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MSc Dissertation

Quantum Fields and Fundamental Forces

Theory Group

The Cosmological Constant Problem Hawking's Three-Form

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Abstract

In 1983 Stephen Hawking proposed a mechanism involving a rank-3 gauge field implying that "the cosmological constant is probably zero". This paper examines the theory through the Einstein-Cartan formalism, casting the field as a 3-form, appearing in the action via a $F \wedge *F$ term where $F = dA$. The equations of motion from A yield an integration constant $*F = c$ that if substituted back into the Euclidean action produces what can be interpreted as a probability distribution via the effective action. The peak is of course found as the effective cosmological constant $\lambda_{eff}(c) \to 0^+$. First, from a review of the associated literature, it is found that the original proposal had an important hole in the argument, caused by substituting a solution into the action before obtaining all the equations of motion. However, this was later shown to not disprove the method in its entirety: a closer look into the correct boundary terms fixed the sign problem caused by the premature substitution. Furthermore, some new variants of this mechanism were found to provide alternate ways to fix the issue, and to generalise the process. An interesting result is a link found between this and Unimodular Gravity, which may yet breathe some new motivation into one or the other theory. Some background and illustrations to the cosmological constant problem are presented first, such as an introduction to Quantum Cosmology, the Sequester and of course Unimodular Gravity. Doubts are also raised about the use of a Euclidean action for a gravitational theory and its probabilistic interpretation, along with possible ways around them.

Contents

1 Introduction

The cosmological constant problem may be one of the greatest discrepancies between experiment and theory in modern Physics. Weinberg's famous review of the issue [1] states a fine-tuning problem of roughly 120 orders of magnitude based on currently known particle physics and upper bounds on the cosmological constant. It is then quite obvious why this topic is of great interest, potentially leading the way to new physics, whether looked at from an observational, experimental or theoretical view. The fact that this discrepancy still remains unsolved is a testament as to how deep-rooted the problem is, potentially requiring some modification or addition to the Standard Model or General Relativity. The first published discussions of the contributions of quantum fluctuations to cosmology were generated by astronomical observations of an accelerating expansion rate of the universe [1]. Authors such as Petrosian, Salpeter, Szekeres, Sklovsky, and Zeldovich began the first discussions of positively curved models with non-zero cosmological constants in the 1960s [2][?][3], revived from previous decades. Even further back, Pauli is believed to have considered the effects of ground state energies on gravity [4], stating that a rough estimate of their effect would place the cosmic horizon closer than the moon! [5] However, Padilla strongly highlights that the problem may be much deeper, centred around radiative instability. Simply considering loop diagrams of a single scalar field gives contributions to the total energy as:

$$\begin{aligned} -V_{vac}\int d^4x \\ V_{vac}\sim \Sigma_{particles}\mathcal{O}(1)m^4_{particles} \end{aligned} \tag{1}$$

where schemes such as dimensional regularisation are performed, and only finite terms are taken into account. Now modifying the measure for general covariance $\int d^4x \to \int d^4x\sqrt{-g}$, one obtains a contribution to the stress-energy tensor of the form:

$$T^{\mu\nu}_{(vac)} = -V_{vac}g^{\mu\nu}. \tag{2}$$

Such a term would lead to a late time de Sitter expansion - as our universe appears to be heading towards - but much earlier than in reality, as previously stated with Dirac. The idea then (naively) is to absorb these fluctuations into the bare cosmological constant term Λ first introduced by Einstein, appearing in the Einstein-Hilbert (E-H) action as [6]:

$$S = \kappa\int d^4x\sqrt{-g}\,(R-2\Lambda) + S_M[g_{\mu\nu},\phi] \tag{3}$$

where $\kappa = 1/(16\pi G)$ with G the gravitational constant, R is the Ricci scalar, and $S_M[g_{\mu\nu},\phi]$ is the matter action, which will contribute to $T^{\mu\nu}$. However, as Padilla points out [4], Λ must be readjusted at every order in the loop diagrams: their contributions do not converge, but in fact lead to an infinite number of fine-tuning! Short of being able to calculate the contribution of each diagram in the infinite expansion, some additional mechanism is needed to "naturally" modulate these contributions. Not to mention the fact that these must not be entirely cancelled, since some residual cosmological constant is clearly present. A point of caution here: many experts on topic have differing views on which presentation of the cosmological constant problem is "the true one" or the more "fundamental" one. What they can all agree on is that something is clearly missing, and I hope the discussion here will contribute in some way to whichever view the reader subscribes to.

Various mechanisms have been proposed in the 60 years since these observations, which are already deftly covered in the 2 reviews mentioned here. Some will be picked out and discussed briefly to give the reader an idea of the landscape of possibilities and their challenges. Quantum Cosmology (QC) is a vast topic, a starting point of which will be covered first, followed by an example of long-distance modifications of gravity. Unimodular Gravity will then be studied, being an interesting theory which is pertinent to the main focus of this paper: Hawking's proposed 3-form A. A brief review of some of the work on this 3-form will be presented, followed by the study of new thoughts and modifications not seen in the literature. First however, some background information will be covered in the the remainder of the Introduction to ensure the reader is up to speed and to lay down any conventions.

1.1 Hamiltonian Constraints and Canonical Quantization

The Hamiltonian dynamics and canonical quantization methods employed in this paper are based on Dirac's famous work on topic. His 1964 *Lectures on Quantum Mechanics* [7] are treated as the gold standard, and reading through at least the first few lectures is highly recommended. Here I cover some of the relevant points and interesting notes that come up repeatedly in the literature around the topic of quantum cosmology/gravity, but are rarely explained. We start with an action and a Lagrangian $S = \int dt L(q, \dot{q})$ defined in configuration space, giving the Euler-Lagrange (E-L) equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n}$ through variation of the action δS subject to endpoints being fixed ($\delta S|_{t_i/t_f} = 0$). We now define the conjugate momenta as $p_n = \frac{\partial L}{\partial \dot{q}_n}$ and perform a Legendre transform so that the Hamiltonian $H = p_n \dot{q}_n - L$ is defined entirely in phase space (q_n, p_n) where summation of repeated indices is always assumed. The equations of motion (EoM) of a quantity g are then given by its Poisson bracket (PB) with the Hamiltonian H:

$$\dot{g} = \{g, H\} \equiv \frac{\partial g}{\partial q_n}\frac{\partial H}{\partial p_n} - \frac{\partial H}{\partial q_n}\frac{\partial g}{\partial p_n} \tag{4}$$

1.1.1 Constraints

Now, the momenta may not be independent of the coordinates; they may be related by some equations called constraints:

$$\phi(q,p)_m = 0\ ,\ m = 1, ..., M. \tag{5}$$

These are called the primary constraints, as opposed to the secondary constraints that emerge from consistency conditions later. The distinction is not particularly fundamental however. The important thing to note here is that these constraints may be added to the Hamiltonian, since they are by definition equal to zero, but will affect the EoM derived from it. This is ok since these constraints are simply allowing for a more general motion, one that includes motion between physically equivalent states. This is like a gauge freedom in a sense, and should be included for the most general motion, since the Hamiltonian is not uniquely determined. The constraints should only be imposed after the Poisson brackets (PB) have been computed, which is denoted as $\phi_m \approx 0$ and is called a "weak equation" under Dirac's terminology. This is just a book-keeping device. Now one must check the consistency conditions given by the constraints being maintained in time through their EoM $\dot{\phi}_m \approx 0$, which may well lead to secondary constraints. A more important distinction is between "First class" objects, whose PB with all constraints vanish ($\{R, \phi_j\} \approx 0 \to R$ is first class), and "second class" objects, which do not. A first class object must then obey $\{R, \phi_j\} = r_{jj'}\phi_{j'}$, where $r_{jj'}$ are undefined coefficients, since constraints are the only independent functions of (q,p) that vanish, by definition. Note that the PB of 2 first class quantities is also first class, which will be an important consistency condition in the quantization. Now one defines the "Extended Hamiltonian" H_E composed of the non-constrained Hamiltonian (which is first class) and the first class constraints:

$$\begin{aligned} H_E &= H + v_j \phi_j \\ &\to \dot{g} \approx \{g, H_E\} \end{aligned} \tag{6}$$

which is just a linear combination of first class objects, with functions v_j that are either arbitrary, or have been determined by a consistency condition. The second class constraints must be dealt with separately, as is laid out in appendix A. An interesting point to note is that the constraints generate infinitesimal symmetry transformations:

$$\Delta g(\delta t) = \epsilon_j \{g, \phi_j\} \tag{7}$$

where $\epsilon_j = \delta t(v_j - v'_{j'})$ is infinitesimal. Now finally, the action may be rewritten as a sum of first class constraints $\tilde{H}_j$, canonical conjugate variables (q,p) and the non-constrained part of the Hamiltonian H, if it is non-zero:

$$S = \int dt \left(\dot{q}_n p_n - v_j \tilde{H}_j - H\right) \tag{8}$$

where the $\tilde{H}_j$ will generate some infinitesimal symmetry transformation (i.e. remaining in the same physical state), and the v_j are arbitrary (unless they are constrained by a consistency condition). Of course this can be recast as a spacetime integral over the Lagrangian density and the coordinates promoted to fields: $q_n \rightarrow \phi(x)$, $p_n \rightarrow \pi(x) = \frac{\delta S}{\delta \phi(x)}$, $\sum_n \rightarrow \int d^4x$ and $\{q_n, p_m\} = \delta_{nm} \rightarrow \{\phi(x), \pi(x')\} = \delta(x - x')$. One final note before passing to quantization: the link between a relativistic theory and the constraint $H \approx 0$. If the Lagrangian is homogeneous of the first degree in the velocities (i.e. $L(\alpha \dot{q}, q) = \alpha L(\dot{q}, q)$), then by Euler's theorem:

$$\begin{aligned} \dot{q}_n \frac{\partial L}{\partial \dot{q}_n} &= L \\ \rightarrow H &= 0. \end{aligned} \tag{9}$$

Meaning our original Hamiltonian also becomes a first-class constraint. Note that foliating spacetime into spacelike surfaces and expressing the Lagrangian as the amount of action δS between two such neighbouring surfaces, divided by some measure of distance $\delta \tau$, necessarily yields a Lagrangian homogeneous to the first order. This is essentially the procedure of the ADM formalism. Now the crux of the argument is that since H_E is made up entirely of first class constraints, the EoM are given by:

$$\dot{g} \approx v_j \{g, \phi_j\} \tag{10}$$

where the v_j are arbitrary, so that the time scale is obviously also arbitrary. Any variable increasing monotonically with t could be used as a measure of time, we say the theory is "time reparametrization invariant". This is a desired feature of any relativistic theory, since no one coordinate system is absolute, and so any manifestly relativistic theory should exhibit $H_E \approx 0$. Furthermore, one may force this behaviour onto any theory through the following procedure: take the time variable as an extra coordinate q_0, and re-express everything in terms of a new time coordinate τ so that:

$$L^* = \frac{dq_0}{d\tau} L\left(q_n, \frac{dq_n/d\tau}{dq_0/d\tau}\right) = L^*\left(q_k, \frac{dq_k}{d\tau}\right) \;,\; k = 0, ..., N. \tag{11}$$

The action is invariant under this change due to the corresponding change in the integration measure, but the Lagrangians are not in general equivalent. L^* will always yield a Hamiltonian which is weakly zero. Lastly note that the measure of distance between two spacelike surfaces $\delta\tau$ is generally taken to be orthogonal to these so that any quantity may be divided into tangential and orthogonal components to the surfaces more easily. If the Hamiltonian is defined at a specific time on one such surface, the orthogonal components drop out.

1.1.2 Dirac's Quantization Prescription

If there are no second class constraints, the procedure at this point is rather straight forward: promote the coordinates to operators $(q, p) \rightarrow (\hat{q}, \hat{p})$, and the Poisson brackets to commutator brackets $\{\cdot, \cdot\} \rightarrow [\cdot, \cdot]$. Then set up a Schrödinger equation of the non-constrained Hamiltonian $i\hbar \frac{d\Psi}{dt} = \hat{H}\Psi$, and impose further conditions as operator equations from the constraints $\hat{\phi}_j \psi = 0$. Now we know from above that the PB of two first class constraints is also a first class constraint, so the following consistency condition must be satisfied: $\left[\hat{\phi}_j, \hat{\phi}_{j'}\right] \Psi = 0$. However, we want this relation to be implied by the previous constraint equations, meaning this commutator must yield a linear combination of known constraints of the form:

$$\left[\hat{\phi}_j, \hat{\phi}_{j'}\right] = \hat{c}_{jj'j''} \hat{\phi}_{j''} \tag{12}$$

and note that all the coefficients $\hat{c}_{jj'j''}$ must appear on the *left*. To achieve this, one should rearrange the operators in the theory if possible. If this is not possible, then one cannot accurately quantize the theory from the classical one (i.e. it is only accurate to order $\mathcal{O}(\hbar)$ and not $\mathcal{O}(\hbar^2)$ for example). Finally, the constraints must also be consistent with the Schrödinger equation as $\left[\hat{\phi}_j, \hat{H}\right] \Psi = 0$, which likewise implies:

$$\left[\hat{\phi}_j, \hat{H}\right] = \hat{b}_{jj'} \hat{\phi}_{j'} \tag{13}$$

where again the correct ordering must be achieved.

1.2 The Einstein-Cartan Action and First-Order Formalism

1.2.1 The Tetrad Basis

At this point the tetrad basis must be introduced, defined using the vierbein e^a_μ and co-veirbein e^μ_a which are local transformations between the general spacetime coordinates given by a metric $g_{\mu\nu}$ and flat spacetime with η_{ab} [8][9][10][11]:

$$\begin{aligned} g_{\mu\nu} &= e^a_\mu e^b_\nu \eta_{ab} \\ \eta_{ab} &= e^\mu_a e^\nu_b g_{\mu\nu}. \end{aligned} \tag{14}$$

This is sometimes called a "non-coordinate basis" since they do not define coordinate transformations (for instance the commutator $[e_a, e_b] \neq 0$ in general ,with $e_a = e^\mu_a \partial_\mu = \partial_a$ a vector). This ends up creating 2 sets of indices: one with Greek letters labelling the coordinates, and one with Latin letters labelling an internal $SO(3,1)$ symmetry. One can switch between these indices by using the tetrad basis matrices e^μ_a, e^a_μ, so long as the indices are those of a tensor. Objects may then be defined in terms of the 1-forms $e^a = e^a_\mu dx^\mu$, which is the basis for the Einstein-Cartan formalism. Note that unless otherwise stated, the product between two forms is the wedge product, so this will generally be omitted. From this and the connection one-form $\Gamma^a_{\ b}$, one can define the torsion 2-form T^a via Cartan's first equation:

$$T^a \equiv De^a = de^a + \Gamma^a_{\ b} e^b \tag{15}$$

where D is the covariant exterior derivative with the connection 1-form as the gauge field. In the defining representation $DV^a = dV^a + \Gamma^a_{\ b} \wedge V^b$, whilst in the adjoint rep $DV^a = dV^a + [\Gamma, V]^a$, where V^a is an r-form and $[v, w] = v \wedge w - (-1)^{pq} w \wedge v$ is the graded commutator for v and w a p- and q-form respectively. A good analogy is Yang-Mills theory where $F = DA = dA + [A, A]$ so that A is the gauge 1-form and $[A, A]$ vanishes if the symmetry generators commute (i.e. $U(1)$). Also note there may be some coupling constant before the second term in the covariant derivative, though this will never meaningfully enter any of the considerations in this paper, and so will be ignored. The precise relation between the tetrad basis and connections is determined by the EoM as well as the equation above. This is a big departure from the E-H method (second order formalism) into what is known as first order formalism, given by the Einstein-Cartan action:

$$S = \frac{\kappa}{2} \int \epsilon_{abcd} e^a e^b \left(R^{cd} - \frac{1}{6} e^c e^d \Lambda \right) + S_M[e^a, \phi] \tag{16}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the fully antisymmetric Levi-Civita tensor, with ϵ_{abcd} its tetrad counterpart.

1.2.2 The Spin Connection

Now, we may define the spin connection, which is the connection needed for the covariant derivative of a spinor field in a general spacetime. First recall the Clifford algebra of the gamma matrices from the Dirac equation in flat spacetime $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where $\{\cdot, \cdot\}$ defines an anticommutator bracket here. One can define their curved spacetime counterparts as $\gamma^\mu \equiv e^\mu_a \gamma^a$ [11], so that the original matrices are now labelled by the internal indices, and the anticommutator may be expressed as:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{17}$$

Ignoring any other local symmetries (such as $U(1)$ which couples the Dirac field ψ to electromagnetism), the kinetic term appearing in the action is of the form $\bar{\psi}\gamma^a \nabla_a \psi$, where $\bar{\psi} = \psi^\dagger \gamma^0$ and ∇_a is a form of covariant derivative so far undefined. Note that under a local Lorentz transformation $\Lambda^a_{\ b}(x)$, the Dirac spinors transform as $\psi_A \to S[\Lambda(x)]_{AB} \psi_B$, where $S[\Lambda(x)]_{AB}$ is the spinor representation of $\Lambda(x)$, with capital Latin letters denoting spinor $SO(4)$ indices (and their position is irrelevant). Thus, for the kinetic term to be invariant, the covariant derivative term must transform as $\nabla_a \psi_A \to S[\Lambda]_{AB} \Lambda_a^{\ b} \nabla_b \psi_B$. This yields a covariant derivative defined in terms of the *spin connection* w^{AB}_a [11][12][13][14]:

$$\begin{aligned} \nabla_a \psi_A &= \partial_a \psi_A + w^{AB}_a \psi_B \\ w^{AB}_a &\equiv \frac{i}{2} \Gamma^b_{\ a}{}^c S^{AB}_{bc} \end{aligned} \tag{18}$$

where $S^{AB}_{ab} = \frac{i}{4}\left[\gamma_a, \gamma_b\right]^{AB}$ is the generator of the $S[\Lambda]_{AB}$ transformations, and $\Gamma^{b}{}_{a}{}^{c}$ are the components of the connection one-form Γ^{bc}. See *Geometry Topology and Physics* by Nakahara [11] for an explicit derivation. As a result, it appears to be common practice to refer to the connection 1-form Γ^{bc} directly as the spin connection (though that is not the case in this paper). Finally the convention $\not\nabla \equiv \gamma^a \nabla_a$ is used. For completeness, I will redefine the *covariant exterior derivative* more generally on an r-form with tetrad and spinor indices V^a_A, that is also coupled to a gauge 1-form A as DV^a_A. In the defining representation of the 1-form connection and the gauge 1-form, this gives:

$$DV^a_A = dV^a_A + \Gamma^a{}_b V^b_A + w_{AB}V^a_B + AV^a_A \tag{19}$$

with $w \equiv w_a e^a$, and a similar equation for the adjoint representation. Of course, if the r-form has no tetrad indices, then the corresponding connection term drops out, and similarly for the spinor indices. Likewise if it is not coupled to A the last term drops out. This is simply a definition to save time. In this paper, the quantities will generally not be coupled to a gauge field A, so for a simple scalar field $D\phi = d\phi$. The "non-form" covariant derivative associated with this would be of course (Assuming V^a_A is a 1-form for simplicity):

$$D_\mu V^a_{\nu A} = \nabla_\mu V^a_{\nu A} + \Gamma^a{}_b V^b_{\nu A} + w_{AB}V^a_{\nu B} + AV^a_{\nu A} \tag{20}$$

with ∇_μ the standard covariant derivative from GR with the Levi-Civita connection (with again differences for adjoints reps).

It is interesting to note that as a consequence of this, spinors imply the need to for non-zero torsion. To see this, first the recovery of the E-H action from the E-C one without a spinor field must be seen:

$$\begin{aligned} S_{EC} &= \frac{\kappa}{2}\int \epsilon_{abcd}e^a e^b \left(R^{cd} - \frac{1}{6}e^c e^d \Lambda\right) + S_M[e^a, \phi, d\phi] \\ &\to S_{EH} = \kappa \int d^4x\sqrt{-g}\,(R - 2\Lambda) + S'_M[g_{\mu\nu}, \phi, \partial_\mu\phi] \end{aligned} \tag{21}$$

where $R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c\Gamma^c{}_b$ is the curvature 2-form given by Cartan's second equation, Λ is the cosmological constant, and any derivatives of scalar fields are simply given by the exterior derivative d (possibly coupled to some gauge field A) so that the matter action is not dependent on the curvature one-form. Now noting that $\delta R^a{}_b = D\delta\Gamma^a{}_b = d\delta\Gamma^a{}_b + [\Gamma, \delta\Gamma]^a{}_b$, and using Cartan's first equation and Stoke's theorem (with $\int D(\cdots)$ being a boundary term and so not entering the EoM), the variation of the E-C action w.r.t the connection one-form is:

$$\begin{aligned} \delta_\Gamma S &= -\kappa \int \epsilon_{abcd}T^a e^b \delta\Gamma^{cd} \\ &\to T^a = 0. \end{aligned} \tag{22}$$

The torsion is thus forced to zero. Variation w.r.t the tetrad one-form e^a then gives the E-C version of the Einstein equations (see Appendix B). Now with this condition (or assuming $T^a = 0$ to begin with) as well as $det(e^a_\mu) \neq 0$, Cartan's theorem states that one may relate the E-H and E-C quantities as:

$$\begin{aligned} \Gamma^a_{\mu b} &= e^a_\nu \nabla_\mu e^\nu_b \\ R^\alpha{}_{\beta\mu\nu} &= e^\alpha_a e^b_\beta R^a{}_{b\mu\nu}. \end{aligned} \tag{23}$$

It can then easily be shown that S_{EC} reduces to S_{EH}. However if the matter action contains any dynamical spinor fields, it will necessarily depend on the one-form connection $S_M = S_M[\psi, \partial_a\psi, \Gamma^a{}_b]$ from a $\not\nabla\psi$ term so that instead of a vanishing Torsion, the EoM from the connection are:

$$T^a \wedge e^b = -\frac{\delta S_M}{\delta\Gamma_{ab}}. \tag{24}$$

This suggests that any full theory of Quantum Cosmology (or even gravity) should likely be formulated in the first order formalism, or some other theory including tetrads (such as Loop Quantum Gravity). Allowing for non-zero torsion may also enable theories of gravity not normally admissible [15][16][17][18][19].

2 Past and Current Attempts at a Solution

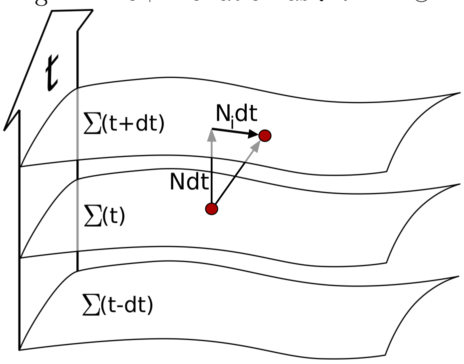
2.1 Quantum Cosmology

One of the favoured avenues of research to solve this problem is to look at the quantum behaviour of gravity over the evolution of the universe. Short of a full theory of verifiable Quantum Gravity, different tricks, guesses and approximations are used. Last year's MSc dissertation by Mateo Pascual *Quantum Time for Quantum Cosmology*, supervised by João Magueijo already covered the ADM formalism and Ashtekar variables in great detail. It is well worth a read, and I will only provide a summary of these topics here for completeness, followed by a "simpler" example of QC which I believe is more illustrative of its characteristics.

2.1.1 ADM Formalism

A good start is generally the ADM formalism, whereby one foliates the 4-dimensional manifold of spacetime along cosmic time into spacelike hypersurfaces $\mathcal{M}(y) = \mathbb{R}(t) \otimes \Sigma(\vec{x})$ [20][21] according to a deformation vector $T^\mu(y) = \dot{y}^\mu = N(y)n^\mu(y) + N^\mu(y)$, where $\{y^\mu\}$ is a coordinate patch on $\mathcal{M}$, while t and $\{x^i\}$ ((i,j,k) being spacial indices) are defined on the embedded submanifolds. n^μ is the unit normal to the surfaces Σ, while N and N^μ can be recognised as the lapse function and shift vector respectively. Pictorially, this is:

Figure 1: 3+1 foliation as $\mathcal{M} = \mathbb{R} \otimes \Sigma$



The following can then be defined:

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \ , \ K_{\mu\nu} = \gamma_{\mu\alpha}\gamma_{\nu\beta}\nabla^\alpha n^\beta \tag{25}$$

so that $\gamma_{\mu\nu}$ gives the 3-metric on the hypersurface (for example if $n_\mu = (1, \vec{0}) \rightarrow \gamma_{00} = 0$ and γ_{ij} is the spacial metric on the surface), and $K_{\mu\nu}$ is the extrinsic curvature in the surfaces. Note that this is all equivalent to splitting the metric into different components as:

$$ds^2 = \left(-N^2 + \gamma_{ij}N^iN^j\right)dt^2 + 2N_i dx^i dt + \gamma_{ij}dx^i dx^j. \tag{26}$$

Now defining all these quantities according to the new 3-metric (i.e. pulling back these quantities onto the surfaces Σ, see last year's dissertation for the details), the E-H action may be written:

$$S = \kappa \int_{\mathbb{R}} dt \int_\Sigma d^3\vec{x}\sqrt{\gamma}N\left(R^{(3)} + K_{ij}K^{ij} - K^2\right) \tag{27}$$

which is the full ADM action, $K = \gamma^{ij}K_{ij}$, and $R^{(3)} = R^{(3)}(\gamma_{ij})$ is the 3-D Ricci scalar on the hypersurfaces Σ. Performing a Legendre transform, the canonical conjugate to γ_{ij} is $\pi^{ij} = \kappa\sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right)$. So that now the action can be rewritten in terms of constraints, as described in the introduction:

$$\begin{aligned} S &= \int_{\mathbb{R}} dt \int_\Sigma d^3\vec{x}\left(\dot{\gamma}^{ij}\pi_{ij} - N_i\tilde{\mathcal{H}}^i - N\tilde{\mathcal{H}}\right) \\ \tilde{\mathcal{H}}^i &= -2\nabla_j\pi^{ji} \\ \tilde{\mathcal{H}} &= \frac{1}{\kappa\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \kappa\sqrt{\gamma}R^{(3)} \end{aligned} \tag{28}$$

where $\tilde{\mathcal{H}}^i \approx 0$ and $\tilde{\mathcal{H}} \approx 0$ are the generators of spacial diffeomorphisms and the Hamiltonian constraint respectively, so that the full (extended) Hamiltonian density is $\mathcal{H} = N_i\tilde{\mathcal{H}}^i + N\tilde{\mathcal{H}}$. ∇_j is the covariant derivative defined on the spacelike surfaces Σ. This is the starting point for many treatments of QC, such will be seen in the Unimodular Gravity section.

2.1.2 Ashtekar Variables

At last we come to one of the most important steps in the development of quantum gravities: the Ashtekar variables. In 1986, Ashtekar introduced his new variables [22] defined on the spacelike hypersurfaces Σ, which vastly simplified the constraint equations. Although this is an intricate topic, I only cover it briefly as most of it does not pertain directly to the rest of the paper (and again, this was already covered extensively last year). However, quantities such as the Chern-Simons invariant will come up later, so I wish to at least illustrate in what context they emerge. Using the tetrad technology from the introduction, the *densitised inverse triad* E^i_a can be introduced [22][23]:

$$\begin{aligned} E^i_a &\equiv \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}e^b_j e^c_k = \sqrt{\gamma}e^i_a \\ \gamma^{ij} &= \frac{1}{\gamma}E^i_a E^j_b \delta^{ab} \end{aligned} \tag{29}$$

and what turns out to be its dual K^a_i as:

$$K_{ij} \equiv K^a_{(i}K^a_{j)}. \tag{30}$$

The (i, j, k) are spacial indices, while the (a, b, c) are internal (tetrad) $SO(3)$ indices, which are generally treated as $SU(2)$ indices in the literature (and by Ashtekar himself) due to the Lie algebra isomorphism $so(3) \approx su(2)$. "Curved" brackets around indices such as (ij) denote symmetrisation, while "square" brackets $[ij]$ encode antisymmetrisation. Now recalling that the extrinsic curvature is symmetric, another constraint must be added: $K^a_{[i}K^a_{j]} = 0 \rightarrow G_{ab} \equiv K_{i[a}E^i_{b]} = 0$, which is better expressed in terms of a new connection as:

$$\begin{aligned} G_a &= \frac{1}{\gamma_{Im}}\mathcal{D}_i E^i_a \\ \mathcal{D}_i E^i_a &\equiv \partial_i E^i_a + \epsilon_{abc}A^b_i E^i_c \\ A^a_i &\equiv \Gamma^a_i + \gamma_{Im}K^a_i \end{aligned} \tag{31}$$

with γ_{Im} the Imirzi parameter, which is a free complex parameter derived from the scale invariance of the $\{E, A\}$ PB [22][24][25], and with $\Gamma^a_i \equiv \frac{1}{2}\epsilon^{abc}\Gamma_{ibc}$ the dualised connection one-form. This new connection A^a_i then is treated as the conjugate momentum instead, and the ADM action can therefore be rewritten in terms of these new Ashtekar variables as:

$$\begin{aligned} S &= \int_{\mathbb{R}} dt \int_{\Sigma} d^3\vec{x}\left(2E^i_a\dot{A}^a_i - \left[\Lambda^a G_a + N_i\tilde{\mathcal{H}}^i + N\tilde{\mathcal{H}}\right]\right) \\ \tilde{\mathcal{H}}_i &= E^j_a F^a_{ij} - (1-\gamma^2_{Im})K^a_i G_a \\ \tilde{\mathcal{H}} &= \frac{E^i_a E^j_b}{\sqrt{det(E^i_a)}}\left(\epsilon^{abc}F^c_{ij} - 2(1-\gamma^2_{Im})K^a_{[i}K^b_{j]}\right) \\ G_a &= \frac{1}{\gamma_{Im}}\mathcal{D}_i E^i_a \end{aligned} \tag{32}$$

where $F^a_{ij} \equiv 2\partial_{[i}A^a_{j]} + \epsilon^{abc}A^b_i A^c_j$ is the curvature of the connection. In fact this action can be used as the starting point for constructing gravity as a gauge theory, where structures such as a metric are merely secondary products, as was done in this excellent introductory review of the basis of Loop Quantum Gravity (LQG) [26]. From this starting point, the Imirzi parameter takes the value $\gamma_{Im} = 1$, which simplifies the constraints yet further, so that the constraint $G_a = \mathcal{D}_i E^i_a$ is analogous to Gauss' law for an electric field E^i_a, $\tilde{\mathcal{H}}_i = E^j_a F^a_{ij}$ generates spacial diffeomorphisms on the conjugate fields, and the Hamiltonian constraint takes the form $\tilde{\mathcal{H}} = \epsilon_{abc}E^{ia}E^{jb}\left(F^c_{ij} + \frac{\Lambda}{3}\epsilon_{ijk}E^{kc}\right)$ (Λ is so far arbitrary). This admits so called *self-dual solutions* that satisfy $F^a_{ij} = -\frac{\Lambda}{3}\epsilon_{ijk}E^{ka}$, and can be solved with $A_{ia} = i\sqrt{\frac{\Lambda}{3}}f(t)\delta_{ia}$. Now the electric field must be the gradient of the

Hamilton-Jacobi function $E^{ia} = -\frac{\delta S(A)}{\delta A_{ia}}$ that can be solved to yield the Cherns-Simons state [27]:

$$S = \frac{2}{3\Lambda} \int Y_{CS}$$
$$Y_{CS} = \frac{1}{2} Tr\left(A \wedge dA + \frac{2}{3} A^3 \right) \tag{33}$$

where Y_{CS} is the famous Chern-Simons invariant [26]. Hideo Kodama then showed in 1990 that the Kodama state is an exact quantum state solution in Lorentzian deSitter spacetime [28]:

$$\Psi_K(A) = \mathcal{N} e^{S_{CS}}. \tag{34}$$

2.1.3 A Simpler Example: The Kodama State from FLRW Spacetime

For simplicity, one may work in mini-superspace (MSS), meaning a homogeneous and isotropic space of 3-metrics. In this context I present an explicit attempt at quantizing gravity, keeping in mind the results here can be obtained from those in the section above. This reduces the ADM metric (26) to the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = -N^2 dt^2 + a(t)^2 \gamma_{ij}(\vec{x}) dx^i dx^j \tag{35}$$

so that $a(t)$ is the expansion factor of the universe. Now, employing the E-H formalism, one defines the curvature k according to the spacial Ricci scalar as $R^{(3)} \equiv 6k$, so that $k = 0, \pm 1$ as usual. The full 4-D Ricci scalar can then be computed as:

$$R = 6\left(\frac{1}{N^2}\frac{\ddot{a}}{a} + \frac{k}{a^2} + \frac{1}{N^2}\left(\frac{\dot{a}}{a}\right)^2 \right) \tag{36}$$

which can be put into the massless E-H action as:

$$S = \frac{3V_c}{8\pi G} \int dt \left(\dot{b}a^2 + Na\left[k + b^2 - \frac{\Lambda}{3}a^2 \right] \right)$$
$$\to S = \frac{3V_c}{8\pi G} \int dt \left(-b\dot{(a^2)} + Na\left[k + b^2 - \frac{\Lambda}{3}a^2 \right] \right) \tag{37}$$

by using $\sqrt{-g} = a^3\sqrt{\gamma}$, $\int d^3x\sqrt{\gamma} \equiv V_c$ (the comoving volume), and defining the new coordinate $b = \frac{\dot{a}}{N}$, as described by Floris Harmanni in their analysis of Higher order Langrangians [29]. The specific choice of b is taken from [18]. Now we wish to find the Hamiltonian of the system, via a Legendre transform. So the following coordinates are picked, with their conjugate momenta and associated Poisson brackets:

$$q_1 = N \to p_1 = 0 \tag{38}$$

$$q_2 = a^2 \to p_2 = -\frac{3V_c}{8\pi G} b \tag{39}$$

$$q_3 = b \to p_3 = 0 \tag{40}$$

So now we have the Hamiltonian:

$$H = -\frac{3V_c}{8\pi G} Na\left[k + b^2 - \frac{\Lambda}{3}a^2 \right]. \tag{41}$$

So from the two first Poisson brackets, the massless Friedmann equations appear (in order):

$$\left(\frac{\dot{a}}{a}\right)^2 + N^2\frac{k}{a^2} = N^2\frac{\Lambda}{3}$$
$$\frac{\ddot{a}}{a} = N^2\frac{\Lambda}{3} \tag{42}$$

However, the first equation above forces the Hamiltonian to simply vanish $H \approx 0$, becoming a constraint as expected. However this is very troublesome for quantizing the theory any further, and any QG theory must attempt to deal with the ramifications of this. For instance, how can there be time evolution without a Hamiltonian? This issue notwithstanding, $H \approx 0$ is generally realised at the quantum level by the Wheeler-de-Witt (WdW) equation [30][31], a constraint equation as prescribed in the introduction:

$$\hat{H}\Psi = 0 \tag{43}$$

where Ψ is the wavefunction of the universe - if such a thing can be defined. Now generally this theory is quantized by promoting $a(t)$ to an operator $\hat{a}$ (converting b back in terms of a), diagonalising w.r.t to this and obtaining the more common version of the Wheeler-de-Witt equation [18]:

$$\left(\frac{\partial^2}{\partial a^2} + \frac{\alpha}{a}\frac{\partial}{\partial a} - U(a)\right)\Psi(a) = 0 \tag{44}$$

where the α term contains effects of operator ordering. However, Magueijo's prescription will be followed instead [18], where he keeps the choice of coordinates a^2 and b (which also implies a choice of ordering):

$$\begin{aligned} \{b, a^2\} &= \frac{4\pi G_N}{3V_c} \\ \rightarrow \left[\hat{b}, \hat{a^2}\right] &= \frac{il_p^2}{3V_c} \end{aligned} \tag{45}$$

from Dirac's canonical quantization, with $l_p = \sqrt{8\pi G_N \hbar}$. Now in the $\Psi(b)$ representation, this commutator may be realised through $\hat{a^2} = -\frac{il_p^2}{3V_c}\frac{\partial}{\partial b}$:

$$\hat{H}\Psi(b) = \left(\frac{i\Lambda l_p^2}{9V_c}\frac{\partial}{\partial b} + k + b^2\right)\Psi(b) = 0 \tag{46}$$

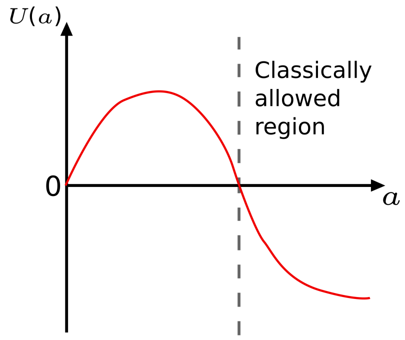
which can be easily solved for:

$$\Psi(b) = \mathcal{N}exp\left(i\frac{9V_c}{\Lambda l_p^2}\left[kb + \frac{b^3}{3}\right]\right) \tag{47}$$

which is just the Fourier dual of the better known Kodama state (34) with a particular ordering (for more details see [18]). This particular variant may in fact dispell some of the criticisms heaped onto the CS state stemming from it not being purely imaginary. Furthermore, it in fact generalises the Kodama state by working from the E-C action to allow for torsion and a dynamical Λ. Explicitly, $b = \dot{a} + aT$, where $T(t)$ is the parity-even part of the torsion, and a new variable c is the parity-odd part of the torsion. In this way, Magueijo is also able to turn Λ into a dynamical variable while keeping a well-behaved theory [17] [18]. Among other things, the hope is to shed some light on the value of the cosmological constant on shell.

Furthermore, one may split the Hamiltonian into a kintetic component $K(a) \propto b^2 = (\dot{a})^2$ and a potential one $U(a) \propto k - \frac{\Lambda}{3}a^2$. For $k = 1$, $\Lambda > 0$ (as may be consistent with current measurements [32][33]), the potential has the following form:

Figure 2: Potential of the Expansion Factor in the E-H MSS Hamiltonian



which raises the possibility of the wavefunction of the Universe "quantum tunnelling out of nothing" from the classically forbidden region to the allowed region. A more general version of this is known as the *Vilenkin wavefunction*, from Alexander Vilenkin's original paper [34] that used a WdW equation of the form $\left(a^{-p}\frac{\partial}{\partial a}a^{p}\frac{\partial}{\partial a} - U(a)\right)\Psi(a) = 0$, where $U(a) = a^2\left(1 - C^2a^2\right)$ and $C = \frac{2\sqrt{2}}{3l_p}G\Lambda$. p encodes the ordering of the operators, and the WKB solutions for the classically allowed ($a \geq C^{-1}$) and classically forbidden ($a < C^{-1}$) regions are respectively [34][35]:

$$\begin{aligned} \Psi^{(1)}_{\pm}(a) &= exp\left(\pm i\int_{C^{-1}}^{a}\sqrt{-U(a')}da' \mp \frac{i\pi}{4}\right) \\ \Psi^{(2)}_{\pm}(a) &= exp\left(\pm\int_{a}^{C^{-1}}|\sqrt{-U(a')}|da'\right) \end{aligned} \tag{48}$$

The tunnelling condition corresponds to the choice of the outgoing wave as $\Psi_T(a > C^{-1}) = \Psi^{(1)}_{-}(a)$.

2.2 Long-distance modifications of gravity - The Sequester

At the other end of the scale, many have tried to make long distance modifications to GR [36][37][38][39]; one may think of the cosmological constant as "the gravitational source of longest wavelength" [4]. As such, many long but finite distance attempts have been made, such as massive gravity or the "Fab Four" action (see Padilla's review for a brief explanation of the latter [4]). However these all run into difficulties when implementing some form of screening mechanism to recover the incredibly accurate measurements of gravitational effects, at the Solar system scale for instance. Indeed, such departures from GR should disappear to one part in 10^5 by that point [4][40]. Furthermore, these theories also have a tendency to struggle to maintain causality, or simply break it [41]. This can be seen from the simple question: how can the theory "know" which contributions to the energy momentum tensor from a source are long wavelength before they have travelled the threshold distance to be classified as long-ranged? This creates an issue for sources of vacuum energy just after the Big-Bang, since the theory does seem to have to "know", to cancel out these long-wavelength modes early on (and almost nothing else), in order to comply with early time observational evidence such as nucleosynthesis. So for a local theory, causality appears to have to be thrown out. However Padilla does mention that causality could only be violated globally by a future boundary condition (just as with a black hole event horizon), which excludes closed timelike curves and so could be consistent with current GR. One should note that to measure the vacuum energy to be cancelled out in this way, contributions over all of spacetime would need to be scanned – in other words we are looking at a global modification of gravity, the "ultimate long distance modification".

In this vein, a proposal for a global modification of gravity known as the Sequester is taken from Padialla's review [4] and summarised here. Here the comological constant Λ is promoted to a dynamical variable (not a spacetime field, but a constant that can be varied over in the action), which we then want to "talk" to the matter fields using another dynamical variable λ. Note that this somewhat parallels to the "coupling field" $\lambda(x)$ introduced in our own proposals below, but the two are different and should not be confused. The action in E-H form is then [4][2][42]:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(R - 2\left[\Lambda - \lambda^4\mathcal{L}_m(\lambda^{-2}g^{\mu\nu}, \Psi)\right]\right) + \sigma\left(\frac{\Lambda}{\lambda^4\mu^4}\right) \tag{49}$$

where $\mathcal{L}_m$ is the matter Lagrangian and σ is the sequestering function (not integrated over) that should be determined by phenomenology, but must be odd and differentiable. Any factors of κ have been absorbed into σ. Note that λ acts as a scaling factor, and can be absorbed by a change of frame as $\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu} \to \sqrt{-g}\lambda^4\mathcal{L}_m(\lambda^{-2}g^{\mu\nu}, \Psi) = \sqrt{-\tilde{g}}\mathcal{L}_m(\tilde{g}^{\mu\nu}, \Psi)$. Now varying Λ, λ and $g^{\mu\nu}$ respectively gives the EoM:

$$\begin{aligned} \frac{1}{\lambda^4\mu^4}\sigma'\left(\frac{\Lambda}{\lambda^4\mu^4}\right) &= \int d^4x\sqrt{-g} \\ \frac{4\Lambda}{\lambda^4\mu^4}\sigma'\left(\frac{\Lambda}{\lambda^4\mu^4}\right) &= \int d^4x\sqrt{-g}\lambda^4\tilde{T} \\ G_{\mu\nu} &= -g_{\mu\nu}\Lambda + \lambda^4\tilde{T}_{\mu\nu} \end{aligned} \tag{50}$$

where $\tilde{T}_{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}}\int d^4x\sqrt{-\tilde{g}}\mathcal{L}_m(\tilde{g}^{\mu\nu}, \Psi)$. Now defining the spacetime average as $\langle Q\rangle = \frac{\int d^4x\sqrt{-g}Q}{\int d^4x\sqrt{-g}}$, the first two equations of motion yield:

$$\Lambda = \frac{1}{4}\langle \lambda^4 \tilde{T} \rangle \tag{51}$$

and the last equation is simply Einstein's equations, if one identifies $T_{\mu\nu} = \lambda^4 \tilde{T}_{\mu\nu}$. Now (120) may be substituted in, yielding:

$$G_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \langle T \rangle. \tag{52}$$

This is reminiscent of the traceless energy momentum tensor discussed in Unimodular Gravity below. Specifically, only the trace of the spacetime average is cancelled here, which happens to contain all the vacuum contributions since $\langle V_{vac} \rangle = V_{vac}$, and $T^{\mu\nu}_{(vac)} = g^{\mu\nu} V_{vac}$. This is a very elegant mechanism and avoids having to deal with replacing the cosmological constant with an integration constant as in Unimodular Gravity (see below). Furthermore, matter coupling globally to the metric via $\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$ ensures that the vacuum energy scales as λ^4 at every order in loop expansions, so that this cancellation of the vacuum energy works at every order. We are only left with the spacetime average of the *local* matter is left as a cosmological constant $\Lambda_{eff} = \frac{1}{4}\langle T_{(local)} \rangle$, since the vacuum energy cancels out – i.e. only the troublesome part has been removed. For more detail on the implications and restrictions of this model, such as requiring a finite spacetime (Big Crunch!), see [4][2][42].

2.3 Unimodular Gravity

What has become known as Unimodular Gravity is a simple trick to make the vacuum energy drop out of the dynamics by performing a restricted variation of the metric so that $|g| = 1$. However, as Padilla points out in his review [4], upon closer inspection it should not help the cosmological problem: $|g| = 1$ is simply a gauge choice for the fully diffeomorphism invariant General Relativity. Explicitly, it changes the symmetry group from the diffeomorphism group $\delta g_{\mu\nu} = \nabla_{(\mu}\zeta_{\nu)}$ to the transverse diffeomorphism group, where additionally $\nabla_\mu \zeta^\mu = 0$[4][43]. So how to reconcile these two observations? There are a few ways of showing this explicitly, depending on how one implements the constraint on g. Padilla presents the use of a Lagrange multiplier λ in a term $\lambda(\sqrt{-g} - 1)$, in which case this new λ's boundary value acts as a cosmological constant, which as a variable becomes radiatively unstable. Another method he shows is *Weyl-Transverse Gravity*, where one defines $g_{\mu\nu} = \frac{f_{\mu\nu}}{|f_{\mu\nu}|^{\frac{1}{4}}}$ and varies $f_{\mu\nu}$ instead. This is my preferred method since it leads directly to the traceless Einstein equations. One can easily check this forces $|g| = 1$. So that the action and EoM read:

$$\begin{aligned} S &= \kappa \int d^4x (R - 2\Lambda(x)) + S_m \\ \rightarrow R_{\mu\nu} &- \frac{1}{4} g_{\mu\nu} R = 8\pi T^{(traceless)}_{\mu\nu} \end{aligned} \tag{53}$$

where $|g| = 1$ was used, with $R = g^{\mu\nu} R_{\mu\nu} = |f|^{1/4} f^{\mu\nu} R_{\mu\nu}$, and any "trace" contributions of the matter action (i.e. anything proportional to $\sqrt{-g}$ but not $g_{\mu\nu}$) along with the cosmological constant Λ were lumped into the scalar $\Lambda(x)$, which drops out of the EoM. Variation of S_m then only gives the traceless part of the full energy momentum tensor, and it is straight-forward to check the trace of the LHS is zero too. Another way to view this result is to define $\frac{1}{4} R = \lambda(x)$, so that the EoM becomes:

$$G_{\mu\nu} = -g_{\mu\nu}\lambda(x) + 8\pi T^{(traceless)}_{\mu\nu} \tag{54}$$

with $\lambda(x)$ so far an unknown scalar field. This is in fact the initial result from using the Lagrange multiplier method mentioned above. Now these obey the Bianchi identity $\nabla_\mu G^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0$, and therefore so must $\lambda(x)$: $\partial_\mu \lambda(x) = 0 \rightarrow \lambda(x) = \Lambda$. The field is thus forced to be a constant and Einstein's equations are recovered, but with Λ an unconstrained constant of integration, similar to those obtained in Hawking's 3-form method which is the focus of this paper. However Λ was defined using the field R appearing in the action so suffers from the same radiative instability as in the previous method, with no apparent mechanism to force it to zero at every order.

Still, it would appear the vacuum contributions have been traded for the extra freedom of a constant of integration (despite it's flaws). So out for interest of its dynamics and in an effort to recast this theory in a manifestly covariant form, Henneaux and Teitelboim's *The Cosmological*

Constant and General Covariance [44] instead provides an elegant Hamiltonian analysis of the theory, for which an overview is presented here. Note however that some extra details are added which are my own best attempts at filling in the gaps of certain steps. Here again the starting point is the ADM formalism, so that in the paper's language the action is:

$$
\begin{aligned}
S &= \int d^4x \left(\pi^{ij}\dot{\gamma}_{ij} - N^i\tilde{\mathcal{H}}_i - \tilde{N}\tilde{\mathcal{H}}_0\right) \\
\tilde{\mathcal{H}}_i &= -2\nabla_j \pi^j_i \\
\tilde{\mathcal{H}}_0 &= \gamma^{-1}\left[\pi_{ij}\pi^{ij} - \frac{1}{2}(\pi^i_i)^2\right] - R^{(3)}
\end{aligned}
\tag{55}
$$

where $\tilde{N} = \sqrt{-g} = N\sqrt{\gamma}$ and $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{H}}_0$ are called the generators of spacial reparametrizations and normal deformations respectively ($\tilde{\mathcal{H}}_0 = \frac{\tilde{\mathcal{H}}}{\sqrt{\gamma}}$ from (28), and the paper ignores the factor $-\kappa$ for simplicity). Henneaux and Teitelboim then note the following Poisson brackets:

$$
\begin{aligned}
\{\tilde{\mathcal{H}}_0(x), \tilde{\mathcal{H}}_0(x')\} &= \left[\left(\frac{1}{\sqrt{\gamma}}\mathcal{H}^i\right)(x) + \left(\frac{1}{\sqrt{\gamma}}\mathcal{H}^i\right)(x')\right]\partial_i\delta(x-x') \\
\{\tilde{\mathcal{H}}_0(x), \mathcal{H}_i(x')\} &= \partial_i(\tilde{\mathcal{H}}_0(x))\delta(x-x') \\
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} &= \mathcal{H}_i(x')\partial_j\delta(x-x') + \mathcal{H}_j(x)\partial_j\delta(x-x').
\end{aligned}
\tag{56}
$$

Now the way of implementing $|g| = 1$ here is to vary γ_{ij}, π^{ij} and N_i but not $\tilde{N}$. This yields the Hamiltonian and the secondary constraint:

$$
\begin{aligned}
H &= \int d^3x \tilde{N}\tilde{\mathcal{H}}_0 + \int d^3x N^i\mathcal{H}_i \\
\{p_i, H\} &= -\frac{\partial H}{\partial N_i} \approx 0 \;\rightarrow\; \mathcal{H}_i \approx 0
\end{aligned}
\tag{57}
$$

for p^i the conjugate momentum to N_i. At this point one may recover the Hamiltonian version of the traceless Einstein equations from $\dot{\pi}_{ij} = \{\pi_{ij}, H\}$ and $\dot{\gamma}_{ij} = \{\gamma_{ij}, H\}$ and the above constraint. The point the paper emphasizes is that this Hamiltonian has further tertiary constraints – i.e. demanding that the secondary constraints be maintained in time – one must impose before analysing the dynamics:

$$
\begin{aligned}
\dot{\mathcal{H}}_i &= \{\mathcal{H}_i, H\} \approx 0 \\
\rightarrow N^j\partial_j\mathcal{H}_i &+ \mathcal{H}_j\partial_i N^j + \tilde{N}\partial_i\tilde{\mathcal{H}}_0 \approx 0
\end{aligned}
\tag{58}
$$

which follows from (56). For arbitrary N^j and fixed $\tilde{N}$ "different from zero at all points", this expression vanishes only if:

$$
\partial_i\tilde{\mathcal{H}}_0 = 0. \tag{59}
$$

There are no quaternary constraints, and one may solve (59) as $\tilde{\mathcal{H}}_0 + 2\Lambda(t) = 0$, with $\Lambda(t)$ a spacial constant. This constraint then corresponds to the Bianchi identity condition $\partial_\mu\lambda(x) = 0$ from Weyl-Transverse Gravity above, for $\mu = i$ (i.e. identifying $\tilde{\mathcal{H}}_0$ with $\lambda(x)$). For the time component, one can make the additional requirement that the Hamiltonian describes a relativistic theory (and so is time-reparametrization invariant):

$$
\begin{aligned}
\{H, H\} \approx 0 \;&\rightarrow\; \partial_t H \approx 0 \\
&\rightarrow \partial_t\tilde{\mathcal{H}}_0 \approx 0 \\
&\rightarrow \tilde{\mathcal{H}}_0 + 2\Lambda \approx 0
\end{aligned}
\tag{60}
$$

where use was made of the constraint (57) and the fact that $\delta\tilde{N} = 0 \rightarrow \partial_t\tilde{N} = 0$ here, and so Λ is now a full spacetime constant. Note how the restricted variation has promoted the Hamiltonian from being a mere constraint $\mathcal{H} \approx 0$ to something marginally less restricted $\mathcal{H} \approx \Lambda$. Also notice the similarity of this free integration constant, and those that will be seen in the next section. As an interesting aside, Henneaux and Teitelboim make the astute observation even when "not written in a manifestly gauge invariant form, theories have a way of telling us that they possess a hidden gauge invariance". In a Hamiltonian method, this is signalled by the appearance of higher generations of constraints. For details on how to know when there are such additional constraints and thus how to properly count degrees of freedom, see [44].

The paper now turns to making the theory manifestly covariant "without changing its physical content". The first step is to incorporate the tertiary constraint into the Hamiltonian, but there remains an issue: the theory is not explicitly invariant under time reparametrizations $t \to f(t)$. Another way of seeing this is that $\partial_t H = -2\partial_t(\Lambda(t)) \int d^3x \tilde{N} = 0$ does not appear to be true in this form, even though it should be from (60). The way to make a non-vanishing Hamiltonian time-reparametrization invariant is to promote the original time t as an additional variable, and re-express everything in terms of a new label t', as described in the introduction. Explicitly, we must return to the action and change $dt = dt'\dot{t}$ (assuming t and t' are "in the same direction"), and making sure this map is invertible $\frac{\partial}{\partial t} = \frac{1}{\dot{t}} \frac{\partial}{\partial t'}$. So now the action reads:

$$S = \int d^4x \left(\pi^{ij}\dot{\gamma}_{ij} - \dot{t}\left[N^i\mathcal{H}_i + \tilde{N}\tilde{\mathcal{H}}_0 \right] \right) \tag{61}$$

where now we have the conjugate variables and Hamiltonian are:

$$\begin{aligned} t \to p_t &= -\left[N^i\mathcal{H}_i + \tilde{N}\tilde{\mathcal{H}}_0 \right] \\ &\to p_t = 2\Lambda(t')\tilde{N} \\ \mathcal{H}' &= \dot{t}p_t + \dot{t}\left[N^i\mathcal{H}_i + \tilde{N}\tilde{\mathcal{H}}_0 \right] \\ &\to \mathcal{H}' = 2\dot{t}p_t + 2\dot{t}\Lambda(t')\tilde{N} \\ &\to \mathcal{H}' = 0, \end{aligned} \tag{62}$$

Henneaux and Teitelboim write this as an additional constraint in terms of the original Hamiltonian density $p_t + \mathcal{H} = 0$. We have now introduced $\Lambda(t')$ as a dynamical variable conjugate to "cosmic time" t [44], so the theory has one global degree of freedom on top of those from g_{ij} and π^{ij}. It is clear to see from the now vanishing Hamiltonian that if one assumes that the wavefuntion of the universe is in an eigenstate of Λ, one recovers the Wheeler-de-Witt equation:

$$\hat{\mathcal{H}}'\Psi(\Lambda) = 0. \tag{63}$$

The next step is to deal with the global variables $t(t')$, $\Lambda(t')$'s breaking of manifest locality. this is best achieved by introducing a time field $\mathcal{T}^0(x, t')$ and its conjugate momentum $\pi_0(x, t')$ so that the parametrization and the constraint $\partial_i\tilde{\mathcal{H}}_0 = 0$ can be can be introduced by writing:

$$\begin{aligned} \pi_0 &= -\tilde{\mathcal{H}}_0 \\ \partial_i\pi_0 &= 0. \end{aligned} \tag{64}$$

The action can then be written:

$$S = \int d^4x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_0\dot{\mathcal{T}}^0 - \tilde{N}\left[\pi_0 + \tilde{\mathcal{H}}_0 \right] - N^i\mathcal{H}_i - \mathcal{T}^i\partial_i\pi_0 \right). \tag{65}$$

so that one varies π^{ij}, g_{ij}, $\mathcal{T}^0$, $\mathcal{T}^i$ and $\tilde{N}$ to obtain the same results. The $\mathcal{T}^i$ are a set of Lagrange multipliers to enforce the constraints above. Now one may define $\mathcal{T}^\mu = \left(\mathcal{T}^0, \mathcal{T}^i\right)$ and $\pi_0 = 2\Lambda$, write π^{ij} in terms of $\dot{g}_{ij}$, to obtain the manifestly covariant action:

$$S = \int d^4x \left(\sqrt{-g}(R - 2\Lambda) + 2\Lambda\partial_\mu\mathcal{T}^\mu \right). \tag{66}$$

which is varied w.r.t. $g_{\mu\nu}(x,t)$, $\Lambda(x,t)$ and $\mathcal{T}^\mu(x,t)$ so that the EoM are:

$$\begin{aligned} G_{\mu\nu} &= -g_{\mu\nu}\Lambda \\ \partial_\mu\Lambda &= 0 \\ \partial_\mu\mathcal{T}^\mu - \sqrt{-g} &= 0. \end{aligned} \tag{67}$$

Which are Einstein's equations with the cosmological constant as a constant of integrations, and one extra EoM. This merely says that the time evolution of the spacial components $\mathcal{T}^i$ is unconstrained, while the time component obeys $\dot{\mathcal{T}}^0 = \partial_i \mathcal{T}^i + \sqrt{-g}$. Defining $T(t') = \int d^3x \mathcal{T}^0$, this may be rewritten as:

$$\begin{aligned} T(t') &= \int d^3x \left(\cancel{\partial_i \mathcal{T}^i} + \sqrt{-g} \right) \\ \rightarrow \int dt' T(t') &= \int d^4x \sqrt{-g} \\ &\rightarrow \Delta T(t') = V \end{aligned} \tag{68}$$

where $V = \int d^4x$ is the invariant 4-D volume. Therefore only the spacial components $\mathcal{T}^i$ are pure gauge, but all of the components are needed to preserve general covariance and locality simultaneously. The overall gauge transformation is obviously:

$$\mathcal{T}^\mu \rightarrow \mathcal{T}^\mu + \epsilon^\mu \tag{69}$$

with the condition $\partial_\mu \epsilon^\mu = 0$. This all may be recast in terms of a 3-form as $A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\mathcal{T}^\sigma$, which is thus dual to the time field:

$$\begin{aligned} S = \int d^4x \sqrt{-g}(R - 2\Lambda) - 2\int A \wedge d\Lambda \\ A \rightarrow A + d\eta \end{aligned} \tag{70}$$

with $A = \frac{1}{3!} A_{\mu\nu\sigma} dx^\mu dx^\nu dx^\sigma$, $\eta = \frac{1}{2}\eta_{\mu\nu} dx^\mu dx^\nu$ and consequently $\epsilon^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\partial_\nu \eta_{\lambda\sigma}$. The last term has the form of the characteristic Chern-Simons term. Note that this term can be rewritten as $+2\int \Lambda F$ for $F = dA$, which is similar to the term appearing in one of the original proposals further down. It will become clear how this term has inherited the role of the free integration constant Λ.

3 Hawking's 3-form

3.1 The Original Proposal

In Hawking's 1983 paper [45], he proposes a mechanism whereby the cosmological constant "is probably zero", meaning it can take any value but has a large probability of vanishing. This makes use of two tools, or assumptions: the presence of a 3-form gauge field $A = \frac{1}{3!} A_{\mu\nu\lambda} dx^\mu dx^\nu dx^\lambda$, and the interpretation of the resulting Euclidean effective action from the path integral as a measure of probability $P(\lambda_{eff}) \propto e^{-S_{eff}[\lambda_{eff}]}$. The Euclidean E-C action is thus:

$$S = \frac{\kappa}{2} \int \epsilon_{abcd} e^a e^b \left(R^{cd} + \frac{1}{6} e^c e^d \Lambda \right) + S_M[e^a, \phi]. \tag{71}$$

and from now on the Euclidean action will always be used, unless otherwise stated. However, for this particular section, parallels with the E-H version will be frequently drawn for clarity. It is well known by now that an antisymmetric abelian gauge field $A_{\mu\nu\lambda}$ can appear in certain models of supergravity [46]. The most apparently relevant to this discussion is N=8 supergravity: $A_{\mu\nu\lambda}$ can appear naturally in 11 dimensions, which when reduced to 4 dimensions is present in the Lagrangian only in terms of its field strength tensor $F_{\mu\nu\rho\lambda}$. Now, following Hawking's method, the following term is added to the E-C action:

$$\int F \wedge *F \tag{72}$$

where $*$ is the spacetime Hodge dual and $F = \frac{1}{4!} F_{\mu\nu\rho\lambda} dx^\mu dx^\nu dx^\rho dx^\lambda$ is related to the gauge field as $F = DA = dA + [A, A]$. In this paper, it will assume that $A_{\mu\nu\lambda}$ is the gauge field of an abelian (i.e. $U(1)$) symmetry (as is consistent with N=8 supergravity), so that simply $F = dA$. The E-H equivalent to this term would be $\frac{1}{4!}\int d^4x \sqrt{g} F_{\mu\nu\rho\lambda} F^{\mu\nu\rho\lambda}$, with $F_{\mu\nu\rho\lambda} = 4\partial_{[\mu} A_{\nu\rho\lambda]}$. The equation of motion (EoM) of A is then:

$$d * F = 0 \tag{73}$$

and since $F_{\mu\nu\rho\lambda}$ is a a fully antisymmetric rank 4 tensor on a 4-dimensional manifold, it must be proportional to the Levi-Civita tensor $F_{\mu\nu\rho\lambda} = c(x)\epsilon_{\mu\nu\rho\lambda}$, where $c(x)$ is a scalar function.

Therefore $\partial_\mu c(x) = 0$, or in other words $c(x) = c$ - a constant, and using $F = \frac{1}{4!} F_{abcd} e^a e^b e^c e^d$ with $F_{abcd} = c(x)\epsilon_{abcd}$:

$$\int F \wedge *F = \frac{c^2}{4!} \int \epsilon_{abcd} e^a e^b e^c e^d \tag{74}$$

or $c^2 \int d^4x \sqrt{g}$ in E-H. Hawking then uses this directly in his action to get (ignoring matter):

$$S = \frac{\kappa}{2} \int \epsilon_{abcd} e^a e^b \left(R^{cd} + \frac{1}{6} e^c e^d \left(\Lambda + \frac{c^2}{2\kappa} \right) \right). \tag{75}$$

The effective cosmological constant would then be $\lambda_{eff}(c) = \Lambda + \frac{c^2}{2\kappa}$, formed of the "bare" cosmological constant Λ containing all the contributions from all the matter fields and anything else that plays a similar role, and an as yet undetermined integration constant c. Solving the Einstein equations in the usual way would then yield $R^{cd} = -\frac{1}{3} e^c e^d \lambda_{eff}(c)$, so that the effective action reads:

$$S_{eff} = -\frac{\kappa \lambda_{eff}(c)}{12} \int \epsilon_{abcd} e^a e^b e^c e^d. \tag{76}$$

Now $\int \epsilon_{abcd} e^a e^b e^c e^d = \int *1 = 4! \int d^4x \sqrt{g} = 4! V_c$, with V_c the comoving volume. If the S^4 solution of a closed universe with $\lambda_{eff}(c) > 0$ is picked, then $V_c = \frac{24\pi^2}{\lambda_{eff}(c)^2}$ [47], so that finally:

$$\begin{aligned} S_{eff} &= -\frac{3\pi}{G \lambda_{eff}(c)} \\ P(c) &\propto e^{-S_{eff}[\lambda_{eff}(c)]}. \end{aligned} \tag{77}$$

In other words, there is a sharp probability peak as $\lambda_{eff}(c) \to 0^+$!

Furthermore, Raphael Bousso's paper *Quantization of Four-form Fluxes and Dynamical Neutralization of the Cosmological Constant* [48] provides an analysis from the view of compactified string theory. It states that $*F$ may be integrated over a zero-dimensional manifold (a point), so that the generalized Dirac quantization condition becomes:

$$*F = \frac{en}{Z} \tag{78}$$

where $n \in \mathbb{Z}$. Rapahel found that compactification of certain M-theory configurations leads to "vacua with discrete but closely spaced values for the cosmological constant", so that $\lambda \approx 0$, but $\lambda \neq 0$. This would agree rather well with observations.

So is that it? Well no, there are 2 issues that need addressing here. The first is a rather subtle question over whether the path integral of the universe can be interpreted in such a probabilistic sense, or even whether a Euclidean action is well posed in this context [1]. The second is more definite, in that results of an EoM should not be substituted back into the action before all the EoM are obtained [47]. I will now cover existing discussions on these topics.

3.2 The Equations of Motion are King - Duff's Rebuttal

A strong argument against Hawking's proposed mechanism came in the form of Duff's paper *The Cosmological Constant is Probably Zero, but the Proof is Probably Wrong* [47], where he correctly argues that one should not substitute a solution into the action before obtaining all the EoM. Although this appears to only change a sign, it is an all important sign: it can be seen from Appendix B (E-C variation) that $\int F \wedge *F$ enters Einstein's equations with a flipped sign. Note that this change is only apparent in E-H once one substitutes $F_{\mu\nu\alpha\beta} \to c(x)\epsilon_{\mu\nu\alpha\beta}$. Thus:

$$\begin{aligned} S &= \frac{\kappa}{2} \int \epsilon_{abcd} e^a e^b \left(R^{cd} + \frac{1}{6} e^c e^d \Lambda \right) + \int F \wedge *F \\ &\to \epsilon_{abcd} e^b \left(R^{cd} + \frac{1}{3} e^c e^d \left(\Lambda - \frac{c^2}{2\kappa} \right) \right) = 0 \end{aligned} \tag{79}$$

so that this is the effective cosmological constant that is measurable $\lambda_{eff}(c) = \Lambda - \frac{c^2}{2\kappa}$, and should have the highest probability of being 0, as opposed to the previous $\Lambda + \frac{c^2}{2\kappa}$. Unfortunately if substituted back into the action and using the same solutions, the sign change yields:

$$S_{eff} = -\frac{3\pi}{G\lambda_{eff}(c)} + \frac{6\pi\Lambda}{G\lambda_{eff}(c)^2}. \tag{80}$$

So for $\Lambda > 0$, this gives the opposite conclusion, since the Λ term will dominate for small λ_{eff} and bring the probability to zero. Furthermore, as seen in Appendix B, the sign flip is independent of the signature, so this still holds in a Lorentzian action - if it could be used for such an interpretation to begin with.

Duff does mention one way of cancelling the sign change, referencing the work of Aurelia, Nicolai and Townsend [46] who supplemented the action with:

$$-2c\int F \tag{81}$$

where c is the integration constant obtained from the $\int F \wedge *F$ term. This is a boundary term, so does not enter the EoM, which are thus the same as above. The difference arises when the results are substituted back into the action which contains $\Lambda + \frac{c^2}{2\kappa} - 2\frac{c^2}{2\kappa} = \Lambda - \frac{c^2}{2\kappa}$, effectively cancelling the sign change and recovering Hawking's result. However as Duff points out, c is no longer an arbitrary integration constant since it enters the action explicitly, defeating the whole purpose of the mechanism.

Nevertheless, this rebuttal is not as airtight as it first seems, as was pointed out in papers by Wu [49] and Duncan and Jensen [50]. Additionally, small modifications can be made that appear to be absent from the literature, and will be presented in section 4.

3.3 Still a Valid Mechanism

These two papers cover a very similar topic and have essentially the same conclusions, but go about showing them in slightly different ways. Duncan and Jensen take a more general approach [50], going through the different possible 4-form terms that can be included in the action, while Wu's paper is more straightforward [49] and focused around the terms that work. The preference is up to personal taste, and I will be mainly following Duncan and Jensen's methodology, with some input from Wu for added detail. On a suitable (smooth, orientable) manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, there are 3 possible 4-form terms that can enter the action:

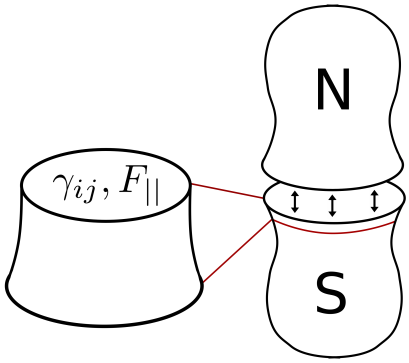
$$\begin{aligned} S_1 &= \int_{\mathcal{M}} F \wedge *F \\ S_2 &= \int_{\mathcal{M}} F \wedge *F - 2c\int_{\partial\mathcal{M}} A \\ S_3 &= \int_{\mathcal{M}} F \wedge *F - 2\int_{\partial\mathcal{M}} A \wedge *F. \end{aligned} \tag{82}$$

S_1 is Hawking's unaltered proposal, and so does not work as was seen, and S_2 is the modification discussed by Duff [47] which reverses the sign problem, but fixes the integration constant. S_3 is the new one proposed by the two papers and seems quite promising. Before looking into how it affects the EoM and the effective action, is there a preliminary argument as to which one should be chosen? It turn out that yes, and this can be seen by looking at the boundary conditions. First, looking at the variation of S_2 and S_3:

$$\begin{aligned} \delta S_2 &= 2\int_{\mathcal{M}} \delta A \wedge d * F + 2\int_{\partial\mathcal{M}} \delta A \wedge (*F - c) \\ \delta S_3 &= 2\int_{\mathcal{M}} \delta A \wedge d * F - 2\int_{\partial\mathcal{M}} A \wedge \delta(*F). \end{aligned} \tag{83}$$

Implicitly in the derivations of the equations of motion, it was assumed that the boundary terms vanish. This is normally achieved by fixing A (or whichever field one is varying) to some constant value on the boundary, but is that justified? When deriving an instanton (Euclidean action around its classical solution), one determines the configuration of the wavefuntion and any fields on the equator of the instanton, by joining together the South hemisphere of the instanton and its time reversal, the North hemisphere. One cannot set a global value for A, but can do so for F (Recall the Dirac monopole) [49]. In the ADM formalism with spacelike foliation, the 3-metric is set on the equator, so pictorially:

Figure 3: On the left: γ_{ij} and $F_{||}$ defined on a timelike foliation of the manifold $\mathcal{M}$, with $F_{||}$ the component of F tangent to the spacial hypersurface. On the right: the North and South hemispheres of the instanton, joined at the equator.



A is only defined locally, on each of the hemispheres above, and so should not be used as boundary data since it is not globally well defined if $*F \neq 0$. Furthermore, the objective is to compare different geometries with different $c = *F$, so $*F$ is starting to appear as a natural replacement. Indeed, looking back at the variations (83), notice that the boundary terms can vanish through other means: for S_2, having $*F = c$ on the boundary eliminates the term, whereas for S_3, $*F$ can simply be set to any constant value on the boundary to get $\delta(*F)|_{\partial\mathcal{M}} = 0$. Now here I have to depart from Duncan and Jensen's view that $*F$ in S_2 "is constrained to equal the coupling [c] in the Lagrangian": it appears to me that selecting this boundary term and then demanding it vanishes is simply a more elaborate way of manually setting the value of c in the Lagrangian. This merely changes the way this arises, which can nevertheless be an interesting mechanism in its own right. S_3 then appears to be the natural choice for the action, and it will now be explicitly shown how this enables Hawking's mechanism to work. The full Euclidean action is then:

$$S = \frac{\kappa}{2}\int \epsilon_{abcd}e^a e^b \left(R^{cd} + \frac{1}{6}e^c e^d \Lambda\right) + \int F \wedge *F - 2\int d(A \wedge *F) \tag{84}$$

with $*F$ set on the boundary, exactly the same EoM as before are recovered:

$$\begin{aligned} \epsilon_{abcd}e^b \left(R^{cd} + \frac{1}{3}e^c e^d \left(\Lambda - \frac{c^2}{2\kappa}\right)\right) &= 0 \\ d * F &= 0 \end{aligned} \tag{85}$$

so that again $\lambda_{eff}(c) = \Lambda - \frac{c^2}{2\kappa}$. Now using the second EoM:

$$-2\int d(A \wedge *F) = -2\int F \wedge *F \tag{86}$$

which effectively flips the sign of $\int F \wedge *F$ just like the $-2c\int F$ term Duff mentioned, returning Hawking's original result with c an undetermined constant free to vary!

$$\begin{aligned} S_{eff} &= -\frac{3\pi}{G\lambda_{eff}(c)} \\ P(c) \to \infty&, \ as\ \lambda_{eff}(c) \to 0^+. \end{aligned} \tag{87}$$

There are no obvious pathologies here, except the question of what happens in the "no boundary" scenario.

Here the boundary terms lose their relevance, so the 4-form term must reduce to $\int F \wedge *F$, but the way to treat this is not as straight forward as one would initially think. Duncan and Jensen remind us of the Dirac monopole where 2 coordinate charts are needed to cover the manifold so that A cannot be globally defined, but is defined locally on each chart such that that the 3-forms are related by a gauge transformation $A \to A + d\lambda$. the rigorous way to go about this is to take advantage of the manifold being closed and compact to use the Hodge decomposition theorem [50]:

$$F = dA + \eta. \tag{88}$$

There is no co-exact form since F is a top form, and $\eta = c * 1$ is a harmonic form which has to be a constant to satisfy $d^\dagger \eta = 0$ (and being a top form must therefore be proportional to the volume form). The 4-form term then becomes:

$$S_F = \int F \wedge *F = \int \tilde{F} \wedge *\tilde{F} + 2c \int \tilde{F} + c^2 \int *1 \tag{89}$$

with the definition $\tilde{F} \equiv dA$. Only the first term enters the EoM, giving the usual $d * \tilde{F} = 0$, with the solution $*\tilde{F} = \tilde{c}$. However this would imply $dA = \tilde{\eta}$, a different harmonic form, and so by the uniqueness of the Hodge decomposition it must be zero $*\tilde{F} = \tilde{F} = 0$. The action then simply reads:

$$S_F = c^2 \int *1 \tag{90}$$

which enters both the EoM <u>and</u> the effective action with no sign change, giving $\lambda_{eff}(c) = \Lambda + \frac{c^2}{2\kappa}$ and again:

$$\begin{aligned} S_{eff} &= -\frac{3\pi}{G\lambda_{eff}(c)} \\ P(c) &\to \infty, \ as \ \lambda_{eff}(c) \to 0^+. \end{aligned} \tag{91}$$

To avoid any confusion, the result of the EoM was not put into the action before obtaining the other EoM. However, this could equally well be done since the result simply removes any $\tilde{F}$ dependence from both the EoM and the action. The action (90) is simply illustrative. So to paraphrase Wu, "The cosmological constant is probably zero, and a proof is possibly right".

3.4 Euclidean Action - Analytic Continuation

It is well known that time ordered vacuum expectation values in a Lorentzian theory can be naturally obtained by analytic continuation from Euclidean correlation functions. Such a Euclidean path integral may have probabilistic interpretations, such as a Boltzmann average. However, can such an argument really be applied to the boundary conditions of the Universe? While the probabilistic interpretation of a path integral over 3-metrics is so far rather up in the air - and perhaps even bordering on philosophy unless and until a rigorous theory for it is uncovered - the literature has some interesting discussions on the use of a Euclidean action in this context. The difficulty here is that analytic continuation over non-trivial topologies is ill-defined at best [50] [51]. At the very least, the process must be treated with care so as to maintain the properties of the path integral from one signature to the other. Duncan and Jensen [50] lay out a way of doing just that for a specific action in a spacetime with closed, homogeneous spacial hypersurfaces, so that the line element is:

$$ds^2 = -N(t)^2 dt^2 + \gamma_{ij}(t) e^i \otimes e^j \tag{92}$$

and the volume element of the 3-sphere $\Omega_3 = e^1 e^2 e^3$ integrates to $\int_{S_3} \Omega_3 = 2\pi^2$, and as before $d^4x\sqrt{-g} = N\sqrt{\gamma} dt \Omega_3$. The Lorentzian action taken is:

$$S_L = -\frac{1}{2} \int F \wedge *F. \tag{93}$$

They then argue that since the spacetime boundaries are slices at constant time, $F = dA_{||}$ where $A_{||} = \frac{1}{3!} A_{ijk} e^i e^j e^k$ (i.e. the objects have been split into orthogonal and tangent components to the hypersurfaces Σ as described in the introduction) and $A_{ijk} = A_{ijk}(t)$ due to homogeneity. Therefore:

$$\begin{aligned} F &= \frac{1}{3!} \dot{A}_{ijk} dt e^i e^j e^k \\ S_L &= \frac{1}{2}(2\pi^2) \int_{t_i}^{t_f} dt \frac{\sqrt{\gamma}}{N} \frac{1}{3!} \dot{A}_{ijk} \dot{A}^{ijk}. \end{aligned} \tag{94}$$

The idea now is to perform a Legendre transform, and analytically continue the action in phase-space. The conjugate momentum is $\pi_{ijk} = 2\pi^2 \frac{\sqrt{\gamma}}{N} \dot{A}_{ijk}$, which is conserved so that $\pi_{ijk} = 2\pi^2 p \epsilon_{ijk}$ with p a constant (this is just the EoM $d * F = 0$). So now the phase-space action is:

$$\begin{aligned} S_L^p &= S_L - \frac{1}{3!}\pi_{ijk}A^{ijk}|_{t_f} + \frac{1}{3!}\pi_{ijk}A^{ijk}|_{t_i} \\ S_L^p &= -\frac{1}{2}\int F \wedge *F + \int d\left(A_{||} \wedge *F\right) \end{aligned} \tag{95}$$

using $\dot{\pi}_{ijk} = 0 \rightarrow \pi_{ijk}\dot{A}^{ijk} = \partial_t\left(\pi_{ijk}A^{ijk}\right)$. This is just the Lorentzian version of the action suggested in the section above by Duncan and Jensen. They also note the that the action is invariant under $A^{ijk} \rightarrow A^{ijk} + \alpha\epsilon^{ijk}$, which gives a conserved current $j = -p\left(\frac{N}{\sqrt{\gamma}}\right)$ and conserved charge:

$$Q = 2\pi^2 p \tag{96}$$

which is a "fixed charge for the vacuum".

A more consistent and rigorous approach to analytic continuation is laid out formally in this review [52], whereby *Picard-Lefschetz theory* is used to deform the integration contour from the real axis into the complex plane such that the path integral is absolutely convergent: the contour of steepest descent. However, this approach creates new problems such as the metric signature not being respected in the path integral, and of course at certain places in the contour the metric is neither Lorentzian nor Euclidean [53]. For our purposes, this simply replaces certain interpretational difficulties with new ones. Although this only scratches the surface of the vast field, there is an interesting way to potentially sidestep these issues in obtaining a probability argument. Vilenkin [34] identified the WdW equation (97) with the Klein Gordon (KG) equation $(\Box - m^2)\phi = 0$ (with $\Box = g_{\mu\nu}\partial^\mu\partial^\nu$ the d'Alembert operator), by writing it in the form:

$$\left(\nabla^2 - U\right)\Psi = 0 \tag{97}$$

The conserved current can easily be obtained from the KG Lagrangian as $j^\mu = \frac{1}{2i}\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$. Similarly, Vilenkin defines the conserved a current:

$$j(a) = \frac{i}{2}a^p\left(\Psi^*(a)\frac{\partial}{\partial a}\Psi(a) - \Psi(a)\frac{\partial}{\partial a}\Psi^*(a)\right). \tag{98}$$

with $\Psi(a)$ the Vilenkin wavefunction (48). For "standard" KG theory, this corresponds to conserved particle number, but Vilenkin interprets it in this case as the *probability flux in superspace*, which we are reminded may take on negative values under non-classical conditions. This appears to give the conserved current for the classically allowed and classically forbidden regions respectively:

$$\begin{aligned} j_\pm^{(1)}(a) &= \mp a^p\sqrt{-U(a)} \\ j_\pm^{(2)}(a) &= 0 \end{aligned} \tag{99}$$

At least in the classically allowed region, this shows an interesting result that is dependent on the details of the potential $U(a)$. This could well be applied to the Hawking's 3-form mechanism, as will be seen in section 4.3.2 further down.

4 New Proposals

In case the mechanisms described above do not sit right with the reader, this paper presents 2 other ways to obtain a Hawking-like result. For those that are already satisfied, these represent a rather different approach, which could arise from distinct processes or motivations, and are therefore of interest in their own right. They both revolve around "coupling" an extra 4-form term to the cosmological constant using a dynamical $\lambda(x)$ field and some general function of this $\sigma(\lambda(x))$.

4.1 How To Avoid Boundary Term: Λ-F Coupling

In a discussion with professor Antonio Padilla about his review mentioned at the start [4], he proposed an action related to the Hawking's, but with some important differences that were later found to highlight a link between Hawking's 3-form and Unimodular Gravity. The E-C version of this action is:

$$S = \frac{\kappa}{2}\int \epsilon_{abcd}e^a e^b \left(R^{cd} + \frac{1}{6}e^c e^d(\Lambda + \lambda(x))\right) + \int \sigma(\lambda(x))F. \tag{100}$$

In E-H the last term is $\int d^x\sqrt{g}\sigma(\lambda(x))\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta}$. Notice the similarity with Heinneaux and Teitelboim's covariant Unimodular Gravity action above (70), for which this appears to be a generalisation. The idea here is for the last term to "talk" to the cosmological constant through $\lambda(x)$, forcing it to a constant value. Using the variation tools from Appendix B, its equations of motion are:

$$\begin{aligned} \epsilon_{abcd}e^b \left(R^{cd} + \frac{1}{3}e^c e^d(\Lambda + \lambda)\right) - \frac{2}{\kappa}\frac{2}{4!}\sigma(\lambda)F_{abcd}e^b e^c e^d = 0 \\ d\sigma(\lambda) = 0 \\ \frac{1}{6}\epsilon_{abcd}e^a e^b e^c e^d + \frac{2}{\kappa}\sigma'(\lambda)F = 0 \end{aligned} \tag{101}$$

remembering there are 3 EoM now due to $\lambda(x)$ (and $\sigma'(\lambda) \equiv \frac{d\sigma}{d\lambda}$). The second one forces $\lambda(x) = c$ to be constant, and so $\sigma(\lambda) = \sigma(c) \equiv \sigma$. The last EoM gives $F = -\frac{\kappa}{12\sigma'}\epsilon_{abcd}e^a e^b e^c e^d$, with $\sigma' \equiv \sigma'(c)$, setting the 4-form to a constant as before. Putting this together, the Einstein equations become:

$$\begin{aligned} \epsilon_{abcd}e^b \left(R^{cd} + \frac{1}{3}e^c e^d \lambda_{eff}(c)\right) = 0 \\ \lambda_{eff}(c) = \Lambda + c + \frac{\sigma}{\sigma'}. \end{aligned} \tag{102}$$

Now substituting this back into the action, it is only $\frac{\sigma}{\sigma'}$ that flips sign, so if c and Λ are absorbed into $\lambda_{eff}(c)$ and the usual S^4 solution with $\int *1 = 4!V_c$ is picked, where $V_c = \frac{24\pi^2}{\lambda_{eff}}$:

$$S_{eff} = -\frac{3\pi}{G\lambda_{eff}(c)} - \frac{6\pi}{G\lambda_{eff}(c)^2}\frac{\sigma}{\sigma'}. \tag{103}$$

No matter the value of c or the form of σ and σ', as long as $\frac{\sigma}{\sigma'} > 0$, this gives a probability peak $P(c) \to \infty$ as $\lambda_{eff}(c) \to 0^+$! There is no need to flip a sign here, and so no need for boundary terms. The mechanism is beautifully simple. An obvious way of insuring $\frac{\sigma}{\sigma'} > 0$ would be to have a positive power of $\lambda(x)$, or a positive exponential.

Now a very interesting point to notice is that for Unimodular Gravity $\sigma(\lambda(x)) \propto \lambda(x)$ (and Λ is lumped into $\lambda(x)$), so that $\lambda_{eff}(c) = 2c$, which leads to:

$$S_{eff} = -\frac{6\pi}{G\lambda_{eff}(c)}. \tag{104}$$

So this does indeed appear to be a generalisation of Unimodular Gravity, and shows a link with Hawking's 3-form through a further generalisation in next section.

4.2 Generalisation: Λ-F^2 Coupling

Taking inspiration from the previous action, a generalisation can be made to the $\int F \wedge *F$ term, by "coupling" it to the cosmological constant in the same way. This exhibits some interesting properties, such as reviving the $-2c\int F$ boundary term as a viable option, but also has some limitations as will be shown. Explicitly, the Euclidean action without any boundary terms is:

$$S = \frac{\kappa}{2}\int \epsilon_{abcd}e^a e^b \left(R^{cd} + \frac{1}{6}e^c e^d(\Lambda + \lambda(x))\right) + \int \sigma(\lambda(x))F \wedge *F \tag{105}$$

giving the following equations of motion:

$$\begin{aligned} \epsilon_{abcd}e^b \left(R^{cd} - \frac{1}{3}e^c e^d(\Lambda + \lambda)\right) - \frac{2}{\kappa}\frac{2}{4!}\sigma(\lambda)F_{abcd}e^b \wedge e^c e^d * F = 0 \\ d\left(\sigma(\lambda) * F\right) = 0 \\ \frac{1}{6}\epsilon_{abcd}e^a e^b e^c e^d + \frac{2}{\kappa}\sigma'(\lambda)F \wedge *F = 0. \end{aligned} \tag{106}$$

These are very similar to the those of the previous action, with an additional $*F$ in each case. Solving the last 2 for $*F = c(x)$ gives, respectively:

$$\begin{aligned} c(x) &= \frac{b}{\sigma(\lambda)} \\ \rightarrow \sigma(\lambda) &= \frac{1}{\frac{4\kappa}{b^2}\lambda + a} \end{aligned} \tag{107}$$

with a, b constants. When put into the first EoM, this produces $\lambda_{eff}(a,b) = \Lambda - \frac{ab^2}{2\kappa}$. This is very similar to the "uncoupled" case, and the resemblance extends to the sign problem when putting these results back into the action

$$\begin{aligned} S_{eff} &= \int \epsilon_{abcd} e^a e^b e^c e^d \left[-\frac{1}{3}\lambda_{eff}(a,b) + \frac{1}{6}\left(\Lambda + 2\lambda(x) + \frac{ab^2}{2\kappa}\right)\right] \\ \rightarrow S_{eff} &= -\frac{3\pi}{G\lambda_{eff}(a,b)} + \frac{1}{4\pi}\left(\int d^4x\sqrt{g}\lambda(x) + \Lambda\int d^4x\sqrt{g}\right). \end{aligned} \tag{108}$$

This looks to have the same issues as without the coupling, so why bother? Well aside from generalising, there are two extra free parameters to tweak: $\lambda(x)$ left over in S_{eff} and c from the previous actions is split into 2 independent parameters a and b.

The first point could be useful if:

$$-\langle\lambda(x)\rangle > \Lambda \tag{109}$$

reusing the spacetime average definition from (120). This would force the second term in S_{eff} to be negative, thus recovering the desired result. However, contriving some function to fit this would appear to just be recasting the assumption that the cosmological constant is zero. This may be useful if some motivation for the form of $\lambda(x)$ could be found, but otherwise advances us no further.

Now the second point does make a difference, in that it allows the use of a boundary term previously discarded. Recall that adding $-2c\int F$ to the action of $\int F \wedge *F$ flipped the sign in the effective action in the right way to recover Hawking's result, but in so doing fixed c. In this new case, one need only fix b from $c(x) = \frac{b}{\sigma(\lambda(x))}$ to achieve the same effect, leaving a free to vary! Even better, a is not squared, so can take on positive or negative values as needed. The total action and results are:

$$\begin{aligned} S = \frac{\kappa}{2}\int \epsilon_{abcd} e^a e^b \left(R^{cd} + \frac{1}{6}e^c e^d(\Lambda + \lambda(x))\right) + \int \sigma(\lambda(x)) F \wedge *F - 2b\int F \\ \rightarrow S_{eff} = -\frac{3\pi}{G\lambda_{eff}(a)}\ , \quad \lambda_{eff}(a) = \Lambda - \frac{ab^2}{2\kappa}. \end{aligned} \tag{110}$$

So this provides an option if one insist on fixing A on the boundary and using this boundary term, despite the concerns raised by Wu [49] and Duncan and Jensen [50] about this only being defined locally. Note that the boundary term preferred by these two can also be adapted for this case, as:

$$-2\int d\left(\sigma(\lambda) A \wedge *F\right) =_{on\ shell} -2\int \sigma(\lambda) F \wedge *F \tag{111}$$

from $d\left(\sigma(\lambda) * F\right) = 0$, and the same result as (110) is recovered but with b not fixed. No other variations on these terms have been found to work, but these two cases already give adequate options for a bounded manifold.

This generalisation appears to even work on a closed manifold. Following the same prescription as previously, with a Hodge decomposition:

$$\begin{aligned} F &= dA + c * 1 \\ \rightarrow \int \sigma(\lambda(x)) F \wedge *F &= \int \sigma(\lambda(x)) \tilde{F} \wedge *\tilde{F} + 2c\int \sigma(\lambda(x))\tilde{F} + c^2 \int \sigma(\lambda(x)) * 1 \end{aligned} \tag{112}$$

with $\tilde{F} = dA$. As opposed to the previous use of the Hodge decomposition, here the changes in the equations of motion must be treated more carefully:

$$\epsilon_{abcd} e^b \left(R^{cd} - \frac{1}{3} e^c e^d \left[\Lambda + \lambda \right] \right) - \frac{4}{4!\kappa} \sigma(\lambda) \left(\tilde{F}_{abcd} * \tilde{F} + 2c\tilde{F}_{abcd} - c^2 \epsilon_{abcd} \right) e^b e^c e^d = 0 \tag{113}$$

$$\sigma'(\lambda) \left(\tilde{F} \wedge * \tilde{F} + 2c\tilde{F} + c^2 * 1 \right) = -\frac{\kappa}{2} \frac{4!}{6} * 1 \tag{114}$$

$$\begin{aligned} d\left(\sigma(\lambda) \left[* \tilde{F} + c \right] \right) &= 0 \\ \rightarrow * \tilde{F} &= \frac{b}{\sigma(\lambda)} - c. \end{aligned} \tag{115}$$

This means that $\tilde{F}$ can be split into 2 pieces, of which one $c * 1$ is a harmonic form. However, from the uniqueness of the Hodge decomposition, $\tilde{F}$ cannot contain a harmonic form, and must therefore be zero $\tilde{F} = * \tilde{F} = 0$. Unless of course $c = 0$ instead, in which case the trivial result of the EoM is $*F = 0$ (this case will be ignored). The other 2 EoM now become

$$\begin{aligned} \sigma'(\lambda) c^2 &= -2\kappa \\ \rightarrow \sigma(\lambda) &= -\frac{2\kappa}{c^2} \lambda + a \end{aligned} \tag{116}$$

$$\begin{aligned} \epsilon_{abcd} e^b \left(R^{cd} - \frac{1}{3} e^c e^d \left[\Lambda + \lambda + \frac{c^2}{2\kappa} \sigma(\lambda) \right] \right) &= 0 \\ \rightarrow \epsilon_{abcd} e^b \left(R^{cd} - \frac{1}{3} e^c e^d \left[\Lambda + \not{\lambda} - \not{\lambda} + \frac{ac^2}{2\kappa} \right] \right) &= 0 \end{aligned} \tag{117}$$

with a an integration constant. Therefore $\lambda_{eff}(a,c) = \Lambda + \frac{ac^2}{2\kappa}$, and putting all these back into the action, the result on a closed manifold is:

$$\begin{aligned} S = \frac{\kappa}{2} \int \epsilon_{abcd} e^a e^b \left(R^{cd} + \frac{1}{6} e^c e^d (\Lambda + \lambda(x)) \right) &+ \int \sigma(\lambda(x)) F \wedge * F \\ \rightarrow S_{eff} = -\frac{3\pi}{G\lambda_{eff}(a)} \ , \quad & \lambda_{eff}(a,c) = \Lambda - \frac{ac^2}{2\kappa} \end{aligned} \tag{118}$$

which is exactly the same result as the boundary case above, but with no constant fixed - unless one counts the harmonic form $c * 1$ as a fixed value, in which case only a is free to vary as above. In fact, if one re-examines the result $*\tilde{F} = 0$, this leads to $\frac{b}{\sigma(\lambda)} = c$. $\lambda(x)$ must therefore be a constant, just as in the $\int \sigma(\lambda(x)) F$ case! Although this does not appear to matter here, it does add some robustness to the results under tweaking of the theory.

4.3 Hamiltonian Analysis of the 3-form

Most quantization schemes found in the literature appear to be of the 4-form or its dual $*F$, in the framework of string theory, or some such overarching theory. Here I will first briefly review the gauge fixing process for the Maxwell field in the framework of differential forms, to then propose a new gauge for the 3-form. The results are however difficult to interpret at best, so the next section gives an unconstrained Hamiltonian analysis of this action. The results here can then be tied together with QC presented at the start of this paper, and a new probability argument obtained using Vilenkin's method and the WdW equation. Note for this whole section, the Lorentzian action is used.

4.3.1 Gauging the 3-form

First a reminder on the analogous Maxwell field is needed, with the 2-form field strength tensor and 1-form gauge field having the usual EoM $d * F = 0$. Now the gauge field is invariant under $A \rightarrow A + d\lambda(x)$, so one may pick a gauge to fix this (or partly fix it). The Lorenz gauge is a common choice for its manifest Lorentz invariance: $d^\dagger A = 0$. This choice may always be reached from an arbitrary field A' where $d^\dagger A' = f(x)$ with $f(x)$ some function. This can be seen by choosing a λ such that:

$$\begin{aligned} A' &= A + d\lambda \\ \rightarrow d^\dagger d\lambda &= f \\ \rightarrow \Delta\lambda &= f \\ \rightarrow \lambda &= \Delta^{-1} f \end{aligned} \tag{119}$$

where $d^\dagger = *d*$ is the adjoint exterior derivative, $\Delta = dd^\dagger + d^\dagger d$ is the *Laplacian operator*, and use was made in the third line of the fact that $d^\dagger f(x) = 0$ for any function. The point is that the Laplacian is always locally invertible, since in flat spacetime it is equivalent to the d'Alembert operator which is always invertible [54]: $\Delta w = -\frac{1}{r!}\Box w_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$ for w an r-form in flat spacetime. This then reduces the EoM to a solvable form as

$$\begin{aligned} d * F = d * dA \propto *d^\dagger dA &= 0 \\ \rightarrow * \left(\Delta A - \cancel{dd^\dagger A}\right) &= 0 \\ \rightarrow \Delta A &= 0. \end{aligned} \tag{120}$$

The wish is then to do a similar thing with the 3-form, but the Lorenz gauge's applicability relied on the gauge changing term λ being a function (0-form). In this case the gauge transformation is $A \rightarrow A + d\eta$ where η is any 2-form. Following the same prescription as above would simply yield $d^\dagger d\eta = \left(\Delta - dd^\dagger\right)\eta = B$ with B some other 2-form. This is not solvable in general, so the Lorenz gauge cannot always be taken. The gauge found to be generally applicable, while making the EoM solvable and leaving the $*F = c$ (c - constant) unchanged is:

$$dd^\dagger A = 0 \tag{121}$$

so that for a 3-form A' that obeys $dd^\dagger A' = B$, a gauge transformation may be chosen so as to return to the gauge condition above:

$$\begin{aligned} A' &= A + d\eta \\ \rightarrow dd^\dagger d\eta &= B \\ \rightarrow \left(\Delta - \cancel{d^\dagger d}\right) d\eta &= B \\ \rightarrow d\eta &= \Delta^{-1} B \\ \rightarrow d\Delta^{-1} B &= 0 \end{aligned} \tag{122}$$

which always has a solution locally (and globally for trivial topologies). Note how this gauge is less restrictive than the Lorenz gauge, which in fact would force $*F = 0$. Now the gauge choice may be rewritten simply as $d^\dagger d\tilde{A} = 0$ with $\tilde{A} = *A$ a 1-form, to make things simpler. In the same vein, the EoM may be rewritten in terms of $\tilde{A}$ and simplified to a solvable form

$$\begin{aligned} d * F = d * dA \propto dd^\dagger \tilde{A} &= 0 \\ \rightarrow \left(\Delta - \cancel{d^\dagger d}\right)\tilde{A} &= 0 \\ \rightarrow \Delta\tilde{A} &= 0. \end{aligned} \tag{123}$$

This is exactly the same EoM as for the Maxwell field! Before solving this however, note that on-shell $\Delta\tilde{A} = dd^\dagger\tilde{A} + \cancel{d^\dagger d\tilde{A}} = dd^\dagger\tilde{A} = 0$, so that $d^\dagger\tilde{A} = c$ is a constant. Now also notice that:

$$\begin{aligned} *F &= d^\dagger\tilde{A} \\ \rightarrow *F &= c \end{aligned} \tag{124}$$

as one would expect. Moving now to flat Minkowski spacetime, the usual plane wave solution for $\tilde{A}$ can be taken, with $p^\mu = (E(\vec{p}), \vec{p})$, $p^2 = 0$ and $E(\vec{p}) = +|\vec{p}|$:

$$\tilde{A}^\mu = \int \frac{d^3\vec{p}}{(2\pi)2E(\vec{p})} \sum_{\lambda=0}^{3} \epsilon^\mu_{(\lambda)}(\vec{p}) \left(a_\lambda(\vec{p}) e^{-ip_\mu x^\mu} + a^\dagger_\lambda(\vec{p}) e^{ip_\mu x^\mu}\right) \tag{125}$$

with $\epsilon^{\mu}_{(\lambda)}(\vec{p})$ polarisation vectors. Now, it is the rank 3 tensor from the 3-form that must be quantized, using $\tilde{A}^{\mu} \propto \epsilon^{\mu\nu\alpha\beta}A_{\nu\alpha\beta}$. Defining $\epsilon^{\mu\nu\alpha}_{(\lambda)}(\vec{p}) = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\beta(\lambda)}(\vec{p})$, we finally have:

$$\hat{A}^{\mu\nu\alpha} = \int \frac{d^3\vec{p}}{(2\pi)2E(\vec{p})} \sum_{\lambda=0}^{3} \epsilon^{\mu\nu\alpha}_{(\lambda)}(\vec{p}) \left(\hat{a}_{\lambda}(\vec{p})e^{-ip_{\mu}x^{\mu}} + \hat{a}^{\dagger}_{\lambda}(\vec{p})e^{ip_{\mu}x^{\mu}}\right) \tag{126}$$

with the gauge constraint forcing $\sum_{\lambda=0}^{3} p_{\mu}p^{[\nu}\epsilon^{\alpha\beta]\mu}_{(\lambda)}(\vec{p}) = 0$, for which the interpretation in unclear, so plane wave solution appears to not be a good ansatz in this case. Quantizing the theory does not then appear to be possible until a solution is found, though quantizing higher spin states famously leads to difficulty and inconsistencies [55][56].

4.3.2 Hamiltonian in Flat Spacetime

First a Hamiltonian (on shell) in flat spacetime is sought, starting from the corresponding Lagrangian $\mathcal{L}_F = -qF_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta} = -4^4 q\partial_{[0}A_{ijk]}\partial^{[0}A^{ijk]}$ (with $q > 0$, but otherwise arbitrary). Explicitly:

$$\mathcal{L}_F = -4q\left(\dot{A}_{ijk}\dot{A}^{ijk} + 3\partial_i A_{jk0}\partial^i A^{jk0} - 6\dot{A}_{ijk}\partial^i A^{jk0} - 6\partial_i A_{jk0}\partial^j A^{ik0}\right). \tag{127}$$

From this, the conjugate momenta are $\pi_{ijk} = -8q\left(\dot{A}_{ijk} - 3\partial_i A_{jk0}\right)$ and $\pi_{0ij} = 0$, so the second is a constraint $\dot{\pi}_{0ij} \approx 0$. The Lagrangian and Hamiltonian therefore take the form:

$$\begin{aligned} \mathcal{L}_F &= -\frac{1}{16q}\pi_{ijk}\pi^{ijk} + 24q\left(\partial_i A_{jk0}\partial^i A^{jk0} + \partial_i A_{jk0}\partial^j A^{ik0}\right) \\ \mathcal{H}_F &= -\frac{1}{16q}\pi_{ijk}\pi^{ijk} + 3\pi_{ijk}\partial^i A^{jk0} - 24q\left(\partial_i A_{jk0}\partial^i A^{jk0} + \partial_i A_{jk0}\partial^j A^{ik0}\right) \end{aligned} \tag{128}$$

and the Equation of motion is then $\dot{\pi}_{ijk} = 0$, with the constraint becoming:

$$\dot{\pi}_{jk0} = 3\partial^i\pi_{ijk} - 48q\left(\partial^i\partial_i A_{jk0} + \partial^i\partial_j A_{ik0}\right) = 0 \tag{129}$$

using integration by parts. Performing integration by parts again, the Hamiltonian can be re-expressed as:

$$\begin{aligned} \mathcal{H}_F &= -\frac{1}{16q}\pi_{ijk}\pi^{ijk} + \frac{3}{2}\pi_{ijk}\partial^i A^{jk0} + \frac{1}{2}\dot{\pi}_{jk0}A^{jk0} \\ &\rightarrow \mathcal{H}_F = -\frac{1}{16q}\pi_{ijk}\pi^{ijk} + \frac{3}{2}\pi_{ijk}\partial^i A^{jk0} \end{aligned} \tag{130}$$

where the constraint was applied in the second equation. The Lagrangian under this classical solution is then:

$$\mathcal{L}_F = -\frac{1}{16q}\pi_{ijk}\pi^{ijk} + \frac{3}{2}\pi_{ijk}\partial^i A^{jk0}. \tag{131}$$

Notice the useful result $\mathcal{L}_F = \mathcal{H}_F$, which is expected since the Lagrangian only has a kinetic term. Now the Euler-Lagrange equations give the result $F^{\mu\nu\alpha\beta} = c\epsilon^{\mu\nu\alpha\beta}$, as seen previously, where c is an integration constant. The Lagrangian on shell is thus $\mathcal{L} = 24qc^2$, so naturally:

$$\mathcal{H}_F = 24qc^2. \tag{132}$$

Additionally one can determine that $\dot{A}^{jk0} = 0$ and $\ddot{A}^{ijk} = 0$, though a general solution to these remains elusive.

4.3.3 FLRW Approximation

The generalisation of the preceding result to curved spacetime appears straightforward: $q \rightarrow q\sqrt{-g}$, since no derivatives or integrals were actually performed in this derivation. Therefore with an FLRW metric this is:

$$\mathcal{H}_F = 24N\sqrt{\gamma}a(t)^3 qc^2. \tag{133}$$

However, does this result hold when performing the Hamiltonian analysis on an action coupled to gravity - i.e. $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F$, where $\mathcal{L}_G$ is the gravity Lagrangian from the E-H action in FLRW spacetime (118)? As seen previously, all the EoM (and Legendre transform obviously) must be obtained (or performed) before any solutions are substituted. To see this, the full Lagrangian is first written out explicitly (with the gravitational sector written in terms of the expansion factor a only, and not b):

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_G + \mathcal{L}_F \\ \mathcal{L}_G &= 6\kappa\sqrt{\gamma}\left(-\frac{1}{N}(\dot{a})^2 a + Nka - \frac{N}{3}\Lambda a^3\right) \\ \mathcal{L}_F &= -4qN\sqrt{\gamma}a^3\left(\dot{A}_{ijk}\dot{A}^{ijk} + 3\partial_i A_{jk0}\partial^i A^{jk0} - 6\dot{A}_{ijk}\partial^i A^{jk0} - 6\partial_i A_{jk0}\partial^j A^{ik0}\right).\end{aligned} \tag{134}$$

Previously when deriving the WdW equation, the spacial 3-metric and its conjugate momentum(γ^{ij}, P_{ij}) were ignored, since they imposed the same constraint as the lapse function N in a matter free FLRW theory. Here, more care must be taken. The conjugate pairs that must be considered are $(N, p_N = 0), (\gamma^{ij}, P_{ij} = 0), (A^{0ij}, \pi_{0ij} = 0), (a, p_a), (A^{ijk}, \pi_{ijk})$, so that the first 3 are constraints, and $p_a = -\frac{12\kappa\sqrt{\gamma}}{N}\dot{a}a$ and $\pi_{ijk} = -8qN\sqrt{\gamma}a^3\left(\dot{A}_{ijk} - 3\partial_i A_{jk0}\right)$. The Legendre transform is exactly the same as the previous cases for the gravity and F sectors:

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_G + \mathcal{H}_F \\ \mathcal{H}_G &= -\frac{N}{24\kappa\sqrt{\gamma}}\frac{p_a^2}{a} - 6\kappa N\sqrt{\gamma}a\left[k - \frac{\Lambda}{3}a^2\right] \\ \mathcal{H}_F &= -\frac{1}{16qN\sqrt{\gamma}a^3}\pi_{ijk}\pi^{ijk} + 3\pi_{ijk}\partial^i A^{jk0} - 24qN\sqrt{\gamma}a^3\left(\partial_i A_{jk0}\partial^i A^{jk0} - 24\partial_i A_{jk0}\partial^j A^{ik0}\right).\end{aligned} \tag{135}$$

Now the $\dot{\pi}_{0ij} \approx 0$ constraint produces the result $\mathcal{H}_F = \mathcal{L}_F$ just as before, so that classically $\mathcal{H}_F = 24qN\sqrt{\gamma}a^3c^2$, with c an integration constant. Combining this result with $\dot{p}_N \approx 0$ yields $\mathcal{H}_G = \mathcal{H}_F$ (see Appendix C for details), meaning $\mathcal{H} \neq 0$. Therefore the Hamiltonian is a constraint, but is not in the correct form to be applied as the WdW equation (i.e. $\hat{H}\Psi = 0$). For this purpose, one should use:

$$\mathcal{H}' \equiv \mathcal{H}_G - \mathcal{H}_F \tag{136}$$

so that $\mathcal{H}' \approx 0$ as required. Meanwhile $\dot{P}_{ij} \approx 0$ and the EoM of $\dot{p}_a$ may be solved to recover the Friedmann equations, with a modified cosmological constant (see Appendix C). Now the following semi-classical Hamiltonian may thus be written:

$$H' = -\frac{2\pi GNV_c}{3}\frac{(p'_a)^2}{a} - \frac{3NV_c}{8\pi G}a\left[k - \frac{a^2}{3}\left(\Lambda - 192\pi Gqc^2\right)\right] \tag{137}$$

where $H' = \int d^3x\mathcal{H}'$, $p'_a = \frac{p_a}{\sqrt{\gamma}}$ and the usual substitutions were made. Quantizing with respect to the conjugate pair (a, p'_a) yields the WdW equation with an effective cosmological constant $\lambda_{eff}(c) = \Lambda - 192\pi qGc^2$ - exactly the one the Einstein equations for this action would produce, and the same as obtained in the path integral method above (whether in Euclidean or Lorentzian signature, and for a particular value of q). Using $V_c = \frac{24\pi^2}{\lambda_{eff}(c)^2}$ as before, this new Hamiltonian yields a potential term $U(a) \propto -\frac{1}{\lambda_{eff}(c)^2}a\left[k - \frac{a^2}{3}\lambda_{eff}(c)\right]$, according to the KG style WdW equation (97). Now, using the probability flux formula for the classically allowed region (98):

$$j_{\pm}^{(1)}(a, c) \propto \mp a^{p+1/2}\sqrt{\frac{1}{\lambda_{eff}(c)^2}\left[k - \frac{a^2}{3}\lambda_{eff}(c)\right]} \tag{138}$$

resulting in $|j_{\pm}^{(1)}(c)| \to \infty, \lambda_{eff}(c) \to 0^+$, agreeing with the results from the path integral method (at least for $k > 0$. For the negative case, one should re-perform the derivation of this probability flux).

5 Conclusions

The ADM formalism was shown to be a useful framework for many theories of Quantum Cosmology or even cosmology in general, such as Unimodular Gravity. In particular the introduction of

Ashtekar variables produced the only accurate quantum state of gravity: the Kodama state [26], describing pure gravity in a deSitter universe. Carrying on down this line, with the use of Wilson loops, one begins to enter the domain of Loop Quantum Gravity [57] [58], which may yet yield some interesting results. At the very least a full theory of quantum gravity will likely make use of some tool equivalent to the tetrad basis, in order to quantize spinors correctly [11]. Returning to the focus of this paper, the hope is that the details of such theories will provide a reason for the vanishing cosmological constant Λ, though as was seen this is a rather difficult endeavour. The global modification of the Sequester appears to achieve the desired result of making Λ drop out of the equations of motion without any obvious pathologies or leaps in logic [4]. It would be interesting to look at any possible motivation for this theory and to look at the details of quantum (or at least semi-classical) behaviour. Perhaps reasons for a vanishing but not quite zero Λ could be found, but this is likely to run into the same difficulties as when quantizing unmodified GR. One theory that I believe has been discarded too readily is Unimodular Gravity: Its na ive initial motivation falls apart quickly [44], but it exhibits some interesting links with Hawking's 3-form mechanism. Perhaps a different motivation for some modified Unimodular Gravity can be attempted. In fact even without this link, it would be enlightening to delve into the behaviour of this theory in the presence of a matter action, and perhaps from the view of first order formalism with a non-zero torsion. In what ways can an invariant volume form be implemented in a tetrad basis, and what would the ramifications be?

Hawking's 3-form mechanism [45] was explored more thoroughly, having gone through apparent disproving [47] to revival [50] [49]. The method does appear to be well founded in both boundary and no-boundary cases, and can easily be modified to avoid the subtleties involved in boundary terms, as was shown in this paper (not to mention the surprising and encouraging link with Unimodular Gravity). It does indeed appear that using a 3-form A in the action as an F^2 or σF term implies a sharp probability peak of finding a vanishing effective cosmological constant $\lambda_{eff} \to 0^+$. Modifications and generalisations such as σF^2 seem to maintain this result. However, the question of the interpretation of the Euclidean action must of course be faced before such a mechanism can go much further. Perhaps these concerns could be sidestepped through a canonical quantization and extraction of some probability density function from the solution, such as Vilenkin performed [34]; the initial results obtained in an FLRW universe appear promising, and a more rigorous treatment of this mechanism - perhaps in the full ADM formalism - would be of great interest. It is a work in progress however, so unless and until a full theory of quantum gravity appears, shortfalls in rigour and interpretational leaps are nigh unavoidable. Nevertheless, the mechanism itself is intriguing, and I believe further investigation is warranted as to why "the cosmological constant is probably zero".

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6 Apendices

6.1 Appendix A - Canonical Quantization

Here I present certain steps and interesting points that did not make it into the introduction of the paper. There will be some repetition and overlap with it, and of course this is still only an overview of Dirac's notes.

6.1.1 From the Lagrangian to the Extended Hamiltonian

Notice how the variation of the Hamiltonian only depends on the variation of p_n and q_n:

$$\begin{aligned}\delta H &= \delta p_n \dot{q}_n + p_n \delta \dot{q}_n - \left(\frac{\partial L}{\partial q_n}\right)\delta q_n - \left(\frac{\partial L}{\partial \dot{q}_n}\right)\delta \dot{q}_n \\ &= \delta p_n \dot{q}_n - \left(\frac{\partial L}{\partial q_n}\right)\delta q_n\end{aligned} \tag{139}$$

using the the E-L equations. So one should convert all the $\dot{q}$'s into p's and q's in the Hamiltonian before attempting to derive any EoM from it (i.e. convert everything to phase space representation). After adding the primary constraints ϕ_m to the Hamiltonian (see introduction), the EoM are given by:

$$\dot{g} = \{g, (H + u_m\phi_m)\} \tag{140}$$

where u_m are arbitrary functions. Now the constraints should only be imposed after the PB have been computed, as noted in the introduction. Now one must check the consistency conditions given by the constraints being maintained in time through their EoM $\dot{\phi}_m \approx 0$. This can either lead to no new information (i.e. reduce to something of the form $0 = 0$), a contradiction (such as $1 = 0$, in which case the Lagrangian is inconsistent), impose a condition on the u_m functions, or reduce to a new constraint equation, independent of the u_m. In the latter case repeat the process until no information can be obtained. These new constraints are referred to as "secondary constraints", but again this distinction is not particularly meaningful. A more important distinction is between "First class" objects, whose PB with all constraints vanish ($\{R, \phi_j\} \approx 0 \rightarrow R$ is first class), and "second class" objects, which do not. A first class object must then obey $\{R, \phi_j\} = r_{jj'}\phi_{j'}$, where $r_{jj'}$ are undefined coefficients, since constraints are the only independent functions of (q, p) that vanish, by definition. Note that the PB of 2 first class quantities is also first class, which will be an important consistency condition in the quantization. Now we recover the "Extended Hamiltonian" H_E from the introduction, composed of the non-constrained Hamiltonian (which is first class) and the first class constraints:

$$\begin{aligned}&H_E = H + v_a\phi_a \\ &\rightarrow \dot{g} \approx \{g, H_E\}.\end{aligned} \tag{141}$$

6.1.2 Symmetry Generators

If one takes the difference of 2 infinitesimally time-evolved quantities with different coefficients v_a and $v_{a'}$, this turns out to be:

$$\Delta g(\delta t) = \epsilon_a\{g, \phi_a\} \tag{142}$$

where $\epsilon_a = \delta t(v_a - v'_{a'})$ is infinitesimal. The constraints ϕ_a are therefore the generating functions of the infinitesimal transformation Δg! Additionally, the difference between a transformation generated by $v_a\phi_a$ and one by $\gamma_{a'}\phi_{a'}$ is $\Delta g = v_a\gamma_{a'}\{g, \{\phi_a, \phi_{a'}\}\}$, so that $\{\phi_a, \phi_{a'}\}$ is also a generating function (and first class, from above).

6.1.3 Second Class Constraints

First notice that 1^{st} and 2^{nd} class constraints can be replaced by any linear combination of themselves, so one must try to arrange them in such a way as to bring as many constraints as possible into the 1^{st} class. Any left over 2^{nd} class constraints will likely have the effect of bringing one or more degrees of freedom out of physical significance, where these should simply be dropped (for example if the constraints are $q_1 \approx 0$, $p_1 \approx 0$, then one should redefined the PB as only running over the coordinates $n = 2, ..., N$, and set $q_1 = p_1 = 0$. In effect writing the first dof out of the theory entirely). Now the way to do this in general is define a matrix of the PB's of all the surviving 2^{nd} class constraints $\chi_s\ ,\ s=1,...,S$:

$$\Delta = \begin{pmatrix} 0 & \{\chi_1,\chi_2\} & \cdots & \{\chi_1,\chi_S\} \\ \{\chi_2,\chi_1\} & 0 & \cdots & \{\chi_2,\chi_S\} \\ \vdots & \vdots & \ddots & \vdots \\ \{\chi_S,\chi_1\} & \{\chi_S,\chi_2\} & \cdots & 0 \end{pmatrix} \tag{143}$$

Dirac then proved that $det\Delta \neq 0$ (even weakly. and in fact this proof implies that the number of χ's that could not be brought into the first class must be even), see Dirac's notes for the full elegant proof [7]. This allows us to define the inverse matrix $c_{ss'} = (\Delta^{-1})_{ss'}$ such that $c_{ss'}\{\chi_{s'},\chi_{s''}\} = \delta_{ss''}$, to then define a new Poisson bracket:

$$\{f,g\}^* = \{f,g\} - \{f,\chi_s\}c_{ss'}\{\chi_{s'},g\} \tag{144}$$

which is still antisymmetric, bilinear and obeys the Jacobi identity. The EoM are still valid under this $\{g,H_E\}^* \approx \{g,H_E\} \to \dot{g} \approx \{g,H_E\}^*$, and a PB involving any second class constraint is always zero $\{g,\chi_s\}^* = 0$. One can therefore simply set all the χ's to zero strongly $\chi_s = 0$, and quantize using this new PB exactly as before $\{\cdot,\cdot\}^* \to [\cdot,\cdot]$.

6.2 Appendix B - Variations

Equations of motion (EoM) are derived many times in this paper, so to avoid clutter, and help any reader new to the topic, the details of the variation of many terms in the action is laid out here.

6.2.1 The Einstein-Hilbert action

First some useful identities are noted, where $g \equiv det(g_{\mu\nu})$:

$$\begin{aligned} \frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu} \quad \to \quad \frac{\delta\sqrt{|g|}}{\delta g^{\mu\nu}} &= -\frac{1}{2}\sqrt{|g|}g_{\mu\nu} \\ \frac{\delta}{\delta g^{\mu\nu}}\left(F_{\alpha\beta\rho\lambda}F^{\alpha\beta\rho\lambda}\right) &= 4F_{\mu\beta\rho\lambda}F_{\nu}^{\ \beta\rho\lambda} \\ \nabla_\mu F^{\mu\nu\rho\lambda} &= \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}F^{\mu\nu\rho\lambda}\right). \end{aligned} \tag{145}$$

Note that these results are independent of the signature of the metric, and $F_{\alpha\beta\rho\lambda}$ here is any fully antisymmetric rank-4 tensor (could be of any rank in the last equation). Further noting that the Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$, and letting $R_{\mu\nu} = R_{\mu\nu}[\Gamma]$, all that is needed to derive the EoM from the Einstein-Hilbert action (plus any extra terms added) is present. The EoM of Γ will be ignored in this paper, which only affects the Ricci tensor to produce the Gibbons-Hawking-York term, which is irrelevant to the discussion herein. Without further ado, the Lorentzian E-H action and its EoM are:

$$S = \kappa\int d^4x\sqrt{g}\,(R-2\Lambda) + S_M[g_{\mu\nu},\phi]$$

$$\begin{aligned} \frac{\delta S}{\delta g^{\mu\nu}} = -\kappa\frac{1}{2}\sqrt{g}g_{\mu\nu}\,(R-2\Lambda) + \kappa\sqrt{g}R_{\mu\nu} + \frac{\delta S_M}{\delta g^{\mu\nu}} &= 0 \\ \to R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= -g_{\mu\nu}\Lambda - \frac{1}{\kappa}\frac{\delta S_M}{\delta g^{\mu\nu}} \\ \to G_{\mu\nu} &= -g_{\mu\nu}\Lambda + T_{\mu\nu}. \end{aligned} \tag{146}$$

Now in the next example, the matter action $S_M[g_{\mu\nu},\phi]$ will contain only the F^2 term for clarity, since this is the case for most of the paper. Any extra terms in the action will end up being proportional to $g_{\mu\nu}$, and so will be absorbed into Λ in the EoM, to create the effective cosmological constant λ_{eff} as discussed. Therefore terms produced from the F^2 will not be lumped into the stress-energy tensor, so that $T_{\mu\nu}=0$.

$$S=\kappa\int d^4x\sqrt{g}\,(R-\Lambda)+\int d^4x\sqrt{g}F_{\mu\nu\rho\lambda}F^{\mu\nu\rho\lambda}$$

$$\begin{aligned}\frac{\delta S}{\delta g^{\mu\nu}}=-\kappa\frac{1}{2}\sqrt{g}g_{\mu\nu}\left(R-2\Lambda+\frac{1}{\kappa}F_{\alpha\beta\rho\lambda}F^{\alpha\beta\rho\lambda}\right)+\kappa\sqrt{g}\left(R_{\mu\nu}+4\frac{1}{\kappa}F_{\mu\beta\rho\lambda}F_{\nu}^{\ \beta\rho\lambda}\right)=0\\ \to G_{\mu\nu}=-g_{\mu\nu}\Lambda+\frac{1}{2\kappa}\left(g_{\mu\nu}F_{\alpha\beta\rho\lambda}F^{\alpha\beta\rho\lambda}-8F_{\mu\beta\rho\lambda}F_{\nu}^{\ \beta\rho\lambda}\right)\end{aligned}\tag{147}$$

$$\begin{aligned}\frac{\delta S}{\delta A_{\nu\rho\lambda}}\delta A_{\nu\rho\lambda}=2\sqrt{g}\partial_{[\mu}(\delta A_{\nu\rho\lambda]})F^{\mu\nu\rho\lambda}\\ \to\frac{\delta S}{\delta A_{\nu\rho\lambda}}=6\partial_\mu(\sqrt{g}F^{\mu\nu\rho\lambda})=0\\ \to\nabla_\mu F^{\mu\nu\rho\lambda}=0.\end{aligned}$$

Any further steps are explained in the relevant sections, making use of ($+/-$on a Riemannian/Lorentzian manifold):

$$\begin{aligned}\epsilon_{\alpha\beta\rho\lambda}\epsilon^{\alpha\beta\rho\lambda}=\pm4!\\ \epsilon_{\mu\beta\rho\lambda}\epsilon_{\nu}^{\ \beta\rho\lambda}=\pm3!g_{\mu\nu}.\end{aligned}\tag{148}$$

6.2.2 The Einstein-Cartan action

Here the derivation of the Einstein equations is more straightforward, one simply needs to vary the action with respect to the co-tetrads e^a, as follows:

$$\begin{aligned}g_{\mu\nu}=e^a_\mu e^b_\nu\eta_{ab}\\ e^a_\mu e^\mu_b=\delta^a_b\ ,\quad e^a_\nu e^\mu_a=\delta^\mu_\nu\\ e^a=e^a_\mu dx^\mu.\end{aligned}\tag{149}$$

The Einstein-Cartan (E-C) action and equations of motion are (recalling $R^a_{\ b}=d\Gamma^a_{\ b}+\Gamma^a_{\ c}\wedge\Gamma^c_{\ b}$, and so does not depend on the tetrad):

$$\begin{aligned}S=\frac{\kappa}{2}\int\epsilon_{abcd}e^ae^b\left(R^{cd}-\frac{1}{6}e^ce^d\Lambda\right)\\ \frac{\delta S}{\delta e^a}=2\epsilon_{abcd}e^b\left(R^{cd}-\frac{2}{6}e^ce^d\Lambda\right)=0\\ \to\epsilon_{abcd}e^b\left(R^{cd}-\frac{1}{3}e^ce^d\Lambda\right)=0.\end{aligned}\tag{150}$$

Now for terms containing F the situation is a little more subtle: $F=dA$ and so has no spacetime dependence $\delta F=0$. However $*F$ is a spacetime dual, and so does have such a dependence. It's variation is computed thus:

$$\begin{aligned}
\delta\int F\wedge *F &= \int F\wedge\delta(*F) = \frac{1}{4!}\int F\wedge\delta(\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta}) \\
&= \frac{1}{4!}\int F\epsilon_{abcd}F^{\mu\nu\alpha\beta}\delta(e^a_\mu e^b_\nu e^c_\alpha e^d_\beta) \\
&= \frac{1}{4!}F\epsilon_{abcd}F^{efgh}\delta(e^a_\mu e^b_\nu e^c_\alpha e^d_\beta)e^\mu_e e^\nu_f e^\alpha_g e^\beta_h \\
&= -\frac{1}{4!}\int F\epsilon_{abcd}F^{efgh}e^a_\mu e^b_\nu e^c_\alpha e^d_\beta\delta(e^\mu_e e^\nu_f e^\alpha_g e^\beta_h) \\
&= -\frac{1}{4!}\int FF^{efgh}\frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\delta(e_e e_f e_g e_h)^{\mu\nu\alpha\beta} \\
&= -\frac{1}{4!}\int F^{efgh}F * \delta(e_e e_f e_g e_h) \\
&= -\frac{1}{4!}\int F^{efgh}\delta(e_e e_f e_g e_h) * F \\
\rightarrow \frac{\delta}{\delta e^a}\left(\int F\wedge *F\right) &= -\frac{4}{4!}F_{abcd}e^b e^c e^d * F
\end{aligned} \tag{151}$$

where use was made of the fact that for an r-form $\omega = \frac{1}{r!}\omega_{\mu\nu\alpha\beta}dx^\mu dx^\nu dx^\alpha dx^\beta$, its Hodge dual is $*\omega = \frac{1}{r!}\omega_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\alpha\beta}$. Note that due to the added antisymmetrisation of the wedge product an additional 4! was added for the correct combinatorics $\delta(e^\mu_e e^\nu_f e^\alpha_g e^\beta_h) = \frac{1}{4!}\delta(e_e\wedge e_f\wedge e_g\wedge e_h)^{\mu\nu\alpha\beta}$. The result has an overall minus sign compared to if one naively varies F instead of $*F$, and is as yet independent of the signature of the metric.

A term in the action that is simply $\int F$ is a boundary term and drops out of the EoM. However if it contains a function $\sigma(\lambda(x))$, this is no longer the case and must contribute to the EoM. But when varying w.r.t e^a, $\delta F = 0$ and $\delta\sigma(\lambda(x)) = 0$ follow naturally, so how can this be? The trick is to note that one can define the 4-form $\sigma = \frac{1}{4!}\sigma(\lambda(x))\epsilon_{\mu\nu\alpha\beta}dx^\mu dx^\nu dx^\alpha dx^\beta$ such that $*\sigma = \sigma(\lambda(x))$, and so:

$$\int\sigma(\lambda(x))F = \int F\wedge *\sigma = \int\sigma\wedge *F. \tag{152}$$

The exact same derivation as (151) can be followed, but with $F\rightarrow\sigma$:

$$\frac{\delta}{\delta e^a}\left(\int\sigma(\lambda(x))F\right) = -\frac{4}{4!}\sigma(\lambda(x))F_{abcd}e^b e^c e^d. \tag{153}$$

Notice again that this result is independent of the signature of the metric.

6.3 Appendix C - 3-form Hamiltonian Analysis in FLRW Spacetime Notes

Here a few extra steps are provided, which were not covered explicitly in the main body of the paper. The constraint $\dot{\pi}_{jk0}\approx 0$ leads to:

$$\begin{aligned}
\dot{\pi}_{jk0} &= 3\partial^i\pi_{ijk} - 48qNa^3\partial^i\left[\sqrt{\gamma}\left(\partial_i A_{jk0} + \partial_j A_{ik0}\right)\right] \\
\rightarrow \mathcal{H}_F &= -\frac{1}{16qN\sqrt{\gamma}a^3}\pi_{ijk}\pi^{ijk} + \frac{3}{2}\pi_{ijk}\partial^i A^{jk0} - \frac{1}{2}\dot{\pi}_{jk0}A^{jk0} \\
\rightarrow \mathcal{L}_F &= -\frac{1}{16qN\sqrt{\gamma}a^3}\pi_{ijk}\pi^{ijk} + \frac{3}{2}\pi_{ijk}\partial^i A^{jk0} + \frac{1}{2}\dot{\pi}_{jk0}A^{jk0}
\end{aligned} \tag{154}$$

where integration by parts was used to obtain each result. Thus under the constraint this produces the stated result $\mathcal{H}_F = \mathcal{L}_F$. Next, computing $\dot{p}_N\approx 0$ yields $\mathcal{H}_G + \tilde{\mathcal{H}}_F = 0$ where:

$$\tilde{\mathcal{H}}_F = N\frac{\partial\mathcal{H}_F}{\partial N} = \frac{1}{16qN\sqrt{\gamma}a^3}\pi_{ijk}\pi^{ijk} - 24qN\sqrt{\gamma}a^3\left(\partial_i A_{jk0}\partial^i A^{jk0} - 24\partial_i A_{jk0}\partial^j A^{ik0}\right) \tag{155}$$

where one can recognise $\tilde{\mathcal{H}}_F = -\mathcal{L}_F$, and thus using the previous result $\mathcal{H}_F = \mathcal{H}_G$, so that the total Hamiltonian is not classically zero. Furthermore $\mathcal{H}_G = \mathcal{L}_F$ leads directly to the first massless

Friedmann equation (42):

$$\begin{aligned}\mathcal{H}_G &= -\frac{6\kappa\sqrt{\gamma}}{N}(\dot{a})^2 a - 6\kappa\sqrt{\gamma}a\left(k - \frac{\Lambda}{3}a^2\right) \\ \mathcal{L}_F &= 24qN\sqrt{\gamma}a^3c^2 \\ \rightarrow \left(\frac{\dot{a}}{a}\right)^2 + N^2\frac{k}{a^2} &= \frac{N^2}{3}\lambda_{eff}(c)\end{aligned} \tag{156}$$

where $\lambda_{eff}(c) = \Lambda - 192\pi Gqc^2$, as stated in the relevant section.

Now, for $\dot{P}_{ij} \approx 0$, note the following results:

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\gamma^{ij}} &= \frac{\partial\mathcal{H}_G}{\partial\gamma^{ij}} + \frac{\partial\mathcal{H}_F}{\partial\gamma^{ij}} = 0 \\ \frac{\partial\mathcal{H}_G}{\partial\gamma^{ij}} &= \frac{1}{2}\gamma_{ij}\mathcal{H}_G + \gamma_{ij}6\kappa N\sqrt{\gamma}a\left(k - \frac{\Lambda}{3}a^2\right) \\ \frac{\partial\mathcal{H}_F}{\partial\gamma^{ij}} &= -\frac{1}{2}\gamma_{ij}\tilde{\mathcal{H}}_F + \tilde{\mathcal{H}}_{ij}\end{aligned} \tag{157}$$

where $\tilde{\mathcal{H}}_{ij} \equiv \left.\frac{\partial\mathcal{H}_F}{\partial\gamma^{ij}}\right|_{\sqrt{\gamma}\text{ held constant}}$ is defined. This is explicitly:

$$\begin{aligned}\tilde{\mathcal{H}}_{ij} = &-\frac{1}{16qN\sqrt{\gamma}a^3}\pi_{ikl}\pi_j^{\ kl} + 3\pi_{ikl}\partial_j A^{kl0} - 24qN\sqrt{\gamma}a^3\left(\partial_i A_{kl0}\partial_j A^{kl0} - 24\partial_i A_{kl0}\partial^k A_j^{\ l0}\right) \\ &-\frac{1}{16qN\sqrt{\gamma}a^3}\pi_{kil}\pi^k_{\ j}{}^l + 3\pi_{kil}\partial^k A_j^{\ l0} - 24qN\sqrt{\gamma}a^3\left(\partial_k A_{il0}\partial^k A_j^{\ l0} - 24\partial_k A_{il0}\partial_j A^{kl0}\right) \\ &-\frac{1}{16qN\sqrt{\gamma}a^3}\pi_{kli}\pi^{kl}{}_j + 3\pi_{kli}\partial^k A^l{}_j{}^0 - 24qN\sqrt{\gamma}a^3\left(\partial_k A_{li0}\partial^k A^l{}_j{}^0 - 24\partial_k A_{li0}\partial^l A^k{}_j{}^0\right).\end{aligned} \tag{158}$$

Notice that $\gamma^{ij}\tilde{\mathcal{H}}_{ij} = 3\mathcal{H}_F$. So taking the trace of (157), and using the result $\tilde{\mathcal{H}}_F = -\mathcal{H}_F$:

$$\mathcal{H}_G + 3\mathcal{H}_F + 12\kappa N\sqrt{\gamma}a\left(k - \frac{\Lambda}{3}a^2\right) = 0. \tag{159}$$

The EoM of $\dot{p}_a$ on the other hand is (using $\tilde{\mathcal{H}}_F = -\mathcal{H}_F$ again):

$$\dot{p}_a = \frac{1}{a}\left(\mathcal{H}_G - 3\mathcal{H}_F\right) + 12\kappa N\sqrt{\gamma}\left(k - \Lambda a^2\right). \tag{160}$$

Using these results, equation (156) and $\mathcal{H} = 2\mathcal{H}_G = 2\mathcal{H}_F$, the second massless Friedmann should also emerge:

$$\frac{\ddot{a}}{a} = \frac{N^2}{3}\lambda_{eff}(c). \tag{161}$$

However, I have not been able to show this explicitly before finishing this paper.