition $\partial_i(A)\tilde{\partial}^i(B) = 0$, where A and B are fields and gauge parameters. The result of strong constraint has reduced the dependency on coordinates from $2D$ dimension into D dimensional subspace. Therefore, fields in DFT with strong constraint are manifestly dependent on D dimensional spacetime.

In [5], the background independent action has been constructed and taken the form

$$S = \int dx\, d\tilde{x}\, e^{-2d}\Big\{ -\frac{1}{4}g^{ik}g^{jl}\mathcal{D}^p\mathcal{E}_{kl}\mathcal{D}_p\mathcal{E}_{ij} + \frac{1}{4}(\mathcal{D}^j\mathcal{E}_{ik}\mathcal{D}^i\mathcal{E}_{jl} + \bar{\mathcal{D}}^j\mathcal{E}_{ki}\bar{\mathcal{D}}^i\mathcal{E}_{lj}) + (\mathcal{D}^i d\bar{\mathcal{D}}^j\mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\mathcal{D}^j\mathcal{E}_{ji}) + 4\mathcal{D}^i d\mathcal{D}_i d\Big\}, \tag{1.3}$$

where derivative $\mathcal{D}_i$ and $\bar{\mathcal{D}}_i$ are defined by

$$\mathcal{D}_i \equiv \frac{\partial}{\partial x^i} - \mathcal{E}_{ik}\frac{\partial}{\partial \tilde{x}_k},\ \bar{\mathcal{D}}_i \equiv \frac{\partial}{\partial x^i} + \mathcal{E}_{ik}\frac{\partial}{\partial \tilde{x}_k}, \tag{1.4}$$

and the field $\mathcal{E}_{ij}$ is defined as $\mathcal{E}_{ij} = g_{ij} + b_{ij}$. In this action, indices are raising and lowering with the metric g_{ij} and each terms is invariant under $O(D,D)$ T-duality group. The gauge transformations of fields are given by

$$\delta_\xi\mathcal{E}_{ij} = \mathcal{D}_i\tilde{\xi}_j - \bar{\mathcal{D}}_j\tilde{\xi}_i + \xi^M\partial_M\mathcal{E}_{ij} + \mathcal{D}_i\xi^k\mathcal{E}_{kj} + \bar{D}_j\xi^k\mathcal{E}_{ik}, \tag{1.5}$$

$$\delta_\xi d = -\frac{1}{2}\partial_M\xi^M + \xi^M\partial_M d, \tag{1.6}$$

where $\xi^M\partial_M = \xi^i\partial_i + \tilde{\xi}_i\tilde{\partial}^i$ and $\partial_M\xi^M = \partial_i\xi^i + \tilde{\partial}^i\tilde{\xi}_i$. However, proving the gauge invariance of this action is so difficult and requires long calculation.

Therefore, in [9], a new action that related to (1.3) has been created from the generalized metric $\mathcal{H}_{MN}$ and field d

$$S = \int dx\, d\tilde{x}\, e^{-2d}\Big\{ 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d + \tfrac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \tfrac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\Big\}. \tag{1.7}$$

The gauge transformation of $\mathcal{H}_{MN}$ and d are given by the generalized Lie derivative

$$\delta_\xi \mathcal{H}_{MN} = \mathcal{L}_\xi \mathcal{H}_{MN} = \xi^P \partial_P \mathcal{H}_{MN} + (\partial_M \xi^P - \partial^P \xi_M)\mathcal{H}_{PN} + (\partial_N \xi^P - \partial^P \xi_N)\mathcal{H}_{MP}, \tag{1.8}$$

$$\delta_\xi (e^{-2d}) = \mathcal{L}_\xi e^{-2d} = \partial_M (\xi^M e^{-2d}). \tag{1.9}$$

From the action (1.7), the $O(D, D)$ structure of each terms is manifest and proving gauge invariant property is simpler than (1.3).

From the string theory point of view, the dimensions of spacetime are 10 and 26 for superstring and bosonic string theory respectively. In order to make a connection with a real world defined in 4 dimensional spacetime, some dimensions should be compactified. The Kaluza-Klein supergravity has been discussed in [11]. Due to the compactification, there exists a mass gap of which size is inversely proportional to the compactified scale. If the energy scale is less than the mass gap, massive modes can be truncated and left us with an effective theory containing only massless states. However, compactification on some internal space might lead to the inconsistent theory after all massive modes are truncated such as Calabi-Yau compactification. Scherk-Scwarz (SS) compactification [12] is one of the consistent compactification [11,13–15]. In SS compactification, the internal manifold is locally isomorphic the group manifold and the fluxes are induced after compactification. These fluxes can be obtained from the twists which can be interpreted as the vielbein on the compact space.

The case in which the compact background is flat 3-torus with non-vanishing H-flux is a good illustrate of the flux compactification [3, 8, 16–18]. By T-duality transformation in one of isometry directions of these backgrounds, H-flux has been transformed into f-flux. These H and f-fluxes have the geometrical meanings which is the three-form flux and Levi-Civita spin connection in the compact space. Moreover, if T-duality is performed in the remaining isometry direction, f-flux can be transformed in to Q-flux which is globally ill-defined. Now, there is the one direction left, however, this is a non-isometry direction. If one performs T-duality transformation in this direction, R-flux will be turned on. However, R-flux is not locally well-defined since T-duality in non-isometry direction exchanges coordinates in that direction with dual coordinates. These problems occur because the metric g_{ij}, the two-form b_{ij} and the dilaton ϕ as well as the concept of dual coordinates are not well-defined on these background. In [19], these problems are solved by replacing g_{ij}, b_{ij} and ϕ by $\tilde{g}_{ij}, \beta^{ij}$ and $\tilde{\phi}$ and using DFT framework as we will explain it more in Chapter 4 and 5.

The main objective of this dissertation is to study the meaning of non-geometric fluxes. By using fields that parameterized in terms of $\tilde{g}_{ij}, \beta^{ij}$ and $\tilde{\phi}$, the definition of fluxes can be obtained [3, 20]. The R-flux can be identified as field strength in the dual theory, and Q-flux can be thought of as the connection corresponding to the winding derivative [16, 21].

This dissertation is organized as follows. In chapter 2, we will follow [2, 4, 7] and review how the $O(D, D)$ structure emerging from the spectrum and level-matching condition in world-sheet prospective. In chapter 3, we introduce the basic knowledge of DFT base on refference [3, 4, 7, 8], such as, $O(D, D)$ representation, the generalized diffeomorphism have been discussed as well as the important of strong constraint for closure of the generalise Lie derivative. In chapter 4, the SS dimensional reduction and gauge symmetry that arise in the effective theory are provided from [3, 8, 22]. Moreover, the example of T-duality on 3-torus is introduced. In chapter 5, by referring to [3, 16, 20, 23], the covariant fluxes are calculated. Furthermore, the geometric interpretation of non-geometic fluxes is discussed and the we will briefly explain the gauged DFT and orbit of fluxes. Computational details are provided in the appendices. In appendix A, we will show how the generalized Lie derivative is reduced into the conventional Lie derivative and two-form gauge transformation when the fields and gauge parameters are independent of the dual coordinates. In appendix B, we will show the commutation relation between two generalized Lie derivatives and show how strong constrain is imposed in order to have a closure relation. In appendix C, the non-vanishing Jacobiator of the generalized Lie derivative that leads to the trivial transformation has been provided. In appendix D, the full calculations of fluxes are provided. In appendix E, we will show that the commutation relation of winding derivative gives the R-flux and Q-flux. Lastly, the detail of constructed of winding connection and its non-covariant part are illustrated in appendix F.

2. Target Space Duality

The intrigue feature of string objects is that strings can wrap along the compact dimension. As a result, it leads to the existence of winding modes that have not been seen in particle theory. Along with the momentum mode, there exists a symmetry that exchanges between momentum modes and winding modes known as "Target Space Duality" or T-Duality. In this chapter, string theory on n-torus background is introduced and this leads to the emergence of T-duality.

2.1. Toroidal compactification

Following from [2], let us consider string theory in D-dimensions with n directions are toroidal compactified. The target space manifold can be expressed as a product between d-dimensional Minkowski space-time and n-torus, such that $\mathbb{R}^{d-1,1} \times T^n$ where $D = n + d$. In this case, the critical string theory, where no Weyl anomalies, is considered. That means it can be either $D = 26$ for bosonic string theory or $D = 10$ for superstring theory. The string action is given by [4]

$$S = -\frac{1}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \{\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\}, \tag{2.1}$$

where $\gamma_{\alpha\beta}$ is a world-sheet metric, $\epsilon^{\alpha\beta}$ is an antisymmetric tensor with $\epsilon^{01} = -1$, G_{ij} is a constant target space metric, and B_{ij} is a constant target space two-form.

In action (2.1), the string coordinates X^i are split into non-compact directions represented by X^μ and compact directions represented by Y^m,

$$X^i = \{Y^m, X^\mu\}, \tag{2.2}$$

where $\mu = 0, \ldots, d-1$ and $m = 1, \ldots, n$.

By using a notation and following from [2, 4], the constant background metric G_{ij} with an inverse metric G^{ij} satisfying $G^{ij}G_{jk} = \delta^i_k$ is written as

$$G_{ij} = \begin{pmatrix} \check{G}_{mn} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \tag{2.3}$$

where $\check{G}_{mn}$ is a flat metric on n-torus T^n and $\eta_{\mu\nu}$ is a Minkowski metric on $\mathbb{R}^{d-1,1}$. Similarly, the constant background two-form B_{ij} is written as

$$B_{ij} = \begin{pmatrix} \check{B}_{mn} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

For later convenience, the background matrix E_{ij} [2] is defined by

$$E_{ij} \equiv G_{ij} + B_{ij} = \begin{pmatrix} \check{E}_{mn} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \tag{2.5}$$

where $\check{E}_{mn} = \check{G}_{mn} + \check{B}_{mn}$.

In this case, it is restricted to the closed string theory, so that the string boundary conditions in compact and non-compact directions are given by

$$Y^m(\sigma + 2\pi) = Y^m(\sigma) + 2\pi w^m, \tag{2.6}$$

$$X^\mu(\sigma + 2\pi) = X^\mu(\sigma), \tag{2.7}$$

respectively, where w^m is known as a winding number and takes an integer value. It represents the number of times that string wraps along Y^m coordinate.

Recall the action (2.1), since the critical string theory is considered, the string world-sheet metric can be chosen such that it is a Minkowski metric in 2-dimension,

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta}. \tag{2.8}$$

By substituting this metric in (2.1), the action becomes

$$\begin{aligned} S &= -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \{\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\}, \\ &= -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \{-\dot{X}^i \dot{X}^j G_{ij} + X'^i X'^j G_{ij} - 2\dot{X}^i X'^j B_{ij}\}, \end{aligned} \tag{2.9}$$

where $\dot{}$ and $'$ represent derivatives with respect to world-sheet time-like coordinate τ and space-like coordinate σ, respectively. The canonical momentum P_i conjugated to the coordinate X^i is defined as

$$P_i = \frac{\delta S}{\delta \dot{X}^i}. \tag{2.10}$$

Therefore, from the action (2.9), the canonical momentum is given by

$$2\pi P_i(\sigma, \tau) = G_{ij}\dot{X}^j(\sigma, \tau) + B_{ij}X'^j(\sigma, \tau). \tag{2.11}$$

A momentum excitation p_i from the canonical momentum is defined by

$$p_i = \int_0^{2\pi} d\sigma P_i. \tag{2.12}$$

Recall that from the Kaluza-Klein theory, the momentum excitation along the compact dimension p_m is quantised and normalised such that it takes an integer value. The reason for the Kaluza-Klein momentum must be quantised is because $\exp\left(ip_m Y^m\right)$ must be a single value function.

The expansion of modes for coordinate X^i is given by [4]

$$X^i(\sigma,\tau) = x^i + w^i\sigma + \tau G^{ij}(p_j - B_{jk}w^k) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\left(\bar{\alpha}^i_n \mathrm{e}^{-in(\tau+\sigma)} + \alpha^i_n \mathrm{e}^{-in(\tau-\sigma)}\right), \tag{2.13}$$

where x^i is the centre of mass of string, α^i_n and $\bar{\alpha}^i_n$ are the n-mode oscillators for right-mover and left-mover, respectively. In this expression, there is no winding number in non-compact directions,

$$w^i = \{w^m, 0\} \tag{2.14}$$

By substituting the coordinate expression (2.13) into the conjugate momentum expression (2.11), it becomes

$$2\pi P_i = p_i + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\left(E_{ij}\bar{\alpha}^i_n \mathrm{e}^{-in(\tau+\sigma)} + E^T_{ij}\alpha^i_n \mathrm{e}^{-in(\tau-\sigma)}\right), \tag{2.15}$$

where E_{ij} is the background matrix defined in (2.5).

2.2. Hamiltonian and level-matching condition

In order to determine the spectrum of the string theory, the Hamiltonian should be determined first and its definition is given by

$$H = \int_0^{2\pi} d\sigma \underline{H}(\sigma,\tau), \tag{2.16}$$

where $\underline{H}(\sigma,\tau)$ is a world-sheet Hamiltonian density given by

$$\underline{H}(\sigma,\tau) = P_i\dot{X}^i + \frac{1}{4\pi}\left(-\dot{X}^i\dot{X}^jG_{ij} + X'^iX'^jG_{ij} - 2\dot{X}^iX'^jB_{ij}\right). \tag{2.17}$$

By substituting the coordinate expression (2.13) and the momentum expression (2.15) into the above equation, the Hamiltonian density becomes

$$4\pi\underline{H} = \left(\, X' \; 2\pi P \,\right)\mathcal{H}(E)\begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \tag{2.18}$$

where $\mathcal{H}(E)$ is a $2D\times 2D$ symmetric matrix and constructed from the metric G_{ij} and the two-form B_{ij}. It is known as the generalized metric and takes the form

$$\mathcal{H}(E) = \begin{pmatrix} G_{ij} - B_{ik}G^{kl}B_{lj} & B_{ik}G^{kj} \\ -G^{ik}B_{kj} & G^{ij} \end{pmatrix}. \tag{2.19}$$

Therefore, the Hamiltonian can be calculated by substituting the expression of coordinate (2.13) and canonical momentum (2.15) into the Hamiltonian density (2.18). The result is

$$H = \frac{1}{2}Z^T\mathcal{H}(E)Z + \frac{1}{2}\sum_{n\neq 0}\left(\bar{\alpha}^i_{-n}G_{ij}\bar{\alpha}^i_n + \alpha^i_{-n}G_{ij}\alpha^j_n\right). \tag{2.20}$$

However the Hamiltonian (2.20) is not in the normal-ordering due to ambiguous order in the second term. By performing normal-ordering and discard the constant from the normal-ordering, the Hamiltonian becomes

$$H = \frac{1}{2}Z^T\mathcal{H}(E)Z + N + \bar{N}, \tag{2.21}$$

where Z is a generalized momentum, that unifies the momentum excitations p_i with the winding modes w^i, and defined by

$$Z = \begin{pmatrix} w^i \\ p_i \end{pmatrix}, \tag{2.22}$$

and N, $\bar{N}$ are number operators for right and left-moving modes, and written by

$$N = \sum_{n>0}\left(\alpha^i_{-n}G_{ij}\alpha^j_n\right), \tag{2.23}$$

$$\bar{N} = \sum_{n>0}\left(\bar{\alpha}^i_{-n}G_{ij}\bar{\alpha}^i_n\right). \tag{2.24}$$

In string theory, the physical state $|\phi\rangle$ satisfies the Virasoro constraints

$$\begin{aligned} L_0 - a|\phi\rangle &= 0, L_m|\phi\rangle &= 0, \quad (2.25) \\ \bar{L}_0 - a|\phi\rangle &= 0, \bar{L}_m|\phi\rangle &= 0, \text{with}\, m > 0. \quad (2.26) \end{aligned}$$

These conditions give rise the level-matching condition which takes the form,

$$L_0 - \bar{L}_0|\phi\rangle = 0. \quad (2.27)$$

After substitute the expression of L_0 and $\bar{L}_0$, the level-matching condition becomes

$$L_0 - \bar{L}_0 = N - \bar{N} - p_i w^i = 0. \quad (2.28)$$

As a result, the level-matching condition gives

$$\begin{aligned} N - \bar{N} &= p_i w^i, \\ &= \frac{1}{2} Z^T \eta Z, \quad (2.29) \end{aligned}$$

where Z is the generalized momentum defined in (2.22) and η is a constant matrix which will play a major role in the next section and defined as

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad (2.30)$$

with $\mathbb{1}$ is an identity $D \times D$ matrix.

2.3. T-duality and $O(n, n, \mathbb{Z})$

From the previous section, the Hamiltonian (2.21) and the level-matching condition (2.29) are obtained. Now let us consider the transformation symmetry of the generalized momentum Z such that

$$Z \to Z = h^T Z', \quad (2.31)$$

where h is a transformation matrix that mixes w^m and p_m after operating on the generalized momentum. The requirement of this transformation is that the level-matching condition and the Hamiltonian are preserved. Therefore, from the level-matching condition and (2.31), it gives

$$\begin{aligned} N - \bar{N} = \frac{1}{2} Z'^T \eta Z' &= \frac{1}{2} Z^T \eta Z \\ &= \frac{1}{2} Z'^T h \eta h^T Z'. \end{aligned} \tag{2.32}$$

From the above relation, the transformation matrix h must preserve η

$$h \eta h^T = \eta. \tag{2.33}$$

That means h is an element of $O(D, D, \mathbb{R})$ group and η is an $O(D, D, \mathbb{R})$ invariant metric. Since we must encounter this group several time in this report, let us introduce the basic feature of this group.

The element h belongs $O(D, D, \mathbb{R})$ group if it preserves the $O(D, D, \mathbb{R})$ invariant metric η

$$O(D, D, \mathbb{R}) = \left\{ h \in GL(2D, \mathbb{R}) : h \eta h^T = \eta \right\}. \tag{2.34}$$

Let a, b, c, and d be $D \times D$ matrices, h can be represented in terms of these matrices such that

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.35}$$

The condition in which h preserves η gives the conditions for a, b, c, and d, namely,

$$a^T c + c^T a = 0, b^T d + d^T b = 0, \text{and } a^T d + c^T b = \mathbb{1}. \tag{2.36}$$

From (2.21), let us consider the first term which is

$$H_0 = \frac{1}{2} Z^T \mathcal{H}(E) Z. \tag{2.37}$$

This term which is invariant under $O(D, D, \mathbb{R})$ induces the transformation property for $\mathcal{H}(E)$

$$\begin{aligned} Z'^T \mathcal{H}(E') Z' &= Z^T \mathcal{H}(E) Z, \\ &= Z'^T h \mathcal{H}(E) h^T Z'. \end{aligned} \tag{2.38}$$

From the above equation, the generalized metric transforms as

$$\mathcal{H}(E') = h \mathcal{H}(E) h^T. \tag{2.39}$$

From (2.39), it leads to the transformation rule for E by the following method.

First, the generalized metric is formulated in terms of a vielbein h_E which is an $O(D,D,\mathbb{R})$ element

$$\mathcal{H}(E) = h_E h_E^T, \tag{2.40}$$

and h_E is defined by

$$h_E = \begin{pmatrix} e & B(e^T)^{-1} \\ 0 & (e^T)^{-1} \end{pmatrix}, \tag{2.41}$$

where e is a vielbein of the metric $G = ee^T$. Next, the action of $O(D,D,\mathbb{R})$ group element h on $D \times D$ matrix F is defined by

$$h(F) = (aF + b)(cF + d)^{-1}. \tag{2.42}$$

From this group action, the background matrix E is obtained from

$$E = h_E(\mathbb{1}) \tag{2.43}$$

From (2.39), the transformed vielbein h'_E is obtained from the original h_E

$$h_{E'} = hh_E. \tag{2.44}$$

Therefore, the transformation rule for E is obtained by

$$E' = h_{E'}(\mathbb{1}) = hh_E(\mathbb{1}) = h(E) = (aE + b)(cE + d)^{-1}. \tag{2.45}$$

In order that the full Hamiltonian is invariant under $O(D,D,\mathbb{R})$ transformation, N, and $\bar{N}$ should be invariant under this transformation. From the transformation rule for E (2.45), the symmetric part of E' is corresponding to G', then we get the relation between G and G' [1]

$$(d + cE)^T G'(d + cE) = G, \tag{2.46}$$
$$(d - cE^T)^T G'(d - cE^T) = G. \tag{2.47}$$

After the transformation of the metric is obtained, and using the commutation relations between the oscillator

$$[\alpha^i_m(E), \alpha^j_n(E)] = [\bar{\alpha}^i_m(E), \bar{\alpha}^j_n(E)] = mG^{ij}\delta_{m+n,0}. \tag{2.48}$$

The transformation rules for α^i_m and $\bar{\alpha}^i_m$ are obtained [1]

$$\alpha_n(E) \rightarrow (d - cE^T)^{-1}\alpha_n(E'), \tag{2.49}$$
$$\bar{\alpha}_n(E) \rightarrow (d + cE)^{-1}\bar{\alpha}_n(E'). \tag{2.50}$$

Therefore, the number operators are invariant. This means the full spectrum is invariant under $O(D, D, \mathbb{R})$.

Moreover, there is another symmetry which is known as the world-sheet parity. The operation of the symmetry flips the sign of the two-form ($B \to -B$) and exchanges the right-moving and left-moving oscillators into each other as

$$\alpha_n \leftrightarrow \bar{\alpha}_n. \tag{2.51}$$

The full Hamiltonian is also invariant under this action.

As we mention before, from the restriction that w^m and p_m take the discrete values due to the boundary condition of n-dimensional toroidal space, so that the symmetry group should be restricted to $O(n, n, \mathbb{Z})$ subgroup of $O(D, D, \mathbb{R})$. This $O(n, n, \mathbb{Z})$ is known as the the T-duality group in string theory. However, it is useful to represent $h \in O(n, n, \mathbb{Z})$ in terms of $O(D, D, \mathbb{R})$ representation and represented as

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.52}$$

with

$$a = \begin{pmatrix} \check{a} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, b = \begin{pmatrix} \check{b} & 0 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} \check{c} & 0 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} \check{d} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{2.53}$$

where $\check{a}$, $\check{b}$, $\check{c}$, and $\check{d}$ are $n \times n$ matrices and can be rearranged in terms of $O(n, n, \mathbb{Z})$ element $\check{h}$ as

$$\check{h} = \begin{pmatrix} \check{a} & \check{b} \\ \check{c} & \check{d} \end{pmatrix}. \tag{2.54}$$

In this report, the representation of $O(D, D)$ and $O(n, n)$ are both used.

2.4. Example of $O(n, n, \mathbb{Z})$ transformation

In previous section, the string theory on the space with n dimension are toroidal compactified background leads to the existence T-duality $O(n, n, \mathbb{Z})$ group. In this section, the examples of the $O(n, n, \mathbb{Z})$ element are provided. At the point, one wonders that every $O(n, n, \mathbb{Z})$ can be used to generate transformation. However, the answer is no because there are some group elements that break the upper triangle of the vielbein (2.41) after transformation. These kinds of group elements do not give the metric and the two-from in the transformed theory, whereas they introduce the bivector β^{ij}. So that in this section, we will focus only on group elements that preserve the upper triangle of (2.41).

Integer theta-parameter shift Θ_{mn}

The first $O(n,n,\mathbb{Z})$ element that we would like to introduce is the theta-parameter shift Θ_{mn}. In the string-world sheet action, the term that correspond to the constant two-form in fact gives the total derivative. That means if the two-form is shifted by the constant integer, it will not contribute to the path integral because it gives only topological contribution. On the other hand, this transformation can be thought of a two-form gauge transformation such that

$$B_{mn} \to B_{mn} + \Theta_{mn}. \tag{2.55}$$

The group elements that correspond to the theta-parameter shift are

$$\check{h}_\Theta = \begin{pmatrix} \mathbb{1} & \Theta \\ 0 & \mathbb{1} \end{pmatrix}, \tag{2.56}$$

where $\Theta_{mn} \in \mathbb{Z}$ and $\Theta_{mn} = -\Theta_{mn}$.

Basis change A

The n-torus T^n is quotient space of $\mathbb{R}^n$ with lattice Λ. The transformation of lattice Λ by $GL(n,\mathbb{Z})$ doesn't change the torus. Thus, the spectrum is invariant under this transformation. The group element of this transformation can be represented as

$$\check{h}_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \tag{2.57}$$

where $A \in GL(n,\mathbb{Z})$.

Factorized duality T_k

The factorized duality T_k is corresponding to the exchange the radius $R_k \to 1/R_k$ along the circle in Y^k direction and leaves the other direction unchanged. This gives rise to the interchange between the winding mode and the momentum mode in this direction,

$$w^k \leftrightarrow p_k. \tag{2.58}$$

In the literature, this transformation is referred to the T-duality along Y^k direction. The group elements that represent this transformation are

$$\check{h}_{T_k} = \begin{pmatrix} \mathbb{1} - e_k & e_k \\ e_k & \mathbb{1} - e_k \end{pmatrix}, \tag{2.59}$$

where e_k is a matrix that has zero component everywhere except kk component. Not only the winding mode and momentum excitation are exchanged, but also some component of the metric and 2-from in the compact direction. This is know as the Bushcer rules [24].

Inversion

The transformation that interchange $R_k \to 1/R_k$ have been previously discussed. If one try to do n successive factorized T-duality in all n-dimensional compact space, it gives the inversion of the background matrix E,

$$\check{E} = \check{G} + \check{B} \to \check{E}' = \check{G}' + \check{B}' = E^{-1}. \tag{2.60}$$

The group element is represented by the $O(n,n)$ invariant metric

$$\check{h}_I = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{2.61}$$

3. Double Field Theory

In the previous chapter, the string theory on the torus background have admitted a new symmetry called T-duality that have not seen in particle theory. This hints an idea about the theory that cooperates with T-duality. Double field theory is a T-duality invariant theory of the low energy sector of string theory on the compact space. In the other word, double field theory is a T-duality symmetrization of the supergravity.

3.1. Supergravity

Before discuss more about double field theory, let us brief the story about the supergravity which is the massless spectrum in String theory. From [3, 25], the common spectra, that are found in closed string theory, consist of the metric tensor g_{ij}, the Kalb-Ramond two-form b_{ij}, and the dilaton ϕ. These states are created from the closed string states

$$\alpha^i_{-1}\bar{\alpha}^j_{-1}|0,p\rangle. \tag{3.1}$$

The symmetric traceless part of (3.1) gives the metric tensor g_{ij}. On the other hand, the antisymmetric part gives the two-form b_{ij}. The trace part is transformed as a scalar called the dilaton ϕ. In the superstring theory, these states are closed string states in the NS-NS sector.

The NS-NS supergravity action takes the form [25]

$$S = \int d^D x\sqrt{g}\mathrm{e}^{-2\phi}\left[R + 4\nabla_i\phi\nabla^i\phi - \frac{1}{12}H^{ijk}H_{ijk}\right], \tag{3.2}$$

where R is a Ricci scalar obtained from the derivative of the metric g_{ij} and H_{ijk} is a three-form field strength of two-form b_{ij} and defined by

$$H_{ijk} = 3\partial_{[i}b_{jk]}, \tag{3.3}$$

and it satisfies the Bianchi identity,

$$\partial_{[i}H_{jkl]} = 0. \tag{3.4}$$

The equations of motion obtained from the action (3.2) are given by [25]

$$R_{ij} - \frac{1}{4}H_i{}^{mn}H_{jmn} + 2\nabla_i\nabla_j\phi = 0, \quad (3.5)$$

$$\nabla^m H_{mij} - 2\left(\nabla^m\phi\right)H_{mij} = 0, \quad (3.6)$$

$$R + 4\left(\nabla_i\nabla^i\phi - \nabla_i\phi\nabla^i\phi\right) - \frac{1}{12}H_{ijk}H^{ijk} = 0. \quad (3.7)$$

From the string point of view, these equations can be derived from the vanishing of β equations at one-loop level and imply the Weyl invariant theory.

The action (3.2) is invariant under local gauge transformations such as diffeomorphisms and two-form gauge transformation.

Diffeomorphisms is an active coordinate transformation. It is generated by the vector field λ^i. The field contents in NS-NS transform as

$$\delta g_{ij} = L_\lambda g_{ij} = \lambda^k\partial_k g_{ij} + g_{kj}\partial_i\lambda^k + g_{ik}\partial_i\lambda^k, \quad (3.8)$$

$$\delta b_{ij} = L_\lambda b_{ij} = \lambda^k\partial_k b_{ij} + b_{kj}\partial_i\lambda^k + b_{ik}\partial_i\lambda^k, \quad (3.9)$$

$$\delta\phi = L_\lambda\phi = \lambda^k\partial_k\phi, \quad (3.10)$$

where L_λ is a Lie derivative along the vector field λ^i and it is defined on the arbitrary vector field V^i as the Lie bracket such that

$$L_\lambda V^i = [\lambda, V]^i = \lambda^j\partial_j V^i - V^j\partial_j\lambda^i. \quad (3.11)$$

The diffeomorphism invariant implies that the laws of physics do not change under coordinate transformation.

Kalb-Ramond two-form gauge transformation is generated by one-form field $\tilde{\lambda}_i$ such that

$$\delta b_{ij} = \partial_i\tilde{\lambda}_j - \partial_j\tilde{\lambda}_i. \quad (3.12)$$

Under this transformation, the three-form H_{ijk} is invariant.

3.2. $O(D, D)$ representation

Double field theory gives a way to think about T-duality invariant theory at the level of supergravity. From the previous chapter, when n-dimensions are n-torus, the T-duality group $O(n, n, \mathbb{Z})$ is emerging from the invariance of the spectrum and the level-matching condition. However, it is useful to think of it as embedded subgroup of $O(D, D)$ group. Therefore, the T-duality group is global $O(D, D)$ group. In order to formulate $O(D, D)$-invariant theory, the $O(D, D)$-invariant action should be constructed and the supergravity degrees of freedom should be rearranged into the $O(D,D)$ tensor.

Generalized metric and scalar

The supergravity fields consist of the metric g_{ij}, the Kalb-Ramond two-form b_{ij}, and the dilaton field ϕ. At this point, one wonders how these fields can be rearranged into $O(D,D)$ tensor. From the previous chapter, we have $O(D,D)$ tensor $\mathcal{H}_{MN}$ called the generalized metric which is constructed from the metric and two-form. Therefore, the metric g_{ij} and the two-form b_{ij} should be combined into the generalized metric $\mathcal{H}_{MN}$, which takes the form

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix}, \tag{3.13}$$

where M, N are $O(D,D)$ curved indices which run from 1 to $2D$. The indices of $O(D,D)$ tensor can be raising or lowering by the $O(D,D)$ invariant metric η_{MN} and η^{MN}, which are defined as

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix}, \eta^{MN} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix}, \tag{3.14}$$

Therefore,

$$\mathcal{H}^{MN} = \eta^{MP}\eta^{NQ}\mathcal{H}_{PQ}, \text{and}\, \mathcal{H}_{MP}\mathcal{H}^{PN} = \delta_M{}^N. \tag{3.15}$$

Moreover from [2], the dilaton ϕ along with the determinant of the metric $g = \det g_{ij}$ can be combined into the $O(D,D)$ singlet d defined as

$$e^{-2d} = \sqrt{g}e^{-2\phi}. \tag{3.16}$$

Generalized coordinates

Not only the field contents should be written in terms of the $O(D,D)$ tensor, but also the coordinates x^i. However, the vector representation of $O(D,D)$ has $2D$ dimensions, whereas the dimensions of coordinates x^i are just D. From the existence of the winding mode in the previous chapter, one can introduce a new set of coordinates $\tilde{x}_i$ that are dual to the winding modes w^i. Therefore, by combining the coordinates x^i with the coordinates $\tilde{x}_i$, the generalized coordinates X^M can be obtained

$$X^M = (\tilde{x}_i, x^i). \tag{3.17}$$

These generalized coordinates induce the generalized derivatives

$$\partial_M = \left(\frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial x^i} \right). \tag{3.18}$$

Additionally, the fields in double field theory should be dependent on the generalized coordinates X^M,

$$\mathcal{H}_{MN}(X), d(X). \tag{3.19}$$

$O(D, D)$ transformation

Since the generalized coordinates are in the fundamental representation of $O(D, D)$ group, under $O(D, D)$ transformation, the generalized coordinates transform as

$$X^M \to h^M{}_N X^N, \tag{3.20}$$

where $h^M{}_N \in O(D, D)$. As a result, this transformation mixes the coordinates x^i with the dual coordinates $\tilde{x}_i$.

Moreover, the generalized fields transform under $O(D, D)$ transformation as

$$\mathcal{H}_{MN}(X^K) \rightarrow h_M{}^P h_N{}^Q \mathcal{H}_{PQ}\left(h^K{}_L X^L\right), \tag{3.21}$$

$$d(X^K) \rightarrow d(h^K{}_L X^L). \tag{3.22}$$

Notice that if h is corresponding to the T-duality in an isometry direction, the transformation (3.21) gives the Buscher rule for g_{ij}, and b_{ij} as [24]

$$\begin{aligned} g_{kk} \to \frac{1}{g_{kk}},\ g_{ki} \to -\frac{b_{ki}}{g_{kk}},\ g_{ij} \to g_{ij} - \frac{g_{ki}g_{kj} - b_{ki}b_{kj}}{g_{kk}}, \\ b_{ki} \to -\frac{g_{ki}}{g_{kk}},\ b_{ij} \to b_{ij} - \frac{g_{ki}b_{kj} - b_{ki}g_{kj}}{g_{kk}}. \end{aligned} \tag{3.23}$$

The transformation (3.21) also includes T-duality transformation along non-isometry direction, which have not been seen in the ordinary supergravity. Since in supergravity limit, the fields are restricted to coordinate x^i and by performing T-duality in non-isometry direction such as x^k, x^k swaps with the dual coordinate $\tilde{x}_k$. That's why the T-duality in non-isometry direction cannot be done in the supergravity framework, however, in DFT framework, it is fine since x^i and $\tilde{x}_i$ are well-defined.

$O(D, D)$ invariant action

After we have all ingredients, which are represented in terms of $O(D, D)$ tensor, the $O(D, D)$ invariant action can be constructed. In [9], the DFT action can be written in terms of the generalized metric, scalar, and derivative such that

$$S = \int d^{2D} X e^{-2d} \mathcal{R}, \tag{3.24}$$

where

$$\begin{aligned}\mathcal{R} = {} & 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & + \tfrac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \tfrac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL},\end{aligned} \tag{3.25}$$

is known as the generalized scalar curvature. Each term in this scalar curvature is an $O(D, D)$ invariant quantity because all indices are totally contracted. Moreover, under generalized gauge transformation in which we will consider later in the generalized Lie derivative section, this curvature scalar is transformed as a scalar. This indicates that $\mathcal{R}$ is a generalized scalar. Additionally, when the fields in action (3.24) are independent of the dual coordinates $\tilde{x}_i$, this action becomes the supergravity action (3.2) [7].

3.3. Strong constraint

From the level-matching condition (2.29), it implies that fields in DFT should satisfy the constraint

$$\eta^{MN}\partial_M\partial_N(A) \;=\; 0, \tag{3.26}$$

$$\tag{3.27}$$

where A are any fields. This constraint is known as the weak constraint. However, when the generalized Lie derivative, which we will discuss in the next section, are considered. The gauge algebra is closed if there is another constraint such that

$$\eta^{MN}\partial_M(A)\partial_N(B) \;=\; 0, \tag{3.28}$$

where A and B are fields and gauge parameters. This constraint (3.28) is called the strong constraint.

As a result of the strong constraint, the field configurations depend only on the D-subspace coordinates, which can be only x^i, or only $\tilde{x}_i$, or the combination of x^i and $\tilde{x}_i$ related by $O(D, D)$ transformation. The subspace is called the totally null subspace [26]. Therefore, DFT with strong constraint is formulated on D-

dimensional totally null subspace. When the section is corresponding to only x^i, this section is called the supergravity frame [3].

3.4. Generalized Lie derivative

From the NS-NS supergravity fields, the metric g_{ij} and the Kalb-Ramond two-form b_{ij} transform under diffeomorphisms (3.10) and two-form gauge transformation (3.11). The supergravity action is invariant under these gauge transformations. Since in DFT, the metric g_{ij} and two-from b_{ij} are unified into the generalized metric $\mathcal{H}_{MN}$, the diffeomorphisms and two-from gauge transformation should be combined and give a generalized gauge transformation.

Recall that the diffeomorphisms and two-form gauge transformation are generated by a vector λ^i and a one-form $\tilde{\lambda}_i$, respectively. These parameters in fact can be combined into a $O(D,D)$ vector called generalized gauge parameters

$$\xi^M = \left(\tilde{\lambda}_i, \lambda^i\right). \tag{3.29}$$

In [7, 9, 23], there is a natural way to combine gauged transformation (3.10) and (3.11) into the generalized gauge transformation with parameter ξ^M. This transformation is known as the generalized Lie derivative and defined as

$$\mathcal{L}_\xi A_M \equiv \xi^P \partial_P A_M + \left(\partial_M \xi^P - \partial^P \xi_M\right) A_P, \tag{3.30}$$
$$\mathcal{L}_\xi B^M \equiv \xi^P \partial_P B^M + \left(\partial^M \xi_P - \partial_P \xi^M\right) B^P, \tag{3.31}$$

where A_M and B^N are generalized vectors. From the generalized Lie derivatives (3.30) and (3.31), the upper and the lower vector indices are treated in the symmetric ways

From this definition, the generalized Lie derivative of the generalized metric $\mathcal{H}_{MN}$ and the $O(D,D)$ singlet e^{-2d} are given by [9]

$$\mathcal{L}_\xi \mathcal{H}_{MN} = \xi^P \partial_P \mathcal{H}_{MN} + \left(\partial_M \xi^P - \partial^P \xi_M\right) \mathcal{H}_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right) \mathcal{H}_{MP}, \tag{3.32}$$
$$\mathcal{L}_\xi \left(e^{-2d}\right) = \partial_M \left(\xi^M e^{-2d}\right). \tag{3.33}$$

From (3.33), e^{-2d} is transformed as a density so that it is the generalized density. Moreover, when the strong constraint is imposed in supergravity frame, the transformation (3.32) reproduces the gauge transformation of the metric g_{ij} and two-form b_{ij} as (see appendix A)

$$\mathcal{L}_\xi g_{ij} = L_\lambda g_{ij}, \tag{3.34}$$
$$\mathcal{L}_\xi b_{ij} = L_\lambda b_{ij} + (\partial_i \tilde{\lambda}_j - \partial_j \tilde{\lambda}_i), \tag{3.35}$$

where L_λ is an ordinary Lie derivation with parameter λ^i. It implies the generalized Lie derivative have unified the ordinary Lie derivative with the two-from gauge transformation.

The generalized Lie derivative of η_{MN} is given by

$$\begin{aligned}\mathcal{L}_\xi \eta_{MN} &= \xi^P \partial_P \eta_{MN} + \left(\partial_M \xi^P - \partial^P \xi_M\right) \eta_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right) \eta_{MP}, \\ &= \partial_M \xi_N - \partial_N \xi_M + \partial_N \xi_M - \partial_M \xi_N, \\ &= 0. \end{aligned}\tag{3.36}$$

As a result, the generalized Lie derivative preserves $O(D, D)$ invariant metric.

Additionally, in DFT, there exists a trivial transformation, which is generated by the generalized derivative of some function χ, such that

$$\xi^M = \partial^M \chi = (\tilde{\partial}_i \chi, \partial^i \chi). \tag{3.37}$$

From this gauge parameters, (3.30) and (3.31) give

$$\mathcal{L}_{\xi=\partial\chi} A_M \quad = \partial^P \chi \partial_P A_M + \left(\partial_M \partial^P \chi - \partial^P \partial_M \chi\right) A_P \quad = 0 \tag{3.38}$$
$$\mathcal{L}_{\xi=\partial\chi} B^N \quad = \partial^P \chi \partial_P B^N + \left(\partial^N \partial_P \chi - \partial_P \partial^N \chi\right) B^P \quad = 0, \tag{3.39}$$

where the first term of each transformations (3.38) and (3.39) vanishes because of the strong constraint.

Moreover, if one consider the generalized Lie derivative of the scalar curvature (3.25), it is transformed as a scalar [9]

$$\mathcal{L}_\xi \mathcal{R} = \xi^M \partial_M \mathcal{R}. \tag{3.40}$$

From the form of the scalar curvature (3.25), each term is an $O(D, D)$ invariant, however, only the full combination of all terms is a generalized scalar. Therefore, the action (3.24) is invariant under the generalized diffeomorphisms.

The commutation relation between the generalized Lie derivative is given by [8–10] (see appendix B)

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1, \xi_2]_C}, \tag{3.41}$$

where the C-bracket $[\ldots,\ldots]_C$ is defined as

$$[\xi_1,\xi_2]_C^M \equiv \xi_1^N\partial_N\xi_2^M - \frac{1}{2}\xi_{1N}\partial^M\xi_2^N - (1\leftrightarrow 2). \tag{3.42}$$

Moreover, from appendix B, it implies that in order to have a closure of transformation (3.41), the strong constraint is necessary. When the strong constraint is imposed in the supergravity frame, the C-bracket has become the Courant bracket [27].

From the definition of C-bracket, one can show that

$$\frac{1}{2}\left(\mathcal{L}_{\xi_1}\xi_2^M - \mathcal{L}_{\xi_2}\xi_1^M\right) = [\xi_1,\xi_2]_C^M\,. \tag{3.43}$$

Moreover, the symmetric part gives

$$\frac{1}{2}\left(\mathcal{L}_{\xi_1}\xi_2^M + \mathcal{L}_{\xi_2}\xi_1^M\right) = \frac{1}{2}\partial^M\left(\xi_1^P\xi_{2P}\right). \tag{3.44}$$

Following from (3.41), it leads to

$$\left[\left[\mathcal{L}_{\xi_1},\mathcal{L}_{\xi_2}\right],\mathcal{L}_{\xi_3}\right] + \text{cyclic} = \mathcal{L}_{[[\xi_1,\xi_2]_C,\xi_3]_C+\text{cyclic}}. \tag{3.45}$$

That means the generalized Lie derivative has a non-vanising Jocabiator given by

$$J^M(\xi_1,\xi_2,\xi_3) = \left[[\xi_1,\xi_2]_C\,,\xi_3\right]_C^M + \text{cyclic}. \tag{3.46}$$

However, this Jacobitor generates trivial gauge transformation since it is proportional to the total derivative [8, 26] (see appendix C)

$$J^M(\xi_1,\xi_2,\xi_3) = \frac{1}{6}\partial^M\left([\xi_1,\xi_2]_C^P\,\xi_{3P} + \text{cyclic}\right). \tag{3.47}$$

Therefore, the general Lie derivative satisfies the Jacobi identity up to the trivial gauge transformation.

4. Dimensional Reduction

The critical dimension of string theory is $D = 26$ for bosonic string theory and $D = 10$ for superstring theory. In order to make a connection with phenomenology, one considers the dimensional reduction. The Kaluza-Klein reduction on n-torus with truncation of massive modes gives $U(1)^{2n}$ gauge symmetry. However, it is more interesting if the non-abelion gauge symmetry is obtained. Scherk-Schwarz compactification provides the low-dimensional theory with non-abelion gauge symmetry, and also the scalar potential. In this chapter, the SS compactification as well as the concept of flux arising from dimensional reduction are discussed.

4.1. Scherk-Schwarz compactification

Following from [12,22], let us review the Scherk-Schwarz compactification. Consider a theory on D-dimensional manifold with n-dimensional compact submanifold χ. Let y^m be a coordinate chart and u^a is a non-vanishing one-from which is defined globally on χ and takes the form

$$u^a = u^a{}_m(y) dy^m, \tag{4.1}$$

where the vielbein $u^a{}_m$ is known as a twist. The internal components of tensor field $A_{ijk\ldots l}$ are dependent on internal coordinates y^m via the vielbein $u^a{}_m$

$$A_{mnp\ldots q}(x, y) = A_{abc\ldots d}(x) u^a{}_m(y) u^b{}_n(y) u^c{}_p(y) \ldots u^d{}_q(y), \tag{4.2}$$

where $A_{abc\ldots d}(x)$ will be a scalar field in effective field theory.

Moreover, the vielbein is satisfying the structure equation,

$$du^a + \frac{1}{2} f^a{}_{bc} u^b \wedge u^c = 0, \tag{4.3}$$

where $f^a{}_{bc}$ is defined by

$$f^a{}_{bc} = 2u_{[c}{}^m \partial_m u^a{}_n u_{b]}{}^n. \tag{4.4}$$

For a group manifold $\mathcal{G}$, there exists a structure equation known as the Maurer-Cartan's structure equation [28]. Therefore, if $f^a{}_{bc}$ is a constant, the compact manifold χ is isomorphic to a group $\mathcal{G}$ with structure constants $f^a{}_{bc}$.

By following from [3,22], consider NS-NS sector of supergravity in D-dimensional theory which we will call a parent theory. It contains the metric g_{ij}, the two-form b_{ij}, and the dilaton ϕ, which all depend on D-dimensional coordinates. Now, the D-dimensional theory is compactified on n-dimensional compact space. Therefore, after truncation all massive modes, we have the d-dimensional effective theory, where $d = D - n$.

The coordinates are divided into coordinates on external space and internal space, such that

$$x^i = (x^\mu, y^m), \tag{4.5}$$

where x^μ and y^m are coordinates in external space and internal space respectively.

The metric can be decomposed into the representation of effective theory, such that

$$ds^2 = \hat{g}_{\mu\nu}(x)dx^\mu dx^\nu + \hat{g}_{ab}(x)\nu^a\nu^b, \tag{4.6}$$

where ν^a is defined as

$$\nu^a = u^a{}_m(y)dy^m + \hat{A}^a{}_\mu(x)dx^\mu. \tag{4.7}$$

Therefore, the metric tensor $g_{ij}(x, y)$ in D-dimensional theory gives rise to the massless modes in d-dimensional effective theory, such as, the metric tensor $\hat{g}_{\mu\nu}(x)$, gauge one-forms $\hat{A}^a{}_\mu(x)$ and scalar fields $\hat{g}_{ab}(x)$, where hatted fields are independent of compact space coordinates. The one-forms $\hat{A}^a{}_\mu$ carry adjoint indices, while the scalar fields $\hat{g}_{ab}$ are in the bi-adjoint representation.

Additionally, the two-form $b_{ij}(x, y)$ can be decomposed as

$$b = \hat{b}_{(2)}(x) + \hat{b}_{(1)a}(x) \wedge \nu^a + \hat{b}_{(0)ab}(x)\nu^a \wedge \nu^b + v, \tag{4.8}$$

where

$$b = b_{ij}(x, y)dx^i \wedge dx^j, \tag{4.9}$$

$$\hat{b}_{(2)}(x) = \hat{b}_{\mu\nu}(x)dx^\mu \wedge dx^\nu, \tag{4.10}$$

$$\hat{b}_{(1)a} \wedge \nu^a = \hat{V}_{a\mu}(x)dx^\mu \wedge \nu^a, \tag{4.11}$$

$$\hat{b}_{(0)ab}\nu^a \wedge \nu^b = \hat{b}_{ab}(x)\nu^a \wedge \nu^b, \tag{4.12}$$

$$v = v_{mn}(y)dy^m \wedge dy^n. \tag{4.13}$$

Therefore, in the effective theory, the two-from $b_{ij}(x, y)$ is broken into the two-form $\hat{b}_{\mu\nu}(x)$, the one-forms $\hat{V}_{a\mu}$ carrying adjoint indices and the scalar fields $\hat{b}_{ab}(x)$

carrying bi-adjoint indices. From the global two-from gauge transformation such as $b \to b+v$, one can introduce gauge transformation by making the gauge parameter depends on the internal coordinates $v \to v(y)$ [3].

The y-dependent information from the internal space are stored in the quantities $u^a{}_m(y)$ and $v_{mn}(y)$, which are called twists. The twists cannot depend on the external coordinates unless the Lorentz invariance in the effective theory will be broken.

At this point let us summarize the field contents in both parent theory and effective theory.

D-dimensional theory		d-dimensional theory
metric $g_{ij}(x,y)$, two-from $b_{ij}(x,y)$ dilaton $\phi(x,y)$	$\to$	metric $\hat{g}_{\mu\nu}(x)$ two-from $\hat{b}_{\mu\nu}(x)$, $2n$ gauge fields; $\hat{A}^a{}_\mu(x)$, $\hat{V}_{a\mu}(x)$ n^2+1 scalar fields; $\hat{g}_{ab}(x)$, $\hat{b}_{ab}(x)$, $\phi(x)$

In addition, the gauge parameters for diffeomorphisms λ^i and two-form gauge transformation $\tilde{\lambda}_i$ are also decomposed in terms of twist $u^a{}_m$ as

$$\begin{aligned} \lambda^i(x,y) &= (\epsilon^\mu(x), u_a{}^m(y)\hat{\Lambda}^a(x)), && (4.14)\\ \tilde{\lambda}_i(x,y) &= (\tilde{\epsilon}_\mu(x), u^a{}_m(y)\hat{\tilde{\Lambda}}_a(x)). && (4.15)\end{aligned}$$

At this point. one wonders how the original gauge transformations have been changed in the effective theory. Let us consider the original diffeomorphisms of a vector field V^i along with the vector field λ^i. The components of vector field V^i are written as

$$V^i(x,y) = (\hat{V}^\mu(x), u_a{}^m(y)\hat{V}^m(x)). \tag{4.16}$$

Therefore, the D-dimensional Lie derivative of this vector with parameters λ^i are given by

$$L_{\lambda^i}V^i = \lambda^j\partial_j V^i - V^j\partial_j\lambda^i. \tag{4.17}$$

Consider the components in the non-compact directions, we get

$$\begin{aligned} L_{\lambda^i}V^\mu &= \lambda^j\partial_j\hat{V}^\mu - \hat{V}^j\partial_j\lambda^\mu,\\ &= \epsilon^\nu\partial_\nu\hat{V}^\mu - \hat{V}^\nu\partial_\nu\epsilon^\mu,\\ &= L_\epsilon\hat{V}^\mu. && (4.18)\end{aligned}$$

This turns out that the Lie derivative of the components in the non-compact direction gives the ordinary Lie derivative in d-dimensional effective theory. Next, consider the component in the compact direction, we get

$$\begin{aligned} L_{\lambda^i} V^m &= \lambda^j \partial_j V^m - V^j \partial_j \lambda^m, \\ &= \epsilon^\nu \partial_\nu V^m + \lambda^n \partial_n V^m - \hat{V}^\nu \partial_\nu \lambda^m - V^n \partial_n \lambda^m, \\ &= u_a{}^m \left(\epsilon^\nu \hat{V}^a - \hat{V}^\nu \partial_\nu \hat{\Lambda}^a + f^a{}_{bc} \hat{\Lambda}^b \hat{V}^c \right). \end{aligned} \tag{4.19}$$

As a results, the transformation is gauged with the structure constant $f^a{}_{bc}$ obtained from the twist $u^a{}_m$.

Moreover, in [22], the two-form gauge transformation in parent theory also induces two-form gauge transformation and non-abelion gauge transformation in the effective theory.

In summary, Scherk-Schwarz compactification gives rise to $2n$ gauge fields in effective theory. The roles of decomposed parameters in (4.15) can be interpreted as

$$\begin{aligned} \epsilon^\mu &\rightarrow \text{diffeomorphism parameter,} \\ \tilde{\epsilon}_\mu &\rightarrow \text{two-form gauge transformation parameter,} \\ \left(\hat{\Lambda}^a, \hat{\tilde{\Lambda}}_a\right) &\rightarrow \text{gauge transformation parameters corresponding to } 2n \text{ gauge fields.} \end{aligned}$$

Gauge fields in effective theory correspond to the non-abelian gauge transformation in which the structure constants or fluxes are obtained from the twists $u^a{}_m$ and v_{mn}, such that [3]

$$H_{abc} = 3\left\{ \partial_{[a} v_{bc]} + f^d{}_{[ab} v_{c]d} \right\}, \tag{4.20}$$

$$f^a{}_{bc} = 2 u_{[c}{}^m \partial_m u^a{}_n u_{b]}{}^n, \tag{4.21}$$

where H_{abc} and $f^a{}_{bc}$ are called two-form flux and metric flux, respectively. Moreover, these fluxes are known as geometric fluxes.

4.2. Geometric flux

What is the physical meaning of these fluxes? Before answer this question, let us find out what is the meaning of the twists $u^a{}_m$ and v_{mn}.

Consider the internal metric $g_{mn}(x, y)$ which can be represented as

$$g_{mn}(x, y) = \hat{g}_{ab}(x) u^a{}_m(y) u^b{}_n(y). \tag{4.22}$$

The $\hat{g}_{ab}(x)$ are scalar fields in effective theory, therefore, when the scalar fields take the vacuum expectation value, that is

$$\hat{g}_{ab} = \delta_{ab}. \tag{4.23}$$

Then, the internal metric becomes

$$g_{mn} = u^a{}_m(y)\delta_{ab}u^b{}_n(y). \tag{4.24}$$

This means that the twist $u^a{}_n(y)$ can be interpreted as a vielbein on the compact space.

Next, consider the internal component of two-form which takes the form

$$b_{mn}(x,y) = \hat{b}_{ab}(x)u^a{}_m(y)u^b{}_n(y) + v_{mn}(y), \tag{4.25}$$

Since $\hat{b}_{ab}(x)$ are also scalar fields in the effective theory, they can take the background value like $\hat{g}_{ab}$, for instance,

$$\hat{b}_{ab} = 0. \tag{4.26}$$

Thus, the internal components of the two-form become

$$b_{mn} = v_{mn}(y), \tag{4.27}$$

which implies that for a frozen background, the twist v_{mn} can be thought as a two-form on the compact manifold.

Since, the twist $u^a{}_m$ can be interpreted as the vielbein on the compact space, the metric flux $f^a{}_{bc}$ is corresponding to the Levi-Civita spin connection on the compact space. The twist v_{mn} can be regarded as the two-form on the compact manifold, then H_{abc} is the H-flux on the compact space. The fluxes $f^a{}_{bc}$ and H_{abc} have the geometrical meaning on the compact space so that they are called geometric flux.

4.3. T-duality chain and non-geometric flux

According to the Buscher rule (3.23), when the T-duality transformation along the isometry direction is performed, some components of the two-from and the metric are exchanged. The metric flux should transform into H-flux, or vice versa. In order to clarify this point, let us consider the case where the compact manifold is a 3-torus [3, 8, 16, 17].

3-torus with H-flux

Let us consider a flat 3-torus with non-vanishing H-flux with the metric and two-from are given by

$$g_{mn}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b_{mn}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Ny^1 \\ 0 & -Ny^1 & 0 \end{pmatrix}. \tag{4.28}$$

In order to investigate that the background is globally well-defined, let us consider the different between the field values at point $y^1 = 0$ and $y^1 = 1$. We see that the metric remains the same, however, for two-form we have

$$b_{mn}(1) - b_{mn}(0) = N_{mn}, \tag{4.29}$$

where the N_{mn} is defined as

$$N_{mn} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & N \\ 0 & -N & 0 \end{pmatrix}. \tag{4.30}$$

Consider two-form gauge transformation with parameter $\tilde{\lambda}_m = (0, Ny^3, 0)$, then we get

$$b'_{mn}(y^1) = b_{mn}(y^1) - N_{mn}. \tag{4.31}$$

Since the two form is well-defined under the coordinate patching

$$b'_{mn}(1) = b_{mn}(0), \tag{4.32}$$

we can conclude that background is globally well-defined.

The metric and the two-from are corresponding to the twists $u^a{}_m$ and v_{mn} given by

$$u^a{}_m(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v_{mn}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Ny^1 \\ 0 & -Ny^1 & 0 \end{pmatrix}. \tag{4.33}$$

The corresponding fluxes of the twist $u^a{}_m$ and v_{mn} are then calculated as

$$H_{123} = N, \text{and } f^1{}_{23} = f^2{}_{31} = f^3{}_{12} = 0. \tag{4.34}$$

Twisted 3-torus without H-flux

Now, let us consider T-duality transformation of the 3-torus background with H-flux. There are two isometry directions which are y^2, or y^3, in this case, the y^3-direction is chosen.

For convenient calculation, the metric and two-form are combined into the generalized metric $\mathcal{H}_{MN}$, which is $O(3,3)$ object. Then, T-duality in y^3-direction is given by $O(3,3)$ factorized duality element

$$h_M{}^N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.35}$$

After transformation

$$\mathcal{H}_{MN} \to h_M{}^P h_N{}^Q \mathcal{H}_{PQ}, \tag{4.36}$$

the metric and the two-from in a new background are given by

$$g_{mn}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+(Ny^1)^2 & Ny^1 \\ 0 & Ny^1 & 1 \end{pmatrix}, b_{mn}(y) = 0. \tag{4.37}$$

This background is known as the twisted torus and it will turn on the metric flux. Let us examine whether this background are globally well-define by considering the metric at $y^1 = 0$ and $y^1 = 1$, we get

$$g_{mn}(y^1=0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g_{mn}(y^1=1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+N^2 & N \\ 0 & N & 1 \end{pmatrix}. \tag{4.38}$$

In order to glue the metric at boundary, let us consider the transformation of the metric by $GL(3,\mathbb{R})$ group elements defined as

$$h_m{}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & N \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.39}$$

Therefore, the transformed metric is obtained as

$$\begin{aligned} g'_{mn}(y) &= h_m{}^p g_{pq}(h_n{}^q)^T \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & N \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+(Ny^1)^2 & Ny^1 \\ 0 & Ny^1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & N & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+N^2(y^1+1)^2 & N(y^1+1) \\ 0 & N(y^1+1) & 1 \end{pmatrix}. \end{aligned} \tag{4.40}$$

So that the diffeomorphism transformation makes the metric globally well-defined since

$$g'_{mn}(0) = g_{mn}(1). \tag{4.41}$$

This is a well-defined background with twists $u^a{}_m$ and v_{mn} taking the form,

$$u^a{}_m(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & Ny^1 & 1 \end{pmatrix}, v_{mn}(y) = 0. \tag{4.42}$$

Therefore, the fluxes are calculated as

$$H_{123} = f^1{}_{23} = f^2{}_{31} = 0, \text{and } f^3_{12} = -N. \tag{4.43}$$

T-duality in y^3-direction ,which is one of isometry directions, changes the background with H-flux to the background with the metric-flux. That means T-duality links two different background together. In fact, in general background, the metric flux and H-flux can be turned on simultaneously.

At this point, there is one isometry direction left, which is y^2-direction, one wonders what kind of a new background, if the T-duality are done in the remaining isometry direction.

Non-geometric background

Consider T-duality transformation of twisted 3-torus background with the metric and the two-from defined in (4.37), the genelized metric in that case is given by

$$\mathcal{H}_{MN}(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -Ny^1 & 0 & 0 & 0 \\ 0 & -Ny^1 & 1+(Ny^1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+(Ny^1)^2 & Ny^1 \\ 0 & 0 & 0 & 0 & Ny^1 & 1 \end{pmatrix}. \tag{4.44}$$

The factoried T-duality transformation in y^2-direction is preformed by the $O(3,3)$ element defined by

$$h_M{}^N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.45}$$

After evaluating the transformation, the generalized metric is obtained as

$$\mathcal{H}_{MN}(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+(Ny^1)^2 & 0 & 0 & 0 & Ny^1 \\ 0 & 0 & 1+(Ny^1)^2 & 0 & -Ny^1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -Ny^1 & 0 & 1 & 0 \\ 0 & Ny^1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.46}$$

This leads to the metric and two-form, which are given by

$$g_{mn}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1+(Ny^1)^2} & 0 \\ 0 & 0 & \frac{1}{1+(Ny^1)^2} \end{pmatrix}, \tag{4.47}$$

$$b_{mn}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{Ny^1}{1+(Ny^1)^2} \\ 0 & \frac{Ny^1}{1+(Ny^1)^2} & 0 \end{pmatrix}. \tag{4.48}$$

In this background, the metric and the two-from cannot be patching by gauge transformation and diffeomorphisms. Therefore, this background are said to be a non-geometric background, where the metric and the two-form are globally ill-defined. This background is related to the Q-flux, and in fact, the patching condition can be done by the stringy transformation called β-transformation, which is represented in terms of $O(3,3)$ element as

$$h^M{}_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -N & 0 & 1 & 0 \\ 0 & N & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.49}$$

This element is not in the part of $O(n,n,\mathbb{Z})$ group elements that are discussed in the chapter 2 because it breaks the upper triangle of the vielbein (2.41). This background is known as T-fold background since it requires T-duality symmetry in order to patch the coordinate together.

Under this transformation, the generalized metric transforms as

$$\begin{aligned} \mathcal{H}'_{MN}(y) &= h_M{}^P h_N{}^Q \mathcal{H}_{PQ}(y), \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+N^2(y^1+1)^2 & 0 & 0 & 0 & N(y^1+1) \\ 0 & 0 & 1+N^2(y^1+1)^2 & 0 & -N(y^1+1) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -N(y^1+1) & 0 & 1 & 0 \\ 0 & N(y^1+1) & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.50}$$

Then we get

$$\mathcal{H}'_{MN}(0) = \mathcal{H}_{MN}(1). \tag{4.51}$$

Therefore, the generalized metric is well-defined, however, the transformation that glues the different patch is given by β-transformation, which is not the part of diffeomorphism and two-form gauge transformation. That is why this background are called the non-geometric background.

If one performs T-duality in the y^1-direction, which is a non-isometry direction, a new background which corresponds to R-flux is occurred. This background is not well-defined even locally, because T-duality in non-isometry direction exchanges the coordinate y^1 with the dual coordinate $\tilde{y}_1$. That means the locality is lost in this background.

As we see, T-duality connects the different backgrounds with different fluxes. This T-duality chain is discussed in [17], and can be summarized as

$$H_{abc} \overset{T_a}{\longleftrightarrow} f^a{}_{bc} \overset{T_b}{\longleftrightarrow} Q_c{}^{ab} \overset{T_c}{\longleftrightarrow} R^{abc}. \tag{4.52}$$

Since, the different background are connected by T-duality, they are said to be on the same orbit. In the other word, compactification on flat torus with flux gives the same effective theory as compactification on twisted tori or non-geometric background since they are related by T-duality. However, there are backgrounds that are not T-duality related to the geometric backgrounds called truly non-geometric backgrounds.

Due to the problem of globally and locally ill-defined issues in backgrounds with Q and R-fluxes, supergravity limit is not suitable for dealing with non-geometric background. So that we will move to double field theory where it is T-duality invariance and defined on the double space coordinates. Therefore, it is free from these problems.

5. Non-Geometric Flux in Double Field Theory

From the previous chapter, when T-duality is performed on torus background of supergravity, H-flux background can be turned into f-flux background. If T-duality is performed again in different isometry direction, the Q-flux is emerged. However, Q-flux background is globally well-defined via β-transformation, which is not the parts of diffeomorphism and two-form gauge transformation group. Moreover, If one performs T-duality in the remaining direction, which is non-isometry direction, the background is changed into R-flux background. In this background, the metric and the two-form are ill-defined even locally. Therefore, supergravity background is not suitable for non-geometric flux. However, these problems are not occurred in double field theory , in which we will see in the following sections.

5.1. Covariant flux

Since in DFT, the coordinates combined with the dual coordinates give rise to the generalized coordinates, the problem with locality ill-defined in R-flux background does not occur. Moreover, double field theory is T-duality invariant theory, patching condition can be done via $O(n,n)$ transformation. That means Q-flux is globally well-defined in DFT.

According to the previous section, the twists $u^a{}_m$ and v_{mn} can be thought of as the vielbein and the two-form in the compact background. In this section, the covariant flux will be defined by a generalized vielbein, which is constructed from the vielbein and two-form. Following from [20], the generalized metric can be parameterized in terms of the generalized vielbein as

$$\mathcal{H}_{MN} = E^A{}_M S_{AB} E^B{}_N, \tag{5.1}$$

where S_{AB} is defined in terms of the Minkowski metric as

$$S_{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta_{ab} \end{pmatrix}. \tag{5.2}$$

In this case, A, B refer to flat indices, whereas M, N refer to curved indices. Moreover, the $O(D,D)$ invariant metric can be constructed as

$$\eta_{MN} = E^A{}_M \eta_{AB} E^B{}_N, \tag{5.3}$$

where the η_{AB} is defined in the same way as η_{MN} by changing curved indices i, j into flat indices a, b. Therefore, from (5.3), it implies that $E^A{}_M$ is also an $O(D,D)$ element.

Next, consider the double Lorentz transformation of the generalized vielbein which is given by

$$\tilde{E}^A{}_M = T^A{}_B E^B{}_M, \tag{5.4}$$

and by requiring that the transformation does not change the generalized metric $\mathcal{H}_{MN}$, it gives a condition for $T^A{}_B$, such that it preserves the double Lorentz metric (5.2)

$$T^A{}_C S^{CD} T^B{}_D = S^{AB}. \tag{5.5}$$

The transformed vielbein (5.4) is also an $O(D,D)$ elements, so that $T^A{}_B$ also preserves the $O(D,D)$ invariant metric

$$T^A{}_C \eta^{CD} T^B{}_D = \eta^{AB}. \tag{5.6}$$

Conditions (5.5) and (5.6) imply that $T^A{}_B$ belongs to the $O(1,D-1)\times O(1,D-1)$ subgroup of $O(D,D)$. Without the dilaton, $E^A{}_M$ is an $O(D,D)$ element parameterized by $2D^2 - D$ parameters. However, $\mathcal{H}_{MN}$ is invariant under gauge transformation $O(1,D-1)\times O(1,D-1)$ which has $D^2 - D$ parameters. Therefore, the moduli space of $E^A{}_M$ is characterized by D^2 parameters.

In supergravity, the generalized vielbein $E^A{}_M$ can be represented in terms of the vielbein $e^a{}_i$ and the two-from b_{ij}, which we can see from (2.41)

$$E^A{}_M = \begin{pmatrix} e_a{}^i & e_a{}^l b_{li} \\ 0 & e^a{}_i \end{pmatrix}. \tag{5.7}$$

However, as we have seen from the previous chapter that the supergravity background defined by the metric g_{ij} and the two-form b_{ij} is not suitable for non-geometric background. For example, the β-transformation, that glues the Q-flux background, breaks the upper triangle and gives rise to the lower triangle part of this vielbein. Thus, the generalized vielbein should be parameterized without fixing gauge that is

$$E^A{}_M = \begin{pmatrix} e_a{}^i & e_a{}^l b_{li} \\ e^a{}_l \beta^{li} & e^a{}_i + e^a{}_l \beta^{lk} b_{ki} \end{pmatrix}. \tag{5.8}$$

where β^{ij} is an antisymmetric bivector.

The $O(D,D)$ element that corresponds to the transformation of this bivector β^{ij} is given by

$$h^M{}_N = \begin{pmatrix} \delta_i{}^j & 0 \\ \theta^{ij} & \delta^i{}_j \end{pmatrix}, \tag{5.9}$$

where θ^{ij} is an antisymmetric constant bivector. This transformation is known as the β-transformation.

The geometric flux and non-geometric flux can be unified into a single $O(D,D)$ tensor known as the covariant flux. The covariant flux can be built with the C-bracket of the generalized vielbeins as

$$\mathcal{F}_{ABC} = [E_A, E_B]_C^L E_{CL}. \tag{5.10}$$

From the definition of C-bracket, the covariant flux is obtained

$$\begin{aligned}\mathcal{F}_{ABC} =& E_A{}^N \partial_N E_B{}^M E_{CM} - \frac{1}{2} E_{AN} \partial^M E_B{}^N E_{CM} - E_B{}^N \partial_N E_A{}^M E_{CM} \\ &+ \frac{1}{2} E_{BN} \partial^M E_A{}^N E_{CM}.\end{aligned} \tag{5.11}$$

By defining Ω_{ABC} such that

$$\Omega_{ABC} = E_A{}^N \partial_N E_B{}^M E_{CM}, \tag{5.12}$$

the covariant flux is given by

$$\mathcal{F}_{ABC} = \Omega_{ABC} - \frac{1}{2}\Omega_{CBA} - \Omega_{BAC} + \frac{1}{2}\Omega_{CAB}. \tag{5.13}$$

Moreover, from the property of the invariant metric η_{AB}

$$\eta_{AB} = E_A{}^M E_{BM} = E_A{}^M \eta_{MN} E_B{}^N, \tag{5.14}$$

and its derivative $\partial_N \eta_{AB} = 0$, Ω_{ABC} is antisymmetric in the last two indices

$$\begin{aligned}\Omega_{ABC} = E_A{}^N \partial_N E_B{}^M E_{CM} =& - E_A{}^N \partial_N E_C{}^M E_{BM}, \\ =& - \Omega_{ACB}.\end{aligned} \tag{5.15}$$

Hence, the covariant flux can be rewritten as

$$\begin{aligned}\mathcal{F}_{ABC} = \ & \Omega_{ABC} - \frac{1}{2}\Omega_{CBA} - \Omega_{BAC} + \frac{1}{2}\Omega_{CAB}, \\ = \ & \Omega_{ABC} + \Omega_{BCA} + \Omega_{CAB}, \\ = \ & 3\Omega_{[ABC]}, \end{aligned} \tag{5.16}$$

which is totally antisymmetric. In terms of $D \times D \times D$ block, it has four independent blocks; $\mathcal{F}_{abc}$, $\mathcal{F}^{a}{}_{bc}$, $\mathcal{F}_{a}{}^{bc}$, and $\mathcal{F}^{abc}$ [20]. These elements are corresponding to the fluxes that we have seen in the previous chapter; H_{abc}, $f^{a}{}_{bc}$, $Q_{a}{}^{bc}$, and R^{abc}.

In order to get full definition of each flux, the generalized vielbein without gauge fixing is used, in other word, we will parameterize the generalized vielbein in terms of the vielbein $e^{a}{}_{i}$, the two-form b_{ij}, and the bivector β^{ij}. However, in a given physical situation, we may fix the gauge, for example, in the supergravity limit, the gauge with vanishing of bivector $\beta^{ij} = 0$ is chosen.

Now, the ingredients for calculating fluxes are ready. First, consider the H-flux (for the full detail of calculation see appendix D).

H-flux

In order to obtain the H-flux, we need Ω_{abc}, which is defined as

$$\Omega_{abc} = e_{a}{}^{i} e_{b}{}^{j} e_{c}{}^{k} \left(\partial_i b_{ij} - b_{mi}\tilde{\partial}^{m} b_{jk}\right) = e_{a}{}^{i} e_{b}{}^{j} e_{c}{}^{k} (D_i b_{jk}), \tag{5.17}$$

note that $D_i \equiv \partial_i + b_{im}\tilde{\partial}^{m}$. Therefore, the H-flux is then calculated as

$$\begin{aligned} H_{abc} = & 3\Omega_{[abc]}, \\ = & 3e_{a}{}^{i} e_{b}{}^{j} e_{c}{}^{k} \left(\partial_{[i} b_{jk]} + b_{[im}\tilde{\partial}^{m} b_{jk]}\right), \\ = & 3e_{a}{}^{i} e_{b}{}^{j} e_{c}{}^{k} D_{[i} b_{jk]}. \end{aligned} \tag{5.18}$$

In the supergravity frame where the strong constraint is imposed $\tilde{\partial}^{m}(\dots) = 0$, the H-flux becomes

$$H_{abc} = 3e_{a}{}^{i} e_{b}{}^{j} e_{c}{}^{k} \partial_{[i} b_{jk]}. \tag{5.19}$$

So, in supergravity point of view, this flux is related to the three-form field strength H_{ijk} by

$$H_{abc} = e_{a}{}^{i} e_{b}{}^{j} e_{c}{}^{k} H_{ijk}. \tag{5.20}$$

f-flux

In order to calculate f-flux, three components of Ω_{ABC} are required. However, by antisymmetric property in the last two indices, two of them are verified

$$\Omega^{a}{}_{bc} = e^{a}{}_{i} e_{b}{}^{j} e_{c}{}^{k} \left\{ \tilde{\partial}^{i} b_{jk} + \beta^{im} D_{m} b_{jk} \right\}, \tag{5.21}$$

$$\Omega_{a}{}^{b}{}_{c} = e_{a}{}^{i} D_{i} e^{b}{}_{j} e_{c}{}^{j} + e_{a}{}^{i} e^{b}{}_{j} \beta^{jk} D_{i} b_{kn} e_{c}{}^{n}. \tag{5.22}$$

Therefore, the f-flux is obtained as

$$\begin{aligned} f^{a}{}_{bc} =& \Omega^{a}{}_{bc} + \Omega_{c}{}^{a}{}_{b} + \Omega_{bc}{}^{a}, \\ =& 2\left(e_{[c}{}^{i} D_{i} e^{a}{}_{j} e_{b]}{}^{j}\right) + e^{a}{}_{i} e_{b}{}^{j} e_{c}{}^{k} \left(\tilde{\partial}^{i} b_{jk} + \beta^{im} H_{mjk}\right). \end{aligned} \tag{5.23}$$

In the supergravity frame with vanishing bivector $\beta^{ij} = 0$, the f-flux takes the from

$$f^{a}{}_{bc} = 2\left(e_{[c}{}^{i} \partial_{i} e^{a}{}_{j} e_{b]}{}^{j}\right). \tag{5.24}$$

This definition matches (4.21) when $u^{a}{}_{m}$ becomes $e^{a}{}_{i}$. Therefore, this f-flux links to the Levi-Civita spin connection.

Q-flux

In this case, two of Ω_{ABC} have to be calculated and are given by

$$\Omega_{a}{}^{bc} = e_{a}{}^{i} e^{b}{}_{j} e^{c}{}_{k} \left(D_{i} \beta^{jk} + \beta^{jm} D_{i} b_{mn} \beta^{kn}\right), \tag{5.25}$$

$$\Omega^{ab}{}_{c} = e^{a}{}_{i} e^{b}{}_{j} e_{c}{}^{k} \beta^{jl} \tilde{D}^{i} b_{lk} + e^{a}{}_{i} e_{c}{}^{n} \left(\tilde{D}^{i} e^{b}{}_{n} + \beta^{ij} b_{jm} (\tilde{\partial}^{m} e^{b}{}_{n} + \tilde{\partial}^{m} e^{b}{}_{k} \beta^{kl} b_{ln})\right). \tag{5.26}$$

where $\tilde{D}^{i} \equiv \tilde{\partial}^{i} + \beta^{im} \partial_{m}$. As a result, the Q-flux is obtained as

$$\begin{aligned} Q_{a}{}^{bc} =& \Omega_{a}{}^{bc} + \Omega^{c}{}_{a}{}^{b} + \Omega^{bc}{}_{a}, \\ =& e_{a}{}^{i} e^{b}{}_{j} e^{c}{}_{k} \left(D_{i} \beta^{jk} + \beta^{jm} \beta^{kn} D_{i} b_{mn} + 2\beta^{[k|l} \tilde{D}^{j]} b_{li}\right) \\ & + 2 e_{a}{}^{i} \left(e^{[b}{}_{j} \tilde{D}^{j} e^{c]}{}_{i} + e^{[b}{}_{j} \beta^{jp} b_{pm} \tilde{\partial}^{m} e^{c]}{}_{i} + e^{[b}{}_{j} \beta^{jp} b_{pm} \tilde{\partial}^{m} e^{c]}{}_{k} \beta^{kl} b_{li}\right). \end{aligned} \tag{5.27}$$

In the supergravity frame with vanishing two-form $b_{ij} = 0$, the Q-flux is given by

$$Q_{a}{}^{bc} = e_{a}{}^{i} e^{b}{}_{j} e^{c}{}_{k} \left(\partial_{i} \beta^{jk}\right) + \beta^{jm} e_{a}{}^{i} (e^{b}{}_{j} \partial_{m} e^{c}{}_{i} - e^{c}{}_{j} \partial_{m} e^{b}{}_{i}). \tag{5.28}$$

R-flux

In this case one component of Ω^{abc} is required and given by

$$\Omega^{abc} = e^a{}_i e^b{}_j e^c{}_k \left\{ \tilde{D}^i\beta^{jk} + \beta^{im} b_{ml}\tilde{\partial}^l\beta^{jk} + \beta^{im}\beta^{jl}\beta^{kn} D_m b_{ln} + \beta^{jl}\beta^{kn}\tilde{\partial}^i b_{ln} \right\}. \tag{5.29}$$

Therefore, the R-flux is obtained as

$$\begin{aligned} R^{abc} =& \Omega^{abc} + \Omega^{bca} + \Omega^{cab}, \\ =& 3e^a{}_i e^b{}_j e^c{}_k \left\{ \tilde{D}^{[i}\beta^{jk]} + \beta^{[i|m} b_{ml}\tilde{\partial}^l\beta^{jk]} + \beta^{[jl}\tilde{\partial}^i b_{ln}\beta^{k]n} + \frac{1}{3}\beta^{im}\beta^{jl}\beta^{kn} H_{mln} \right\}. \end{aligned} \tag{5.30}$$

In the supergravity limit with vanishing two-form $b_{ij} = 0$, the R-flux is given by

$$R^{abc} = 3^a{}_i e^b{}_j e^c{}_k \beta^{[im}\partial_m\beta^{jk]}. \tag{5.31}$$

From the definitions of fluxes that are obtained, the H-flux and f-flux can be calculated from the vielbein and the two-form. However, in supergravity limit, Q-flux and R-flux are also well-defined in terms of the bivector instead of the two-form [21].

Since the H-flux and f-flux are related to the three-from field strength and the Levi-Civita spin connection, Q-flux and R-flux may have the geometrical meaning.

5.2. Geometry of non-geometric flux

In this section, the geometrical meaning of Q-flux and R-flux are discussed base on [16]. Since Q-flux and R-flux are not well-defined in terms of the metric g_{ij}, the two-form b_{ij}, and the dilaton ϕ, the generalized metric should be parameterized in terms of a new metric $\tilde{g}_{ij}$, a bivector β^{ij}, and a new dilaton $\tilde{\phi}$

$$\mathcal{H}_{MN} = \begin{pmatrix} \tilde{g}^{ij} - \beta^{il}\tilde{g}_{lk}\beta^{kj} & -\beta^{il}\tilde{g}_{lj} \\ \tilde{g}_{ik}\beta^{kj} & \tilde{g}_{ij} \end{pmatrix}, e^{-2d} = \sqrt{\tilde{g}}e^{-2\tilde{\phi}}, \tag{5.32}$$

where $\tilde{g}$ is the determinate of the metric $\tilde{g}^{ij}$. The strong constraint is hold in the general way such that

$$\partial_k\tilde{\partial}^k A = \partial_k A\tilde{\partial}^k B = 0, \tag{5.33}$$

where A and B are fields and gauge parameters.

Equations (5.27) and (5.30) give the relation between Q-flux and R-flux in flat indices labelled by a, b, c and curved indices labelled by i, j, k. Therefore, R-flux should be a tensor because it is related to R^{ijk} by

$$R^{abc} = e^a{}_i e^b{}_j e^c{}_k R^{ijk}, \tag{5.34}$$

where R^{ijk} is given by

$$R^{ijk} = 3\left(\tilde{D}^{[i}\beta^{jk]}\right). \tag{5.35}$$

In the frame where two-from is vanishing and the strong constraint is imposed in arbitrary frame, the Q-flux (5.27) is given by

$$Q_a{}^{bc} = e_a{}^i e^b{}_j e^c{}_k \partial_i \beta^{jk} + 2 e_a{}^i e^{[b}{}_j \tilde{D}^j e^{c]}{}_i. \tag{5.36}$$

However, there is no direct relation between flat indices and curved indices in this case. So Q-flux should be a connection and the Q-flux in curved indices is given by

$$Q_i{}^{jk} = \partial_i \beta^{jk}. \tag{5.37}$$

Let us consider the R-flux first because it is easier to determine. The R-flux is given in (5.35), and with the definition of the derivative $\tilde{D}^i$ it becomes

$$R^{ijk} = 3\left(\tilde{\partial}^{[i}\beta^{jk]} + \beta^{[il}\partial_i \beta^{jk]}\right). \tag{5.38}$$

As we mention before, in the supergravity limit, it takes the form

$$R^{ijk} \ = 3\beta^{[il}\partial_l \beta^{jk]}. \tag{5.39}$$

Since it is constructed from the ordinary derivative rather than the covariant derivative, it should not be a tensor. However, from the symmetric property of the Levi-Civita connection, this definition can be rewritten in terms of the covariant derivative instead of the partial derivative as

$$R^{ijk} = 3\beta^{[il}\nabla_l \beta^{jk]}. \tag{5.40}$$

This is a well-defined tensor in the supergravity limit.

Conversely, if the frame is chosen in another way such that the derivative with respect to the coordinates vanishing $\partial_i(\dots) = 0$. The R-flux becomes

$$R^{ijk} = 3\tilde{\partial}^{[i}\beta^{jk]}. \tag{5.41}$$

R-flux in this form reminds us the structure of the three-form flux H_{ijk} in the supergravity limit, which takes the form

$$H_{ijk} = 3\partial_{[i}b_{jk]}. \tag{5.42}$$

Therefore, R^{ijk} should play role of three-form field strength in the dual theory.

Next, let us consider the role of Q-flux. By performing the generalized Lie derivative of the generalized metric (5.32) along the generalized vector $\xi^M = (0, \xi^i)$, the metric and the bivector transform as

$$\mathcal{L}_\xi \tilde{g}_{ij} = L_\xi \tilde{g}_{ij}, \tag{5.43}$$

$$\mathcal{L}_\xi \beta^{ij} = L_\xi \beta^{ij} - (\tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i). \tag{5.44}$$

In this case, the generalized Lie derivative is identified as the gauge transformation with parameter ξ^i

$$\delta_\xi(\dots) = \mathcal{L}_\xi(\dots). \tag{5.45}$$

The transformation is said to be covariant if and only if the gauge transformation is equivalent to the ordinary Lie derivative

$$\delta_\xi(\dots) = L_\xi(\dots). \tag{5.46}$$

If the gauge transformations of fields are not covariant transformations, there exists a non-covariant part

$$\Delta_\xi(\dots) = (\delta_\xi - L_\xi)(\dots). \tag{5.47}$$

Hence, from (5.43) and (5.44), the metric transforms covariantly, on the contrary, the bivector has a non-covariant part given by the winding derivative of the gauge parameter ξ^i

$$\Delta_\xi \tilde{g}_{ij} = 0, \tag{5.48}$$

$$\Delta_\xi \beta^{ij} = -(\tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i). \tag{5.49}$$

Next, consider the gauge transformation of the dilaton $\tilde{\phi}$, which is coveriant under the transformation

$$\delta_\xi \tilde{\phi} = \xi^i \partial_i \tilde{\phi}. \tag{5.50}$$

On the other hand, the winding derivative of the dilaton $\tilde{\partial}^i\tilde{\phi}$ is not covariant

$$\begin{aligned}\delta_\xi(\tilde{\partial}^i\tilde{\phi}) =&\tilde{\partial}^i(\delta_\xi\tilde{\phi}),\\ =&\tilde{\partial}^i(\xi^j\partial_j\tilde{\phi}),\\ =&(\tilde{\partial}^i\xi^j)\partial_j\tilde{\phi}+\xi^j\partial_j(\tilde{\partial}^i\tilde{\phi}).\end{aligned} \tag{5.51}$$

By adding terms

$$(\partial_j\xi^i)\tilde{\partial}^j\tilde{\phi}+(\tilde{\partial}^j\xi^i)\partial_j\tilde{\phi}, \tag{5.52}$$

which vanish due to the strong constraint, into (5.51), it becomes

$$\begin{aligned}\delta_\xi(\tilde{\partial}^i\tilde{\phi}) =&\xi^j\partial_j(\tilde{\partial}^i\tilde{\phi})-(\partial_j\xi^i)\tilde{\partial}^j\tilde{\phi}+(\tilde{\partial}^i\xi^j-\tilde{\partial}^j\xi^i)\partial_j\tilde{\phi},\\ =&L_\xi(\tilde{\partial}^i\tilde{\phi})+(\tilde{\partial}^i\xi^j-\tilde{\partial}^j\xi^i)\partial_j\tilde{\phi}.\end{aligned} \tag{5.53}$$

That means a non-covariant part of $\tilde{\partial}^i\tilde{\phi}$ is given by

$$\Delta_\xi(\tilde{\partial}^i\tilde{\phi})=(\tilde{\partial}^i\xi^j-\tilde{\partial}^j\xi^i)\partial_j\tilde{\phi}. \tag{5.54}$$

The non-convariant parts of β^{ij} (5.49) and $\tilde{\partial}^i\tilde{\phi}$ (5.54) imply $\tilde{D}^i\tilde{\phi}=\tilde{\partial}^i\tilde{\phi}+\beta^{ij}\partial_j\tilde{\phi}$ transforms covariantly

$$\begin{aligned}\Delta_\xi(\tilde{D}^i\tilde{\phi}) =&\Delta_\xi(\tilde{\partial}^i\tilde{\phi})+(\Delta_\xi\beta^{ij})\partial_j\tilde{\phi}+\beta^{ij}\Delta_\xi(\partial_j\tilde{\phi}),\\ =&(\tilde{\partial}^i\xi^j-\tilde{\partial}^j\xi^i)\partial_j\tilde{\phi}-(\tilde{\partial}^i\xi^j-\tilde{\partial}^j\xi^i)\partial_j\tilde{\phi},\\ =&0.\end{aligned} \tag{5.55}$$

This derivative has a non-vanishing commutation relation, which is given by Q-flux and R-flux (see appendix E.)

$$\left[\tilde{D}^i,\tilde{D}^j\right]=R^{ijk}\partial_k+Q_k{}^{ij}\tilde{D}^k, \tag{5.56}$$

where R^{ijk} and $Q_k{}^{ij}$ are defined in (5.35) and (5.37) respectively. Furthermore, R^{ijk} is covariant under gauge transformation

$$\Delta_\xi R^{ijk}=0. \tag{5.57}$$

Therefore, R^{ijk} is obviously a covariant tensor.

However, for a vector V^i, the derivative $\tilde{D}^i$ of the vector is not covariant

$$\Delta_\xi(\tilde{D}^iV^j)=-\tilde{D}^i\partial_k\xi^jV^k. \tag{5.58}$$

A new covariant derivative of the vector is then defined as

$$\tilde{\nabla}^i V^j = \tilde{D}^i V^j - \check{\Gamma}_k{}^{ij} V^k, \tag{5.59}$$

where the non-covariant part of $\check{\Gamma}_k{}^{ij}$ is required such that

$$\Delta_\xi \check{\Gamma}_k{}^{ij} = -\tilde{D}^i \partial_k \xi^j. \tag{5.60}$$

Since the antisymmetric part of the connection is not covariant

$$\Delta_\xi \check{\Gamma}_k{}^{[ij]} = -2\tilde{D}^{[i} \partial_k \xi^{j]} \neq 0, \tag{5.61}$$

this means the antisymmetric part of this connection is not a tensor and cannot be chosen to be zero like the case of the Levi-Civita connection. Therefore, both symmetric and antisymmetric parts must be evaluated. Firstly let us consider the symmetric part of the connection. By requiring the metric compatibility

$$\tilde{\nabla}^i \tilde{g}^{jk} = \tilde{D}^i \tilde{g}^{jk} - \check{\Gamma}_l{}^{ij} \tilde{g}^{lk} - \check{\Gamma}_l{}^{ik} \tilde{g}^{jl} = 0, \tag{5.62}$$

the symmetric part of the connection is (see appendix F)

$$\check{\Gamma}_k{}^{(ij)} = \tilde{\Gamma}_k{}^{ij} - \tilde{g}_{mk} (\check{\Gamma}_l{}^{[jm]} \tilde{g}^{li} + \check{\Gamma}_l{}^{[im]} \tilde{g}^{lj}), \tag{5.63}$$

where $\tilde{\Gamma}_k{}^{ij}$ is analogous to the Levi-Civita connection with the metric is $\tilde{g}^{ij}$ and the derivative $\tilde{D}^i$

$$\tilde{\Gamma}_k{}^{ij} = \frac{1}{2} \tilde{g}_{km} \left(\tilde{D}^j \tilde{g}^{im} + \tilde{D}^i \tilde{g}^{mj} - \tilde{D}^m \tilde{g}^{ij} \right). \tag{5.64}$$

In addition, the commutation relation of the covariant derivative on the dilaton $\tilde{\phi}$ is given by

$$\begin{aligned} \left[\tilde{\nabla}^i, \tilde{\nabla}^j \right] \tilde{\phi} &= \tilde{\nabla}^i (\tilde{D}^j \tilde{\phi}) - (i \leftrightarrow j), \\ &= \tilde{D}^i \tilde{D}^j \tilde{\phi} - \check{\Gamma}_k{}^{ij} \tilde{D}^k \tilde{\phi} - (i \leftrightarrow j), \\ &= \left[\tilde{D}^i, \tilde{D}^j \right] \tilde{\phi} - 2\check{\Gamma}_k{}^{[ij]} \tilde{D}^k \tilde{\phi}. \end{aligned} \tag{5.65}$$

By using the commutation relation (5.56), the commutation relation of the covariant derivatives on the dilaton become

$$\left[\tilde{\nabla}^i, \tilde{\nabla}^j \right] \tilde{\phi} = R^{ijk} \partial_k \tilde{\phi} + (Q_k{}^{ij} - 2\check{\Gamma}_k{}^{[ij]}) \tilde{D}^k \tilde{\phi}. \tag{5.66}$$

Since the R^{ijk} is a covariant tensor, and the commutation relation should give the covariant tensor, this condition requires the second term of the above expression vanishes. Therefore, the antisymmetric part of the connection can be determined

$$\check{\Gamma}_k{}^{[ij]} = \frac{1}{2} Q_k{}^{ij}. \tag{5.67}$$

Thus the expression of the connection is given by

$$\check{\Gamma}_k{}^{ij} = \tilde{\Gamma}_k{}^{ij} - \tilde{g}_{mk}\tilde{g}^{l(i}Q_l{}^{j)m} + \frac{1}{2}Q_k{}^{ij}. \tag{5.68}$$

From the above relation, the Q-flux can be regarded as a winding covariant derivative and the result in (5.68) is satisfying the requirement of the connection (5.60).

Moreover, the Q-flux in flat indices (5.36) can be evaluated by the following procedure. First, consider the covariant derivative on the vielbein

$$\begin{aligned} \check{\Gamma}_a{}^{bc}e^a{}_i =& \tilde{\nabla}^b e^c{}_i, \\ =& e^b{}_i \tilde{\nabla}^i e^c{}_k, \\ =& e^b{}_i \left(\tilde{D}^i e^c{}_k + \check{\Gamma}_k{}^{ij} e^c{}_j \right). \end{aligned} \tag{5.69}$$

As a result, the connection in flat indices takes the form

$$\check{\Gamma}_a{}^{bc} = e_a{}^i e^b{}_j \left(\tilde{D}^j e^c{}_i + \check{\Gamma}_i{}^{jk} e^c{}_k \right). \tag{5.70}$$

By using the same relation in (5.67), the Q-flux in flat indices is obtained

$$\begin{aligned} Q_a{}^{bc} =& \check{\Gamma}_a{}^{bc} - \check{\Gamma}_a{}^{cb}, \\ =& e_a{}^k e^b{}_i e^c{}_j \left(\check{\Gamma}_k{}^{ij} - \check{\Gamma}_k{}^{ji} \right) + e_a{}^i \left(e^b{}_j \tilde{D}^j e^c{}_i - e^c{}_j \tilde{D}^j e^b{}_i \right), \\ =& e_a{}^k e^b{}_i e^c{}_j (Q_k{}^{ij}) + 2 e_a{}^i \left(e^{[b}{}_j \tilde{D}^j e^{c]}{}_i \right). \end{aligned} \tag{5.71}$$

This is the same expression as (5.36).

5.3. Gauged Double Field Theory

From [3, 16, 20], let us consider the double field theory with n-dimensions are compactified. Upon the compactification, the $O(D, D)$ double Lorentz structure in the parent theory is broken, namely,

$$O(D, D) \to O(d, d) \times O(n, n). \tag{5.72}$$

The $O(d,d)$ is the double Lorentz group in the effective theory and $O(n,n)$ is the global symmetry group. As we mention in chapter 3, with all the massive modes truncated, the massless modes are

1. The vielbein $e^{\alpha}{}_{\mu}$.
2. The two-form $b_{\mu\nu}$ or the bivector $\beta^{\mu\nu}$.
3. $2n$ gauge fields $A^{\check{M}}{}_{\mu}$.
4. n^2 scalar fields $\Phi_{\check{M}\check{N}}$ and the dilaton ϕ.

The gauge fields $A^{\check{M}}{}_{\mu}$, which are the $O(n,n)$ vector representation, consist of the gauge fields arising from the metric and the two-form or bivector in the parent theory. Moreover, the scalar fields $\Phi_{\check{M}\check{N}}$ are constructed from the scalar fields $\hat{g}_{ab}$ and $\hat{b}_{ab}$.

In this case, it is known as the ungauged theory because the gauge symmetry is $U(1)^{2n}$ so it has $2n$ gauge field. The ungauged theory is also invariant under the $O(n,n)$ global group [3], however, it is not interesting because it consists of many problems. For example, the scalar potential vanishes and is also flat in any direction, so that the moduli space is then degenerate and there is no freedom for choosing the expectation value. Moreover, the vanishing scalar potential leads to the absence of cosmological constant, which is contradict to the observation showing the acceleration of the Universe. Due to the abelian gauge symmetry $U(1)^{2n}$, this theory cannot describe the standard model because the interactions between particles are governed in terms of the non-abelian gauge symmetry.

However, the abelian structure can be promoted to the non-abelian group, which is a subgroup of $O(n,n)$. This method is known as the embedding tensor formalism [16, 20, 23, 29]. Recall that the group generators of $O(n,n)$ global symmetry are represented by $(t_\alpha)_{\check{M}}{}^{\check{N}}$, the embedding tensors are given by $\Theta^{\alpha}_{\check{M}}$. Therefore, the gauge group generators are given by $\Theta^{\alpha}_{\check{M}}(t_\alpha)_{\check{N}}{}^{\check{L}}$. This embedding tensor have governed the detail of gauge group the effective theory, and the fluxes $\mathcal{F}_{\check{M}\check{N}\check{L}}$ arising from compactification are components of $\Theta^{\alpha}_{\check{M}}(t_\alpha)_{\check{N}}{}^{\check{L}}$.

Despite the abelian gauge symmetry can be promoted in to non-abelian gauge symmetry, the $O(n,n)$ global symmetry in this case are broken into a subgroup that leaves $\mathcal{F}_{\check{M}\check{N}\check{L}}$ invariant. Actually, the $O(n,n)$ symmetry links the different configurations of fluxes together because the fluxes $\mathcal{F}_{\check{M}\check{N}\check{L}}$ can be thought of as covariant tensor. Therefore, if fluxes transform into new fluxes via $h \in O(n,n)$

$$\mathcal{F}_{\check{M}\check{N}\check{L}} \to h_{\check{M}}{}^{\check{P}} h_{\check{N}}{}^{\check{Q}} h_{\check{L}}{}^{\check{R}} \mathcal{F}_{\check{P}\check{Q}\check{R}}, \tag{5.73}$$

gauge fields and scalar fields can be redefined via $h \in O(n,n)$

$$A^{\check{M}}{}_{\mu} \to h^{\check{M}}{}_{\check{N}} A^{\check{N}}, \tag{5.74}$$

$$\Phi_{\check{M}\check{N}} \to h_{\check{M}}{}^{\check{P}} h_{\check{N}}{}^{\check{Q}} \Phi_{\check{P}\check{Q}}. \tag{5.75}$$

As a result, the overall action is invariant under $O(n,n)$ [3,16]

$$S\left[\mathcal{F}, A, \Phi\right] = S'\left[h(\mathcal{F}), h(A), h(\Phi)\right]. \tag{5.76}$$

That means the two theories that are related by $O(n,n)$ are the same theory. Therefore, it is convenient to classify the orbit of fluxes rather the configuration of fluxes.

The orbits of fluxes that can be transformed into the geometric fluxes are called orbit of geometric fluxes, that we see from the example in chapter 4. In that case, only one flux is turned on, for example, on the torus the H-flux is presented. When the T-duality are performed in the isometry direction, H-flux has become f-flux. If T-duality is done again in the another isometry direction, f-flux is then transformed into Q-flux. Eventually, Q-flux can be turned into R-flux via T-duality in the remaining direction. This is an example of the orbits of geometric fluxes, where the only one flux can be turned on at the same time. Moreover, the cases, where multiple fluxes are turned on at the same time and T-duality cannot eliminate the non-geometric fluxes are called the orbit of truly non-geometric fluxes. They are investigated in [20] and they lead to the interesting in inflation problem in [30].

6. Conclusion

When the string theory is formulated on n-torus space, the result leads to the existence of new modes called the winding modes and the T-duality symmetry relates these winding modes to the momentum modes. The T-duality gives rise to the $O(n,n)$ structure preserving the spectrum and level-matching condition in the compact space. When cooperating T-duality into symmetry of the action, double field theory is emerging. Since it includes the momentum modes as well as the winding modes, the field contents in double field theory should depend on $2D$-dimension, which correspond to the coordinates dual to momentum and winding modes. However, with the requirement of the closure of the generalized Lie derivative, the strong constraint must be imposed. Consequently, the configurations of fields then depend only on D-dimensional null subspace of $2D$-dimensional manifold.

The field contents in double field theory can be parameterized in several ways. For example, the generalized metric $\mathcal{H}_{MN}$ which is a $O(D,D)$ representation, can be expressed in terms of the symmetric tensor (metric tensor) and antisymmetric tensor (either a two-form or a bivector). This theory leads to the NS-NS sector in the supergravity theory and the unification of the Lie derivative with two-form gauge transformation.

Upon Scherk-Schwarz compactification, fluxes have been induced in the effective theory. Fluxes can be transformed into other fluxes by T-duality transformation. However, there are some fluxes, which are not well-defined globally, such as Q-flux, and even locally such as R-flux. These problems arise because of the need of stringy symmetry in order to glue coordinate patching in Q-flux, and the dependence of dual coordinates in R-flux. On the other hand, these problems are not occurred in double field theory framework because T-duality symmetry is implanted in double field theory, so that glueing coordinate patches via T-duality is acceptable. Moreover, due to the dependence of both coordinates and dual coordinates, R-flux is locally well-defined in this framework.

By using the covariant fluxes calculated from the generalized vielbein without gauge fixing, the meaning of non-geometric fluxes such as Q-flux and R-flux can be determined in terms of the geometrical quantities. The R-flux can be identified with covariant field strength in analogous to to the H-flux, but using bivector instead of two-form. Similarly, Q-flux can be identified with the connection of the winding derivative.

Fluxes in the effective theory are required because they give rise to the scalar potential in effective theory, which leads to the vacuum configuration as well as the cosmological constant. Moreover, fluxes in effective theory also play the role of a structure constant of the gauge group, which is a subgroup of $O(n,n)$ and leads to the non-abelion gauge symmetry. Even if the $O(n,n)$ symmetry is broken due to the existence of fluxes, the action is still invariant because fluxes are the covariant $O(n,n)$ tensor along with the redefining fields via $O(n,n)$ element, therefore, the action is manifestly invariant. This implies the classification of the theory by the gauge orbits rather than the configuration of fluxes. The geometric flux and non-geometric flux can be presented at the same time, and this correspond to the non-geometric flux space, which is interesting and will be studied in the future work.

A. The Generalized Lie derivative of $\mathcal{H}_{MN}$

Let us consider the generalized Lie derivative of the generalized metric with parameter $\xi^M = (\tilde{\lambda}_i, \lambda^i)$ [9]. In this case, the strong constraint is imposed in the supergravity frame such that $\tilde{\partial}^m(\dots) = 0$. The generalized Lie derivative of the generalized metric is given by

$$\mathcal{L}_\xi \mathcal{H}_{MN} = \xi^P \partial_P \mathcal{H}_{MN} + \left(\partial_M \xi^P - \partial^P \xi^M\right) \mathcal{H}_{PN} + \left(\partial_N \xi^P - \partial^P \xi^N\right) \mathcal{H}_{MP}, \quad \text{(A.1)}$$

where $\mathcal{H}_{MN}$ is defined as

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{ij} - b_{ik} g^{kl} b_{lj} & b_{ik} g^{kj} \\ -g^{ik} b_{kj} & g^{ij} \end{pmatrix}. \quad \text{(A.2)}$$

Therefore, let us consider the generalized Lie derivative of the component $\mathcal{H}^{ij}$, which is given by

$$\begin{aligned} \mathcal{L}_\xi \mathcal{H}^{ij} =& \xi^P \partial_P \mathcal{H}^{ij} + (\tilde{\partial}^i \xi^P) \mathcal{H}_P{}^j - (\partial^P \lambda^i) \mathcal{H}_{Pj} + (\tilde{\partial}^j \xi^P) \mathcal{H}^i{}_P - (\partial^P \lambda^j) \mathcal{H}^i{}_P, \\ =& \tilde{\lambda}_k \tilde{\partial}^k \mathcal{H}^{ij} + \lambda^k \partial_k \mathcal{H}^{ij} + (\tilde{\partial}^i \tilde{\lambda}_k) \mathcal{H}^{kj} + (\tilde{\partial}^i \lambda^k) \mathcal{H}_k{}^j - (\tilde{\partial}^k \lambda^i) \mathcal{H}_{kj} - (\partial_k \lambda^i) \mathcal{H}^{kj} \\ & (\tilde{\partial}^j \tilde{\lambda}_k) \mathcal{H}^{ik} + (\tilde{\partial}^j \lambda^k) \mathcal{H}^i{}_k - (\tilde{\partial}^k \lambda^j) \mathcal{H}^i{}_k - (\partial_k \lambda^j) \mathcal{H}^{ik}. \end{aligned} \quad \text{(A.3)}$$

When the strong constraint is imposed, this equation becomes

$$\mathcal{L}_\xi \mathcal{H}^{ij} = \lambda^k \partial_k \mathcal{H}^{ij} - (\partial_k \lambda^i) \mathcal{H}^{kj} - (\partial_k \lambda^j) \mathcal{H}^{ik}. \quad \text{(A.4)}$$

By substituting the definition of $\mathcal{H}_{MN}$, this gives

$$\mathcal{L}_\xi g^{ij} = \lambda^k \partial_k g^{ij} - (\partial_k \lambda^i) g^{kj} - (\partial_k \lambda^j) g^{ik} = L_\lambda g^{ij}. \quad \text{(A.5)}$$

This is a normal Lie derivative of the inverse metric tensor g^{ij} along the vector field λ^i.

Next, let us consider the generalized Lie derivative of $\mathcal{H}_{ij}$, which is given by

$$
\begin{aligned}
\mathcal{L}_\xi \mathcal{H}_{ij} =& \xi^P \partial_P \mathcal{H}_{ij} + (\partial_i \xi^P - \partial^P \tilde{\xi}_i)\mathcal{H}_{Pi} + (\partial_j \xi^P - \partial^P \lambda_j)\mathcal{H}_{iP}, \\
=& \tilde{\lambda}_k \partial^k \mathcal{H}_{ij} + \lambda^k \partial_k \mathcal{H}_{ij} + (\partial_i \tilde{\lambda}_k)\mathcal{H}^k{}_j + (\partial_i \lambda^k)\mathcal{H}_{kj} - (\tilde{\partial}^k \tilde{\lambda}_i)\mathcal{H}_{kj} - (\partial_k \tilde{\lambda}_i)\mathcal{H}^k{}_j \\
& + (\partial_i \tilde{\lambda}_k)\mathcal{H}_i{}^k + (\partial_j \lambda^k)\mathcal{H}_{ik} - (\tilde{\partial}^k \tilde{\lambda}_j)\mathcal{H}_{ik} - (\partial_k \tilde{\lambda}_j)\mathcal{H}_i{}^k. \qquad (A.6)
\end{aligned}
$$

In supergravity frame, it becomes

$$
\mathcal{L}_\xi \mathcal{H}_{ij} = \lambda^k \partial_k \mathcal{H}_{ij} + (\partial_i \lambda^k)\mathcal{H}_{kj} - (\partial_k \tilde{\lambda}_i)\mathcal{H}^k{}_j + (\partial_i \tilde{\lambda}_k)\mathcal{H}_i{}^k + (\partial_j \lambda^k)\mathcal{H}_{ik} - (\partial_k \tilde{\lambda}_j)\mathcal{H}_i{}^k. \qquad (A.7)
$$

Consider the left-hand side of (A.7), which is given by

$$
\begin{aligned}
\text{LHS} =& \mathcal{L}_\xi (g_{ij} - b_{ik} g^{kl} b_{lj}) \\
=& \mathcal{L}_\xi g_{ij} - (\mathcal{L}_\xi b_{ik})\, g^{kl} b_{lj} - b_{ik} \left(\mathcal{L}_\xi g^{kl}\right) b_{lj} - b_{ik} g^{kl} \left(\mathcal{L}_\xi b_{lj}\right). \qquad (A.8)
\end{aligned}
$$

On the other hand, the right-hand side of (A.7) takes the form

$$
\begin{aligned}
\text{RHS} =& \lambda^p \partial_p (g_{ij} - b_{ik} g^{kl} b_{lj}) + (\partial_i \tilde{\lambda}_k)(-g^{kl} b_{lj}) + (\partial) i \lambda^k)(g_{kj} - b_{kl} g^{lp} b_{pj}) \\
& - (\partial_k \tilde{\lambda}_k)(-g^{kl} b_{lj}) + (\partial_j \tilde{\lambda}_k)(b_{il} g^{lk}) + (\partial_j \lambda^k)(g_{ik} - b_{il} g^{lp} b_{pk}) - (\partial_k \tilde{\lambda}_j)(b_{il} g^{lk}), \\
=& \lambda^p \partial_p g_{ij} - (\lambda^p \partial_p b_{ik}) g^{kl} b_{lj} - b_i k(\lambda^p \partial_p g^{kl}) b_{lj} - b_{ik} g^{kl} (\lambda^p \partial_p b_{lj}) + (\partial_i \lambda^k) g_{kj} \\
& + (\partial_i \lambda^k) g_{ik} - (\partial_i \tilde{\lambda}_k) g^{kl} b_{lj} + (\partial_k \tilde{\lambda}_i) g^{KL} b_{lj} - (\partial_i \lambda^q) b_{qk} g^{kl} b_{lj} - (\partial_k \lambda^q) b_{Iq} g^{kl} b_{lj} \\
& - (\partial_k \tilde{\lambda}_j) b_{il} g^{lk} - (\partial_j \lambda^K) b_{il} g^{lp} b_{pk} + (\partial_k \lambda^q) b_{iq} g^{kl} b_{lj}, \\
=& L_\lambda g_{ij} - \left\{ L_\lambda b_{ik} + (\partial_i \tilde{\lambda}_k - \partial_k \tilde{\lambda}_i) \right\} g^{kl} b_{lj} - b_{ik} \left\{ L_\lambda g^{kl} \right\} b_{lj} \\
& - b_{ik} g^{kl} \left\{ L_\lambda + (\partial_l \tilde{\lambda}_j - \partial_j \tilde{\lambda}_l \right\}. \qquad (A.9)
\end{aligned}
$$

Therefore, the generalized Lie derivative of the metric and the two-from when the strong constraint is hold in the supergravity frame are given as

$$
\begin{aligned}
\mathcal{L}_\xi g_{ij} &= L_\lambda g_{ij}, \qquad (A.10) \\
\mathcal{L}_\xi b_{ij} &= L_\lambda b_{ij} + (\partial_i \tilde{\lambda}_j - \partial_j \tilde{\lambda}_i). \qquad (A.11)
\end{aligned}
$$

B. Commutation Relation of Generalized Lie Derivative

Suppose that the commutation relation of the generalized Lie derivative on a generalized vector V^M is given by

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}]\, V^M = \mathcal{L}_{\xi_{12}} V^M. \tag{B.1}$$

The right-hand side of (B.1) is given by (3.31)

$$\mathrm{RHS} = \xi_{12}^P \partial_P V^M + \left(\partial^M \xi_{12P} - \partial_P \xi_{12}^M\right) V^P. \tag{B.2}$$

On the other hand, the left-hand side is given by

$$\begin{aligned}
\mathrm{LHS} =& \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} V^M - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1} V^M, \\
=& \mathcal{L}_{\xi_1}\left\{\xi_2^P \partial_P V^M + \left(\partial^M \xi_{2P} - \partial_P \xi_2^M\right) V^P\right\} - (1 \leftrightarrow 2), \\
=& \left(\mathcal{L}_{\xi_1}\xi_2^P\right)\partial_P V^M + \xi_2^P\left(\mathcal{L}_{\xi_1}\partial_P V^M\right) + \left(\mathcal{L}_{\xi_1}\partial^M\xi_{2P}\right)V^P + \left(\partial^M\xi_{2p}\right)\left(\mathcal{L}_{\xi_1}V^P\right) \\
& - \left(\mathcal{L}_{\xi_1}\partial_P\xi_2^M\right)V^P - \left(\partial_P\xi_2^M\right)\left(\mathcal{L}_{\xi_1}V^P\right) - (1 \leftrightarrow 2), \\
=& \left\{\xi_1^N\partial_N\xi_2^P + (\partial^P\xi_{1N} - \partial_N\xi_1^P)\xi_2^N\right\}\partial_P V^M \\
& + \xi_2^P\left\{\xi_1^N\partial_N(\partial_P V^M) + (\partial_P\xi_1^N - \partial^N\xi_{1P})\partial_N V^M + (\partial^M\xi_{1N} - \partial_N\xi_1^M)\partial_P V^N\right\} \\
& + V^P\left\{\xi_1^N\partial_N(\partial^M\xi_{2P}) + (\partial^M\xi_{1N} - \partial_N\xi_1^M)(\partial^N\xi_{2P}) + (\partial_P\xi_1^N - \partial^N\xi_{1P})(\partial^M\xi_{2N})\right\} \\
& + (\partial^M\xi_{2P})\left\{\xi_1^N\partial_N V^P + (\partial^P\xi_{1N} - \partial_N\xi_1^P)V^N\right\} \\
& - V^P\left\{\xi_1^N\partial_N\partial_P\xi_2^M + (\partial_P\xi_1^N - \partial^N\xi_{1P})(\partial_N\xi_2^M) + (\partial^M\xi_{1N} - \partial_N\xi_1^M)(\partial_P\xi_2^N)\right\} \\
& - (\partial_P\xi_2^M)\left\{\xi_1^N\partial_N V^P + (\partial^P\xi_{1N} - \partial_N\xi_1^P)V^B\right\} - (1 \leftrightarrow 2), \\
=& \left\{\xi_1^N\partial_N\xi_2^P - \xi_2^N\partial_N\xi_1^P\right\}\partial_P V^M \\
& + V^P\left\{\xi_1^N\partial_N\partial^M\xi_{2P} + \partial_P\xi_1^N\partial^M\xi_{2N} - \partial^N\xi_{1P}\partial^M\xi_{2N} - \xi_1^N\partial_N\partial_P\xi_2^M\right. \\
& \qquad + \partial_N\xi_1^M\partial_P\xi_2^N - \xi_2^N\partial_N\partial^M\xi_{1P} - \partial_P\xi_2^N\partial^M\xi_{1N} + \partial^N\xi_{2P}\partial^M\xi_{1N} \\
& \qquad + \partial^N\xi_{2P}\partial^M\xi_{1N} + \xi_2^N\partial_N\partial_P\xi_1^M - \partial_N\xi_2^M\partial_P\xi_1^N \\
& \qquad \left. - \partial_N\xi_1^M\partial^N\xi_{2P} + \partial_N\xi_2^M\partial^N\xi_{1P}\right\},
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\xi_1^N \partial_N \xi_2^P - \xi_2^N \partial_N \xi_1^P\right\} \partial_P V^M \\
&\quad + \partial^M \left(\xi_{1N}\partial^N \xi_{2P} - \xi_{2N}\partial^N \xi_{1P} - \frac{1}{2}\xi_{1N}\partial_P \xi_2^N + \frac{1}{2}\xi_{2N}\partial_P \xi_{1N}\right) V^P \\
&\quad - \partial_P \left(\xi_{1N}\partial^N \xi_2^M - \xi_{2N}\partial^N \xi_1^M - \frac{1}{2}\xi_{1N}\partial^M \xi_2^N + \frac{1}{2}\xi_{2N}\partial^M \xi_{1N}\right) V^P \\
&\quad - \left(\partial_N \xi_1^M \partial^N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P}\right) V^P
\end{aligned} \tag{B.3}
$$

Therefore, we get

$$
\begin{aligned}
\text{LHS} = &\left(\xi_1^N \partial_N \xi_2^P - \xi_2^N \partial_N \xi_1^P - \frac{1}{2}\xi_1^N \partial^P \xi_{2N} + \frac{1}{2}\xi_2^N \partial^P \xi_{1N}\right) \partial_P V^M \\
&+ \partial^M \left(\xi_{1N}\partial^N \xi_{2P} - \xi_{2N}\partial^N \xi_{1P} - \frac{1}{2}\xi_{1N}\partial_P \xi_2^N + \frac{1}{2}\xi_{2N}\partial_P \xi_{1N}\right) V^P \\
&- \partial_P \left(\xi_{1N}\partial^N \xi_2^M - \xi_{2N}\partial^N \xi_1^M - \frac{1}{2}\xi_{1N}\partial^M \xi_2^N + \frac{1}{2}\xi_{2N}\partial^M \xi_{1N}\right) V^P \\
&- \left(\partial_N \xi_1^M \partial^N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P}\right) V^P \\
&+ \frac{1}{2}\xi_1^N \partial^P \xi_{2N}\partial_P V^M - \frac{1}{2}\xi_2^N \partial^P \xi_{1N}\partial_P V^M
\end{aligned} \tag{B.4}
$$

As a result, the parameter ξ_{12}^P is given by

$$
\xi_{12}^P = [\xi_1, \xi_2]_C^P. \tag{B.5}
$$

This shows that

$$
[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}]\, V^M = \mathcal{L}_{[\xi_1,\xi_2]_C} V^M + \Delta F^M, \tag{B.6}
$$

where

$$
\Delta F^M = \frac{1}{2}\xi_1^N \partial^P \xi_{2N}\partial_P V^M - \frac{1}{2}\xi_2^N \partial^P \xi_{1N}\partial_P V^M - \left(\partial_N \xi_1^M \partial^N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P}\right) V^P. \tag{B.7}
$$

In order for the closure of this transformation, or in other word, LHS of (B.1) is equal to RHS, some constraint must be imposed. This constraint is known as the strong constraint, and when it is hold, it leads to the vanishing of ΔF^M.

Therefore, the commutation relation of the generalized Lie derivative with strong constraint takes the form

$$
[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]_C}. \tag{B.8}
$$

C. Jacobiator of the Generalized Lie Derivative

By following from [26], the jacobiator can be written as

$$J^M(\xi_1,\xi_2,\xi_3) = \left[[\xi_1,\xi_2]_C\,,\xi_3\right]_C^M + \left[[\xi_2,\xi_3]_C\,,\xi_1\right]_C^M + \left[[\xi_3,\xi_1]_C\,,\xi_2\right]_C^M\,. \tag{C.1}$$

First, let us consider

$$\left[[\xi_1,\xi_2]_C\,,\xi_3\right]_C^M\,. \tag{C.2}$$

From the properties of the generalized Lie derivative (3.41) and (3.43), we then get

$$\begin{aligned}\left[[\xi_1,\xi_2]_C\,,\xi_3\right]_C^M &= \frac{1}{2}\left(\mathcal{L}_{[\xi_1,\xi_2]_C}\xi_3^M - \mathcal{L}_{\xi_3}\left[\xi_1,\xi_2\right]_C^M\right)\,, &\text{(C.3)}\\ &= \frac{1}{2}\left(\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2}\xi_3^M - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}\xi_3^M\right) + \frac{1}{4}\left(\mathcal{L}_{\xi_3}\mathcal{L}_{\xi_2}\xi_1^M - \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_1}\xi_2^M\right)\,. &\text{(C.4)}\end{aligned}$$

From (C.3), then the Jacobiator takes the form

$$\begin{aligned}J^M(\xi_1,\xi_2,\xi_3) =& \left[[\xi_1,\xi_2]_C\,,\xi_3\right]_C^M + \left[[\xi_2,\xi_3]_C\,,\xi_1\right]_C^M + \left[[\xi_3,\xi_1]_C\,,\xi_2\right]_C^M\,,\\ =&\frac{1}{2}\left(\mathcal{L}_{[\xi_1,\xi_2]_C}\xi_3^M + \mathcal{L}_{[\xi_2,\xi_3]_C}\xi_1^M + \mathcal{L}_{[\xi_3,\xi_1]_C}\xi_2^M\right.\\ &\left.-\mathcal{L}_{\xi_3}\left[\xi_1,\xi_2\right]_C^M - \mathcal{L}_{\xi_1}\left[\xi_2,\xi_3\right]_C^M - \mathcal{L}_{\xi_2}\left[\xi_3,\xi_1\right]_C^M\right)\,. \end{aligned}\tag{C.5}$$

Next let us consider

$$\mathcal{L}_{\xi_3}\left[\xi_1,\xi_2\right]_C^M + \mathcal{L}_{\xi_1}\left[\xi_2,\xi_3\right]_C^M + \mathcal{L}_{\xi_2}\left[\xi_3,\xi_1\right]_C^M\,. \tag{C.6}$$

By using (3.43), these terms can be rewritten as

$$\begin{aligned}&\mathcal{L}_{\xi_3}\left[\xi_1,\xi_2\right]_C^M + \mathcal{L}_{\xi_1}\left[\xi_2,\xi_3\right]_C^M + \mathcal{L}_{\xi_2}\left[\xi_3,\xi_1\right]_C^M\\ &= \left(\mathcal{L}_{\xi_3}\mathcal{L}_{\xi_1}\xi_2^M - \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_2}\xi_1^M + \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2}\xi_3^M - \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_3}\xi_2^M + \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_3}\xi_1^M - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}\xi_3^M\right)\,,\\ &= \frac{1}{2}\left(\mathcal{L}_{[\xi_1,\xi_2]_C}\xi_3^M + \mathcal{L}_{[\xi_2,\xi_3]_C}\xi_1^M + \mathcal{L}_{[\xi_3,\xi_1]_C}\xi_2^M\right)\,.\end{aligned}\tag{C.7}$$

Therefore, from (C.5) and (C.7), the Jacobiator is obtained as

$$J^M(\xi_1, \xi_2, \xi_3) = \frac{1}{4}\left(\mathcal{L}_{[\xi_1,\xi_2]_C}\xi_3^M + \mathcal{L}_{[\xi_2,\xi_3]_C}\xi_1^M + \mathcal{L}_{[\xi_3,\xi_1]_C}\xi_2^M\right). \tag{C.8}$$

Alternatively, if one starts from (C.4), the Jacobiator can be written as

$$J^M(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big(\mathcal{L}_{\xi_1}\left[\xi_2, \xi_3\right]_C^M + \mathcal{L}_{\xi_2}\left[\xi_3, \xi_1\right]_C^M + \mathcal{L}_{\xi_3}\left[\xi_1, \xi_2\right]_C^M\Big). \tag{C.9}$$

From the symmetric property of the generalized Lie derivative (3.44) along with (C.8) and (C.9), the Jacobiator can be obviously rearranged such that it is written in terms of total derivative

$$J^M(\xi_1, \xi_2, \xi_3) = \frac{1}{6}\partial^M\left(\xi_1^P[\xi_2, \xi_3]_{CP} + \xi_2^P[\xi_3, \xi_1]_{CP} + \xi_3^P[\xi_1, \xi_2]_{CP}\right). \tag{C.10}$$

D. Covariant Flux Calculation

By following from [20] and the generalized vielbein is parameterized in terms of the vielbein $e^a{}_i$, the two-from b_{ij}, and the bivector β^{ij}. In the other word, the gauge fixing is not imposed such that

$$E^A{}_M = \begin{pmatrix} E_a{}^i & E_{ai} \\ E^{ai} & E^a{}_i \end{pmatrix} = \begin{pmatrix} e_a{}^i & e_a{}^l b_{li} \\ e^a{}_l \beta^{li} & e^a{}_i + e^a{}_l \beta^{lk} b_{ki} \end{pmatrix}. \tag{D.1}$$

Let us begin with the simplest flux that is H_{abc}.

H-flux

The definition of H-flux is given by

$$H_{abc} = 3\Omega_{[abc]}. \tag{D.2}$$

There is only one component of Ω_{ABC} needed to be calculated in this case. That is

$$\begin{aligned}
\Omega_{abc} =& E_a{}^M \partial_M E_b{}^N E_{cN}, \\
=& E_a{}^m \partial_m E_b{}^n E_{cn} + E_{am} \tilde{\partial}^m E_b{}^n E_{cn} + E_a{}^m \partial_m E_{bn} E_c{}^n + E_{am} \tilde{\partial}^m E_{bn} E_c{}^n, \\
=& e_a{}^m \partial_m e_b{}^n (e_c{}^j b_{jn}) + (e_a{}^k b_{km}) \tilde{\partial}^m e_b{}^n (e_c{}^j b_{jn}) + e_a{}^m \partial_m (e_b{}^j b_{jn}) e_c{}^n \\
&+ (e_a{}^k b_{km}) \tilde{\partial}^m (e_b{}^j b_{jn}) e_c{}^n, \\
=& e_a{}^m \partial_m e_b{}^n e_c{}^j b_{jn} + e_a{}^k b_{kn} \tilde{\partial}^m e_b{}^n e_c{}^j b_{jn} + e_a{}^m \partial_m e_b{}^j b_{jn} e_c{}^n + e_a{}^m e_b{}^j \partial_m b_{jn} e_c{}^n \\
&+ e_a{}^k b_{km} \tilde{\partial}^m e_b{}^j b_{jn} e_c{}^n + e_a{}^k b_{km} e_b{}^j \tilde{\partial}^m b_{jn} e_c{}^n, \\
=& e_a{}^i e_b{}^j e_c{}^k (\partial_i b_{jk} + b_{im} \tilde{\partial}^m b_{jk}), \\
=& e_a{}^i e_b{}^j e_c{}^k (D_i b_{jk}),
\end{aligned} \tag{D.3}$$

where the derivative D_m is defined as

$$D_m \equiv \partial_m + b_{ml} \tilde{\partial}^l. \tag{D.4}$$

Therefore, H-flux is easily obtained

$$H_{abc} = 3\Omega_{[abc]} = 3e_a{}^i e_b{}^j e_c{}^k (\partial_{[i} b_{jk]} + b_{[i|m}\tilde{\partial}^m b_{jk]}) = 3e_a{}^i e_b{}^j e_c{}^k D_{[i} b_{jk]}. \tag{D.5}$$

f-flux

The f-flux can be obtained by

$$f^a{}_{bc} = \Omega^a{}_{bc} + \Omega_c{}^a{}_b + \Omega_{bc}{}^a. \tag{D.6}$$

However, from the antisymmetric property of Ω_{ABC}, we have

$$\Omega_{bc}{}^a = -\Omega_b{}^a{}_c. \tag{D.7}$$

Therefore, the $\Omega^a{}_{bc}$ and $\Omega_a{}^b{}_c$ need to be evaluated.

$$\begin{aligned}
\Omega^a{}_{bc} =& E^{aM}\partial_M E_b{}^N E_{cN}, \\
=& E^{am}\partial_m E_b{}^n E_{cn} + E^a{}_m \tilde{\partial}^m E_b{}^n E_{cn} + E^{am}\partial_m E_{bn} E_c{}^n + E^a{}_m \tilde{\partial} E_{bn} E_c{}^n, \\
=& (e^a{}_i \beta^{im})\partial_m (e_b{}^j b_{jn}) e_c{}^n + (e^a{}_m + e^a{}_i \beta^{ij} b_{jm})\tilde{\partial}^m e_b{}^n (e_c{}^k b_{kn}) \\
& + (e^a{}_i \beta^{im})\partial_m (e_b{}^j b_{jn}) e_c{}^n + (e^a{}_m + e^a{}_i \beta^{ij} b_{jm})\tilde{\partial}^m (e_b{}^k b_{kn}) e_c{}^n, \\
=& e^a{}_i \beta^{im}\partial_m e_b{}^n e_c{}^n b_{kj} + e^a{}_m \tilde{\partial}^m e_b{}^n e_c{}^k b_{kn} + e^a{}_i \beta^{ij} b_{jm}\tilde{\partial}^m e_b{}^n e_c{}^k b_{kn} \\
& + e^a{}_i \beta^{im}\partial_m e_b{}^j b_{jn} e_c{}^n + e^a{}_i \beta^{im} e_b{}^j \partial_m b_{jn} e_c{}^n + e^a{}_m \tilde{\partial}^m e_b{}^k b_{kn} e_c{}^n \\
& + e^a{}_m e_b{}^k \tilde{\partial}^m b_{kn} e_c{}^n + e^a{}_i \beta^{ij} b_{jm}\tilde{\partial}^m e_b{}^k b_{kn} e_c{}^n + e^a{}_i \beta^{ij} b_{jm} e_b{}^k \tilde{\partial}^l b_{kl} e_c{}^n, \\
=& e^a{}_i e_b{}^j e_c{}^k \left\{ \tilde{\partial}^i b_{jk} + \beta^{im}(\partial_m b_{jk} - b_{lm}\tilde{\partial}^l b_{jk}) \right\}, \\
=& e^a{}_i e_b{}^j e_c{}^k \left\{ \tilde{\partial}^i b_{jk} + \beta^{im} D_m b_{jk} \right\}.
\end{aligned} \tag{D.8}$$

$$
\begin{aligned}
\Omega_a{}^b{}_c =& E_a{}^M \partial_M E^{bN} E_{cN}, \\
=& E_a{}^m \partial_m E^{bn} E_{cn} + E_{am} \tilde{\partial}^m E^{bn} E_{cn} + E_a{}^m \partial_m E^b{}_n E_c{}^n + E_{am} \tilde{\partial}^m E^b{}_n E_c{}^n, \\
=& e_a{}^m \partial_m (e^b{}_j \beta^{jn})(e_c{}^k b_{kn}) + (e_a{}^i b_{im}) \tilde{\partial}^m (e^b{}_j \beta^{jn})(e_c{}^k b_{kn}) \\
& + e_a{}^m \partial^m (e^b{}_n + e^b{}_j \beta^{jk} b_{kn}) e_c{}^n + (e_a{}^i b_{im}) \tilde{\partial}^m (e^b{}_n + e^b{}_j \beta^{jk} b_{kn}) e_c{}^n, \\
=& e_a{}^m \partial_m e^b{}_j \beta^{jn} e_c{}^k b_{kn} + e_a{}^m e^b{}_j \partial_m \beta^{jn} e_c{}^k b_{kn} + e_a{}^i b_{im} \tilde{\partial}^m e^b{}_j \beta^{jn} e_c{}^k b_{kn} \\
& + e_a{}^i b_{im} e^b{}_j \tilde{\partial}^m \beta^{jn} e_c{}^k b_{kn} + e_a{}^m \partial_m e^b{}_n e_c{}^n + e_a{}^m \partial_m e^b{}_j \beta^{jk} b_{kn} e_c{}^k \\
& + e_a{}^m e^b{}_j \partial_m \beta^{jk} b_{kn} e_c{}^n + e_a{}^m e^b{}_j \beta^{jk} \partial_m b_{kn} e_c{}^n + e_a{}^i b_{im} \tilde{\partial}^m e^b{}_n e_c{}^n \\
& + e_a{}^i b_{im} \tilde{\partial}^m e^b{}_j \beta^{jk} b_{kn} e_c{}^n + e_a{}^i b_{im} e^b{}_j \tilde{\partial}^m \beta^{jk} b_{kn} e_c{}^n + e_a{}^i b_{im} e^b{}_j \beta^{jk} \tilde{\partial}^m b_{kn} e_c{}^n, \\
=& e_a{}^i \partial_i e^b{}_j e_c{}^j + e_a{}^i b_{im} \tilde{\partial}^m e^b{}_j e_c{}^j + e_a{}^i e^b{}_j \beta^{jk} (\partial_i b_{kn} + b_{im} \tilde{\partial}^m b_{kn}) e_c{}^n, \\
=& e_a{}^i D_i e^b{}_j e_c{}^j + e_a{}^i e^b{}_j \beta^{jk} D_i b_{kn} e_c{}^n.
\end{aligned} \tag{D.9}
$$

Thus, $f^a{}_{bc}$ is then obtained as

$$
\begin{aligned}
f^a{}_{bc} =& \Omega^a_{bc} + \Omega_c{}^a{}_b - \Omega_b{}^a{}_c, \\
=& e^a{}_i e_b{}^j e_c{}^k \left\{ \tilde{\partial}^i b_{jk} + \beta^{im} D_m b_{jk} \right\} + e_c{}^i D_i e^a{}_j e_b{}^j + e_c{}^i e^a{}_j \beta^{jk} D_i b_{kn} e_b{}^n \\
& - e_b{}^i D_i e^a{}_j e_c{}^j - e_b{}^i e^a{}_j \beta^{jk} D_i b_{kn} e_c{}^n \\
=& 2 \left(e_{[c}{}^i D_i e^a{}_j e_{b]}{}^j \right) + e^a{}_i e_b{}^j e_c{}^k \left(\tilde{\partial}^i b_{jk} + \beta^{im} H_{mjk} \right).
\end{aligned} \tag{D.10}
$$

Q-flux

The Q-flux is defined as

$$
Q_a{}^{bc} = \Omega_a{}^{bc} + \Omega^{bc}{}_a + \Omega^c{}_a{}^b. \tag{D.11}
$$

Therefore, there are two components of Ω_{ABC} evaluated since $\Omega^c{}_a{}^b = -\Omega^{cb}{}_a$

$$
\begin{aligned}
\Omega_a{}^{bc} =& E_a{}^M \partial^M E^{bN} E^c{}_N, \\
=& E_a{}^m \partial^m E^{bn} E^c{}_n + E_{am} \tilde{\partial}^m E^{bn} E^c{}_n + E_a{}^m \partial_m E^b{}_n E^{cn} + E_{am} \tilde{\partial}^m E^b{}_n E^{cn}, \\
=& e_a{}^m \partial^m (e^b{}_i \beta^{in})(e^c{}_n + e^c{}_j \beta^{jk} b_{kn}) + (e_a{}^i b_{im}) \tilde{\partial}^m (e^b{}_j \beta^{jn})(e^c{}_n + e^c{}_k \beta^{kl} b_{ln}) \\
& + e_a{}^m \partial_m (e^b{}_n + e^b{}_i \beta^{ij} b_{jn})(e^c{}_k \beta^{kn}) + (e_a{}^i b_{im}) \tilde{\partial}^m (e^b{}_n + e^b{}_j \beta^{jk} b_{kn})(e^c{}_l \beta^{ln})
\end{aligned}
$$

$$
\begin{aligned}
=&e_a{}^m\partial_m e^b{}_i\beta^{in}e^c{}_n + e_a{}^m\partial_m e^b{}_i\beta^{in}e^c{}_j\beta^{jk}b_{kn} + e_a{}^m e^b{}_i\partial_m\beta^{in}e^c{}_n \\
&+ e_a{}^m e^b{}_i\partial_m\beta^{in}e^c{}_j\beta^{jk}b_{kn} + e_a{}^i b_{im}\tilde{\partial}^m e^b{}_j\beta^{jn}e^c{}_n + e_a{}^i b_{im}\tilde{\partial}^m e^b{}_j\beta^{jn}e^c{}_k\beta^{kl}b_{ln} \\
&+ e_a{}^i b_{im}e^b{}_j\tilde{\partial}^m\beta^{jn}e^c{}_n + e_a{}^i b_{im}e^b{}_j\tilde{\partial}^m\beta^{jn}e^c{}_k\beta^{kl}b_{ln} + e_a{}^m\partial_m e^b{}_n e^c{}_k\beta^{kn} \\
&+ e_a{}^m\partial_m e^b{}_i\beta^{ij}b_{jn}e^c{}_k\beta^{kn} + e_a{}^m e^b{}_i\partial_m\beta^{ij}b_{jn}e^c{}_k e^c{}_k\beta^{kn} + e_a{}^m e^b{}_i\beta^{ij}\partial_m b_{jn}e^c{}_k\beta^{kn} \\
&+ e_a{}^i b_{im}\tilde{\partial}^m e^b{}_j e^c{}_l\beta^{ln} + e_a{}^i b_{im}\tilde{\partial}^m e^b{}_j\beta^{jk}b_{kn}e^c{}_l\beta^{ln} + e_a{}^i b_{im}e^b{}_j\tilde{\partial}^m\beta^{jk}b_{kn}e^c{}_l\beta^{ln} \\
&+ e_a{}^i b_{im}e^b{}_j\beta^{jk}\tilde{\partial}^m b_{kn}e^c{}_l\beta^{ln} \\
=&e_a{}^i e^b{}_j e^c{}_k\left(\partial_i\beta^{jk} + b_{im}\tilde{\partial}^m\beta^{jk} + \beta^{jm}\partial_i b_{mn}\beta^{kn} + \beta^{jm}b_{il}\tilde{\partial}^l b_{mn}\beta^{kn}\right), \\
=&e_a{}^i e^b{}_j e^c{}_k\left(D_i\beta^{jk} + \beta^{jm}D_i b_{mn}\beta^{kn}\right).
\end{aligned}
\tag{D.12}
$$

$$
\begin{aligned}
\Omega^{ab}{}_c =&E^{aM}\partial_M E^{bN}E_{cN} \\
=&E^{am}\partial_m E^{bn}E_{cn} + E^a{}_m\tilde{\partial}^m E^{bn}E_{cn} + E^{am}\partial_m E^b{}_n E_c{}^n + E^a{}_m\tilde{\partial}^m E^b{}_n E_c{}^n \\
=&(e^a{}_i\beta^{im})\partial_m(e^b{}_j\beta^{jn})(e_c{}^k b_{kn}) + (e^a{}_m + e^a{}_i\beta^{ij}b_{jm})\tilde{\partial}^m(e^b{}_k\beta^{kn})(e_c{}^l b_{ln}) \\
&+ (e^a{}_i\beta^{im})\partial_m(e^b{}_n e^b{}_j\beta^{jk}b_{kn})e_c{}^n + (e^a{}_m + e^a{}_i\beta^{ij}b_{jm})\tilde{\partial}^m(e^b{}_n e^b{}_k\beta^{kl})e_c{}^n \\
=&e^a{}_i\beta^{im}\partial_m e^b{}_j\beta^{jn}e_c{}^k b_{kn} + e^a{}_i\beta^{im}e^b{}_j\partial_m\beta^{jn}e_c{}^k b_{kn} + e^a{}_m\tilde{\partial}^m e^b{}_k\beta^{kn}e_c{}^l b_{ln} \\
&+ e^a{}_m e^b{}_k\tilde{\partial}^m\beta^{kn}e_c{}^l b_{ln} + e^a{}_i\beta^{ij}b_{jm}\tilde{\partial}^m e^b{}_k\beta^{kn}e_c{}^l b_{ln} + e^a{}_i\beta^{ij}b_{jm}e^b{}_k\tilde{\partial}^m\beta^{kn}e_a{}^l b_{ln} \\
&+ e^a{}_i\beta^{im}\partial_m e^b{}_n e_c{}^n + e^a{}_i\beta^{im}\partial_m e^b{}_j\beta^{jk}b_{kn}e_c{}^n + e^a{}_i\beta^{im}e^b{}_j\partial_m\beta^{jk}b_{kn}e_c{}^n \\
&+ e^a{}_i\beta^{im}e^b{}_j\beta^{jk}\partial_m b_{kn}e_c{}^n + e^a{}_m\tilde{\partial}^m e^b{}_n e_c{}^n + e^a{}_m\tilde{\partial}^m e^b{}_k\beta^{kl}b_{ln}e_c{}^n \\
&+ e^a{}_m e^b{}_k\tilde{\partial}^m\beta^{kl}b_{ln}e_c{}^n + e^a{}_m e^b{}_k\beta^{kl}\tilde{\partial}^m b_{ln}e_c{}^n + e^a{}_i\beta^{ij}b_{jm}\tilde{\partial}^m e^b{}_n e_c{}^n \\
&+ e^a{}_i\beta^{ij}b_{jm}\tilde{\partial}^m e^b{}_k\beta^{kl}b_{ln}e_c{}^l + e^a{}_i\beta^{ij}b_{jm}e^b{}_k\tilde{\partial}^m\beta^{kl}b_{ln}e_c{}^n \\
&+ e^a{}_i\beta^{ij}b_{jm}e^b{}_k\beta^{kl}\tilde{\partial}^m b_{ln}e_c{}^n \\
=&e^a{}_i e_c{}^n\left(\tilde{\partial}^i e^b{}_n + \beta^{im}\partial_m e^b{}_n + \beta^{ij}b_{jm}\tilde{\partial}^m e^b{}_n + \beta^{ij}b_{jm}\tilde{\partial}^m e^b{}_k\beta^{kl}b_{ln}\right) \\
&+ e^a{}_i e^b{}_j e_c{}^n\left(\beta^{im}\beta^{jk}\partial_m b_{kn} + \beta^{jl}\tilde{\partial}^i b_{ln}\right), \\
=&e^a{}_i e^b{}_j e_c{}^k\beta^{jl}\tilde{D}^i b_{lk} + e^a{}_i e_c{}^n\left(\tilde{D}^i e^b{}_n + \beta^{ij}b_{jm}(\tilde{\partial}^m e^b{}_n + \tilde{\partial}^m e^b{}_k\beta^{kl}b_{ln})\right),
\end{aligned}
\tag{D.13}
$$

where $\tilde{D}^i$ is defined by

$$
\tilde{D}^i \equiv \tilde{\partial}^i + \beta^{ij}\partial_j. \tag{D.14}
$$

Thus, the Q-flux is obviously obtained as

$$\begin{aligned}
Q_a{}^{bc} =& \Omega_a{}^{bc} + \Omega^c{}_a{}^b + \Omega^{bc}{}_a, \\
=& \Omega_a{}^{bc} - \Omega^{bc}{}_a - \Omega^{cb}{}_a, \\
=& e_a{}^i e^b{}_j e^c{}_k \left(D_i \beta^{jk} + \beta^{jm} D_i b_{mn} \beta^{kn} + \beta^{kl} \tilde{D}^j b_{li} - \beta^{jl} \tilde{D}^k b_{li} \right) \\
&+ e^b{}_j e_a{}^i \left(\tilde{D}^j e^c{}_i + \beta^{jp} b_{pm} (\tilde{\partial}^m e^c{}_i + \tilde{\partial}^m e^c{}_k \beta^{kl} b_{li}) \right) \\
&- e^c{}_j e_a{}^i \left(\tilde{D}^j e^b{}_i + \beta^{jp} b_{pm} (\tilde{\partial}^m e^b{}_i + \tilde{\partial}^m e^c{}_k \beta^{kl} b_{li}) \right), \\
=& e_a{}^i e^b{}_j e^c{}_k \left(D_i \beta^{jk} + \beta^{jm} \beta^{kn} D_i b_{mn} + 2\beta^{[k|l} \tilde{D}^{j]} b_{li} \right) \\
&+ 2 e_a{}^i \left(e^{[b}{}_j \tilde{D}^j e^{c]}{}_i + e^{[b}{}_j \beta^{jp} b_{pm} \tilde{\partial}^m e^{c]}{}_i + e^{[b}{}_j \beta^{jp} b_{pm} \tilde{\partial}^m e^{c]}{}_k \beta^{kl} b_{li} \right).
\end{aligned} \tag{D.15}$$

R-flux

The R-flux is defined as

$$R^{abc} = 3\Omega^{[abc]}. \tag{D.16}$$

There is only one component of Ω_{ABC} calculated for R-flux.

$$\begin{aligned}
\Omega^{abc} =& E^{aM} \partial_M E^{bN} E^c{}_n, \\
=& E^{am} \partial_m E^{bn} E^c{}_n + E^a{}_m \tilde{\partial}^m E^{bn} E^c{}_n + E^{am} \partial_m E^b{}_n E^{cn} + E^a{}_m \tilde{\partial}^m E^b{}_n E^{cn}, \\
=& (e^a{}_i \beta^{im}) \partial_m (e^b{}_j \beta^{jn})(e^c{}_n + e^c{}_k \beta^{kl} b_{ln}) \\
&+ (e^a{}_m + e^a{}_i \beta^{ij} b_{jm}) \tilde{\partial} (e^b{}_k \beta^{kn})(e^c{}_n + e^c{}_l \beta^{lp} b_{pn}) \\
&+ (e^a{}_i \beta^{im}) \partial_m (e^b{}_n + e^b{}_j \beta^{jk} b_{kn})(e^c{}_l \beta^{ln}) \\
&+ (e^a{}_m + e^a{}_i \beta^{ij} b_{jm}) \tilde{\partial}^m (e^b{}_n + e^b{}_k \beta^{kl} b_{ln})(e^c{}_p \beta^{pn}) \\
=& e^a{}_i \beta^{im} \partial_m e^b{}_j \beta^{jn} e^c{}_n + e^a{}_i \beta^{im} e^b{}_j \partial_m \beta^{jn} e^c{}_n + e^a{}_i \beta^{im} \partial_m e^b{}_j \beta^{jn} e^c{}_k \beta^{kl} b_{ln} \\
&+ e^a{}_i \beta^{im} e^b{}_j \partial_m \beta^{jn} e^c{}_k \beta^{kl} b_{ln} + e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_n + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_n \\
&+ e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_l \beta^{lp} b_{pn} + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_l \beta^{lp} b_{pn} + e^a{}_i \beta^{ij} b_{jm} \tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_n \\
&+ e^a{}_i \beta^{ij} b_{jm} e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_n + e^a{}_i \beta^{ij} b_{jm} \tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_l \beta^{lp} b_{pn} \\
&+ e^a{}_i \beta^{ij} b_{jm} e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_l \beta^{lp} b_{pn} + e^a{}_i \beta^{im} \partial_m e^b{}_j e^c{}_l \beta^{ln} + e^a{}_i \beta^{im} \partial_m e^b{}_j \beta^{jk} b_{kn} e^c{}_l \beta^{ln} \\
&+ e^a{}_i \beta^{im} b_{jm} \partial_m \beta^{jk} b_{kn} e^c{}_l \beta^{ln} + e^a{}_i \beta^{im} e^b{}_j \beta^{jk} \partial_m b_{kn} e^c{}_l \beta^{ln} + e^a{}_m \tilde{\partial} e^b{}_n e^c{}_p \beta^{pn} \\
&+ e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kl} b_{ln} e^c{}_p \beta^{pn} + e^a{}_m e^b{}_k \tilde{\partial} \beta^{kl} b_{ln} e^c{}_p \beta^{pn} + e^a{}_m e^b{}_k \beta^{kl} \tilde{\partial}^m b_{ln} e^c{}_p \beta^{pn} \\
&+ e^a{}_i \beta^{ij} b_{jm} \tilde{\partial} e^b{}_n e^c{}_p \beta^{pn} + e^a{}_i \beta^{ij} b_{jm} \tilde{\partial}^m e^b{}_k \beta^{kl} b_{ln} e^c{}_p \beta^{pn} + e^a{}_i \beta^{ij} b_{jm} e^b{}_k \tilde{\partial}^m \beta^{kl} b_{ln} e^c{}_p \beta^{pn}
\end{aligned}$$

$$
\begin{aligned}
&+ e^a{}_i \beta^{ij} b_{jm} e^b{}_k \beta^{kl} \tilde{\partial}^m b_{ln} e^c{}_p \beta^{pn} \\
=& e^a{}_i e^b{}_j e^c{}_k \left\{ \tilde{\partial}^i \beta^{jk} + \beta^{im} \partial_m \beta^{jk} + \beta^{il} b_{lm} \tilde{\partial}^l \beta^{jk} + \beta^{im} \beta^{jl} \beta^{kn} \partial_m b_{ln} + \beta^{jl} \beta^{kn} \tilde{\partial}^i b_{ln} \right. \\
&\left. + \beta^{im} \beta^{jl} \beta^{kn} b_{mp} \tilde{\partial}^b b_{ln} \right\} \\
=& e^a{}_i e^b{}_j e^c{}_k \left\{ \tilde{D}^i \beta^{jk} + \beta^{im} b_{ml} \tilde{\partial}^l \beta^{jk} + \beta^{im} \beta^{jl} \beta^{kn} D_m b_{ln} + \beta^{jl} \beta^{kn} \tilde{\partial}^i b_{ln} \right\} . \qquad \text{(D.17)}
\end{aligned}
$$

Finally, the R-flux is obtained

$$
\begin{aligned}
R^{abc} =& 3\Omega^{[abc]}, \\
=& 3 e^a{}_i e^b{}_j e^c{}_k \left\{ \tilde{D}^{[i} \beta^{jk]} + \beta^{[i|m} b_{ml} \tilde{\partial}^l \beta^{jk]} + \beta^{im} \beta^{jl} \beta^{kn} D_{[m} b_{ln]} + \beta^{[jl} \tilde{\partial}^i b_{ln} \beta^{k]n} \right\}, \\
=& 3 e^a{}_i e^b{}_j e^c{}_k \left\{ \tilde{D}^{[i} \beta^{jk]} + \beta^{[i|m} b_{ml} \tilde{\partial}^l \beta^{jk]} + \beta^{[jl} \tilde{\partial}^i b_{ln} \beta^{k]n} + \frac{1}{3} \beta^{im} \beta^{jl} \beta^{kn} H_{mln} \right\} . \qquad \text{(D.18)}
\end{aligned}
$$

E. Commutation relation of $\tilde{D}^i$

Consider the commutation relation of the derivative $\tilde{D}^i$ on the dilaton $\tilde{\phi}$

$$\begin{aligned}
\left[\tilde{D}^i, \tilde{D}^j\right]\tilde{\phi} =& \tilde{D}^i\tilde{D}^j\tilde{\phi} - (i \leftrightarrow j),\\
=& \tilde{\partial}^i\tilde{\partial}^j\tilde{\phi} + \tilde{\partial}^i\beta^{jk}\partial_k\tilde{\phi} + \beta^{jk}\partial_k\tilde{\partial}^i\tilde{\phi} + \beta^{il}\partial_l\tilde{\partial}^i\tilde{\phi}\\
&+ \beta^{il}(\partial_l\beta^{jk})\partial_k\tilde{\phi} + \beta^{il}\beta^{jk}\partial_l\partial_k\tilde{\phi} - (i \leftrightarrow j),\\
=& \tilde{D}^i\beta^{jk}\partial_k\tilde{\phi} + \tilde{D}^j\beta^{ki}\partial_k\tilde{\phi}.
\end{aligned} \tag{E.1}$$

By adding terms

$$\tilde{\partial}^k\beta^{ij}\partial_k\tilde{\phi} + \partial_l\beta^{ij}\tilde{\partial}^l\tilde{\phi}, \tag{E.2}$$

that vanish due to the strong constraint into (E.1) then the commutation relation becomes

$$\begin{aligned}
\left[\tilde{D}^i, \tilde{D}^j\right]\tilde{\phi} &= \left(\tilde{D}^i\beta^{jk} + \tilde{D}^j\beta^{ki} + \tilde{D}^k\beta^{ij}\right)\partial_k\tilde{\phi} + \partial_l\beta^{ij}\tilde{\partial}^l\tilde{\phi} - \beta^{kl}\partial_l\beta^{ij}\partial_k\tilde{\phi},\\
&= \left(\tilde{D}^i\beta^{jk} + \tilde{D}^j\beta^{ki} + \tilde{D}^k\beta^{ij}\right)\partial_k\tilde{\phi} + (\partial_l\beta^{ij})(\tilde{\partial}^l\tilde{\phi} + \beta^{lk}\partial_k\tilde{\phi}).
\end{aligned} \tag{E.3}$$

By using the definition of R-flux (5.35) and Q-flux (5.37), the commutation relation eventually becomes

$$\left[\tilde{D}^i, \tilde{D}^j\right]\tilde{\phi} = R^{ijk}\partial_k\tilde{\phi} + Q_k{}^{ij}\tilde{D}^k\tilde{\phi}. \tag{E.4}$$

F. The Connection of Winding Derivative

From the requirement that the covariant derivative on the metric is vanishing, it gives the metric compatibility as

$$\tilde{\nabla}^i \tilde{g}^{jk} = \tilde{D}^i \tilde{g}^{jk} - \check{\Gamma}_l{}^{ij} \tilde{g}^{lk} - \check{\Gamma}_l{}^{ik} \tilde{g}^{jl} = 0. \tag{F.1}$$

By redefining indices of (F.1), three equations are obtained

$$\tilde{D}^i \tilde{g}^{jk} - \check{\Gamma}_l{}^{ij} \tilde{g}^{lk} - \check{\Gamma}_l{}^{ik} \tilde{g}^{jl} = 0, \tag{F.2}$$

$$\tilde{D}^j \tilde{g}^{ki} - \check{\Gamma}_l{}^{jk} \tilde{g}^{li} - \check{\Gamma}_l{}^{ji} \tilde{g}^{kl} = 0, \tag{F.3}$$

$$\tilde{D}^k \tilde{g}^{ij} - \check{\Gamma}_l{}^{ki} \tilde{g}^{lj} - \check{\Gamma}_l{}^{ij} \tilde{g}^{il} = 0. \tag{F.4}$$

By combining (F.2) and (F.4), then subtracting with (F.3), the result is

$$2\tilde{g}^{jl} \check{\Gamma}_l{}^{(ki)} = \tilde{D}^i \tilde{g}^{jk} + \tilde{D}^k \tilde{g}^{ij} - \tilde{D}^j \tilde{g}^{jl} - 2\left(\check{\Gamma}_l{}^{[ij]} \tilde{g}^{lk} + \check{\Gamma}_l{}^{[kj]} \tilde{g}^{li} \right). \tag{F.5}$$

As a result, the symmetric part of this connection is given by

$$\check{\Gamma}_k{}^{(ij)} = \frac{1}{2} \tilde{g}_{km} \left(\tilde{D}^j \tilde{g}^{mi} + \tilde{D}^i \tilde{g}^{jm} - \tilde{D}^m \tilde{g}^{ij} \right) - \tilde{g}_{mk} \left(\check{\Gamma}_l{}^{[jm]} \tilde{g}^{li} + \check{\Gamma}_l{}^{[im]} \tilde{g}^{lj} \right). \tag{F.6}$$

By defining the $\tilde{\Gamma}_m{}^{ij}$ such that

$$\tilde{\Gamma}_k{}^{ij} = \frac{1}{2} \tilde{g}_{km} \left(\tilde{D}^j \tilde{g}^{mi} + \tilde{D}^i \tilde{g}^{jm} - \tilde{D}^m \tilde{g}^{ij} \right), \tag{F.7}$$

the symmetric part of the connection then becomes

$$\check{\Gamma}_k{}^{(ij)} = \tilde{\Gamma}_k{}^{ij} - \tilde{g}_{mk} \left(\check{\Gamma}_l{}^{[jm]} \tilde{g}^{li} + \check{\Gamma}_l{}^{[im]} \tilde{g}^{lj} \right). \tag{F.8}$$

The antisymmetric part is given by

$$\check{\Gamma}_k{}^{[ij]} = \frac{1}{2} Q_k{}^{ij}. \tag{F.9}$$

Therefore, the full expression of the connection takes the form

$$\check{\Gamma}_k{}^{ij} = \tilde{\Gamma}_k{}^{ij} - \frac{1}{2}\tilde{g}_{mk}\left(\tilde{g}^{li}Q_l{}^{jm} + \tilde{g}^{lj}Q_l{}^{im}\right) + \frac{1}{2}Q_k{}^{ij}. \tag{F.10}$$

In order the prove that the expression (F.10) leads to the non-vanishing part of the connection such that

$$\Delta_\xi \check{\Gamma}_k{}^{ij} = -\tilde{D}^i \partial_k \xi^j, \tag{F.11}$$

let us consider the following pieces. The first one is $Q_k{}^{ij}$

$$\begin{aligned}
\delta_\xi Q_k{}^{ij} =& \delta_\xi(\partial_k \beta^{ij}), \\
=& \partial_k(\delta_\xi \beta^{ij}), \\
=& \partial_k(L_\xi \beta^{ij} - \tilde{\partial}^i \xi^j + \tilde{\partial}^j \xi^i), \\
=& L_\xi(\partial_k \beta^{ij}) - (\partial_k \partial_l \xi^j)\beta^{il} - (\partial_k \partial_l \xi^i)\beta^{lj} - (\partial_k \tilde{\partial}^i \xi^j) + (\partial_k \tilde{\partial}^j \xi^i).
\end{aligned} \tag{F.12}$$

Therefore, the non-covariant part of $Q_k{}^{ij}$ is given by

$$\Delta_\xi Q_k{}^{ij} = -\tilde{D}^i \partial_k \xi^j + \tilde{D}^j \partial_k \xi^i. \tag{F.13}$$

The second piece is $\tilde{\Gamma}_k{}^{jk}$. However it is easier for considering the gauge transformation of $\tilde{D}^i \tilde{g}^{jk}$ first

$$\begin{aligned}
\delta_\xi(\tilde{D}^i \tilde{g}^{jk}) =& \delta_\xi(\tilde{\partial}^i \tilde{g}^{jk} + \beta^{il}\partial_l \tilde{g}^{jk}), \\
=& \tilde{\partial}^i \delta_\xi \tilde{g}^{jk} + (\delta_\xi \beta^{il})\partial_l \tilde{g}^{jk} + \beta^{il}\partial_l(\delta^\xi \tilde{g}^{jk}), \\
=& \tilde{\partial}^i (L_\xi \tilde{g}^{jk}) + (L_\xi \beta^{il} - \tilde{\partial}^i \xi^l + \tilde{\partial}^l \xi^i)\partial_l \tilde{g}^{jk} + \beta^{il}\partial_l L_\xi \tilde{g}^{jk}, \\
=& (\tilde{\partial}^i \xi^l)\partial_l \tilde{g}^{jk} + \xi^l \partial_l \tilde{\partial}^i \tilde{g}^{jk} - (\tilde{\partial}^i \partial_l \xi^j)\tilde{g}^{lk} - (\partial_l \xi^k)\tilde{\partial}^i \tilde{g}^{lk} - (\tilde{\partial}^i \partial_l \xi^k)\tilde{g}^{jl} \\
& - (\partial_l \xi^k)\tilde{\partial}^i \tilde{g}^{jl} + L_\xi \beta^{il}\partial_l \tilde{g}^{jk} + \beta^{il}(\partial_l \xi^m)\partial_m \tilde{g}^{jk} + \beta^{il}\xi^m \partial_m \partial_l \tilde{g}^{jk} \\
& - \beta^{il}(\partial(l\partial_m \xi^j)\tilde{g}^{mk} - \beta^{il}(\partial_l \partial_m \xi^k)\tilde{g}^{jm} - \beta^{il}(\partial_m \xi^k)\partial_l \tilde{g}^{jm}, \\
=& L_\xi(\tilde{\partial}^i \tilde{g}^{jk}) + (L_\xi \beta^{il})\partial_l \tilde{g}^{jk} + \beta^{il} L_\xi(\partial_l \tilde{g}^{jk}) \\
& - (\tilde{\partial}^i \partial_l \xi^j)\tilde{g}^{lk} - (\tilde{\partial}^i \partial_l \xi^k)\tilde{g}^{jl} - \beta^{il}(\partial_l \partial_m \xi^j)\tilde{g}^{mk} - \beta^{il}(\partial_l \partial_m \xi^k)\tilde{g}^{jm}.
\end{aligned} \tag{F.14}$$

The non-covariant part of $\tilde{D}^i \tilde{g}^{jk}$ takes the form

$$\Delta_\xi(\tilde{D}^i \tilde{g}^{jk}) = -\tilde{g}^{mk}(\tilde{D}^i \partial_m \xi^j) - \tilde{g}^{mj}(\tilde{D}^i \partial_m \xi^k). \tag{F.15}$$

As a result, the non-covariant part of the connection is then obtained as

$$
\begin{aligned}
\Delta_\xi \check{\Gamma}_k{}^{ij} =& \Delta_\xi \tilde{\Gamma}_k{}^{ij} - \frac{1}{2}\tilde{g}_{mk}\tilde{g}^{li}\Delta_\xi Q_l{}^{jm} - \frac{1}{2}\tilde{g}_{mk}\tilde{g}^{lj}\Delta_\xi Q_l{}^{im} + \frac{1}{2}\Delta_\xi Q_k{}^{ij}, \\
=& \frac{1}{2}\Big\{ -\tilde{g}^{lj}(\tilde{D}^i\partial_l\xi^m) - \tilde{g}^{lm}(\tilde{D}^i\partial_l\xi^j) - \tilde{g}^{li}(\tilde{D}^j\partial_l\xi^m) - \tilde{g}^{lm}(\tilde{D}^j\partial_l\xi^i) \\
& + \tilde{g}^{lj}(\tilde{D}^m\partial_l\xi^i) + \tilde{g}^{li}(\tilde{D}^m\partial_l\xi^j)\Big\} \\
& - \frac{1}{2}\tilde{g}_{mk}\tilde{g}^{li}(-\tilde{D}^j\partial_l\xi^m + \tilde{D}^m\partial_l\xi^j) - \frac{1}{2}\tilde{g}_{mk}\tilde{g}^{lj}(-\tilde{D}^i\partial_l\xi^m + \tilde{D}^m\partial_l\xi^i) \\
& + \frac{1}{2}(-\tilde{D}^i\partial_k\xi^j + \tilde{D}^j\partial_k\xi^i), \\
=& -\tilde{D}^i\partial_k\xi^j.
\end{aligned}
\tag{F.16}
$$

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