Imperial College London

Department of Physics

Asymptotic Safety in Gravity and its Obstacles

Author:
Liheng Yao

Supervisor:
Kellogg S. Stelle

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Abstract

Einstein's theory of gravity is known to be perturbatively unrenormalisable in the covariant formulation. However, it was proposed that such a theory may be governed by a non-Gaussian fixed point in theory space in the UV limit, rendering it asymptotically safe. In this dissertation, the perturbative unrenormalisability of Einstein gravity is illustrated using the simple example of the one-loop scalar corrections to the graviton propagator. Then, the Wilsonian renormalisation group is briefly introduced, and some methods used in the asymptotic safety programme are elaborated. Recent progress in the field is described, and major open problems are identified, with a focus on issues concerning the violation of unitarity and causality, and the effectiveness of running couplings in finite truncations of theory space.

Acknowledgments

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1 Introduction

Einstein's theory of general relativity is our best theory of gravitation. It describes gravitational interactions as emergent from objects moving along geodesics in a curved spacetime manifold, whose curvature is induced by the presence of mass and energy. However, advances in theory and experiment since the beginning of the twentieth century seem to indicate that physical processes are quantum in nature. All known matter and their interactions, with the exception of gravity, are well described by the standard model of particle physics, which is a quantum field theory. As a result, the right-hand side of Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

is a fundamentally quantum object. Thus, it is most natural to attempt to rewrite the left-hand side that describes the curvature of spacetime into a quantum theory as well. In addition, Einstein's theory notably produces singularities at the centre of a black hole and at the Big Bang, which is considered a nuisance by many.

Unfortunately, the reconciliation of quantum mechanics with Einstein's theory of general relativity is one of the long-standing open problems in physics. When one attempts to quantise gravity in the same manner as one does the other interactions, difficulties arise in the form of perturbative unrenormalisability. Absorption of divergences in loop expansions of Einstein gravity requires the introduction of counterterms with higher momentum dependencies, which is equivalent to adding higher-derivative terms to the action. As more loops are calculated, more higher-derivative terms are required, and each such term contains a finite part in its coupling constant that needs to be determined through experiment. As a result, general relativity, when quantised in the usual way, requires an infinite number of experiments to fix the coupling constants of its counterterms, and has no predictive power. Thus, general relativity is widely regarded to be a low-energy (infrared, IR) effective theory valid up to the Planck scale, at which it will have to be ”completed” by a fundamental high-energy (ultraviolet, UV) quantum theory of gravity. A candidate of such a theory is string theory.

However, in 1976, Weinberg proposed [7] a generalised renormalisability condition based on Wilson’s renormalisation group (RG) formulation [8]. He claimed that a theory can be considered UV complete if it lies on a finite-dimensional UV critical surface of a nontrivial fixed point for the RG flow, and the perturbative unrenormalisability of the theory only means that it does not lie in the UV critical surface of the Gaussian fixed point at the origin, and does not completely rule out its UV completeness. If a theory satisfies these criteria, it can be considered asymptotically safe, in analogy to the asymptotically freedom of QCD which lies in the UV critical surface of the fixed point at the origin.

For the renormalisation group procedures to be well defined, the theory must be formulated with all possible interaction terms allowed by its symmetries considered. It would be unrealistic to consider the infinite number of allowed couplings, so in practice finite truncations are used to probe the RG behaviour of gravitational theories, in the hope that as the number of terms added to the truncations increases, the obtained fixed point and its UV critical dimension converge to some finite values. A large number of such truncations have been investigated [9] using various methods, and fixed points in agreement with Weinberg’s criteria have been identified in all of them. For instance, the asymptotic freedom of one of the couplings in quadratic gravity has been exploited so that a UV fixed point in the quadratic truncation is located using perturbation theory [10]. Moreover, non-perturbative methods such as the functional renormalisation group equation have been used to investigate the RG flow of various other truncations [9]. However, the convergence of the locations, and more importantly, the dimensions of the UV critical surfaces of the fixed points remain an open problem.

There are also open foundational problems that form obstacles in the formulation of asymptotically safe gravity. For example, any truncation longer than the Einstein-Hilbert action would inevitably contain higher derivative terms, which are well known to give rise to unphysical propagating ghost states. There are various proposed resolutions of this problem, but none of them appear to be entirely convincing. The effectiveness and universality of running coupling constants in finite truncations is also uncertain.

Hence, the asymptotic safety scenario in gravity is still an active ongoing area of research, with significant progress in the past few decades, but also with many open problems to answer. This dissertation aims to provide an elementary introduction to the subject, including motivations, key methods used, recent progress and open problems.

2 Covariant quantisation of gravity

Through a covariant treatment of gravity, calculating the one-loop scalar correction to a graviton propagator, the pathology that arises when quantising Einstein gravity, namely the necessity of introducing higher-derivative counterterms at each loop level, becomes apparent.

2.1 The graviton propagator

Classically, Einstein gravity is described by the usual Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R \tag{2}$$

where $\kappa = \sqrt{8\pi G}$, G is the usual gravitational constant, and the cosmological constant is discarded for the time being. Einstein gravity enjoys an invariance under diffeomorphisms of the form

$$g_{\mu\nu} \to g'_{\mu\nu} \equiv g_{\mu\nu} + g_{\alpha\nu}\partial_\mu \xi^\alpha + g_{\mu\alpha}\partial_\nu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu} \tag{3}$$

where ξ^α are the components of an infinitesimal vector field. This invariance stems from the fact that there is no preferred coordinate system in general relativity, and can be seen as a gauge invariance. Thus, it could be shown that at each point in spacetime, there is a local neighbourhood where the geometry can be described by the Minkowski metric $\eta_{\mu\nu}$, which justifies the decomposition of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

where $h_{\mu\nu}$ is the dynamical fluctuation of the metric that is to be quantised. However, it was noted by Fadeev and Popov [11] that if $h_{\mu\nu}$ is not sufficiently small, its quantum effects would change the signature of the metric tensor $g_{\mu\nu}$. Here, this complication is ignored, and $h_{\mu\nu}$ is assumed to be small. The diffeomorphism invariance induces a similar invariance of $h_{\mu\nu}$ under the gauge variation

$$h_{\mu\nu} \to h'_{\mu\nu} \equiv h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{5}$$

In order to derive the Feynman rules, the theory is quantised in the usual path integral formulation, taking into account the gauge invariance mentioned above. Thus, in addition to the classical Einstein-Hilbert action, gauge-fixing and the related Fadeev-Popov ghost terms need to be added. The resultant path integral will be of the form

$$Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}C^\mu \mathcal{D}\bar{C}_\mu \exp[S_{\rm EH}(h) + S_{\rm gf}(h) + S_{\rm gh}(h, C, \bar{C})] Z_{\rm matter} \tag{6}$$

where the fluctuation metric $h_{\mu\nu}$ is integrated over, but the raising and lowering of the spacetime indices are performed using the background Minkowski metric $\eta_{\mu\nu}$, and C and $\bar{C}$ are the usual Fadeev-Popov ghosts.

First, the Lagrangian is expanded in powers of $h_{\mu\nu}$. For the derivation of the propagator, expansion to the quadratic order is sufficient. Following the procedure of Veltman [6], the background metric is first assumed to be a general metric that is not necessarily $\eta_{\mu\nu}$. The full metric is relabelled $\bar{g}_{\mu\nu}$, and the background is labelled $g_{\mu\nu}$, such that the decomposition is of the form:

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{7}$$

Then, up to quadratic order in $h_{\mu\nu}$, the classical part of the Lagrangian can be decomposed as

$$\mathcal{L}_{\rm cl} = \sqrt{-\bar{g}}[R + \underline{R} + \underline{\underline{R}}] \tag{8}$$

where R is constructed using the background metric, and $\underline{R}$ is the part of the Ricci scalar linear in $h_{\mu\nu}$, while $\underline{\underline{R}}$ is the part of the Ricci scalar quadratic in $h_{\mu\nu}$. Following the expansions carried out in Appendix A, and using the shorthand $h = h^\mu_\mu$, one arrives at

$$\mathcal{L}_{\text{cl}} = -\frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{4}(\partial_\mu h)^2 - \frac{1}{2}\partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2}\partial^\alpha h_{\alpha\nu}\partial_\mu h^{\mu\nu} \tag{9}$$

As general relativity has gauge invariance in the form of (5), a gauge fixing term $\mathcal{L}_{\text{gf}} = -\frac{1}{2}G_\mu^2$ needs to be added to the Lagrangian to prevent redundant functional integration. The gauge that produces the simplest form for the propagator is the de Donder gauge

$$G_\mu = \partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h \tag{10}$$

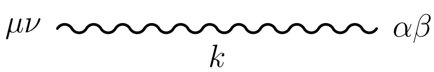
which simplifies the part of the Lagrangian relevant to the graviton propagator to

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{gf}} \\ &= -\frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{8}(\partial_\mu h)^2 \\ &= -\frac{1}{2}\partial_\lambda h_{\alpha\beta} V^{\alpha\beta\mu\nu}\partial^\lambda h_{\mu\nu} \end{aligned} \tag{11}$$

where

$$V^{\alpha\beta\mu\nu} = \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu} - \frac{1}{4}\eta^{\alpha\beta}\eta^{\mu\nu} \tag{12}$$

Taking its inverse, one arrives at the graviton propagator



$$D_{\mu\nu\alpha\beta}(k) = \frac{\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\beta\mu}\eta_{\alpha\nu} - \eta_{\mu\nu}\eta_{\alpha\beta}}{k^2 + i\epsilon} \tag{13}$$

It is helpful at this point to examine the dimensionality of all terms involved. The Lagrangian above is to the second order of the derivative of the metric. The metric itself has to be dimensionless, in natural units the act of differentiation gives a mass dimension of 1, the integration measure is of dimension -4, and the action needs to be overall dimensionless, so κ has to be of dimension -1. In order to rewrite the kinetic term (11) into one that does not need a coupling constant, the coupling constant can be absorbed into a redefinition $h_{\mu\nu} \to \kappa h_{\mu\nu}$, so that $h_{\mu\nu}$ now has mass dimension 1. Then, it becomes apparent that for the action to be dimensionless, any operator with mass dimension larger than 4, such as ones to the order $(\partial h)^2 h$, must have a coupling with negative mass dimension. Unfortunately, the full expansion of the Ricci scalar contains an abundance of such terms. Hence, Einstein gravity is superficially power-counting unrenormalisable.

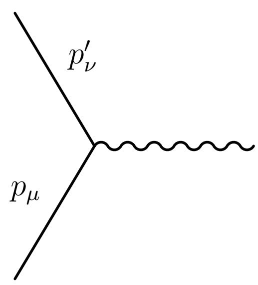
2.2 One-loop scalar corrections to the graviton propagator

The superficial unrenormalisablity can be confirmed by calculating the one-loop scalar correction to the graviton propagator. Matter is coupled to gravity through its stress-energy tensor $T_{\mu\nu}$. Using the redefined dimensionful metric, the interaction term in the Lagrangian is of the form $\frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}$. For a massive scalar, the stress-energy tensor is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2) \tag{14}$$

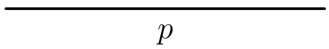
It is immediately apparent this interaction is power-counting unrenormalisable. One proceeds to show this by calculating the one-loop scalar correction to the graviton propagator.

The scalar-scalar-graviton vertex can be read off the interaction term to be



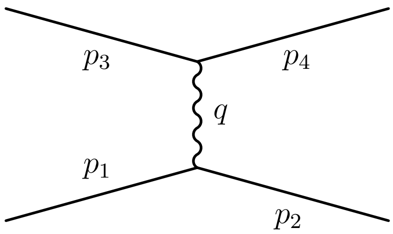
$$V_{\mu\nu} = \frac{i\kappa}{2} [(p_\mu p'_\nu + p'_\mu p_\nu) - \eta_{\mu\nu} (p \cdot p' - m^2)] \tag{15}$$

With the usual scalar propagator



$$D(p) = \frac{1}{p^2 - m^2 + i\epsilon} \tag{16}$$

As a sanity check, the tree-level scalar-scalar scattering amplitude is



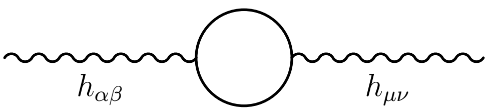
$$\begin{aligned} i\mathcal{M} =& \frac{i\kappa}{2}[(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu}(p_1 \cdot p_2 - m^2)] \\ & \times i\frac{\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\beta\mu}\eta_{\alpha\nu} - \eta_{\mu\nu}\eta_{\alpha\beta}}{q^2} \\ & \times \frac{i\kappa}{2}[(p_3^\alpha p_4^\beta + p_4^\alpha p_3^\beta - \eta^{\alpha\beta}(p_3 \cdot p_4 - m^2)] \end{aligned} \tag{17}$$

Taking the non-relativistic limit $p_\mu \to (m, \mathbf{0})$, this amplitude can be read off to be

$$\mathcal{M} = -\kappa^2 \frac{m^4}{q^2} \tag{18}$$

which can be "Fourier transformed" to obtain the usual gravitational potential energy for two equal masses $V(r) = -Gm^2/r$.

Then, one proceeds to calculate the one-loop scalar correction to the graviton propagator



$$-i\Pi^{(1)}_{\alpha\beta\mu\nu}(q) = \int \frac{d^4k}{(2\pi)^4}\frac{i\kappa}{2}\left[k_\alpha(k+q)_\beta + k_\beta(k+q)_\alpha\right]\frac{i}{k^2}\frac{i}{(k+q)^2} \times \frac{i\kappa}{2}\left[k_\mu(k+q)_\nu + k_\nu(k+q)_\mu\right] \tag{19}$$

Following the procedures of Appendix B, it can be shown that there is a divergent term of the form

$$\sim \frac{\kappa^2}{(4\pi)^2}(q_\alpha q_\beta q_\mu q_\nu)\int_0^1 dx\; x^2(1-x)^2\left(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi)\right) \tag{20}$$

The q^4 dependence cannot be absorbed into any existing operator, as the Ricci scalar is up to the second derivative of the metric, and the absorption of the above divergence requires a fourth-derivative counterterm. As a result, higher-order counterterms not present in the original action must be introduced to cancel out divergences in gravitational interactions.

2.3 One-loop finiteness of pure gravity

Using gravity coupled to scalar fields and deducting the contribution due to the scalars, 't Hooft and Veltman [12] showed explicitly that pure gravity in the absence of matter and vacuum energy is finite at one-loop order. Here, one can skip the explicit calculations by postulating that the one-loop corrections to pure gravitational processes also carry divergences that are of the order p^4 (which is indeed the case, as shown by 't Hooft and Veltman). Then, the fourth-derivative counterterms allowed by the diffeomorphism invariance of the theory are proportional to R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. However, the counterterm involving $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ can be absorbed into the other two terms by observing that the Euler characteristic

$$\chi = \frac{1}{8\pi^2}\int d^4x\sqrt{-g}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) \tag{21}$$

is a topological invariant.

Einstein's field equations in the absence of matter and vacuum energy are uniquely solved by $R = R_{\mu\nu} = 0$. Hence, for the quantum theory to reproduce the classical theory on-shell, the entire counter-Lagrangian has to vanish, which implies that at one-loop order, Einstein gravity can be written as a finite quantum theory.

This convenient cancellation of the counter-Lagrangian gave hope that similar cancellations might also occur at higher loop orders. Unfortunately, Goroff and Sagnotti [13] showed conclusively that such coincidences do not happen at two-loop order for pure gravity. In addition, as soon as matter is introduced, the on-shell conditions no longer stipulate that $R = R_{\mu\nu} = 0$, and one-loop gravity is once again unrenormalisable. In general, absorption of divergences at all loop orders requires the introduction of an infinite number of higher-derivative operators into the counter-Lagrangian. Each of these counterterms contains a finite part that has to be experimentally determined, removing any predictive power from naively quantised general relativity.

3 The asymptotic safety programme

The perturbative unrenormalisability of Einstein gravity does not conclusively rule out UV completeness of the covariant formulation of quantum gravity. Perturbative expansions using Feynman diagrams probe only the neighbourhood where the coupling constants are small, and give little to no information about the couplings far from the free theory. In fact, the action of renormalisation was considered a mathematical trick of "sweeping divergences under the rug" until Wilson [8] introduced the concept of the renormalisation group, which gave the procedure physical meaning.

3.1 The Wilsonian renormalisation group

The Wilsonian renormalisation group (RG) is based on the idea that descriptions of the physical world is dependent on the scales at which one probes it. Consider a physical system described at some momentum scale k by some function Z of the complete set of operators $\{\psi_i\}$ allowed by the symmetries of the physical system described, and their coupling constants $\{g_i\}$. Then, an RG action of coarse-graining and rescaling of the theory to a new scale k' will shift the couplings by $\{g_i\} \to \{\tilde{g}_i\}$. Thus, the couplings can be written as functions of the scale $g_i(k)$, where the flow of the couplings due to shifts in the scale is given by the β function

$$\beta_i[g_i(k)] = k\partial_k g_i(k) \tag{22}$$

To illustrate this more concretely, consider a general scalar field theory in d dimensional Euclidian space, with the action

$$S[\phi] = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \cdots \right) \tag{23}$$

The theory is taken to be effective up to some momentum scale k_0, which is infinite for a “fundamental” theory. The momentum cutoff can be made manifest by regularising the theory so that the path integral is only integrated up to the cutoff, or equivalently by stipulating that the fields are finitely supported in momentum space, so that

$$\phi(\boldsymbol{p}) = \begin{cases} 0 & |\boldsymbol{p}| > k_0 \\ \phi(\boldsymbol{p}) & |\boldsymbol{p}| < k_0 \end{cases} \tag{24}$$

The partition function is written as

$$Z_{k_0} = \int \mathcal{D}\phi \; \mathrm{e}^{-S_{k_0}[\phi]} \tag{25}$$

Now, if one wants to extract from this theory an effective theory valid up to a lower momentum scale k, one needs to integrate out modes with momenta $k < |\boldsymbol{p}| < k_0$. The procedure is as follows.

The field can be split into two components, the high energy (UV) modes that will be integrated out

$$\phi^+(\boldsymbol{p}) = \begin{cases} 0 & |\boldsymbol{p}| > k_0 \\ \phi(\boldsymbol{p}) & k < |\boldsymbol{p}| < k_0 \\ 0 & |\boldsymbol{p}| < k \end{cases} \tag{26}$$

and the low energy (IR) modes that will be rescaled to be the degrees of freedom for the IR effective theory

$$\phi^-(\boldsymbol{p}) = \begin{cases} 0 & |\boldsymbol{p}| > k \\ \phi(\boldsymbol{p}) & |\boldsymbol{p}| < k \end{cases} \tag{27}$$

The partition function then becomes

$$Z_{k_0} = \int \mathcal{D}\phi^- \mathcal{D}\phi^+ \; e^{-S_{k_0}\left[\phi^+ + \phi^-\right]} = \int \mathcal{D}\phi^- \; \mathrm{e}^{-S_k\left[\phi^-\right]} \tag{28}$$

where one wishes to absorb the effect of the UV modes into an effective action that depends only on the IR modes.

$$\mathrm{e}^{-S_k[\phi^-]} = \int \mathcal{D}\phi^+ \; \mathrm{e}^{-S_{k_0}[\phi^+ + \phi^-]} \tag{29}$$

This action can in general be evaluated by functionally integrating over the high momentum modes, treating the low momentum modes as external sources. The action (23) is already the most general action that can be written down, so the effective action can only be of the same form, but with different couplings.

$$S_k[\phi^-] = \int d^d x \left(\frac{1}{2} Z_\phi (\nabla \phi^-)^2 + \frac{1}{2} m^2 (\phi^-)^2 + \frac{\lambda}{4!} (\phi^-)^4 + \cdots \right) \tag{30}$$

Thus, at a lower energy scale, one sees a different theory with different values for the couplings, which have absorbed the effects of higher-energy modes beyond the cutoff of the theory. However, the original theory contains an implicit dependence on the cutoff scale k_0 different from the cutoff k of the effective theory, which means that the momenta in the IR theory are in fact to scale with the cutoff in the UV theory. What one really wants is a theory defined up to the same scale as the UV theory, with merely a smaller scope of effectiveness. In such a theory, taking the momentum to k should be equivalent to taking the momentum to k_0 in the UV theory. To achieve this, the positions and momenta in the IR theory must be rescaled as

$$\boldsymbol{x} \to \boldsymbol{x'} = \frac{k}{k_0} \boldsymbol{x}, \quad \boldsymbol{p} \to \boldsymbol{p'} = \frac{k_0}{k} \boldsymbol{p} \tag{31}$$

This induces a rescaling of the integration measure and the gradient operator

$$d^d x = \left(\frac{k_0}{k} \right)^d d^d x', \quad \nabla = \frac{k}{k_0} \nabla' \tag{32}$$

In addition, the overall scaling of the field has no physical significance. Thus, the scale of the field must be manually fixed. It is customary to choose the coupling of the kinetic term to be $1/2$, which is achieved through a field rescaling

$$\phi'(\boldsymbol{x'}) = (\frac{k_0}{k})^{\frac{d-2}{2}} \sqrt{Z_\phi} \phi^-(\boldsymbol{x}) \tag{33}$$

The effective action is now

$$S_k[\phi'] = \int d^d x' \left(\frac{1}{2}(\nabla'\phi')^2 + \frac{1}{2}m^2(k)(\phi')^2 + \frac{\lambda(k)}{4!}(\phi')^4 + \cdots \right) \tag{34}$$

This form is exactly analogous with the original action, but now with coupling constants running with the energy scale. The above procedure is the renormalisation group, which in general consists of three steps:

1. "Coarse grain" the theory by integrating out modes with momenta above an imposed cutoff.

2. Rescale spacetime and momenta so that the action is now to scale with the new cutoff.

3. Rescale the fields so that the effective and UV actions are of forms that are directly comparable.

In practice, these procedures are carried out in a continuous flow, integrating out an infinitesimal momentum shell at each time. It should be noted that despite the name, the renormalisation group procedure is not a group action, as the coarse graining at each step causes one to lose information about the physics at momenta higher than the cutoff, making the procedure irreversible.

As shown above, the RG procedure induces a flow in the typically infinite dimensional space of coupling constants, with each theory corresponding to a point in that space, which is hence also known as theory space. In analogy to dynamical systems, such flows also yield fixed points where the RG procedure does not change the couplings of the theory. In other words, the β function vanishes at these points, and these theories are known as conformal field theories, with manifest scale invariance. The infinitesimal RG flow around these points give useful information about the behaviour of theories in their neighbourhoods. Around a fixed point g_* with couplings

$\{g_{i*}\}$, the β functions can be linearly expanded in the couplings to be

$$\beta_i[g_i(k)] = k\partial_k g_i(k) = \sum_j B_{ij}(g_j(k) - g_{j*}) \tag{35}$$

where the stability matrix B_{ij} encodes the behaviour of the RG flow in the vicinity of the fixed point. This matrix can be diagonalised to yield the eigenvalues and eigenvectors

$$\sum_j B_{ij} V_j^J = -\theta^J V_i^J \tag{36}$$

With the complete set of eigenvectors $\{V^J\}$, the RG equation (35) can be solved as

$$g_i(k) = g_{i*} + \sum_J C^J V_i^J \left(\frac{k_0}{k}\right)^{\theta^J} \tag{37}$$

where $\{C^J\}$ are the integration constants, and k_0 is some initial energy scale. In general, B_{ij} will not be diagonal, and the eigenvalues will be complex. However, for any physically relevant solution, the C^Js will be such that they cancel out contributions from the imaginary part of the θ^Js. Then, the stability of the fixed point in response to RG action can be read off the sign of $\mathrm{Re}(\theta^J)$.

1) $\mathrm{Re}(\theta^J) > 0$: The fixed point is repulsive to small RG perturbations in the direction of V^J. An RG trajectory emanates from the fixed point in this direction. As the RG action is always from the UV to the IR, the flow in this direction is relevant to the observed physics at low energy. Thus, this direction is a relevant direction.

2) $\mathrm{Re}(\theta^J) < 0$: The fixed point is attractive to small RG perturbations in the direction of V^J. An RG trajectory flows into the fixed point in this direction, making that trajectory irrelevant to low-energy physics. Thus, this direction is an irrelevant direction.

3) $\mathrm{Re}(\theta^J) = 0$: More information is needed to determine the RG flow in the direction of V^J. This direction is a marginal direction.

For low-energy physics, the relevant directions are of interest. The RG flow emanating from these directions around a fixed point span a manifold known as the unstable manifold or the UV critical surface, whose dimensionality is given by the number of orthogonal relevant directions around that fixed point.

For any theory space, there is always a trivial fixed point located at the origin known as the Gaussian fixed point (GFP). At this point, all couplings vanish, and $\mathrm{Re}(\theta^J)$ in the directions of the couplings are exactly equal to the mass dimensions of the couplings. This can be shown using the example of the scalar field (34) given above.

Consider some RG action on the theory where the effect of the coarse-graining is negligibly small, and the rescaling dominates. Then, Z_ϕ can be taken to be unity in (33) so that the scalar field is rescaled by a factor of $(\frac{k_0}{k})^{\frac{d-2}{2}}$. For a general interaction term at some power n of the field, the rescaling is realised as

$$\begin{aligned} d^d x \; g_n(k_0) \; \phi^n &= \left[(\frac{k_0}{k})^d d^d x' \right] g_n(k_0) \left[(\frac{k}{k_0})^{n\frac{d-2}{2}} (\phi')^n \right] \\ &= d^d x' \; g_n(k) \; (\phi')^n \end{aligned} \tag{38}$$

where

$$g_n(k) = \left(\frac{k_0}{k} \right)^{d+n\frac{2-d}{2}} g_n(k_0) \tag{39}$$

Instead of finding the β functions, which is trivial at this point, one observes that in the vicinity of the GFP where couplings vanish, a small perturbation of some coupling g_n can only grow under an RG action when the power $d + n(2-d)/2$ is positive. If this power is negative, a small "turn-on" of the coupling would be re-attracted to the GFP, rendering the coupling irrelevant. This power is exactly equal to the mass dimension of the coupling.

This gives physical meaning to the statement from perturbation theory that theories with couplings with negative mass dimensions are unrenormalisable: they do not lie in the UV critical surface of the GFP. The direct opposite of this would be an asymptotically free theory, which lies in the UV critical surface of the GFP, so that at higher energy scales the theory tends towards the free theory. As perturbative expansions assume weak coupling, they in fact form the linearisation of theory space around the GFP. A coupling with negative mass dimension lies in the irrelevant direction of the GFP, but this leaves open the possibility that the UV behaviour of the theory is instead governed by a non-trivial non-Gaussian fixed point (NGFP) away from the origin.

3.2 Weinberg's criterion

Thus, one sees that the perturbative unrenormalisability does not completely rule out possibilities of UV completion of gravity in the covariant formulation. This motivated Weinberg [7] to propose a conjecture that there exists a physical infinite-cufoff limit of quantum gravity if:

1) The theory space contains a non-trivial NGFP with a finite-dimensional UV critical surface.

2) Every RG trajectory that does not emanate from this fixed point develop unphysical behaviour such as diverging couplings at the UV limit.

If both of these two criteria are confirmed, then gravity can be considered asymptotically safe, in analogy to the asymptotic freedom of theories such as QCD. The finiteness of the dimensionality of the UV critical surface distinguishes asymptotically safe gravity from an effective field theory, as the latter requires an increasing number of couplings that need to be experimentally fixed as the energy scale increases, making the theory increasingly ineffecient at higher energies, and removing any predictive power at the UV limit. On the contrary, asymptotically safe gravity can be considered fundamental, as even in the limit when the energy scale diverges, the theory still only contains a finite number of relevant constants that can be determined experimentally.

3.3 Gravity in $2 + \epsilon$ dimensions

The first example hinting at asympotically safe gravity, mentioned in Weinberg's original article, is gravity in $2 + \epsilon$ dimensions. In exactly 2 dimensions, Newton's constant becomes dimensionless, so that Einstein gravity is power-counting perturbatively renormalisable. However, in 2 dimensions, the action also happens to be proportional to the topologically invariant Euler characteristic, making the entire theory trivial. Thus, to study gravity in 2 dimensions, one needs to first perform the necessary calculations in an expansion of $2+\epsilon$ dimensions, then take the limit $\epsilon \to 0$.

In the ϵ expansion, Newton's constant gains a mass dimension $-\epsilon$, so instead one considers the dimensionless parameter

$$g_0(k) = k^\epsilon G_0 \tag{40}$$

to be the bare coupling constant. Then, the finite part of this bare coupling is labelled $g(k)$. In analogy to dimensional regularisation, this coupling admits poles as $\epsilon \to 0$. Thus, the bare coupling can be written in a Laurent expansion

$$g_0(k) = g(k) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu} b_\nu[g(k)] \tag{41}$$

where the constants b_ν can only depend on k through $g(k)$ as k is an artificially introduced scale, so there are no dimensionful parameters that k can be compared to. Then the operation $k\partial_k$ can be performed on both sides of the equation to yield

$$\epsilon g + b_1(g) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu} b_{\nu+1}(g) = \beta(g) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu} \frac{\partial b_\nu}{\partial g} \beta(g) \tag{42}$$

which can be rearranged into

$$\beta(g) = \left(\epsilon g + b_1(g) + \sum_{\nu=1}^{\infty} \epsilon^{-\nu} b_{\nu+1}(g) \right) \left(1 + \sum_{\nu=1}^{\infty} \epsilon^{-\nu} \frac{\partial b_\nu}{\partial g} \right)^{-1} \tag{43}$$

Taking a binomial expansion and retaining only the terms analytic as $\epsilon \to 0$

$$\beta(g) = \epsilon g + b_1(g) - g\,\frac{\partial b_1}{\partial g} \tag{44}$$

For small g, it is expected that $b_1 = bg^2 + \mathcal{O}(g^3)$, so that

$$\beta(g) = \epsilon g - bg^2 \tag{45}$$

Literature differ on the exact value of b, but all agree that b is positive [1]. Thus, the RG flow contains a NGFP at

$$g_* = \frac{\epsilon}{b} \tag{46}$$

This result is clearly obtained in the limit $\epsilon \ll 1$, and the validity of its extension to $\epsilon = 2$ is highly questionable. Nonetheless, it provides a hint that a NGFP may exist in the theory space of four-dimensional gravity.

3.4 Perturbative treatment of quadratic gravity

As shown in the calculations in Appendix B, the problematic divergence in gravitational interactions at one loop order is quartic in the external momenta, which requires fourth-derivative counterterms to absorb. Thus, a candidate of a perturbatively renormalisable theory of gravity would be one containing higher derivative terms. It was shown by Stelle [14] that a fourth-derivative theory with the action quadratic in the Ricci scalar

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \beta R^2 + \alpha R_{\mu\nu}R^{\mu\nu}\right) \tag{47}$$

is indeed perturbatively renormalisable to all loop orders, with significant caveats that will be discussed later. Using the topological invariance of the Gauss-Bonnett term E which is the integrand of the Euler characteristic given in (21), and the fact that the square of the Weyl tensor is given by

$$C^2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = E + 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 \tag{48}$$

the term proportional to the square of the Ricci tensor can be absorbed into a term proportional to the square of the Weyl tensor. Hence, the action of quadratic gravity can be rewritten (with the cosmological constant included) as

$$S = \int d^4x\sqrt{-g}\left(\tilde{\Lambda} - \frac{1}{\kappa^2}R + \frac{1}{2s}C^2 - \frac{\omega}{3s}R^2\right) \tag{49}$$

It was proven [15][16] that the coupling s of the C^2 term is asymptotically free. Recall that in the quantisation of EH gravity, before evaluating the Feynman diagrams, the metric was rescaled by $h_{\mu\nu} \to \kappa h_{\mu\nu}$ so that the kinetic term has the usual coupling of $1/2$. Then, the gravitational vertices would carry a factor of κ. The same treatment, extended to quadratic gravity, would mean that any vertex contained in the action carries positive powers of s, so that perturbative expansions are effectively expansions in powers of s. Thus, asymptotic freedom of this coupling implies that perturbation theory well describes the behaviour of the theory in the UV limit, and if a NGFP exists in that limit, it should be visible through perturbation theory. This argument is exactly analogous to the argument for the validity of the description of high-energy strong interactions using perturbative QCD due to the asymptotic freedom of the coupling in that theory.

This motivated Niedermaier [17][10] to search for such a fixed point using perturbation theory. As the dimensionful nature of the couplings dictate that there is power law running of the couplings, the conventional method of dimensional regularisation and minimal subtraction, which sees only logarithmic divergences, is no longer effective. Hence, a new nonminimal subtraction ansatz is introduced to take care of the power law divergences, so that at one loop order, the bare couplings now become

$$\tilde{\Lambda}_0 = \mu^4 \frac{2\lambda}{g_N} \left\{ 1 + \frac{\hbar}{(4\pi)^2} \left[a_{10} + a_{11} \ln\left(\frac{\Lambda_{\rm UV}}{\mu}\right) + a_{12} \left(\frac{\Lambda_{\rm UV}}{\mu}\right)^2 + a_{13} \left(\frac{\Lambda_{\rm UV}}{\mu}\right)^4 \right] \right\} \quad (50)$$

$$\kappa_0^2 = \mu^{-2} g_N \left\{ 1 + \frac{\hbar}{(4\pi)^2} \left[b_{10} + b_{11} \ln\left(\frac{\Lambda_{\rm UV}}{\mu}\right) + b_{12} \left(\frac{\Lambda_{\rm UV}}{\mu}\right)^2 \right] \right\} \quad (51)$$

where μ is the renormalisation scale, g_N and λ are the dimensionless renormalised gravitational and cosmological "constants", and $\Lambda_{\rm UV}$ is the UV cutoff. For the dimensionless couplings s and ω, conventional minimal subtraction suffice, so that

$$s_0 = s \left\{ 1 + \frac{\hbar}{(4\pi)^2} \left[c_{10} + c_{11} \ln\left(\frac{\Lambda_{\rm UV}}{\mu}\right) \right] \right\} \quad (52)$$

$$\omega_0 = \omega \left\{ 1 + \frac{\hbar}{(4\pi)^2} \left[d_{10} + d_{11} \ln\left(\frac{\Lambda_{\rm UV}}{\mu}\right) \right] \right\} \quad (53)$$

The metric is again decomposed into $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the total metric in the original action, $g_{\mu\nu}$ is the static background metric, with which raising and lowering operations are performed, and $h_{\mu\nu}$ is the dynamical quantum metric, which is functionally integrated over in the path integral. The field renormalisation is then

$$\bar{g}^0_{\mu\nu} = \bar{g}_{\mu\nu} \left[1 + \frac{\hbar}{(4\pi)^2} \ln\left(\frac{\Lambda_{\rm UV}}{\mu}\right) \xi g_N \right] \quad (54)$$

where ξ can in general be a function of s/g_N, λ or ω.

In addition, when the theory is renormalised at the cutoff scale, the renormalised couplings should coincide with the bare ones. This yields the condition

$$\begin{aligned} \kappa_0^2 &= \Lambda_{\rm UV}^{-2} g_N(\mu = \Lambda_{\rm UV}) \\ \tilde{\Lambda}_0 &= \Lambda_{\rm UV}^4 \left(\frac{2\lambda}{g_N}\right)(\mu = \Lambda_{\rm UV}) \end{aligned} \quad (55)$$

that is only satisfied when

$$a_{10} + a_{12} + a_{13} = 0, \quad b_{10} + b_{12} = 0 \quad (56)$$

For pure gravity, the general form of the one-loop effective action containing the logarithmic and power law divergences at one loop level can be written down as

$$\Gamma^{(1)}_{\text{div}} = -\frac{1}{(4\pi)^2}\int d^4x\sqrt{-g}\,\Big[\Lambda^4_{\text{UV}}\Upsilon_1 + \Lambda^2_{\text{UV}}\left(\Upsilon_2 R + \mu^2\Upsilon_3\right) + \ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)\left(\zeta_1 C^2 + \zeta_2 R^2 + \mu^2\zeta_4 R + \mu^4\zeta_5\right)\Big] \tag{57}$$

where g is the determinant of the static background metric, and the Υs and ζs are loop-counting parameters which are real valued functions of s/g_N, λ or ω.

Substituting the bare couplings and field (50)-(54) into the action (49) and expanding to first order in $\hbar$, the divergent part of the one-loop effective action can be calculated to be

$$\begin{aligned}\Gamma^{(1)}_{\text{div}} = \int d^4x\sqrt{-g}\frac{\hbar}{(4\pi)^2}\Bigg\{&\frac{g_N\xi}{2}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)\left(\frac{2\lambda\mu^4}{g_N} - \frac{\mu^2}{g_N}R + \frac{1}{2s}C^2 - \frac{\omega}{3s}R^2\right)\\ &+ \frac{2\lambda\mu^4}{g_N}\left[a_{11}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right) + a_{12}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)^2 + a_{13}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)^4\right]\\ &+ \frac{\mu^2}{g_N}\left[b_{11}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right) + b_{12}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)^2\right]R\\ &+ \frac{1}{2s}c_{11}\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)C^2\\ &+ \frac{\omega}{3s}(d_{11}-c_{11})\ln\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)R^2\Bigg\}\end{aligned} \tag{58}$$

By comparing the forms of (57) and (58) and using the relations (56), the relations between all the Latin coefficients (except c_{10} and d_{10} which do not enter the β functions) and the Greek coefficients can be determined. Operating on both sides of the relations (50)-(53) with the operator $\mu\partial_\mu$ and expanding to first order in the loop-counting parameter $\hbar$ while rewriting the Latin coefficients as the Greek ones, the β

functions can then be calculated to be

$$\begin{aligned}
\mu\partial_\mu s &= -\frac{\hbar}{(4\pi)^2}2\zeta_1 s^2 \\
\mu\partial_\mu \omega &= -\frac{\hbar}{(4\pi)^2}s\left(3\zeta_2 + 2\omega\zeta_1\right) \\
\mu\partial_\mu \mathrm{g}_N &= 2\mathrm{g}_N + \frac{\hbar}{(4\pi)^2}\mathrm{g}_N^2\left[\zeta_4 + \xi + 2\Upsilon_2\right] \\
\mu\partial_\mu \lambda &= -2\lambda + \frac{\hbar}{(4\pi)^2}\frac{\mathrm{g}_N}{2}\left[\zeta_5 + 4\lambda\zeta_4 + \Upsilon_3 + 4\lambda\Upsilon_2 + 4\Upsilon_1 - (2\lambda\xi + 2\lambda\zeta_4 - \Upsilon_3)\right]
\end{aligned} \tag{59}$$

The determination of the RG flow then depends on the evaluation of the one-loop effective action to extract the Greek coefficients. First, a gauge-fixing term needs to be added to the action. Niedermaier used a three-parameter minimal harmonic gauge

$$\begin{aligned}
S_{\text{gf}} &= \frac{1}{2s}\int d^4x\sqrt{-g}\chi_\mu Y^{\mu\nu}\chi_\nu \\
\chi_\mu &= \nabla^\nu h_{\mu\nu} + b_1\nabla_\mu h \\
Y^{\mu\nu} &= -g^{\mu\nu}\nabla^2 - (b_2 - 1)\nabla^\mu\nabla^\nu + R^{\mu\nu} \\
b_1 &= -\frac{1}{4}\frac{1+4\omega}{1+\omega}, \quad b_2 = \frac{2}{3}(1+\omega)
\end{aligned} \tag{60}$$

The ghost action in this gauge is

$$\begin{aligned}
S_{\text{gh}} &= \int d^4x\sqrt{-g}\bar{C}_\mu\Delta^\mu_\nu C^\nu \\
\Delta^{\mu\nu} &= -\left[g^{\mu\nu}\nabla^2 + (1+2b_1)\nabla^\mu\nabla_\nu + R^{\mu\nu}\right]
\end{aligned} \tag{61}$$

Then, the total action will be of the form

$$S_{\text{tot}} = S + S_{\text{gf}} + S_{\text{gh}} \tag{62}$$

Formally the one-loop effective action is of the form

$$\Gamma^{(1)} = \frac{1}{2}\operatorname{Tr}\ln\mathcal{H} - \frac{1}{2}\operatorname{Tr}\ln Y - \operatorname{Tr}\ln\Delta \tag{63}$$

where $\mathcal{H}$ is the Hessian of the operator $2s(S + S_{\text{gf}})$. The factor of $2s$ arises from a rescaling of $h_{\mu\nu} \to \sqrt{2s}h_{\mu\nu}$ when evaluating the functional integral, which is in complete analogy with similar operations in Einstein gravity and Yang-Mills theory. Explicitly,

$$\frac{\delta^2}{\delta \bar{h}_{\mu\nu}(x)\delta \bar{h}_{\alpha\beta}(y)}\left[2s\left(S+S_{\rm gf}\right)\right]=\mathcal{H}\,\delta_{\mu\alpha}\delta_{\nu\beta}\delta^{(d)}(x-y) \tag{64}$$

where $\bar{h}_{\mu\nu}$ is the vacuum expectation value of the quantum metric $h_{\mu\nu}$. As conventional dimensional regularisation fails to detect power law divergences, a background covariant operator regularisation scheme [18] is employed in combination with heat kernel methods introduced in Appendix C. For a self-adjoint operator $\mathbf{A}$ with order $2r$ on a d dimensional Riemannian manifold, the prescription is to replace $\ln \mathbf{A}$ with $F_{k^r,\Lambda^r_{\rm UV}}(A)$ acting as an integral operator obtained by averaging the heat kernel

$$F_{k^r,\Lambda^r_{\rm UV}}(A)(x,y)=\int_0^\infty d\tau\,\tilde{F}_{k^r,\Lambda^r_{\rm UV}}(\tau)A(\tau;x,y) \tag{65}$$

where $A(\tau;x,y)$ is the heat kernel of the operator $\mathbf{A}$. The function $\tilde{F}_{k^r,\Lambda^r_{\rm UV}}$ is the inverse Laplace transform of $F_{k^r,\Lambda^r_{\rm UV}}$, which depends parametrically on an IR cutoff k and the UV cutoff $\Lambda_{\rm UV}$ through

$$F_{k^r,\Lambda^r_{\rm UV}}(z)=f\left(\frac{z}{k^2}\right)-f\left(\frac{z}{\Lambda^2_{\rm UV}}\right) \tag{66}$$

for a suitable f so that for $k=\Lambda_{\rm UV}$ all regularised Gaussian integrals are reduced to unity. Here, one uses

$$f(y)=-\ln\left(1+\frac{1}{y}\right)+\frac{1}{1+y}+\frac{1}{2(1+y)^2} \tag{67}$$

Moreover, for a flat background the heat kernel can be decomposed in momentum space to

$$A(\tau;x,y)=\int\frac{d^dp}{(2\pi)^d}e^{ip(x-y)}e^{-\tau A(p)} \tag{68}$$

where there is a spectral decomposition

$$A(p)=\sum_j\lambda_j(p)\Pi_j\left(\frac{p}{\sqrt{p^2}}\right) \tag{69}$$

for a set of orthogonal projectors Π_j. This gives

$$\mathrm{Tr}\, F_{k^r,\Lambda_{\mathrm{UV}}^r}(A) = \sum_j m_j \int \frac{d^d p}{(2\pi)^d} F_{k^r,\Lambda_{\mathrm{UV}}^r}[\lambda_j(p)] \tag{70}$$

where $m_j = \mathrm{tr}\, \Pi_j$ is the multiplicity of λ_j.

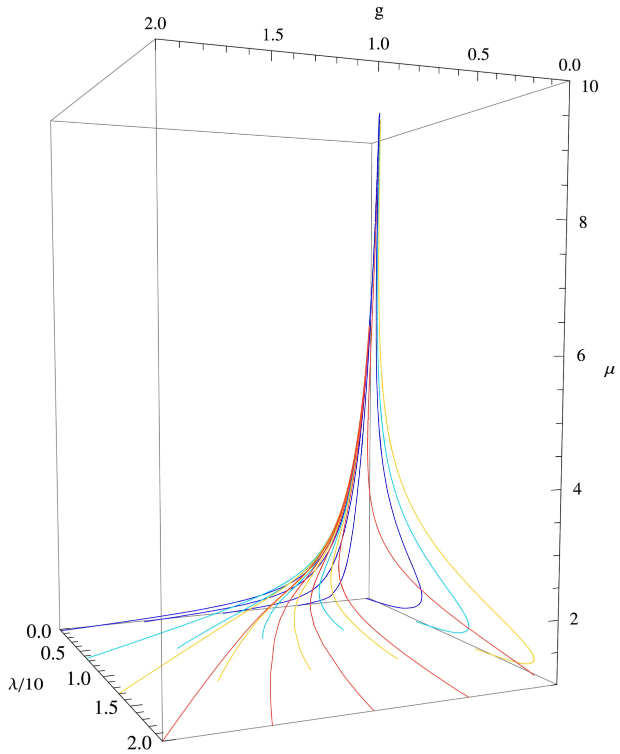


Figure 1: The RG flow of g_N and λ in quadratic gravity [10].

The UV divergences correspond to the behaviour of the heat kernel at small τ which corresponds to the early time expansion in four dimensions

$$\langle x|e^{-\tau\mathbf{A}}|x\rangle \sim \frac{1}{(4\pi)^2}\frac{\Gamma(\frac{2}{r})}{r}\sum_{n=0}^{\infty}\tau^{\frac{n-2}{r}}E_{2n}(x) \tag{71}$$

Then the divergent part of $\mathrm{Tr}\ln\mathbf{A}$ can be regularised as

$$\begin{aligned}\mathrm{Tr}\, F_{k^r,\Lambda_{\mathrm{UV}}^r}(A) = \frac{1}{(4\pi)^2}\frac{\Gamma(\frac{2}{r})}{r}\int d^d x\sqrt{-g}\,[2rE_4(x)\ln\Lambda_{\mathrm{UV}} + q_{1/r}E_1(x)\Lambda_{\mathrm{UV}}^2 \\ + q_{2/r}E_0(x)\Lambda_{\mathrm{UV}}^4 + \mathcal{O}(1)\,]\end{aligned} \tag{72}$$

where $q_n = (2n)^{-1}\Gamma(3-n)$ is related to the choice of $f(y)$, the derivation of which is given by [10].

What remains is the calculation of the heat kernel coefficients of the operators $\mathcal{H}$, Y and Δ. The coefficients of the latter two are well tabulated, but the heat kernel of $\mathcal{H}$ is highly non-trivial in curved backgrounds, so that Niedermaier [10] evaluated it on a flat background. The resultant numerical results of the non-trivial fixed points of the couplings are

$$g_N^* \approx 1.3697, \quad \lambda^* \approx 0.9451, \quad \omega^* \approx -0.0228 \tag{73}$$

where the value of ω^* agrees with previous results [16][15] and the rough position of the fixed point is confirmed by later nonperturbative treatments [19]. The overall RG flow is illustrated in Figure 1, showing that the fixed point is indeed repulsive to RG action, in line with the requirements of asymptotic safety.

3.5 The functional renormalisation group equation

Despite the effectiveness of perturbation theory in quadratic gravity, to gain a more complete view of the RG trajectory in general truncations of theory space, non-perturbative tools have to be used. The most commonly used such tool is the functional renormalisation group equation (FRGE) first introduced by Wetterich [20] and adapted for gravitational scenarios by Reuter [21].

The starting point is the generating functional of the connected Green's functions. As usual, the metric consists of a fixed background and a fluctuating part $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, so that the generating functional is

$$\begin{aligned}\exp\left(W_k\left[t^{\mu\nu},\sigma^{\mu},\bar{\sigma}_{\mu},\beta^{\mu\nu},\tau_{\mu}\right]\right)\\ =\int\mathcal{D}h_{\mu\nu}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu}\exp\big(S(h)+S_{\text{gf}}(h)+S_{\text{gh}}(h,C,\bar{C})\\ +\Delta S_k(h,C,\bar{C})+S_{\text{source}}\big)\end{aligned} \tag{74}$$

where S, S_{gf} and S_{gh} are the usual classical, gauge fixing and ghost actions. The source term is given by

$$S_{\text{source}}=\int d^dx\sqrt{-g}\left(t^{\mu\nu}h_{\mu\nu}+\bar{\sigma}_{\mu}C^{\mu}+\sigma^{\mu}\bar{C}_{\mu}+\beta^{\mu\nu}\mathcal{L}_C\bar{g}_{\mu\nu}+\tau_{\mu}C^{\nu}\partial_{\nu}C^{\mu}\right) \tag{75}$$

where the last two terms are required by BRST symmetry, which is not of importance in the scope of this dissertation. The crucial term relevant to the nonperturbative evaluation of the RG flow is the mode suppression term ΔS_k, which is bilinear in the metric and ghost. The term is introduced to suppress the modes with momenta lower than k, and is in general of the form

$$\Delta S_k=\frac{\kappa^2}{2}\int\mathrm{d}^dx\sqrt{-g}h_{\mu\nu}\mathcal{R}_k^{\text{grav}}(g)^{\mu\nu\rho\sigma}h_{\rho\sigma}+\sqrt{2}\int\mathrm{d}^dx\sqrt{-g}\bar{C}_{\mu}\mathcal{R}_k^{\text{gh}}(g)C^{\mu} \tag{76}$$

where the operators have the general structure $\mathcal{R}_k(g)=\mathcal{Z}_k k^2R^{(0)}(-\nabla^2/k^2)$. The dimensionless shape function $R^{(0)}(-\nabla^2/k^2)$ is chosen to interpolate smoothly between $R^{(0)}(0)=1$ and $R^{(0)}(\infty)=0$. Examples of such functions are shown in Figure 2.

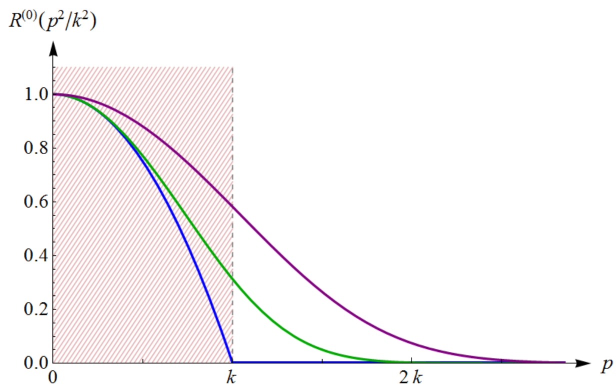


Figure 2: Typical shapes of the mode suppressing function $R^{(0)}$ [22].

Suppressing the terms with low momenta in the functional integral prevent them from being integrated out. Thus, lowering the IR cutoff scale k is equivalent to integrating out UV modes starting from a lower momentum scale, which is in the spirit of an RG action. The term $\mathcal{Z}_k$ is different for the ghost and graviton, and is in general a number for the ghost, and a tensor constructed from the background metric for the graviton.

The expectation values of the graviton and ghost fields are then given by

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta W_k}{\delta t^{\mu\nu}}, \quad \xi^\mu = \frac{1}{\sqrt{-g}}\frac{\delta W_k}{\delta \bar{\sigma}_\mu}, \quad \bar{\xi}_\mu = \frac{1}{\sqrt{-g}}\frac{\delta W_k}{\delta \sigma^\mu} \tag{77}$$

Then, one can take the Legendre transform to obtain the effective action

$$\widetilde{\Gamma}_k[\bar{h},\xi,\bar{\xi},\beta,\tau] = \int d^dx\sqrt{-g}(t^{\mu\nu}\bar{h}_{\mu\nu} + \bar{\sigma}_\mu\xi^\mu + \sigma^\mu\bar{\xi}_\mu) - W_k\left[t,\sigma,\bar{\sigma},\beta,\tau\right] \tag{78}$$

which as usual gives rise to the source-field relations

$$\frac{\delta\widetilde{\Gamma}_k}{\delta\bar{h}_{\mu\nu}} = \sqrt{-g}t^{\mu\nu}, \quad \frac{\delta\widetilde{\Gamma}_k}{\delta\bar{\xi}_\mu} = -\sqrt{-g}\sigma^\mu, \quad \frac{\delta\widetilde{\Gamma}_k}{\delta\xi^\mu} = -\sqrt{-g}\bar{\sigma}_\mu \tag{79}$$

The effective average action (EAA) can hence be defined as

$$\Gamma_k[\bar{h},\xi,\bar{\xi},\beta,\tau] = \widetilde{\Gamma}_k[\bar{h},\xi,\bar{\xi},\beta,\tau] - \Delta S_k[\bar{h},\xi,\bar{\xi}] \tag{80}$$

where the mode suppression term is now inserted with the classical fields. For convenience, the quantum fields are labelled $\varphi = (h, C, \bar{C})$, the classical fields, with implicit k dependence, are labelled $\bar{\varphi}_k = (\bar{h},\xi,\bar{\xi})$, and the sources are labelled $J = (t,\sigma,\bar{\sigma})$. One then observes that at fixed values of $\bar{\varphi}_k$

$$\begin{aligned} k\partial_k\widetilde{\Gamma}_k &= -k\partial_k W_k = k\partial_k\langle\Delta S\rangle \\ &= \frac{1}{2}\int d^dx\int d^dy\ \langle\varphi(x)\varphi(y)\rangle\ k\partial_k\mathcal{R}_k(x,y) \end{aligned} \tag{81}$$

where $\mathcal{R}_k(x,y) = \mathcal{R}_k(-\nabla^2/k^2)\delta^{(d)}(x-y)$, and the angular brackets denote expectation values. The expectation value of the product of the fields at two points is related to the definition of the two-point correlation function as

$$\langle\varphi(x)\varphi(y)\rangle = G(x,y) + \bar{\varphi}_k(x)\bar{\varphi}_k(y) \tag{82}$$

Substituting this into (81), one has

$$k\partial_k\widetilde{\Gamma}_k = \frac{1}{2}\,\mathrm{Tr}[Gk\partial_k\mathcal{R}_k] + k\partial_k\Delta S_k[\bar{\varphi}_k] \tag{83}$$

However, the two-point correlation function is also given by the inverse of the Hessian of the effective action with respect to the classical fields

$$\begin{aligned} G^{-1}(x,y) = \widetilde{\Gamma}_k^{(2)}(x,y) &= \frac{\delta^2\widetilde{\Gamma}_k}{\delta\bar{\varphi}_k(x)\delta\bar{\varphi}_k(y)} \\ &= \Gamma_k^{(2)}(x,y) + \mathcal{R}_k(x,y) \end{aligned} \tag{84}$$

Hence, one arrives at the functional renormalisation group equation (FRGE)

$$\begin{aligned} k\partial_k\Gamma_k[\bar{\varphi}_k] &= \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right] \\ &= \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{\bar{h}\bar{h}} (k\partial_k\mathcal{R}_k)_{\bar{h}\bar{h}}\right] \\ &\quad - \frac{1}{2}\,\mathrm{Tr}\left[\left\{\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{\bar{\xi}\xi} - \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}_{\xi\bar{\xi}}\right\} (k\partial_k\mathcal{R}_k)_{\bar{\xi}\xi}\right] \end{aligned} \tag{85}$$

The cancellation of the $k\partial_k\Delta S_k$ term is one of the main motivations for the definition of the EAA.

With the FRGE, one can then attempt to probe the entire RG flow by choosing the background field and cutoff function. Ideally, one should investigate the flow in the infinite-dimensional theory space

$$S = \int d^4x\sqrt{-g}\left\{\Lambda + \frac{1}{2\kappa^2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \dots\right\} \tag{86}$$

but this is clearly unrealistic, so in practice one would always take some sort of finite truncation. In general, early-time heat kernel expansions or numerical methods such as Monte Carlo simulations are employed to probe the fixed points within these subspaces. Most notably, the Einstein-Hilbert truncation was found to yield a repulsive NGFP at $g_N^* \approx 0.403$, $\lambda^* \approx 0.330$ [23]. The RG flow in the two-dimensional theory subspace is plotted in Figure 3.

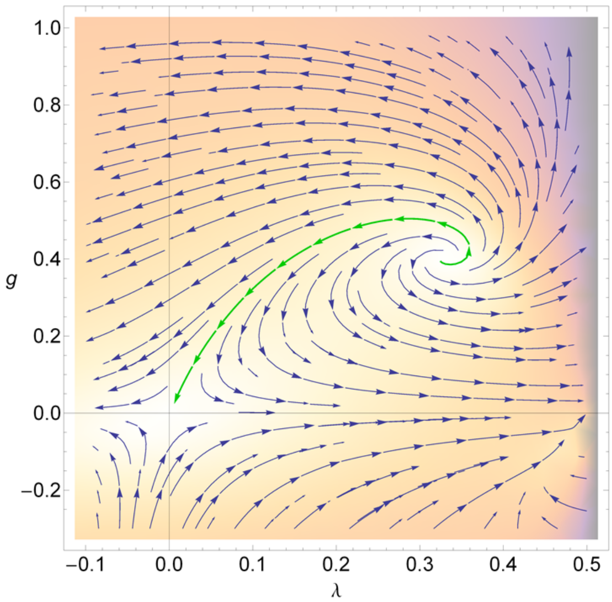


Figure 3: The RG flow of the Einstein-Hilbert truncation [23].

A large assortment of other truncations have being investigated with various backgrounds and cutoffs. A recent list of publications containing such results can be found in [9], which includes the Einstein- Hilbert action, quadratic gravity, gravity with the two-loop counterterm, actions containing up to 71 powers of the Ricci scalar, actions containing a single trace of up to 35 Ricci tensors, and actions composed with polynomial functions of $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. In addition, systems coupled with matter were also investigated extensively [24]. Remarkably, fixed points obeying the requirements of asymptotic safety have being found in all cases [9], strongly hinting at the existence of such a fixed point in the full theory space.

4 Obstacles to asymptotically safe gravity

Despite the positive prospects of the existence of an asymptotically safe theory of gravity, various complications remain to be addressed. Most notably, the exact location of the fixed point is dependent on the choice of the background metric, the cutoff function and the truncation used. In particular, there is an implicit assumption that including higher and higher order terms in the truncation would eventually cause the calculated fixed point to converge towards the one in the infinite-dimensional theory space, and current results appear to be encouraging [9], but unlike for the GFP, where the eigenvectors of the stability matrix point exactly in the directions of the axes in theory space, for a NGFP such eigenvectors may in general point in directions that are linear combinations of an infinite number of couplings, making it near impossible to determine if any chosen truncation contain sufficient information about the RG flow in vicinity of the NGFP in full theory space. In addition to these technical difficulties, there exist a number of foundational questions that pose serious challenges to the asymptotic safety programme.

4.1 Unitarity, causality and analyticity

For any truncation of theory space containing more terms than the Einstein-Hilbert one, there inevitably exist higher-derivative terms in the classical action. The inclusion of higher-derivative terms in the Lagrangian is already problematic on the classical level due to Ostrogradsky's instability theorem [25], which states that any non-degenerate Lagrangian dependent on time derivatives higher than one corresponds to a linearly unstable Hamiltonian that contains at least one term linear in some conjugate momentum, so that it is unbounded from below. To see this, consider a general Lagrangian $L(x, \dot{x}, \cdots, x^{(N)})$, where the Euler-Lagrange equations are

$$\sum_{i=0}^{N} \left(-\frac{d}{dt}\right)^{i} \frac{\partial L}{\partial x^{(i)}} = 0 \tag{87}$$

Ostrogrdsky chose to span the phase space with the $2N$ canonical variables

$$X_i = x^{(i-1)}, \qquad P_i = \sum_{j=i}^{N} \left(-\frac{d}{dt}\right)^{j-i} \frac{\partial L}{\partial x^{(j)}} \tag{88}$$

Non-degeneracy requires that one can always solve for the $x^{(i)}$s in terms of the P_is and the X_is, so that there exists a function $\mathcal{A}$ such that

$$\left.\frac{\partial L}{\partial q^{(N)}}\right|_{x^{(i-1)}=X_i,\, x^{(N)}=\mathcal{A}} = P_N \tag{89}$$

For a general N, the Hamiltonian is

$$\begin{aligned} H &= \sum_{i=1}^{N} P_i x^{(i)} - L \\ &= P_1 X_2 + P_2 X_3 + \cdots + P_{N-1} X_N + P_N \mathcal{A} - L\left(X_1, \ldots, X_N, \mathcal{A}\right) \end{aligned} \tag{90}$$

Clearly, except for the special case $N = 1$, the Hamiltonian has linear dependencies on at least one conjugate momentum, and can thus be taken to arbitrarily low values by choosing arbitrarily low values for the conjugate momenta. As the Hamiltonian corresponds to energy for natural systems, such a theory clearly cannot describe real-world physics.

In quantum theories, this problem manifests itself in the existence of propagating states that violate unitarity. The existence of such ghost states in higher derivative theories of gravity was first shown by Stelle [14], who proved that a theory with the action (47) is perturbatively renormalisable to all loop orders at the expense of propagating spin-2 ghost states. Stelle showed rigorously that the graviton propagator in such a theory has momentum dependency $\propto p^{-4}$, which is the reason for its perturbative renormalisability as it is able to absorb the divergences in the numerator. Consider a general p^{-4} propagator, which can be decomposed in a partial fraction

$$D(p) = \frac{1}{p^2 - p^4/M^2} = \frac{1}{p^2} - \frac{1}{p^2 - M^2} \tag{91}$$

where M^2 can be interpreted as the mass of the ghost state, and can in general be negative, which implies tachyons that further complicate the situation. Clearly, if the same usual Feynman prescription $+i\epsilon$ is used for both terms in the partial fraction, the second term will yield a negative residue due to its negative sign. Recall that a propagator can be written in the Källén-Lehmann representation

$$D(p) = \frac{1}{p^2 - m^2 + i\epsilon} + \int_{4m^2}^{\infty} d\mu^2 \frac{\rho(\mu^2)}{p^2 - \mu^2 + i\epsilon} \tag{92}$$

where m is the mass of the free theory, and the density function $\rho(\mu^2)$ is a sum of some norm states $| \langle 0|\phi|n\rangle |^2$. A negative residue of the propagator means that $\rho(\mu^2)$ is negative, violating unitarity, and implies that some physical process has negative probability.

One might salvage unitarity by using a different Feynman prescription $-i\epsilon$ for the negative-sign propagator, so that it now has the pole structure shown in Figure 4b.

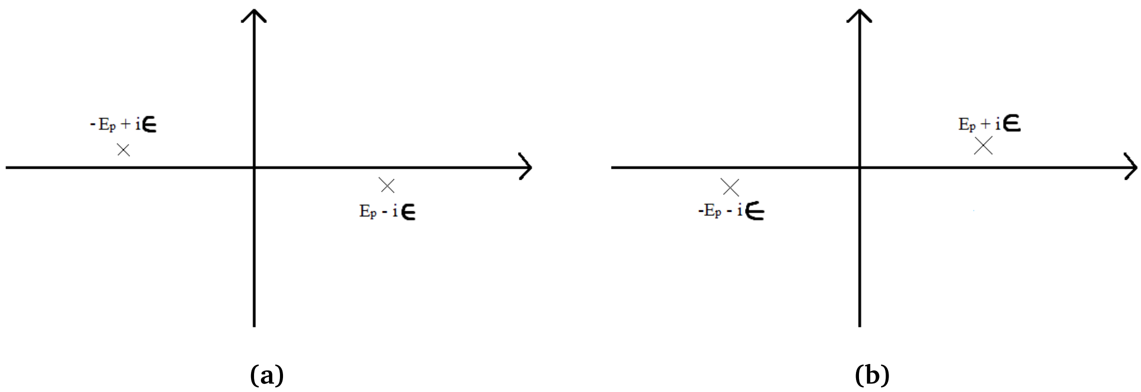


Figure 4: (a) The pole structure for the $+i\epsilon$ prescription; (b) The pole structure for the $-i\epsilon$ prescription.

Now, the Feynman propagator of the ghost state is of the form

$$D_F(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{-i}{p^2 - M^2 - i\epsilon} \tag{93}$$

Consider the forward-propagating part of this propagator, which corresponds to $x^0 - y^0 > 0$. The p_0 coordinate is as usual integrated over a left-handed semicircular contour in the lower half of the complex plane so that the contribution from the arc vanishes when the radius of the semicircle is taken to infinity. Hence, using Cauchy's theorem, the forward propagator picks up the residue from the pole at $-E_p$, so that

$$D_F^{(+)}(x,y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_p} \exp\left[-iE_p(y^0 - x^0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})\right] \tag{94}$$

The residue is now indeed positive, but at the expense of the state propagating backwards in time (equivalent to a negative energy state propagating forward in time), violating causality. Hence, it could be seen that for higher-derivative theories one is forced to make a choice between unitarity and causality.

There are proposed ways to circumvent this problem. For the specific example of quadratic gravity, Niedermaier [10] calculated the spectrum of the Hessian which amounts to the inverse propagator, and found that in an area of theory space around the vicinity of the NGFP the eigenvalues are strictly positive, so that there are no ghost states. This is by no means general, as the perturbative treatment employed exploits the asymptotic freedom of the Weyl-squared coupling, and contains little to no information about the RG flow far away from the UV fixed point. However, it hints at the possibility that some theory would follow trajectories in theory space where the Hessian has a strictly positive spectrum, so that the ghost states are never turned on.

For general quantum theories, self-energy corrections need to be added to the propagator. Donoghue and Menezes [26] argued that the self-energy corrections cause the denominator of the propagator to develop an imaginary part, which will be non-negative to satisfy unitarity. For example, after quantum corrections the scalar part of the propagator of quadratic gravity takes the approximate form

$$D(p) = \frac{1}{p^2} - \frac{1}{p^2 - m_r^2 - i\gamma(p^2)} \tag{95}$$

where the real part is absorbed into m_r and γ is the imaginary part, so that the pole in the ghost propagator is now located at $p_0 = \pm\sqrt{E_p^2 + i\gamma} \approx \pm(E_p + i\gamma/2E_p)$.

This results in the same pole structure as the one shown in Figure 4b, but with ϵ replaced with a natural imaginary part $\gamma/2E_p$ that is not taken to vanish. The resultant forward Feynman propagator evaluated using the usual contour integral will yield the same result as (94), but with E_p replaced by $(E_p + i\gamma/2E_p)$, so that

$$D_F^{(+)}(x,y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{e^{-iE_p(y^0-x^0)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{2(E_p + i\gamma/2E_p)} e^{-\frac{\gamma}{2E_p}(x^0-y^0)} \tag{96}$$

This propagator still violates causality, but for large enough γ it will decay very rapidly so that causality is only violated microscopically. For this particular ghost state, the timescale of the decay is proportional to the inverse Planck scale, making the ghost effectively undetectable [26]. This violation of micro-causality is exactly analogous to Dirac's cure [27] for the linear runaway mode of a point-charge electron in classical electrodynamics, and is also a general feature of Lee-Wick type theories [28].

In the context of asymptotic safety, an additional complication arise. All functional integrals mentioned in this dissertation are evaluated in Euclidean space, which is connected to the physical Lorentzian space by a Wick rotation continuously rotating the time coordinate from the imaginary axis back to the real axis in the complex plane. This rotation is only mathematically valid if the physical amplitudes retain their analytic properties along the path of the rotation, which stipulates that it does not sweep over poles of the propagator. For the usual Feynman prescription in common QFTs, the poles are always in the second and fourth quadrant in the complex plane, so that they are not swept over by the Wick rotation. However, as shown above, quantum corrections cause the propagator of quadratic gravity to develop poles in the first and third quadrants, which are swept over by the Wick rotation, undermining the robustness of such an operation. Moreover, it was shown that addition of even higher derivative terms would only introduce more such ghost poles [29], further worsening the situation.

Alternative cures to the unitarity problem include a proposal that the propagator is an entire function with a single pole at vanishing momentum [30] which would inevitably require knowledge of the behaviour of the theory in the infinite dimensional theory space, or including higher spacial derivative terms while keeping two orders of time derivatives, violating Lorentz invariance [31].

4.2 The running of gravitational couplings

It is already shown in earlier sections that in the context of the Wilsonian renormalisation group, the couplings of the general theory can seen as running in an infinite dimensional theory space. However, for finite truncations of the theory, the definition of running dimensionful couplings becomes ambiguous, as the RG procedure inevitably causes them to receive contributions from operators that were turned off in the original truncation, as clearly shown in the case of Einstein gravity, where each additional loop introduces terms two derivative orders higher into the counter Lagrangian. As the forms of these contributions are dependent on the physical process used to probe the momentum dependency of the couplings, the definition of running dimensionful couplings is not universal.

At the perturbative level, Donoghue and Anber [32] were able to demonstrate this explicitly by calculating to one-loop order the running Newton's constant using the graviton vacuum polarisation, graviton-graviton scattering and the gravitational scattering of identical massless scalars and non-identical relativistic scalars. The resultant forms of the running Newton's constant turn out to be different for each process. For instance, consider the scattering of non-identical scalars $A + B \to A + B$. Using the usual Mandelstam variables s, t and u and taking the relativistic limit $s \gg m^2$, one finds the tree-level amplitude

$$\mathcal{M}_{\text{tree}} = \frac{i\kappa^2 su}{4t} \tag{97}$$

and the one-loop amplitude

$$\begin{aligned}\mathcal{M}_{1-\text{loop}} = i\frac{\kappa^4}{(4\pi)^2}\Big[&\frac{1}{16}\left(s^4 I_4(s,t) + u^4 I_4(u,t)\right) + \frac{1}{8}\left(s^3 + u^3 + tsu\right) I_3(t) \\ &- \frac{1}{8}\left(s^3 I_3(s) + u^3 I_3(u)\right) - \frac{1}{240}\left(71us - 11t^2\right) I_2(t) \\ &+ \frac{1}{16}\left(s^2 I_2(s) + u^2 I_2(u)\right)\Big]\end{aligned} \tag{98}$$

where I_4, I_3 and I_2 are the scalar box, triangle and bubbled diagrams respectively, with the form

$$\begin{aligned}I_4(s,t) &= \frac{1}{st}\left\{\frac{2}{\epsilon^2}\left[(-s)^{-\epsilon} + (-t)^{-\epsilon}\right] - \ln^2\left(\frac{-s}{-t}\right) - \pi^2\right\} \\ &= \frac{1}{st}\left\{\frac{4}{\epsilon^2} - \frac{2\ln(-s) + 2\ln(-t)}{\epsilon} + 2\ln(-s)\ln(-t) + \text{ finite }\right\} \\ I_3(s) &= \frac{1}{\epsilon^2}(-s)^{-1-\epsilon} = -\frac{1}{s}\left(\frac{1}{\epsilon^2} - \frac{\ln(-s)}{\epsilon} + \frac{\ln^2(-s)}{2}\right) \\ I_2(s) &= \frac{1}{\epsilon(1-2\epsilon)}(-s)^{-\epsilon} = \left(\frac{1}{\epsilon} - \ln(-s) + \text{ finite }\right)\end{aligned} \tag{99}$$

where $\epsilon = (4-d)/2$ is introduced due to dimensional regularisation. The overall amplitude contains IR divergences, which can be eliminated [33] by subtracting

$$\mathcal{M}_{\text{IR}} = \frac{\kappa^2}{2(4\pi)^2}\frac{(-s)^{1-\epsilon} + (-t)^{1-\epsilon} + (-u)^{1-\epsilon}}{\epsilon^2}\mathcal{M}_{\text{tree}} \tag{100}$$

to obtain the hard amplitude

$$\begin{aligned}\mathcal{M}_h = i\frac{\kappa^4}{(4\pi)^2}\Big[&\frac{1}{8}\left(\frac{s^3}{t}\ln(-s)\ln(-t) + \frac{u^3}{t}\ln(-u)\ln(-t)\right) \\ &- \frac{1}{16t}\left(s^3 + u^3 + tsu\right)\ln(-t) + \frac{1}{16}\left(s^2\ln^2(-s) + u^2\ln^2(-u)\right) \\ &+ \frac{us}{16t}\left(s\ln^2(-s) + t\ln^2(-t) + u\ln^2(-u)\right) \\ &+ \frac{1}{240}\left(71us - 11t^2\right)\ln(-t) - \frac{1}{16}\left(s^2\ln(-s) + u^2\ln(-u)\right)\Big]\end{aligned} \tag{101}$$

To obtain the total amplitude that can be used to calculate the running coupling, the renormalisation point is chosen to be $s = 2E^2$, $t = u = -E^2$ so that the total amplitude is

$$\mathcal{M} = \frac{i\kappa^2 E^2}{2}\left[1 - \frac{\kappa^2 E^2}{10(4\pi)^2}\left((19 + 10\ln 2)\ln\left(\frac{E^2}{\mu^2}\right) + 5\left(\pi^2 - (\ln 2 - 1)\ln 2\right)\right)\right] \tag{102}$$

which yields the running Newton's constant

$$G(E) = G_0\left[1 - \frac{\kappa^2 E^2}{10(4\pi)^2}\left((19 + 10\ln 2)\ln\left(\frac{E^2}{\mu^2}\right) + 5\left(\pi^2 - (\ln 2 - 1)\ln 2\right)\right)\right] \tag{103}$$

However, the amplitude of the process $A + A \to B + B$ is linked to this amplitude by $s \leftrightarrow t$, which becomes

$$\mathcal{M} = \frac{i\kappa^2 E^2}{8}\left[1 + \frac{\kappa^2 E^2}{10(4\pi)^2}\left(9\ln\left(\frac{E^2}{\mu^2}\right) - 5\pi^2 + (19 + 5\ln 2)\ln 2\right)\right] \tag{104}$$

This yields another definition for the running coupling, which is now

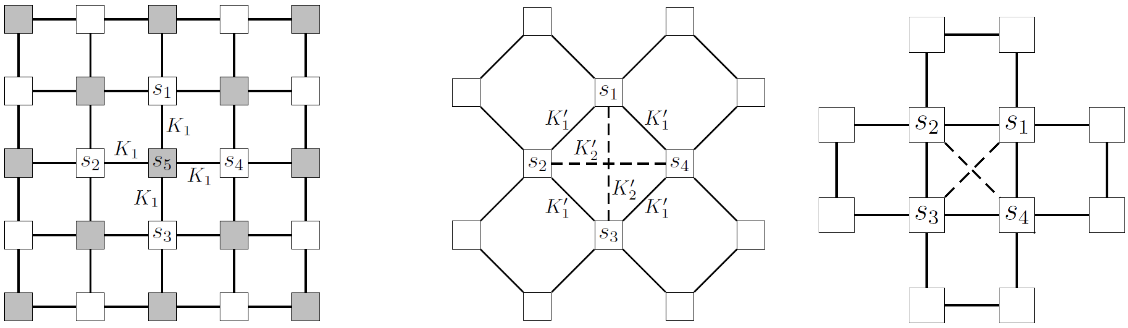
$$G(E) = \frac{G_0}{4}\left[1 + \frac{\kappa^2 E^2}{10(4\pi)^2}\left(9\ln\left(\frac{E^2}{\mu^2}\right) - 5\pi^2 + (19 + 5\ln 2)\ln 2\right)\right] \tag{105}$$

One immediately sees that the gravitational couplings obtained from the two processes are completely different. In general, different processes will yield different expressions for the running gravitational coupling due to them receiving different contributions from implicit higher-derivative operators.

There is no reason to believe that this behaviour is absent in the non-perturbative regime. For a non-perturbative example, one could invoke the two-dimensional Ising model on a square lattice for N spins in the absence of external fields, whose partition function is simply

$$Z(K_1, N) = \sum_{\{s_i\}} \exp\left(K_1 \sum_{\langle ij\rangle}^{N} s_i s_j\right) \tag{106}$$

where K_1 is the nearest-neighbour coupling and only distinct nearest-neighbour pairs are summed over. Then, Kadanoff's real-space renormalisation group procedure [34] illustrated in Figure 5 is carried out.



(a) **(b)** **(c)**

Figure 5: The real-space renormalisation group procedure [35]. (a) Every second spin shown in dark grey is decimated. (b) All decimated spins in the original lattice are summed out, generating nearest-neighbour coupling K_1', next-nearest-neighbour coupling K_2' and quadruple coupling K_3'. (c) The lattice spacing is rescaled by a factor of $\sqrt{2}$ and rotated $45°$ to obtain the renormalised theory.

Collecting the decimated spins in one term, the partition function is now

$$\begin{aligned} Z(K_1, N) &= \sum_{\text{remaining}} \sum_{\text{decimated}} \cdots \exp\left(K_1 s_5 \left[s_1 + s_2 + s_3 + s_4\right]\right) \cdots \\ &= \sum_{\text{remaining}} \cdots 2\cosh\left(K_1 \left[s_1 + s_2 + s_3 + s_4\right]\right) \cdots \end{aligned} \tag{107}$$

where the sums over the decimated spins such as $s_5 = \pm 1$ are performed. However, one notes that the sum of four spins has 3 allowed values due to symmetry. By considering all possible spin combinations, one has

$$\begin{aligned} 2\cosh\left(K_1 \left[s_1 + s_2 + s_3 + s_4\right]\right) = \exp\big(&K_0' + K_1' \left[s_1 s_2 + s_1 s_4 + s_2 s_3 + s_3 s_4\right] \\ &+ K_2' \left[s_1 s_3 + s_2 s_4\right] + K_3' s_1 s_2 s_3 s_4\big) \end{aligned} \tag{108}$$

where the new couplings are functions of K_1. Then it becomes clear that after a single RG action, the renormalised partition function now has two new interactions turned on. The same would be expected to happen in finite truncations of a gravitational theory, so that the connection between the running of a coupling calculated in a finite truncation and the running of that coupling in the infinite dimensional theory space requires further investigation. The fact that running dimensionful couplings receive contributions from other interation terms is the key motivation for Wilson to formulate the renormalisation group in an infinite dimensional theory space that contains all possible interactions.

In addition to these issues, some awkward technical nuances may also arise. For example, if more than one NGFPs that satisfy Weinberg's criteria are discovered, the theory may be ill-equipped to choose between them in the absence of physical predictions within the scope of currently available experiments.

Even more fundamentally, some claim that the quantisation of gravity in the covariant formulation itself is problematic. For instance, Fadeev and Popov [11] noted that for a fluctuation metric that is not small enough, the quantum effects could change its signature. Moreover, Hawking [36] found that the gravitational path integral is dominated by virtual gravitational instantons. Thus, in the words of Gibbons and Hawking [37], one might argue that:

> "Attempts to quantize gravity ignoring the topological possibilities and simply drawing Feynman diagrams around flat space have not been very successful. It seems to me that the fault lies not with the pure gravity or supergravity theories themselves but with the uncritical application of perturbation theory to them. In classical relativity we have found that perturbation theory has only limited range of validity. One cannot describe a black hole as a perturbation around flat space. Yet this is what writing down a string of Feynman diagrams amounts to."

It should be noted that the above quote is somewhat inaccurate, and is only included to provide a perspective, as Duff [38] had already obtained the Schwarzschild solution from tree-level quantum contributions in 1973.

5 Discussion

As demonstrated, the asymptotic safety programme in gravity has made significant progress over the past few decades, but certain key questions remain open. Comments on certain technicalities are in place. Firstly, for longer truncations, the non-Gaussian fixed points are rarely unique. In the polynomial $f(R)$ truncation investigated up to R^{70}, the addition of additional terms often introduces new fixed points [39], while a truncation containing a sum of traces of up to 35 Ricci tensors contains two non-trivial fixed points [40]. In each example, a particular fixed point, whose coordinates in the cosmological constant-Newton's constant plane is in the neighbourhood of the fixed point Reuter found for the Einstein-Hilbert truncation, is identified and chosen as the fixed point corresponding to the one in the infinite-dimensional theory space. A more rigorous mechanism of choosing between multiple fixed points would be desirable.

Secondly, there is currently no convincing indication of the finiteness of the dimensionality of the UV critical surface, which is the key criterion that differentiates between a fundamental theory and an effective one. The ideal way to show this finiteness is to show that the number of relevant directions at the NGFP, after reaching a certain value, stops increasing even when new operators are added into the truncation. This is indeed true for the polynomial $f(R)$ truncation, whose Reuter fixed point has exactly three relevant directions even when terms up to R^{70} are added [39]. However, for a truncation containing a sum of traces of the Ricci tensor, it was shown that the Reuter fixed point has up to four relevant directions [40]. The increase in the dimensionality of the UV critical surface due to the inclusion of a different class of terms is a negative sign for fundamental asymptotically safe gravity, and longer truncations containing a greater variety of terms need to be investigated to see if this trend continues.

It should be noted that many of the problems mentioned above disappear if an effective field theory (EFT) approach is taken. In fact, if Einstein gravity is treated as an EFT, its unrenormalisability is no longer a fundamental issue, as EFTs are not required to be UV complete. In addition, the ghost issue with higher derivative theories dissolves if the theory is treated as an effective one that is to be taken over by some more fundamental UV theory, as one can simply claim that the range of effectiveness of the EFT is up to the mass scale of the ghost. For instance, in the asymptotic safety scenario for matter couplings it was proposed [41] that if the transition scale into the asymptotically safe scaling regime occurs lower than the Planck scale, then the asymptotically safe theory can act as an intermediate effective theory between a UV fundamental theory such as string theory and the IR regime governed by the standard model.

In summary, the asymptotic safety scenario in gravity is still an active area of research with significant obstacles to overcome, and progress in this field would shed light on the nature of gravitational interactions, even if the fundamental theory of quantum gravity is not described by such a scenario.

A Quadratic expansion of the Einstein-Hilbert action

Throughout this section, Veltman's [6] notation and procedure are followed. An overbar indicates "with respect to the full metric", lack of bars indicate "with respect to the background metric", an underbar indicates "linear in $h_{\mu\nu}$", and double underbars indicate "quadratic in $h_{\mu\nu}$". Raising and lowering operations are performed using the background metric. Terms above quadratic order are omitted, and commas refer to covariant differentiation.

The expression to be calculated is

$$\mathcal{L}_{\mathbf{cl}} = \sqrt{-\bar{g}}[R + \underline{R} + \underline{\underline{R}}] \tag{109}$$

Starting from

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} = g_{\mu\alpha}(\delta^\alpha_\nu + h^\alpha_\nu) \tag{110}$$

one has the inverse metric

$$\bar{g}^{\mu\nu} = g^{\mu\alpha}(\delta^\nu_\alpha - h^\nu_\alpha + h^\beta_\alpha h^\nu_\beta) \tag{111}$$

Then using the identity

$$\sqrt{-\bar{g}} = \exp\left(\frac{1}{2}\,\mathrm{Tr}[\ln(-\bar{g})]\right) = \sqrt{-g}\exp\left(\frac{1}{2}\,\mathrm{Tr}[\ln(\delta^\alpha_\nu + h^\alpha_\nu)]\right) \tag{112}$$

and expanding to the appropriate order, one has

$$\sqrt{-\bar{g}} = \sqrt{-g}(1 + \frac{1}{2}h^\alpha_\alpha - \frac{1}{4}h^\alpha_\beta h^\beta_\alpha + \frac{1}{8}(h^\alpha_\alpha)^2) \tag{113}$$

Then expanding the Ricci scalar step by step

$$\begin{aligned}
\bar{\Gamma}^\alpha_{\mu\nu} &= \Gamma^\alpha_{\mu\nu} + \underline{\Gamma}^\alpha_{\mu\nu} + \underline{\underline{\Gamma}}^\alpha_{\mu\nu} \\
\underline{\Gamma}^\alpha_{\mu\nu} &= \frac{1}{2}\left(h^\alpha_{\nu,\mu} + h^\alpha_{\mu,\nu} - h_{\mu\nu}^{;\alpha}\right)
\end{aligned}$$

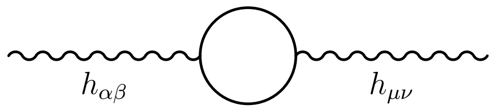
$$
\begin{aligned}
\underline{\underline{\Gamma}}^{\alpha}_{\mu\nu} &= -\frac{1}{2}h^{\alpha\gamma}\left(h_{\gamma\nu,\mu}+h_{\mu\gamma,\nu}-h_{\mu\nu,\gamma}\right)\\
\underline{\Gamma}^{\alpha}_{\mu\alpha} &= \frac{1}{2}h^{\alpha}_{\alpha,\mu}, \quad \underline{\underline{\Gamma}}^{\alpha}_{\mu\alpha} = -\frac{1}{2}h^{\alpha}_{\beta}h^{\beta}_{\alpha,\mu}\\
\bar{R}^{\mu}_{\nu\alpha\beta} &= R^{\mu}_{\nu\alpha\beta}+\underline{R}^{\mu}_{\nu\alpha\beta}+\underline{\underline{R}}^{\mu}_{\nu\alpha\beta}\\
\underline{R}^{\mu}_{\nu\alpha\beta} &= \frac{1}{2}\left(h^{\mu}_{\beta,\nu\alpha}-h^{,\mu}_{\nu\beta,\alpha}-h^{\mu}_{\alpha,\nu\beta}+h^{,\mu}_{\nu\alpha,\beta}\right)+\frac{1}{2}R^{\mu}_{\gamma\alpha\beta}h^{\gamma}_{\nu}+\frac{1}{2}R^{\gamma}_{\nu\beta\alpha}h^{\mu}_{\gamma}\\
\underline{\underline{R}}^{\mu}_{\nu\alpha\beta} &= \nabla_{\alpha}\underline{\underline{\Gamma}}^{\mu}_{\nu\beta}-\nabla_{\beta}\underline{\underline{\Gamma}}^{\mu}_{\nu\alpha}+\underline{\Gamma}^{\gamma}_{\beta\nu}\underline{\Gamma}^{\mu}_{\gamma\alpha}-\underline{\Gamma}^{\gamma}_{\alpha\nu}\underline{\Gamma}^{\mu}_{\gamma\beta}\\
\underline{R}_{\nu\alpha} &= \frac{1}{2}\left(h^{\beta}_{\beta,\nu\alpha}-h^{\beta}_{\nu,\alpha\beta}-h^{\beta}_{\alpha,\nu\beta}+h^{,\beta}_{\nu\alpha,\beta}\right)\\
\underline{\underline{R}}_{\nu\alpha} &= -\frac{1}{2}\nabla_{\alpha}\left(h^{\beta}_{\mu}h^{\mu}_{\beta,\nu}\right)+\frac{1}{2}\nabla_{\beta}\left\{h^{\beta}_{\gamma}\left(h^{\gamma}_{\nu,\alpha}+h^{\gamma}_{\alpha,\nu}-h^{,\gamma}_{\alpha\nu}\right)\right\}\\
&\quad+\frac{1}{4}\left(h^{\gamma}_{\beta,\nu}+h^{\gamma}_{\nu,\beta}-h^{,\gamma}_{\beta\nu}\right)\left(h^{\beta}_{\gamma,\alpha}+h^{\beta}_{\alpha,\gamma}-h^{,\beta}_{\gamma\alpha}\right)\\
&\quad-\frac{1}{4}\left(h^{\gamma}_{\alpha,\nu}+h^{\gamma}_{\nu,\alpha}-h^{,\gamma}_{\nu\alpha}\right)h^{\beta}_{\beta,\gamma}\\
\underline{R} &= h^{\beta,\alpha}_{\beta,\alpha}-h^{\beta,\alpha}_{\alpha,\beta}-R^{\alpha}_{\nu}h^{\nu}_{\alpha}\\
\underline{\underline{R}} &= -\frac{1}{2}\nabla_{\alpha}\left(h^{\beta}_{\mu}h^{\mu,\alpha}_{\beta}\right)+\frac{1}{2}\nabla_{\beta}\left\{h^{\beta}_{\nu}\left(2h^{\nu\alpha}_{,\alpha}-h^{\alpha,\nu}_{\alpha}\right)\right\}\\
&\quad+\frac{1}{4}\left(h^{\nu}_{\beta,\alpha}+h^{\nu}_{\alpha,\beta}-h^{,\nu}_{\beta\alpha}\right)\left(h^{\beta,\alpha}_{\nu}+h^{\beta\alpha}_{,\nu}-h^{\alpha,\beta}_{\nu}\right)\\
&\quad-\frac{1}{4}\left(2h^{\nu\alpha}_{,\alpha}-h^{\alpha,\nu}_{\alpha}\right)h^{\beta}_{\beta,\nu}-\frac{1}{2}h^{\nu\alpha}h^{\beta}_{\beta,\nu\alpha}\\
&\quad+\frac{1}{2}h^{\nu}_{\alpha}\nabla_{\beta}\left(h^{\beta,\alpha}_{\nu}+h^{\beta\alpha}_{,\nu}-h^{\alpha,\beta}_{\nu}\right)+h^{\nu}_{\beta}h^{\beta}_{\alpha}R^{\alpha}_{\nu}
\end{aligned}
$$

Omitting total derivatives and taking $g_{\mu\nu}\to\eta_{\mu\nu}$, the background Ricci tensor and scalar, both multiples of the derivative of the metric, vanish, and covariant derivatives turn into partial derivatives. Hence, using $\sqrt{-\eta}=1$ one can take the expanded Lagrangian to be

$$
\begin{aligned}
\mathcal{L}_{\mathbf{cl}} &= -\frac{1}{4}\partial_{\nu}h^{\beta}_{\alpha}\partial^{\nu}h^{\alpha}_{\beta}+\frac{1}{4}\partial_{\mu}h^{\alpha}_{\alpha}\partial^{\mu}h^{\beta}_{\beta}-\frac{1}{2}\partial_{\beta}h^{\alpha}_{\alpha}\partial^{\mu}h^{\beta}_{\mu}+\frac{1}{2}\partial^{\alpha}h^{\nu}_{\beta}\partial_{\nu}h^{\beta}_{\alpha}\\
&= -\frac{1}{4}(\partial_{\mu}h_{\alpha\beta})^{2}+\frac{1}{4}(\partial_{\mu}h)^{2}-\frac{1}{2}\partial_{\mu}h\partial_{\nu}h^{\mu\nu}+\frac{1}{2}\partial^{\alpha}h_{\alpha\nu}\partial_{\mu}h^{\mu\nu}
\end{aligned}
\tag{114}
$$

where in the last line the shorthand $h=h^{\mu}_{\mu}$ is used, and the last terms in the two lines are equal up to two total derivatives.

B One-loop scalar corrections to the graviton propagator



$$
\begin{aligned}
i\Pi^{(1)}_{\alpha\beta\mu\nu}(q) = \int \frac{d^4k}{(2\pi)^4}\frac{i\kappa}{2}\left[k_\alpha(k+q)_\beta + k_\beta(k+q)_\alpha\right]\frac{i}{k^2}\frac{i}{(k+q)^2} \\
\times \frac{i\kappa}{2}\left[k_\mu(k+q)_\nu + k_\nu(k+q)_\mu\right]
\end{aligned} \tag{115}
$$

Using a Feynman parameter, the denominator of the integrand becomes

$$
\begin{aligned}
\frac{1}{k^2(k+q)^2} &= \int_0^1 \frac{dx}{(k^2+2xk\cdot q+xq^2)^2} \\
&= \int_0^1 \frac{dx}{(l^2+x(1-x)q^2)^2}
\end{aligned} \tag{116}
$$

where $l = k+xq$. Taking into account that that parity of the Wick-rotated integration measure will prohibit integrand terms that are odd powers of l, the numerator can be reorganised in terms of l to be

$$
\begin{aligned}
&4l_\alpha l_\beta l_\mu l_\nu + 4x(x-1)(q_\alpha q_\beta l_\mu l_\nu + q_\mu q_\nu l_\alpha l_\beta) \\
&+ (1-2x)^2(q_\alpha q_\mu l_\beta l_\nu + q_\alpha q_\nu l_\beta l_\mu + q_\beta q_\mu l_\alpha l_\nu + q_\beta q_\nu l_\alpha l_\mu) \\
&+ 4x^2(x-1)^2 q_\alpha q_\beta q_\mu q_\nu
\end{aligned} \tag{117}
$$

where the term of interest is the one quartic in the external momentum.

Then, performing a Wick rotation by substituting $l_0 = il_{E0}$, one obtains

$$
i\Pi^{(1)}_{\alpha\beta\mu\nu}(q) \sim i\kappa^2(q_\alpha q_\beta q_\mu q_\nu)\int_0^1 dx\; x^2(x-1)^2 \int \frac{d^4l_E}{(2\pi)^4}\frac{1}{(l_E^2+\Delta)^2} \tag{118}
$$

where $\Delta = -x(1-x)q^2$. The last integral in (118) can be evaluated using dimensional regularisation, where $d = 4 - \epsilon$

$$\int \frac{d^d l_E}{(2\pi)^4} \frac{1}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \xrightarrow{d \to 4} \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi)\right) \tag{119}$$

Thus, the amplitude of interest is

$$i\Pi^{(1)}_{\alpha\beta\mu\nu}(q) \sim \frac{i\kappa^2}{(4\pi)^2}(q_\alpha q_\beta q_\mu q_\nu) \int_0^1 dx\; x^2(x-1)^2 \left(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi)\right) \tag{120}$$

C The heat kernel method

Consider a self-adjoint differential operator Δ in d dimensions. One can define the heat kernel

$$K(\tau; x, y; \Delta) = \langle x|e^{-\tau\Delta}|y\rangle \tag{121}$$

which satisties the heat equation

$$(\partial_\tau + \Delta)K(\tau; x, y; \Delta) = 0 \tag{122}$$

subject to the initial condition

$$\lim_{\tau\to 0} K(\tau; x, y; \Delta) = \delta(x - y) \tag{123}$$

Then it becomes clear that the heat kernel $K(\tau; x, y; \Delta)$ is the Green's function of the operator $(\partial_\tau + \Delta)$.

Consider the canonical example where $\Delta_0 = -\nabla_\mu\nabla^\mu + m^2$, then the heat kernel is the Green's function of a forced heat equation

$$K(\tau; x, y; \Delta_0) = \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-\frac{(x-y)^2}{4\tau} - \tau m^2\right) \tag{124}$$

An important use of heat kernel methods in QFT is related to its ability to calculate one-loop effective actions. For a theory whose action is defined by

$$\frac{\delta^2 S}{\delta\bar{\phi}(x)\delta\bar{\phi}(y)} = \Delta\, \delta^{(d)}(x - y) \tag{125}$$

with some Hessian Δ, where $\bar{\phi}$ is the vacuum expectation value of the quantum field ϕ, the one-loop effective action is given by

$$\begin{aligned} \Gamma^{(1)} &\sim \mathrm{Tr}\ln\Delta \\ &= \int d^d x\sqrt{-g}\,\langle x|\ln\Delta|x\rangle \end{aligned} \tag{126}$$

However, one could use the identity that

$$\ln \Delta = \int_0^\infty \frac{d\tau}{\tau} e^{-\tau\Delta} + \text{const.} \tag{127}$$

so that

$$\begin{aligned}\Gamma^{(1)} &\sim \int_0^\infty \frac{d\tau}{\tau} \int d^d x \sqrt{-g} \, \langle x | e^{-\tau\Delta} | x \rangle \\ &= \int_0^\infty \frac{d\tau}{\tau} \int d^d x \sqrt{-g} K(\tau; x, x; \Delta)\end{aligned} \tag{128}$$

This expression can be divergent at both large τ and small τ, but only the small τ limit is relevant to the UV behaviour of the theory. Thus, the heat kernel can be expanded in an early-time expansion

$$\begin{aligned}K(\tau; x, x; \Delta) &= K(\tau; x, x; \Delta_0)(a_0 + a_1\tau + a_2\tau^2 + \cdots) \\ &= \frac{e^{-\tau m^2}}{(4\pi\tau)^{d/2}} (a_0 + a_1\tau + a_2\tau^2 + \cdots)\end{aligned} \tag{129}$$

where the coefficients a_n can in general be looked up for common operators Δ.

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