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Introduction to Mirror Symmetry in Aspects of Topological String Theory

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Abstract

Under the compactification by the Calabi-Yau threefold, the string theory shows there is duality called mirror symmetry, which implies there is an isomorphism between two string theories under the compactifications of two topologically different internal manifolds. By twisting the topological string theory in two methods, the twisted theories named A-model and B-model have an isomorphism to each other under the mirror symmetry.

Contents

1 Introduction

Nowadays, there are two pillars of modern physics, which are Einstein's general theory of relativity and the quantum field theory with the standard model. However these two theories cannot be unified together properly to become "the theory of everything". There are several attempts of such great grand unification, and superstring theory is one of them and it seems to be the most successful one [6][7]. The superstring theory is a theory in which the elementary bosonic particles are considered to be the vibration modes of the 1-dimensional string moving in the spacetime, and the fundamental fermions are the super partners of these bosons under the supersymmetry. The superstring theory predicts the existence of gravity and is also able to fit into the quantum mechanical theory we have already obtained. Apart for these, there are also some particular predictions it made, one of the most famous prediction is that our universe should be in a spacetime with 10 or 11 dimensions (we will focus on the 10-dimensional case, and the 11-dimensional theory is called M-theory [15]). It shows that the 6 extra spatial dimensions should be conpactified in a very small scale, and the corresponding compact space is called a Calabi-Yau manifold. Calabi in 1957 first conjectured the existence of such kind of manifolds [22] and then it was proved by Yau in 1977 [1][21].

There is a symmetry relation among the Calabi-Yau manifolds, which is called the mirror symmetry [11]. It implies a duality between the string theories with two topologically different Calabi-Yau manifolds. The mirror symmetry first came to people's sight in 1989 with the work of Greene and Plesser [2] and Candelas, Lyker and Schimmrigk [3]. Mirror symmetry also gives an isomorphism between two superstring theories, type-IIA and type-IIB, under different internal space [12], and it can be understand in aspect of T-duality [4]. Apart from that the mirror symmetry would also play important role in the topological string theory. In topological string theory, we can twist the theory in two different ways to obtain two theories, A-model and B-model. In 1991, Witten found that the mirror symmetry can also make the duality between two twisted models [5], and it links the complex structure on B-model to the Kählar structure on A-model.

In this dissertation, we will discuss what the Calabi-Yau compactification is and some corresponding properties of such compact manifold. Then the topological string theory will be introduced, and also the two models obtained by twisting the topological theory and their related knowledge will be given. Finally, the mirror symmetry will be explained and we will look at this duality in a few different aspects.

2 Calabi-Yau Compactification

In the superstring theory, the conformal invariance of the SCFT coupled to a worldsheet theory requires a 10-dimensional spacetime rather than 4-dimensional one in which we feel in the ordinary life, so a way of compactification on the spacetime manifold is needed. Therefore the whole spacetime should be a manifold which can be expressed in the form of $M_{1,3} \times M_6$, where $M_{1,3}$ is the 4-dimensional Minkowski spacetime manifold and M_6 is the 6-dimensionl compact manifold for the spatial compactification. There are some requirements on M_6. Firstly, the compact manifold should be a vacuum solution of the Einstein's field equation, which also means the manifold should be Ricci-flat. On the other hand, M_6 is also required to preserve some supersymmetries rather than break them all [18][19], so it implies that the manifold needs to be a Kählar manifold. All the requirements leads that manifold M_6 should be a compact Ricci-flat 6-dimensional Kählar manifold, that is to say, we need a Calabi-Yau three-fold for the extra spatial dimensions.

In this chapter, we will introduce the complex manifolds and the Kählar manifolds, which are the keys to give the definition of the Calabi-Yau manifolds, and the corresponding knowledge of the differential manifold for each case will be introduced as well (such as complex differential forms, cohomology group, homology, Hodge diamond and etc.)[8][13][29][30][31]. After all the necessary knowledge is given, the definition of the Calabi-Yau manifolds will be illustrated, and some of its properties (such as topological invariant, moduli spaces [9][10] and etc.) will also be introduced.

2.1 Complex Manifolds

Before we define a complex manifold, we need to first define a holomorphic map on $\mathbb{C}^m$. A complex function $:\mathbb{C}^m \to \mathbb{C}$ is holomorphic, if $f = f_1 + if_2$ satisfies the Cauchy-Riemann relations for each $z^\mu = x^\mu + iy^\mu$,

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu}\ , \qquad \frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu} \tag{2.1.1}$$

Then an m-dimensional complex manifold M is defined as the following axioms:

(i) M is a topological space.

(ii) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$.

(iii) $\{U_i\}$ is a family of open sets which covers M. The map φ_i is a homeomorphism from U_i to an open subset U of $\mathbb{C}^m$. [Hence, M is even dimensional.]

(iv) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ji} = \varphi_j \circ \varphi_i^{-1}$ from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$ is holomorphic.

The number m is the complex dimension of M, and is denoted as $\dim_{\mathbb{C}} M = m$, and its real dimension is $2m$. The Axioms ensure that calculus on the complex manifold can exist without the dependence of the any chosen coordinates, and the manifold is differentiable. Complex manifold can also preserve its orientation, so they are also orientable. We can locate a point p on M by using the coordinate $z^\mu = \varphi(p) = x^\mu + iy^\mu$ in a chart (U, φ), where $1 \leq \mu \leq m$. Then the tangent space T_pM of complex manifold M is naturally spanned by $2m$ vectors

$$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m}\right\} \tag{2.1.2}$$

and its co-tangent space T_p^*M is spanned by

$$\{dx^1, \dots, dx^m, dy^1, \dots, dy^m\}. \tag{2.1.3}$$

Let us define another $2m$ basic vecoters:

$$\frac{\partial}{\partial z^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right) \tag{2.1.4.a}$$

$$\frac{\partial}{\partial \bar{z}^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right), \tag{2.1.4.b}$$

and corresponding $2m$ one-forms are defined as:

$$dz^\mu \equiv dx^\mu + idy^\mu \tag{2.1.5.a}$$

$$d\bar{z}^\mu \equiv dx^\mu - idy^\mu \tag{2.1.5.b}$$

These are called the holomorphic bases (ones without bar) and anti-holomorphic bases (ones with bar). By using these vector and co-vector bases, we can then define a so-called almost complex structure which will play an essential role in the following sections.

We define a real tensor field of type $(1,1)$ on point p of a complex manifold M, $J_p : T_pM \to T_pM$ such that

$$J_p\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\mu}, \qquad J_p\left(\frac{\partial}{\partial y^\mu}\right) = -\frac{\partial}{\partial x^\mu}. \tag{2.1.6}$$

Note that

$$J_p^2 = -i\ d_{T_pM} \tag{2.1.7}$$

J_p is the almost complex structure of M at point p,and it corresponds to the multiplication of $\pm i$ [17]. This structure is also independent of the charts chosen, which can be proven by finding its action of the overlapping parts of any two charts in a complex manifold [8]. The almost complex structure can be only defined globally on a complex manifold.

By acting the almost complex structure to the bases that we defined in (2.1.4), we can easily find that the bases $\partial/\partial z^{\mu}$ and $\partial/\partial \bar{z}^{\mu}$ are both the unit eigenvectors of J_p with eigenvalues $+i$ and $-i$ respectively.

$$J_p\left(\frac{\partial}{\partial z^{\mu}}\right) = i\frac{\partial}{\partial z^{\mu}} \qquad J_p\left(\frac{\partial}{\partial \bar{z}^{\mu}}\right) = -i\frac{\partial}{\partial \bar{z}^{\mu}} \tag{2.1.8}$$

Therefore we can diagonalise J_p as a matrix in the (anti-)holomorphic bases:

$$J_p = idz^{\mu} \otimes \frac{\partial}{\partial z^{\mu}} - id\bar{z}^{\mu} \otimes \frac{\partial}{\partial \bar{z}^{\mu}} = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}, \tag{2.1.9}$$

where I_m is the $m \times m$ identity matrix.

For a tangent space $T_pM^{\mathbb{C}}$ of a complex manifold M, we can decompose it into two disjoint tangent spaces

$$T_pM^{\mathbb{C}} = T_pM^{+} \oplus T_pM^{-} \tag{2.1.10}$$

where T_pM^{+} is the tangent space with holomorphic bases and T_pM^{-} is the one with anti-holomorphic bases. The vectors in each tangent space are the eigenvectors of J_p with the eigenvalues $\pm i$ respectively, so we could give the definition:

$$T_pM^{\pm} = \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm iZ\}. \tag{2.1.11}$$

Therefore a complex tangent vector $Z \in T_pM^{\mathbb{C}}$ can be uniquely decomposed as $Z = Z^{+} + Z^{-}$, where $Z^{\pm} \in T_pM^{\pm}$, and Z^{+} is defined as a holomorphic vector and Z^{-} is defined as an anti-holomorphic vector. We need to note that $T_pM^{\pm}$ are in the same dimensions, and their dimensions are both half of $\dim_{\mathbb{C}} T_p M^{\mathbb{C}} = \dim_{\mathbb{C}} M$.

Until here, we have defined an almost complex structure in a complex manifold. It is obvious that every complex manifold has an almost complex structure, as J_p can be directly related to the multiplication by $\pm i$, but an almost complex manifold, which is defined as a differential manifold with an almost complex structure, is not always a complex manifold [8]. It needs some restrictions.

We then define a tensor field of rank (1, 2) on an almost complex manifold, $N : T_p M \times T_p M \to T_p M$:

$$N(X, Y) \equiv [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \tag{2.1.12}$$

which is called Nijenhuis tensor, and $[X, Y]$ is the Lie bracket of two vectors. If we have a vanishing Nijenhuis tensor on a manifold M, the almost complex structure J is integrable. Then there is another theorem, which can be proved with complicated mathematical concepts [14], showing that: (Newlander and Nirdnberg 1957) Let (M, J) be a $2m$-dimensinal almost complex manifold. If J is integrable, the manifold M is a complex manifold with the almost complex structure J. Hence we can say a vanishing Nijenhuis tensor lead an almost complex manifold to be a complex manifold.

2.2 Complex Differential Forms

Same as real manifolds, complex manifolds also have differential forms. If we have a complex manifold M with $\dim_{\mathbb{C}} M = m$, a complex q-form $\omega \in \Omega_p^q(M)^{\mathbb{C}}$ at point p can be defined as $\omega = \eta + i\zeta$, where η and ζ are both real q-forms $\eta, \zeta \in \Omega_p^q(M)^{\mathbb{R}}$ at p, and its complex conjugate is $\overline{\omega} = \eta - i\zeta$. Then we can easily find that the exterior derivative of the q-form ω is

$$d\omega = d\eta + id\zeta. \tag{2.2.1}$$

We can also define the complex q-form in the holomorphic and anti-holomorphic bases, which would be more helpful for the later description. Let the q-form $\omega \in \Omega_p^q(M)^{\mathbb{C}}$ $(q \leq 2m)$ and r, s be positive integers such that $r + s = q$. Then we take q vectors $V_i \in T_pM^{\mathbb{C}}$ $(1 \leq i \leq q)$ in either T_pM^+ or T_pM^- spaces. If $\omega(V_1, \ldots, V_q) = 0$ unless r of V_i are in T_pM^+ and s of V_i are in T_pM^-, ω is said to be an (r,s)-form or a form of bidegree (r,s), $\omega \in \Omega_p^{r,s}(M) = \Omega_p^q(M)^{\mathbb{C}}$. Then we can see that the bases of an (r,s)-form are just r holomorphic basic one-forms dz and s anti-holomorphic basic one-forms $d\bar{z}$, which can be found in (2.1.5). Therefore the (r,s)-form ω can be written as:

$$\omega = \frac{1}{r!\,s!}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_s}. \tag{2.2.2}$$

All the components in the form are totally anti-symmetric in μ and ν respectively.

A complex q-form ω can be uniquely decomposed by disjoint (r,s)-forms:

$$\omega = \sum_{r+s=q} \omega^{(r,s)} \tag{2.2.3}$$

where $\omega^{(r,s)} \in \Omega^{r,s}$, and its set can also be decomposed as:

$$\Omega^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}(M). \tag{2.2.4}$$

Then the exterior derivative of an (r,s)-form can be found by changing the coordinate system to the (anti-)holomorphic one on (2.2.1):

$$d\omega = \frac{1}{r!\,s!}\left(\frac{\partial}{\partial z^{\lambda}}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}dz^{\lambda} + \frac{\partial}{\partial \bar{z}^{\lambda}}\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}d\bar{z}^{\lambda}\right) \times (dz^{\mu_1}\wedge\ldots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\ldots\wedge d\bar{z}^{\nu_s}). \qquad (2.2.5)$$

Here we can see that the new complex $(q+1)$-form $d\omega$ is actually a mixture of an $(r+1,s)$-form and an $(r,s+1)$-form. Then we can separate the exterior derivative operator into two parts:

$$d = \partial + \bar{\partial} \qquad (2.2.6)$$

where ∂ is the operator mapping $\Omega^{r,s}(M)$ into $\Omega^{r+1,s}(M)$, and $\bar{\partial}$ is the one mapping $\Omega^{r,s}(M)$ into $\Omega^{r,s+1}(M)$, and both of operators are call the Dolbeault operators [8], and because all three operators in (2.2.6) are nilpotent, we can easily find the following relation:

$$\partial\partial\omega = \bar{\partial}\bar{\partial}\omega = \left(\bar{\partial}\partial + \partial\bar{\partial}\right)\omega = 0 \qquad (2.2.7)$$

The de Rham cohomology in a complex manifold is similar to the one for real manifolds, but with respect to each Dolbeault operator. The $\bar{\partial}$-closed (r,s)-form, i.e. $\omega \in \Omega^{r,s}(M), \bar{\partial}\omega = 0$, is called the (r,s)-cocycle denoted by $Z^{r,s}_{\bar{\partial}}(M)$, and the $\bar{\partial}$-exact (r,s)-forms, i.e. $\omega \in \Omega^{r,s}(M)$, $\eta \in \Omega^{r,s-1}(M), \bar{\partial}\eta = \omega$, is called the (r,s)-coboundary denoted by $B^{r,s}_{\bar{\partial}}(M)$. Then the corresponding cohomology group is

$$H^{r,s}_{\bar{\partial}}(M) \equiv Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M) \qquad (2.2.8)$$

which is a complex vector space and is called the (r,s)th $\bar{\partial}$-cohomology group. The de Rham cohomology for ∂ is extended in the same way as the one for $\bar{\partial}$ as shown above.

2.3 Kählar Manifolds

For a complex manifold M we can extend the metric g to act on the complex tangent spaces

$$g: T_pM^{\mathbb{C}} \times T_pM^{\mathbb{C}} \to \mathbb{C}. \tag{2.3.1}$$

Taking two vectors $Z = X + iY$ and $W = U + iV$ on the same tangent space $T_pM^{\mathbb{C}}$ at point p, we can have

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + i\big[g_p(X, V) + g_p(Y, U)\big]. \tag{2.3.2}$$

For the (anti-)holomorphic coordinates, the components of the metric are simply $g_{\mu\nu}(p) = g_p\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right)$, $g_{\mu\bar{\nu}}(p) = g_p\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right)$ and etc. .

Before we give the definition of the Kählar manifolds, we need first to define what a Hermitian manifold is. If the Riemannian metric g of a complex manifold M satisfies the following restriction:

$$g_p\big(J_pX, J_pY\big) = g_p(X, Y). \tag{2.3.3}$$

where point $p \in M$, vectors $X, Y \in T_pM^{\mathbb{C}}$ and J_p is the almost complex structure of the manifold, we say the metric g is a Hermitian metric, and the pair (M, g) we called Hermitian manifold. It can be proven that the components of the metric with holomorphic indices will vanish

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = g\left(J_p\frac{\partial}{\partial z^\mu}, J_p\frac{\partial}{\partial z^\nu}\right) = -g_{\mu\nu} = 0, \tag{2.3.4}$$

and it is same for $g_{\bar{\mu}\bar{\nu}}$. Therefore the Hermitian metric only takes the form with mixed indices:

$$g = g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu \tag{2.3.5}$$

Hence we can find the geometry of a Hermitian manifold through its metric g. The connection on the manifold can be found by its definition, for example:

$$\nabla_\mu \frac{\partial}{\partial z^\nu} = \Gamma^\lambda{}_{\mu\nu}(z) \frac{\partial}{\partial z^\lambda} \tag{2.3.6}$$

where ∇_μ is the covariant derivative and $\Gamma^\lambda{}_{\mu\nu}$ is the connection. We can obtain the connection with pure anti-holomorphic indices $\Gamma^{\bar\lambda}{}_{\bar\mu\bar\nu}$, in the same way. As the manifold should have the metric compatibility, we need relations that $\nabla_\kappa g_{\mu\bar\nu} = \nabla_{\bar\kappa} g_{\mu\bar\nu} = 0$ to be true. Therefore we can find the connection can be written as

$$\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\bar\sigma}\partial_\mu g_{\bar\sigma\nu} \qquad \Gamma^{\bar\lambda}{}_{\bar\mu\bar\nu} = g^{\bar\lambda\sigma}\partial_{\bar\mu} g_{\bar\nu\sigma}, \tag{2.3.7}$$

and the connections with mixed indices are all vanished. These connections are called the Hermitian connection. Because the only non-vanished connections are the two with pure indices, we can see that a holomorphic tangent space on a Hermitian manifold can only be parallel transported in holomorphic directions, and it is also true for the anti-holomorphic cases. The Hermitian manifolds are not necessarily torsion free, similar to the connections, the only non-vanishing torsion are also with pure indices, $T^\lambda{}_{\mu\nu}$ and $T^{\bar\lambda}{}_{\bar\mu\bar\nu}$. Then for the curvature of the manifold, we can use the definition the Riemannian curvature tensors:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{2.3.8}$$

to find that the only non-vanishing components are $R^\kappa{}_{\lambda\bar\mu\nu}$, $R^\kappa{}_{\lambda\mu\bar\nu}$, $R^{\bar\kappa}{}_{\bar\lambda\bar\mu\nu}$, $R^{\bar\kappa}{}_{\bar\lambda\mu\bar\nu}$. Due to the symmetry of switching the last two indices of the tensor, the only two independent components of the Riemannian curvature tensor are $R^\kappa{}_{\lambda\bar\mu\nu}$, and $R^{\bar\kappa}{}_{\bar\lambda\mu\bar\nu}$, and they are the complex conjugate of each other. Then we can find the tensors can be written as:

$$R^{\kappa}{}_{\lambda\bar{\mu}\nu} = \partial_{\bar{\mu}}\Gamma^{\kappa}{}_{\nu\lambda} = \partial_{\bar{\mu}} g^{\bar{\xi}\kappa}\partial_{\nu} g_{\lambda\bar{\xi}} \tag{2.3.9.a}$$

$$R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = \partial_{\mu}\Gamma^{\bar{\kappa}}{}_{\bar{\nu}\bar{\lambda}} = \partial_{\mu} g^{\xi\bar{\kappa}}\partial_{\bar{\nu}} g_{\xi\bar{\lambda}}. \tag{2.3.9.b}$$

By contracting the first upper and the first lower indices of the Riemann tensor, we can define a anti-symmetric rank 2 tensor:

$$\mathfrak{R}_{\mu\bar{\nu}} \equiv R^{\kappa}{}_{\kappa\mu\bar{\nu}} = -\partial_{\bar{\nu}}\left(g^{\bar{\xi}\kappa}\partial_{\mu} g_{\kappa\bar{\xi}}\right) = -\partial_{\bar{\nu}}\partial_{\mu}\log G \tag{2.3.10}$$

where $G \equiv \det\left(g_{\mu\bar{\nu}}\right) = \sqrt{g}$. We define the Ricci form by

$$\mathfrak{R} \equiv i\mathfrak{R}_{\mu\bar{\nu}} dz^{\mu} \wedge d\bar{z}^{\nu} = i\partial\bar{\partial}\log G, \tag{2.3.11}$$

and it is a real form. We can find $\mathfrak{R}$ is also a closed form, i.e. $d\mathfrak{R} \propto d\partial\bar{\partial}\log G = -\frac{1}{2}d^2\left(\partial - \bar{\partial}\right)\log G = 0$, but it does not imply it is also an exact form. $\mathfrak{R}$ also defines a the a non-trivial element called the first Chern class, i.e. $c_1(M) \equiv [\mathfrak{R}/2\pi] \in H^2(M;\mathbb{R})$ [20]. It is also the reason why we say a Ricci flat Kähler manifold has a vanishing first Chern class, i.e. $\mathfrak{R} = 0 \Rightarrow c_1(M) = 0$.

Then we also need to give a definition of another concept, Kähler form. In a Hermitian manifold (M, g), we define a tensor field Ω at point $p \in M$, such that

$$\Omega_p(X, Y) = g_p\left(J_p X, Y\right) \qquad X, Y \in T_p M. \tag{2.3.12}$$

We can find that it is an anti-symmetric tensor field acting on two vectors to give a number, so it is a 2-form field, and we call it Kähler form. We can extend the domain of the Kähler form from $T_p M$ to $T_P M^{\mathbb{C}}$, it shows that it would be a $(1,1)$-form:

$$\Omega\left(\frac{\partial}{\partial z^{\mu}},\frac{\partial}{\partial \bar{z}^{\nu}}\right)=g_p\left(J_p\frac{\partial}{\partial z^{\mu}},\frac{\partial}{\partial \bar{z}^{\nu}}\right)=ig_{\mu\bar{\nu}}=-\ \Omega\left(\frac{\partial}{\partial \bar{z}^{\nu}},\frac{\partial}{\partial z^{\mu}}\right) \qquad (2.3.13)$$

where the cases with pure indices will also vanish just like the metric, i.e. $g_{\mu\nu}=g_{\overline{\mu}\overline{\nu}}=0$. Then we can write Ω

$$\Omega=ig_{\mu\bar{\nu}}dz^{\mu}\otimes d\bar{z}^{\nu}-ig_{\bar{\nu}\mu}d\bar{z}^{\nu}\otimes dz^{\mu}=ig_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}^{\nu}, \qquad (2.3.14)$$

or
$$\Omega=-J_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}_{\nu} \qquad (2.3.15)$$

as $J_{\mu\bar{\nu}}=g_{\mu\bar{\lambda}}J^{\bar{\lambda}}{}_{\bar{\nu}}=-ig_{\mu\bar{\nu}}$, and it is easy to show that Ω is a real form by finding that its complex conjugate is itself.

On a Hermitian manifold with complex dimension m and the Kähler form Ω, we have a nowhere vanishing $2m$-form,

$$\underbrace{\Omega\wedge\ldots\wedge\Omega}_{m}=\Omega^{m}\,.$$

To prove this, we can use the orthonormal basis:

$$\Omega(\,\hat{e}_i\,,J\hat{e}_j)=\delta_{ij} \qquad \Omega(\hat{e}_i\,,\hat{e}_j)=\ \Omega(J\hat{e}_i\,,J\hat{e}_j)=0 \qquad (2.3.16)$$

Then it follows that

$$\begin{aligned}&\underbrace{\Omega\wedge\ldots\wedge\Omega}_{m}(\hat{e}_1,J\hat{e}_1,\ldots,\hat{e}_m,J\hat{e}_m)\\&=\sum_{p}\Omega(\hat{e}_{P_1},J\hat{e}_{P_1})\ldots\Omega(\hat{e}_{P_m},J\hat{e}_{P_m})\\&=m!\,\Omega(\hat{e}_1,J\hat{e}_1)\ldots\Omega(\hat{e}_m,J\hat{e}_m)=m!\end{aligned} \qquad (2.3.17)$$

where P is the permutation of m objects. It shows that Ω^{m} is a nowhere vanishing real form on the Hermitian manifold and it serves as the volume form. Therefore it again shows that the manifold we are talking about is an orientable manifold, and it fits the fact that the compact internal space should be orientable. Then by using this non-vanishing top form, we can decompose it into two separate parts: a holomorphic volume m-form and an anti-holomorphic volume m-form, which will be unique and non-vanishing on a Calabi-Yau manifold.

After introducing Hermitian manifold and Kähler form, we can give the definition of the Kähler manifold: Kähler manifold is the Hermitian manifold (M, g) with a closed Kähler form, which means $d\Omega = 0$, and the metric satisfying above condition is called Kähler metric. Then we extend the condition:

$$\begin{aligned} d\Omega &= \left(\partial + \bar{\partial}\right) i g_{\mu\bar{\nu}}\, dz^{\mu} \wedge d\bar{z}^{\nu} \\ &= i\partial_{\lambda} g_{\mu\bar{\nu}} dz^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu} + i\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} dz^{\bar{\lambda}} \wedge dz^{\mu} \wedge d\bar{z}^{\nu} \\ &= \frac{1}{2} i\left(\partial_{\lambda} g_{\mu\bar{\nu}} - \partial_{\mu} g_{\lambda\bar{\nu}}\right) dz^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu} \\ &\quad + \frac{1}{2} i\left(\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} - \partial_{\bar{\nu}} g_{\bar{\lambda}\mu}\right) d\bar{z}^{\lambda} \wedge dz^{\mu} \wedge d\bar{z}^{\nu} = 0 \end{aligned} \tag{2.3.18}$$

where we can find the relations:

$$\partial_{\lambda} g_{\mu\bar{\nu}} = \partial_{\mu} g_{\lambda\bar{\nu}} \qquad \partial_{\bar{\lambda}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\bar{\lambda}\mu}. \tag{2.3.19}$$

Therefore for a given chart (U_i, φ_i) on a Kähler manifold M, one can write the components of the metric g in the following form:

$$g_{\mu\bar{\nu}} = \partial_{\mu}\partial_{\bar{\nu}}\mathcal{K}_i \tag{2.3.20}$$

where $\mathcal{K}_i \in \mathcal{F}(U_i)$ is called the Kähler potential of the metric. As the metric can be expressed differently in different chart, the Kähler potential can only be written in the form of (2.3.20) locally rather than globally. The potential is also not unique, because we can change the potential by adding (anti-)holomorphic functions without changing its metric. Then on a compact Kähler manifold without boundary, the form Ω^m (the wedge product over m Kähler forms) is closed but not exact.

2.4 Holonomy Group of Kähler Manifolds

We can recall that the holonomy group of a Riemannian manifold M of $\dim_{\mathbb{R}} M = m$ is a subgroup of $O(m)$. Then because the Kähler manifolds that we have studied in this topic is also orientable which means the holonomy group should keep the orientation of the parallel transported vectors, the holonomy group of Kähler manifolds with m real dimensions should be a subgroup of $SO(m)$. Furthermore it follows from the index structure of the connection of the Kähler manifolds that under parallel transport elements of TM^+ and TM^- do not mix, and the lengths of the vectors should be preserved under the parallel transports. Hence the holonomy group of the Kähler manifolds should be a subgroup of $U(n)$ where n is the complex dimension of the manifold. The elements of TM^+ and TM^- transform in $U(n)$ respectively. Then we may recall the definition of the holonomy group, which is the transformation of a vector on the manifold after a parallel transports around a loop and should also be a Lie group. Now we may take a parallel transport around an infinitesimal loop on the manifold, so we can find the infinitesimal transformation of the vector, which is also corresponding to the Lie algebra of the holonomy group, $\mathfrak{u}_+(n) \oplus \mathfrak{u}_-(n)$. Assuming there is a parallel transport of a holomorphic vector V around an infinitesimal loop enclosed by the sides δ^l and $\epsilon^{\bar{k}}$, we can find the change on the vector can be expressed as

$$\delta V^i = \epsilon^{\bar{k}} \delta^l R^i{}_{j\bar{k}l} V^j. \tag{2.4.1}$$

Due to the index structure of the Riemann tensor, only the effects from the loops with mixed indices are non-vanishing in (2.4.1). The matrix $\epsilon^{\bar{k}} \delta^l R^i{}_{j\bar{k}l}$ should be the element of the Lie algebra $\mathfrak{u}(n)$. If we take the trace of the matrix, it will become $\epsilon^{\bar{k}} \delta^l \mathfrak{R}_{\bar{k}l}$ which is just proportional to the Ricci form. Then it can generates a $\mathfrak{u}(1)$ part decomposed from the Lie algebra $\mathfrak{u}(n)$, so we can have the decomposition $\mathfrak{u}(n) \simeq \mathfrak{u}(1) \oplus \mathfrak{su}(n)$[9], where $\mathfrak{su}(n)$ implies that the holonomy group on a Ricci-flat Kähler manifold M with $\dim_{\mathbb{C}} M = n$ is a subgroup of $SU(n)$ [36].

2.5 $\bar{\partial}$-cohomology Groups and Hodge Numbers

Reminding that the (r,s)th $\bar{\partial}$-cohomology group is defined by (2.2.8). Similarly to the cohomology group of the real manifolds, $H^{r,s}_{\bar{\partial}}(M)$ is a complex vector space. The element $[\omega] \in H^{r,s}_{\bar{\partial}}(M)$ is an equivalence class of $\bar{\partial}$-closed forms of bidegree (r,s) differing the form ω by a $\bar{\partial}$-exact form:

$$[\omega] = \left\{ \eta \in \Omega^{r,s}(M) \middle| \bar{\partial}\eta = 0, \omega - \eta = \bar{\partial}\psi, \psi \in \Omega^{r,s-1}(M) \right\}. \tag{2.5.1}$$

One reason why we need to study the $\bar{\partial}$-cohomology group is that the groups can measure some topological properties of the complex manifolds.

On a Hermitian manifold M with $\dim_{\mathbb{C}} M = m$, a complex Hodge star $\star$ is a mapping: $\Omega^{r,s}(M) \to \Omega^{m-r,m-s}$, and we define the inner product as

$$(\alpha, \beta) \equiv \int_M \alpha \wedge \bar{\star} \beta \tag{2.5.2}$$

where $\alpha, \beta \in \Omega^{r,s}(M)$ and $\bar{\star}$ is the complex conjugate of $\star$ satisfying that $\bar{\star} \beta \equiv \overline{\star \beta} = \star \bar{\beta}$. The inner product helps us to define the adjoint Dolbeault operators. Like the adjoint exterior derivative $d^\dagger$, the adjoint Dolbeault operators are defined as $\partial^\dagger : \Omega^{r,s}(M) \to \Omega^{r-1,s}(M)$ and $\bar{\partial}^\dagger : \Omega^{r,s}(M) \to \Omega^{r,s-1}(M)$ such that

$$(\alpha, \partial\beta) = (\partial^\dagger \alpha, \beta) \qquad (\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger \alpha, \beta). \tag{2.5.3}$$

We can see that $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$, and both of the adjoint Dolbeault operators are also nilpotent, i.e. $(\partial^\dagger)^2 = (\bar{\partial}^\dagger)^2 = 0$. Then we can define the corresponding Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$ on the Hermitian manifolds

$$\Delta_\partial \equiv (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial \tag{2.5.4.a}$$

$$\Delta_{\bar{\partial}} \equiv (\bar{\partial} + \bar{\partial}^\dagger)^2 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \tag{2.5.4.b}$$

An (r, s)-form ω is called a ∂-harmonic form if $\Delta_\partial \omega = 0$, and it is called the $\bar{\partial}$-harmonic form if $\Delta_{\bar{\partial}} \omega = 0$. The ∂-harmonic form is also ∂-closed and ∂-co-closed, i.e. $\partial\omega = \partial^\dagger \omega = 0$, and it has the same relation for the $\bar{\partial}$-harmonic forms. Furthermore, we can have the Hodge's theorem in the complex version, and the (r, s)-forms can be decomposed into three orthogonal terms:

$$\Omega^{r,s}(M) = \bar{\partial}\Omega^{r,s-1}(M) \oplus \bar{\partial}^\dagger \Omega^{r,s+1}(M) \oplus Harm_{\bar{\partial}}^{r,s}(M) \tag{2.5.5}$$

where $Harm_{\bar{\partial}}^{r,s}(M)$ is the set of $\bar{\partial}$-harmonic (r, s)-forms. We can express a (r, s)-forms as

$$\omega = \bar{\partial}\alpha + \bar{\partial}^\dagger \beta + \gamma \tag{2.5.6}$$

where $\alpha \in \Omega^{r,s-1}(M)$, $\beta \in \Omega^{r,s+1}(M)$ and $\gamma \in Harm_{\bar{\partial}}^{r,s}(M)$. Then the (r, s)-forms on a

Hermitian manifold can also decomposed in the holomorphic way.

In the Hermitian manifold, we define the Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$ separately, and they are also different indeed. However, if the manifold is a Kähler manifold, the Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$ are the same [17].

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}} \tag{2.5.7}$$

We define the complex dimension of $\bar{\partial}$-cohomology group $H^{r,s}_{\bar{\partial}}(M)$ as the Hodge number $h^{r,s}$, and we can construct a Hodge diamond for the cohomology groups of all possible bidegrees. For a Kähler manifold with 2 complex dimension, we can build the Hodge diamond as

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} \tag{2.5.8}$$

There are 9 Hodge numbers for this manifold. If we have $\dim_{\mathbb{C}} M = m$, there would be $(m+1)^2$ of them. However, the Hodge numbers are not all independent to each other, and we have two relations for the numbers if the manifold is Kähler:

$$h^{r,s} = h^{s,r} \tag{2.5.9.a}$$

$$h^{r,s} = h^{m-r,m-s} \tag{2.5.9.b}$$

The relation (2.5.9.a) is due to the fact that the relation in (2.5.7) make the Laplacians on the Kähler manifold are the same as its complex conjugates. Therefore for any harmonic (r,s)-form, there exists a corresponding (s, r)-form and vice versa, so the Hodge numbers are the same for opposite bidegrees. Then the relation (2.5.9.b) is due to the Poincare duality, which shows that the cohomology group $H_{\bar{\partial}}^{m-r,m-s}(M)$ is the dual vector space to $H_{\bar{\partial}}^{r,s}(M)$, and hence they have the same dimensions [8].

The relations in (2.5.9) make the Hodge diamond of the Kählar manifolds symmetric in both vertical and horizontal directions. It allows us to use less independent numbers to parameterize the Hodge diamond. For example, we can just use 4 Hodge numbers in (2.5.8) instead of 9 to parameterize the diamond. Furthermore, the Hodge numbers in the Kählar manifolds also have a close relation with the Betti numbers. This relation makes the Hodge diamond be able to describe some topological properties of the Kählar manifolds, and it is not true for just general Hermitian manifold. The theorem describing this relation is: Let M be a Kähler manifold with $\dim_{\mathbb{C}} M = m$ and $\partial M = \emptyset$. Then the Betti numbers b^p $(1 \leq p \leq 2m)$ satisfy

$$b^p = \sum_{r+s=p} h^{s,r} \tag{2.5.10.a}$$

$$b^{2p-1} \text{ is even } (1 \leq p \leq m) \tag{2.5.10.b}$$

$$b^{2p} \geq 1 \qquad (1 \leq p \leq m). \tag{2.5.10.c}$$

Then we can find the Euler characteristic of the Kähler manifold is

$$\chi = \sum_{p} (-1)^p b^p = \sum_{r,s} (-1)^{r+s} h^{r,s} \tag{2.5.11}$$

2.6 Calabi-Yau Manifolds

A m dimensional Calabi-Yau manifold is defined to be a compact, complex, Kähler manifold which has $SU(m)$ holonomy, where m is the complex dimension of the manifold. Also, the previous sections showed that the $SU(m)$ holonomy implies the corresponding Kähler manifold should be Ricci-flat, and therefore it has a vanishing first Chern class. Chern classes are topological invariants of a manifold. Specifically, we can get the classes through Taylor expansion of the Chern form

$$\begin{aligned} c(M) &= 1 + \sum_j c_j(M) = \det\left(1 + \frac{i\mathcal{R}}{2\pi}\right) \\ &= 1 + \mathrm{tr}\frac{i\mathcal{R}}{2\pi} + \mathrm{tr}\left(\frac{i\mathcal{R}}{2\pi} \wedge \frac{i\mathcal{R}}{2\pi} - 2\left(\mathrm{tr}\frac{i\mathcal{R}}{2\pi}\right)^2\right) + \cdots \end{aligned} \tag{2.6.1}$$

where $\mathcal{R}$ is the matrix valued curvature 2-form, i.e. $\mathcal{R} = R^k{}_{li\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. Then we can find the first Chern class $c_1(M) = \mathrm{tr}\, i\mathcal{R}/2\pi$ is equal to $[\mathfrak{R}\backslash 2\pi]$. Furthermore, due to the Ricci-flat condition, there is a unique nowhere vanishing holomorphic $(m, 0)$-form on the Calabi-Yau manifold which implies the Hodge number $h^{m,0} = 1$. Also, by using this non-vanishing $(m, 0)$-form, we can make a isomorphism mapping $(0, p)$-forms to $(0, m - p)$-forms, where $1 \leq p \leq m$. Then the $SU(m)$ holonomy group does not allow a $(0, p)$-form to be harmonic, which means the corresponding coholonomy is trivial, $h^{0,p} = 0$. It makes the Hodge diamond of a Cababi-Yau three-fold even simpler, we can see:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{2.6.2}
$$

where there is only two independent Hodge number in the diamond due to the vertical and horizontal symmetry relations, and we can just look at $h^{1,1}$ and $h^{2,1}$. Then the Euler characteristic of the Calabi-Yau manifold is

$$\chi = 2(h^{1,1} - h^{2,1}) \tag{2.6.3}$$

2.7 Moduli Spaces of Calabi-Yau Manifolds

According to the Yau's theorem, The Calabi-Yau manifold is a Ricci-flat Kähler manifold. Thus it seems that we are able to perturb the metric $g_{\mu\nu}$ of the manifold to a new metric $g_{\mu\nu} + \delta g_{\mu\nu}$ without changing the Ricci flat condition,

$$R_{\mu\nu}(g) = 0 \quad \Longrightarrow \quad R_{\mu\nu}(g + \delta g) = 0. \tag{2.7.1}$$

Apparently we can easily achieve such a perturbation by just changing the coordinate system, but these are not the case we are interesting about. We can impose a coordinate condition to the system to fix the choice of coordinate, and we can write the condition as

$$\nabla^{\mu} \delta g_{\mu\nu} = 0, \tag{2.7.2}$$

which is pretty similar to fixing a gauge. Then we can re-express the perturbation (2.7.1) together with the coordinate condition as:

$$\nabla^\rho \nabla_\rho \delta g_{\mu\nu} - 2R_{\mu\ \nu}^{\ \rho\ \sigma} \delta g_{\rho\sigma} = 0 \qquad (2.7.3)$$

By using the (anti-)holomorphic coordinate system, we can see that the perturbation can be applied by δg_{ij} in pure indices and $\delta g_{i\bar{j}}$ in mixed indices, because of the index structure of the metric and the Riemann tensor of the Kähler manifold. Then we can look at these two ways of perturbation:

(1) $\delta g_{i\bar{j}}$: Under the metric perturbation with mixed indices, it remains the original index structure of g and keeps the metric still Hermitian. The condition in (2.7.3) with $\delta g_{i\bar{j}}$ is equivalent to $(\Delta\delta g)_{i\bar{j}} = 0$, and we view $\delta g_{i\bar{j}}$ as the components of a (1,1)-form. Such (1,1)-forms also correspond to the changes of the Kähler form Ω and therefore the Kähler class $[\Omega]$ of the manifold. Because the (1,1)-form is also harmonic, it is uniquely associated to an element of $H_{\bar{\partial}}^{1,1}(M)$. Hence we can expand $\delta g_{i\bar{j}}$ in the basis of real (1,1)-forms with dimension $h^{1,1}$:

$$\delta g_{i\bar{j}} = \sum_{\alpha=1}^{h^{1,1}} \tilde{t}^\alpha b_{i\bar{j}}^\alpha \qquad \tilde{t}^\alpha \in \mathbb{R} \qquad (2.7.4)$$

According to Yau's theorem, there is a Ricci-flat Kähler metric for any $[\Omega + \delta\Omega]$. Hence, the perturbation with mixed indices can make the new metric still be Kähler metric and also deform the Kähler structure (which is parameterized by the Kähler class) at the same time. Therefore, we say the perturbation expressed in (2.7.4) is the Kähler moduli space of the Calabi-Yau manifold.

(2) δg_{ij}: Under the metric perturbation with pure indices, it will not keep the Hermitian structure in the manifold and the Kähler metric is also not preserved. The condition (2.7.2) with δg_{ij} is equivalent to

$$\Delta_{\bar{\partial}}\delta g^{i} = \left(\bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial}\right)\delta g^{i} \tag{2.7.5}$$

where $\delta g^{i} = \delta g^{i}{}_{\bar{j}} d\bar{z}^{\bar{j}} = g^{i\bar{k}}\delta g_{\bar{k}\bar{j}} d\bar{z}^{\bar{j}}$. It is a harmonic $(1,0)$-form associated to an element of cohomology group $H^{1,0}_{\bar{\partial}}(M)$. As the transformation makes the metric not Kähler any more, we need to change the coordinate system so that the metric becomes Kähler again. However, we cannot use a holomorphic transformation to remove the pure-index perturbation, because such transformation does not change type of the indices. In this step, it seems that this coordinate transformation could violate the fixed coordinate condition (2.7.2) we have mentioned before, but it is not (the new metric cannot be obtained from the old one by just a diffeomophism). Then there is a unique non-vanishing holomorphic $(3,0)$-form Ω (here is not the Kähler form) in the Calabi-Yau manifold, and we can use it to define an isomorphism between $H^{1,0}_{\bar{\partial}}(M)$ and $H^{2,1}(M)$ by defining the complex $(2,1)$-forms

$$\Omega_{ijk}\delta g^{k}_{\bar{l}} dz^{i} \wedge dz^{j} \wedge d\bar{z}^{l} \tag{2.7.6}$$

which is still harmonic. Then we can expand the $(2,1)$-form in a basis with dimension $h^{2,1}$, such that

$$\Omega_{ijk}\delta g^{k}_{\bar{l}} = \sum_{\alpha=1}^{h^{2,1}} t^{\alpha} b^{\alpha}_{ij\bar{l}} \qquad t^{\alpha} \in \mathbb{C} \tag{2.7.7}$$

which we called the complex structure moduli space, and it tells the complex structure deformation of the manifold.

3 Topological String Theory

After discussing the Calabi-Yau compactification of the superstring theory, a branch of the string theory will be introduced, and it is called the topological string theory. Rough speaking, it is a "simplified" version of the string theory, as it only discusses how the topological properties affect the theory without caring the exact form of metric of the manifolds (for both worldsheet and the target space). As mentioned in the last chapter, mostly the Calabi-Yau manifold has both Kähler structure and complex structure. One reason that we introduce the topological string theory is that we can twist it in two different ways to obtained two models called A-model and B-model, and these models will depend on the Kähler structure and the complex structure of the target Calabi-Yau space respectively. It may link two models to the two moduli spaces separately.

In this chapter, the supersymmetric $N = (1,1)$ non-linear sigma model and some corresponding symmetry generator will be introduced. Then two ways of twisting the theory, A-twist and B-twist, will be given. Some concepts of cohomological field theory will be introduced as well [35], so that we could illustrate the two theories obtained by two way of twisting, A-model and B-model [23][24] [25][26][27]. At the end, how the theories couples to the gravity to become string theories will be briefly talked.

3.1 Superspaces

Supersymmetry makes the symmetry between bosons and fermions, and d its generators usually transform as spin 1/2 fermions under Lorentz group. For an $N = 1$ supersymmetric theory, it has one such supersymmetry generator, which is called supercharge, for each bosonic dimension. An

$N = p > 1$ supersymmetric theory has p supercharges for each dimension, and the number p here is the multiple of 2, such as 2, 4, 8, and so on (maximal number is 32). For the case $N = (2,2)$, it is the $N = 2$ supersymmetry theory on the complex manifold, as the Lorentz group for each fundamental spin-1/2 supersymmetry generators can be split up into two components which transform with opposite charges (holomorphic and anti-holomorphic). Its makes the two fundamental supercharges become two irreducible positive charges and two irreducible negative charges. To describe each theory in two dimensions, we need to use superspaces, four fermionic coordinates $\theta^{\pm}, \bar{\theta}^{\pm}$ (in Grassmann number) such that under Lorentz transformation $SO(2) = U(1)$ if original coordinate $z \mapsto e^{i\alpha} z$ then

$$\theta^{\pm} \mapsto e^{\pm\frac{i\alpha}{2}} \theta^{\pm}, \quad \bar{\theta}^{\pm} \mapsto e^{\pm\frac{i\alpha}{2}} \bar{\theta}^{\pm} \tag{3.1.1}$$

Therefore we can have a superfield which we can expend to have both bosonic and fermionic parts rather than just bosonic part. By taking the Taylor expansion on superfeild Φ with respect to the fermionic variables, we have

$$\Phi(z, \bar{z}, \theta^{\pm}, \bar{\theta}^{\pm}) = \phi(z, \bar{z}) + \psi_{+}(z, \bar{z})\theta^{+} + \psi_{-}(z, \bar{z})\theta^{-} + \cdots. \tag{3.1.2}$$

Then we can have an integral such that

$$S = \int d^2 z d^4 \theta \, \mathcal{K}\left(\Phi^i, \bar{\Phi}^i\right) \tag{3.1.3}$$

where $\mathcal{K}$ is the Kähler potential. The potential, as mentioned before, is only locally defined and non-unique. By integral the potential over the all the fermionic coordinate, we can get the the corresponding Lagrangean density.

Then in the $N = (2,2)$ theory, the supercharges of it are $Q_\pm$ and $\bar{Q}_\pm$ are all nilpotent, i.e. $Q_\pm^2 = \bar{Q}_\pm = 0$, and they have the anti-commutation relations that:

$$\{Q_\pm, \bar{Q}_\pm\} = H \pm P, \tag{3.1.4}$$

where H and P are the Hamiltonian and momentum operator respectively. Then the rest of anti-commutations relations are all vanishing.

3.2 Supersymmetric Nonlinear Sigma Model

We first need to know what a Supersymmetric nonlinear sigma model with $N = (2,2)$ in two dimensions is defined as [5]. It maps a Riemann surface Σ (which is also the worldsheet of the string) to Riemannian manifold (which is the target space) M with the metric g such that $\Phi: \Sigma \to M$. As the target space M we discuss in this dissertation is a complex manifold, we need to use the coordinate $z, \bar{z}$ on the worldsheet Σ and (anti-)holomorphic coordinate $\phi^i = \overline{\phi^{\bar{\imath}}}$ on M (but we also denote ϕ^I as the real coordinates for convenience), where locally we can describe Φ in a function $\phi^i(z,\bar{z})$. Then Let K and $\bar{K}$ be the canonical and anti-canonical line bundles on the Riemann surface Σ (which are the bundle of one-forms of type (1,0) and (0,1) respectively), and $K^{1/2}$ and $\bar{K}^{1/2}$ are the square roots of these bundles (the square roots here roughly means that the transformation acting on $K^{1/2}$ is also the square root of the same transformation on K, and it also corresponds to the nature of fermions). Let TM be the complexified tangent bundle of M, and it can be decomposed as $TM = TM^+ \oplus TM^-$. Then the fermi fields of the model are ψ_+ and ψ_-, where ψ_+ can be projected in $K^{1/2} \otimes \Phi^*(TM^+)$ and $K^{1/2} \otimes \Phi^*(TM^-)$ denoted as ψ_+^i and $\psi_+^{\bar{\imath}}$ respectively and ψ_- can be projected in $\bar{K}^{1/2} \otimes$

$\Phi^*(TM^+)$ and $\overline{K}^{1/2} \otimes \Phi^*(TM^-)$ denoted as ψ_-^i and $\psi_-^{\bar{\imath}}$ respectively. Then the action is written as

$$S = 2t\int_\Sigma d^2z\left(\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\psi_-^{\bar{\imath}}D_z\psi_-^i g_{\bar{\imath}i} + i\psi_+^{\bar{\imath}}D_{\bar{z}}\psi_+^i g_{\bar{\imath}i} \right.$$
$$\left. + R_{i\bar{\imath}j\bar{\jmath}}\psi_+^i\psi_+^{\bar{\imath}}\psi_-^j\psi_-^{\bar{\jmath}}\right) \tag{3.2.1}$$

where dz^2 is $-idz \wedge d\bar{z}$, t is the coupling constant, $R_{i\bar{\imath}j\bar{\jmath}} = g_{i\bar{k}}R^{\bar{k}}{}_{\bar{\imath}j\bar{\jmath}}$ is the Riemann tensor of the target space M and $D_{\bar{z}}$ is the $\bar{\partial}$ operator on $K^{1/2} \otimes \Phi^*(TM)$ constructed by using the pullback of the Levi-Civita connection on TM. It can be expressed as

$$D_z\psi_+^i = \frac{\partial}{\partial\bar{z}}\psi_+^i + \frac{\partial\phi^j}{\partial\bar{z}}\Gamma^i{}_{jk}\psi_+^k \tag{3.2.2}$$

where $\Gamma^i{}_{jk}$ is the affine connection of the target space M in holomorphic indices. Then it is similar for D_z (∂ operator on $\overline{K}^{1/2} \otimes \Phi^*(TM)$ with Levi-Civita connection $\Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}}$ on M)

Then the supersymmetries of the model can be generated by infinitesimal transformations:

$$\delta\phi^i = i\alpha_-\psi_+^i + i\alpha_+\psi_-^i \tag{3.2.3.a}$$

$$\delta\phi^{\bar{\imath}} = i\tilde{\alpha}_-\psi_+^{\bar{\imath}} + i\tilde{\alpha}_+\psi_-^{\bar{\imath}} \tag{3.2.3.b}$$

$$\delta\psi_+^i = -\tilde{\alpha}_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma^i{}_{jm}\psi_+^m \tag{3.2.3.c}$$

$$\delta\psi_+^{\bar{\imath}} = -\alpha_-\partial_z\phi^{\bar{\imath}} - i\tilde{\alpha}_+\psi_-^{\bar{\jmath}}\Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{m}}\psi_+^{\bar{m}} \tag{3.2.3.d}$$

$$\delta\psi_-^i = -\tilde{\alpha}_+\partial_{\bar{z}}\phi^i - i\alpha_-\psi_+^j\Gamma^i{}_{jm}\psi_-^m \tag{3.2.3.e}$$

$$\delta\psi_-^{\bar{\imath}} = -\alpha_+\partial_{\bar{z}}\phi^{\bar{\imath}} - i\tilde{\alpha}_-\psi_+^{\bar{\jmath}}\Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{m}}\psi_-^{\bar{m}} \tag{3.2.3.f}$$

where $\alpha_\pm$ and $\tilde{\alpha}_\pm$ are the infinitesimal fermionic parameters (α_- and $\tilde{\alpha}_-$ are in sections of $\overline{K}^{1/2}$, then α_+ and $\tilde{\alpha}_+$ are in sections of $K^{1/2}$).

3.3 R-Symmetries

In 2 dimensions, the Lorentz group is $SO(2) = U(1)$. Then as mentioned before, the group for 2 fundamental supercharges can be spilt up into two $U(1)$ associating to θ and $\bar{\theta}$ with plus notations and minus notations, and we denote these two groups as $U(1)_R$ and $U(1)_L$ (In section 3.1, it was said to be in positive charge and negative charge), and the corresponding symmetries under these group transformations are called the R-symmetries. Then we make two new R-symmetries groups from the $U(1)_{R/L}$, and this modification is made by twisting the worldsheet Lorentz group either by vector or axial symmetries:

$$U(1)_V = U(1)_L + U(1)_R, \qquad U(1)_A = U(1)_L - U(1)_R \tag{3.3.1}$$

The A-model, we will discuss in later sections, can be obtained by twisting with $U(1)_V$ connection, and it can be considered on any Kähler manifold. Then the B-model is obtained by twisting with $U(1)_A$ connection, and it is only well defined on a Calabi-Yau manifold. Then the action of the vector symmetry $U(1)_V$ and axial symmetry $U(1)_A$ on the fermionic coordinate can be expressed as

$$R_V(\alpha)\!: \ (\theta^+, \bar{\theta}^+) \mapsto \left(e^{-i\alpha}\theta^+, e^{i\alpha}\bar{\theta}^+\right), \quad (\theta^-, \bar{\theta}^-) \mapsto \left(e^{-i\alpha}\theta^-, e^{i\alpha}\bar{\theta}^-\right) \tag{3.3.2.a}$$

$$R_A(\beta)\!: \ (\theta^+, \bar{\theta}^+) \mapsto \left(e^{-i\beta}\theta^+, e^{i\beta}\bar{\theta}^+\right), \quad (\theta^-, \bar{\theta}^-) \mapsto \left(e^{i\beta}\theta^-, e^{-i\beta}\bar{\theta}^-\right) \tag{3.3.2.b}$$

Then the transformation can be applied to the superfield Φ:

$$R_V = e^{i\alpha F_V}\!: \Phi(x, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\alpha q_V}\Phi\left(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm\right) \tag{3.3.3.a}$$

$$R_A = e^{i\beta F_A}\!: \Phi(x, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\beta q_A}\Phi\left(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm\right) \tag{3.3.3.b}$$

where F_V and F_A are the operators which generate the transformations, and q_V and q_A are the corresponding Noether charges of the each R-symmetry. Let M be the generator of the two-dimensional Lorentz transformation $SO(1,1)$, and we can then take the x^0 direction to be $-ix'^0$ and remain the x^1 same as before. Then the new generator $M_E = iM$ can be obtained, and it generates a compact Euclidean rotation group which is isomorphic to $U(1)_E$. Then we can find the commutation relations between the supersymmetry algebras and the generators:

$$[M_E, Q_{\mp}] = \mp Q_{\pm}, \qquad [M_E, \bar{Q}_{\pm}] = \mp \bar{Q}_{\pm}, \tag{3.3.4.a}$$

$$[F_V, Q_{\pm}] = -Q_{\pm}, \qquad [F_V, \bar{Q}_{\pm}] = \bar{Q}_{\pm}, \tag{3.3.4.b}$$

$$[F_A, Q_{\pm}] = \mp Q_{\pm}, \qquad [F_A, \bar{Q}_{\pm}] = \pm \bar{Q}_{\pm}, \tag{3.3.4.c}$$

It shows that $Q_{\pm}$ and $\bar{Q}_{\pm}$ have the opposite charges under the R-symmetries.

3.4 Twisting Supersymmetric Sigma Model

The twisting here is equivalent to changing the Euclidean rotation group $U(1)_E$ by the generators of the $U(1)$ R-symmetry groups, and it defines the new generators of the Euclidean rotation group $U(1)_{E'}$ as $M_E' = M_E + R$. This twisting can make some supercharge operator scalar with respect to the new Euclidean rotation group [23].

Then the $N = (2.2)$ theory can be twisted in 2 different ways. The first case is that instead of taking ψ_+^i and $\psi_+^{\bar{\imath}}$ to be the sections of $K^{1/2} \otimes \Phi^*(TM^+)$ and $K^{1/2} \otimes \Phi^*(TM^-)$ respectively, we can take them to be sections of $\Phi^*(TM^+)$ and $K \otimes \Phi^*(TM^-)$ respectively, and this kind of twist is called a $+$ twits. Then the second case is that we twist ψ_+^i and $\psi_+^{\bar{\imath}}$ in the way that ψ_+^i become the sections of $K \otimes \Phi^*(TM^+)$ and $\psi_+^{\bar{\imath}}$ becomes the sections of

$\Phi^*(TM^-)$, and this is called the $-$ twist. Similarly, we can also twist ψ_-^i and $\psi_-^{\bar{\imath}}$ in such ways. Under the $+$ twist, ψ_-^i and $\psi_-^{\bar{\imath}}$ will be taken to be in the sections of $\Phi^*(TM^+)$ and $\bar{K} \otimes \Phi^*(TM^-)$ respectively. Then under the $-$ twist, they will be twisted to be the sections of $\bar{K} \otimes \Phi^*(TM^+)$ and $\Phi^*(TM^-)$ respectively. It can be found that the kinetic terms of the fermions of the Lagrangean, e.g. $\psi_-^{\bar{\imath}} D_z \psi_-^i g_{\bar{\imath}i}$, do not change under each twist, even though we have twisted $\psi_\pm^i$ and $\psi_\pm^{\bar{\imath}}$ from fermion terms to the boson-like terms. Then we can have the A-twist and B-twist by combining $\pm$ twists in different ways. The A-twist is the kind of twist in which we do the $-$ twist for ψ_+ and do the $+$ twist for ψ_-, and it is denoted as $(-,+)$. Then the B-twist is an orthogonal twist to the first one, and it is doing the $+$ twist for both ψ_+ and ψ_-, which is denoted as $(+,+)$. There is also a variant that is possible, and that is to twist only ψ_+ or only ψ_- and leave the other untwisted. These are called the half-twisting.

The A- and the B-twists, as mentioned before, change the original Euclidean rotation group $U(1)_E$ into two new groups such that

$$A\text{-twist:} \qquad M_{E'} = M_E + F_V \tag{3.4.1.a}$$

$$B\text{-twist:} \qquad M_{E'} = M_E + F_A \tag{3.4.1.b}$$

We denote the generator of the Euclidean group modified by the A-twist as M_A and denote the one modified by the B-twist as M_B. Then we can find the new commutation relations that

$$[M_A, Q_+] = -2Q_+ \qquad [M_B, Q_+] = -2Q_+ \tag{3.4.2.a}$$

$$[M_A, Q_-] = 0 \qquad [M_B, Q_-] = 2Q_+ \tag{3.4.2.b}$$

$$[M_A, \bar{Q}_+] = 0 \qquad [M_B, \bar{Q}_+] = 0 \tag{3.4.2.c}$$

$$[M_A, \bar{Q}_-] = 2Q_+ \qquad [M_B, \bar{Q}_-] = 0 \tag{3.4.2.d}$$

In the relations above, one can find that some supersymmetry operators become scalar, which is what we expected before, and the others becomes like spin-1 spinors. Then we can define

$$Q_A = Q_- + \bar{Q}_+ \tag{3.4.3.a}$$

$$Q_B = \bar{Q}_- + \bar{Q}_+ \tag{3.4.3.b}$$

where Q_A and Q_B are scalar, nilpotent operators which can be used to define two different cohomological theories, which are called the A-model and the B-model respectively (Two models can also be defined by $(+,-)$ and $(-,-)$ twists, but it will leave $\bar{Q}_A$ and $\bar{Q}_B$ scalar and nilpotent.). Furthermore, we call an operator ϕ a chiral operator or (c,c) operator if ϕ is Q_B-closed:

$$[Q_B, \phi] = 0 \tag{3.4.4}$$

and similarly, we call the operator ϕ a twisted chiral operator or (a,c) operator if

$$[Q_A, \phi] = 0 \tag{3.4.5}$$

3.5 Cohomological Field Theory

As mentioned in section 3.4, both of the A-model and B-model are Topological cohomological field theory, so the concepts of cohomological field theory is needed to be introduced before we go to the A- and B- models parts. cohomological field theories are the field theories that possess a very special type of symmetry. There are serval requirements needed to define a cohomological theory. Firstly, the fermionic symmetry operator Q should squares to zero, i.e. $Q^2 = 0$, which is exactly satisfied by theory we are interested in. Then the second property is that the physical operators O_i in the theory should be closed under the action of the operator Q.

$$\{Q, O_i\} = 0 \tag{3.5.1}$$

This relation is also called that the operator is Q-closed. Both the first and the second properties are corresponding to the BRST symmetry and quantization. Then third requirement is that theory needs the Q-symmtry to be not spontaneously broken, which means the vacuum is symmetric. Then there will be an equivalence relation that:

$$O_i \sim O_i + \{Q, \Lambda\} \tag{3.5.2}$$

where Λ is an arbitrary operator function. As the vacuum is symmetric, the vacuum quantum state should satisfy that

$$Q|0\rangle = 0. \tag{3.5.3}$$

Then by using this relation we can find that the expectation value of the Q-exact operator $\{Q, \Lambda\}$ should be zero:

$$\begin{aligned} &\langle 0|O_{i_1} \dots O_{i_j}\{Q, \Lambda\}O_{i_{j+1}} \dots O_{i_n}|0\rangle \\ &\qquad = \langle 0|O_{i_1} \dots O_{i_j}(Q\Lambda - \Lambda Q)O_{i_{j+1}} \dots O_{i_n}|0\rangle \end{aligned} \tag{3.5.4}$$

and each term will vanishes separately

$$\begin{aligned}
&\langle 0|O_{i_1}\dots O_{i_j}(Q\Lambda)O_{i_{j+1}}\dots O_{i_n}|0\rangle \\
&\qquad\qquad = \pm\langle 0|O_{i_1}\dots QO_{i_j}\Lambda O_{i_{j+1}}\dots O_{i_n}|0\rangle \\
&\qquad\qquad = \pm\langle 0|QO_{i_1}\dots O_{i_j}\Lambda O_{i_{j+1}}\dots O_{i_n}|0\rangle = 0.
\end{aligned} \tag{3.5.5}$$

In this property, we can find that the physical operators are Q-cohomology class through the equivalence relation. Then the final requirement is that the energy-momentum tensor $T_{\mu\nu}$ of the theory should not only be Q-closed but also Q-exact:

$$T_{\mu\nu} \equiv \frac{\delta S}{\delta h^{\mu\nu}} = \{Q, G_{\mu\nu}\} \tag{3.5.6}$$

where $G_{\mu\nu}$ is some operator, and $h^{\mu\nu}$ is the metric. A direct consequence of this last property is that the correlation functions do not depend on the metric, and the proof is that

$$\begin{aligned}
\frac{\delta}{\delta h^{\mu\nu}}\langle O_1\dots O_n\rangle &= \frac{\delta}{\delta h^{\mu\nu}}\left(\int D\phi O_1\dots O_n e^{iS[\phi]}\right) \\
&= i\int D\phi O_1\dots O_n \frac{\delta S[\phi]}{\delta h^{\mu\nu}} e^{iS[\phi]} \\
&= i\langle O_1\dots O_n\{Q, G_{\mu\nu}\}\rangle = 0
\end{aligned} \tag{3.5.7}$$

where $D\phi$ is the measure of the path integral, and the Q symmetry acting on the last line shows the metric independence. Then there is a practical way to ensure (3.5.6) which is to use a Q-exact Lagrangean:

$$L = \{Q, V\} \tag{3.5.8}$$

where V is an operator. We can find this kind of Lagrangean make the calculation easier as it makes correlation function independent of the Planck' constant. It can be proven by put the Planck's constant back to the quantum measure:

$$\exp\frac{i}{\hbar}\left\{Q,\int_M V\right\}. \tag{3.5.9}$$

and consequently the same method in (3.5.7) can be used to show that

$$\frac{d}{d\hbar}\langle O_1 \dots O_n\rangle = 0. \tag{3.5.10}$$

Therefore one can find that the independence of $\hbar$ can make the calculation exactly in a classical limit.

Recall the equation (3.5.6), we can find the momentum operator by integrating the energy-momentum tensor field over a spatial hypersurface.

$$P_\mu = \{Q, G_\mu\} \tag{3.5.11}$$

where G_μ is a fermionic operator. Then we may consider the operator

$$O_\mu^{(1)} = i\{G_\mu, O^{(0)}\} \tag{3.5.12}$$

where $O^{(0)}$ is a scalar physical Q-closed operator, i.e.$\left\{Q, O^{(0)}\right\} = 0$. By doing some calculations We can find that

$$\frac{d}{dx^\mu}O^{(0)} = i\left[P_\mu, O^{(0)}\right] = \{Q, O_\mu^{(1)}\} \tag{3.5.13}$$

where we can write momentum operator in the form in (3.5.11) and then use the Jacobi identity to obtain the result of (3.5.13). Then we can define the operator $O_\mu^{(1)}$ to be a component of a one-form operator $O^{(1)} = O_\mu^{(1)}dx^\mu$, so that we can rewrite (3.5.13) as

$$dO^{(0)} = \{Q, O^{(1)}\}. \tag{3.5.14}$$

Then we can find that the integration of this equation over a closed curve $\gamma \subset M$ is zero by using the Stoke's theorem. We can repeat the above method again and again so that we can obtain a whole tower of p-form operators:

$$\begin{aligned}
\{Q, O^{(0)}\} &= 0 \\
\{Q, O^{(1)}\} &= dO^{(0)} \\
\{Q, O^{(2)}\} &= dO^{(1)} \\
&\cdots\cdot \\
\{Q, O^{(n)}\} &= dO^{(n-1)} \\
0 &= dO^{(n)}
\end{aligned} \tag{3.5.15}$$

The integrals of $O^{(p)}$, where $1 \leq p \leq n$, over a p-dimensional submanifold of M give us large classes of non-local physical operators. As the "top-form" operator needs to be integrated over the whole manifold, by using the Stoke's theorem again, we will have

$$\{Q, \textstyle\int_M O^{(n)}\} = 0. \tag{3.5.14}$$

which implies that we are free to add terms $t^\mu O_\mu^{(n)}$ for any coupling constant t^μ to the Lagrangean without breaking the fact that the theory is cohomological.

3.6 A-model

In the A-model, we regard ψ_-^i and $\psi_+^{\bar{\imath}}$ as the sections of $\Phi^*(TM^+)$ and $\Phi^*(TM^-)$ and regard the ψ_+^i and $\psi_-^{\bar{\imath}}$ as the section of $K \otimes \Phi^*(TM^+)$ and $\bar{K} \otimes \Phi^*(TM^-)$, and the scalar fields ϕ^i and $\phi^{\bar{\imath}}$ are still the same as the untwisted theory. To classify the twisted fermionic fields, we express the field in a new way:

$$\psi_+^i \equiv \psi_z^i \in K \otimes \Phi^*(TM^+) \tag{3.6.1.a}$$

$$\psi_-^i \equiv \chi^i \in \Phi^*(TM^+) \tag{3.6.1.b}$$

$$\psi_+^{\bar{\imath}} \equiv \chi^{\bar{\imath}} \in \Phi^*(TM^-) \tag{3.6.1.c}$$

$$\psi_-^{\bar{\imath}} \equiv \psi_{\bar{z}}^{\bar{\imath}} \in \bar{K} \otimes \Phi^*(TM^-) \tag{3.6.1.d}$$

In term of these renamed variables the action is

$$\begin{aligned} S = 2t \int_\Sigma d^2z \Big(\frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i\psi_{\bar{z}}^{\bar{\imath}} D_z \chi^i g_{\bar{\imath}i} + i\psi_z^i D_{\bar{z}} \chi^{\bar{\imath}} g_{\bar{\imath}i} \\ + R_{i\bar{\imath}j\bar{\jmath}} \psi_z^i \chi^{\bar{\imath}} \psi_-^j \chi^j \psi_{\bar{z}}^{\bar{\jmath}} \Big) \end{aligned} \tag{3.6.2}$$

Then we act the fermionic symmetry $\delta = \bar{\epsilon}_- \bar{Q}_+ + \epsilon_+ Q_-$ on the fields so that we can have the following relations

$$\delta\phi^i = \epsilon_+ \chi^i \qquad \delta\phi^{\bar{\imath}} = \bar{\epsilon}_- \chi^{\bar{\imath}} \tag{3.6.3.a}$$

$$\delta\psi_z^i = 2i\bar{\epsilon}_- \partial_z \phi^i + \epsilon_+ \Gamma^i{}_{jk} \psi_z^j \chi^k \qquad \delta\chi^i = 0 \tag{3.6.3.b}$$

$$\delta\psi_{\bar{z}}^{\bar{\imath}} = -2i\epsilon_+ \partial_{\bar{z}} \phi^{\bar{\imath}} + \bar{\epsilon}_- \Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}} \psi_{\bar{z}}^{\bar{\jmath}} \chi^{\bar{k}} \qquad \delta\chi^{\bar{\imath}} = 0 \tag{3.6.3.c}$$

As mentioned before, A-model is the topological cohomological theory, we can write the Lagrangean in the form of $\{Q_A, V\}$ and we can also find that the theory is independent of t. However, the action of theory is actually rewritten as

$$S' = it \int_\Sigma d^2z \{Q_A, V\} + t \int_\Sigma \Phi^*(\Omega) \tag{3.6.4}$$

where

$$V = g_{i\bar{\imath}} (\psi_z^i \partial_{\bar{z}} \phi^{\bar{\imath}} + \partial_z \phi^i \psi_{\bar{z}}^{\bar{\imath}}) \tag{3.6.5}$$

and Ω is the Kähler form of the target space, i.e. $\Omega = -ig_{i\bar{j}}dz^i \wedge d\bar{z}^j$.

$$t\int_{\Sigma}\Phi^*(\Omega) = 2tg_{i\bar{\imath}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{\imath}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{\imath}}) \tag{3.6.6}$$

The Lagrangean can be partly written in the Q-exact form, and the other term is related to the Kähler structure of the target manifold. On the second term of the action, we can make the pullback Φ^* on the Kähler form Ω back act on the worldsheet, and then it can be found that the it becomes important that Ω is a closed form so that the integral will only depend on the cohomology class of $\Phi(\Sigma)$. We denote the cohomology class as $\beta \in H_2(M)$, and the integral of the Kähler form over it is written as $\Omega \cdot \beta$. Then it shows that

$$\int_{\Sigma}\Phi^*(\Omega) = \int_{\Phi(\Sigma)}\Omega = \Omega \cdot \beta \geq 0. \tag{3.6.7}$$

If an anti-symmetric tensor field such as the Kalb–Ramond field, i.e. the B-field, is non-vanishing, we can replace the original real Kähler form Ω by a complexified Kähler form $\Omega_c = \Omega + iB = \left(b_{i\bar{j}} + ig_{i\bar{j}}\right)dz^i \wedge d\bar{z}^j$.

Then we can find the correlation relations of the A-model can be written as

$$\langle O_1 \dots O_n\rangle = e^{-it\,\Omega\cdot\beta}\int_M D\phi D\chi D\psi e^{-it\{Q_A,\int V\}}O_1 \dots Q_n \tag{3.6.8}$$

where $D\phi$, $D\chi$ and $D\psi$ are the corresponding measures of the path integral. We can find that the correlation function of these physical operators does not have dependence of the metric on the term V, but it only depends on the metric of target space M via the Kähler form Ω from the term $e^{-it\,\Omega\cdot\beta}$ by using the similar method in (3.5.7). That is to say, the theory does not depend on any structure which appears only in V, but it depend on the structure in rest term on the action (For this case, it is the second term in (3.6.4).). Therefore it can be found that the A-model is independent of the choice of the complex structure, but it clearly depends on the choice of the Kähler class of the target space. It means that the theory is “half-topological” with respect to the target space, as it depends on “half ” of the moduli of the Calabi-Yau manifold. In conclusion, the A-model theory only topologically depends on the Kähler moduli of the Calabi-Yau target space. Furthermore, by taking the derivative with respect to the coupling constant t and using the similar method in (3.5.10), the second factor in (3.6.10) (the Q_A-exact part) is also independent of the choice of t so that we can calculate it exactly by taking the classical $t \to \infty$ limit, and therefore we can somehow reduce the theory to a weak coupling theory.

3.7 B-model

In the B-model, we will do the same method as for the A-model. For this model, we twist ψ_+^i and ψ_-^i to be the section of $\bar{K} \otimes \Phi^*(TM^+)$ and $K \otimes \Phi^*(TM^-)$ respectively, and we also regard both $\psi_+^{\bar{\imath}}$ and $\psi_-^{\bar{\imath}}$ as the section of $\Phi^*(TM^-)$. Then the field ϕ^i and $\phi^{\bar{\imath}}$ remain the same as before.

$$\psi_+^i \in K \otimes \Phi^*(TM^+) \tag{3.7.1.a}$$

$$\psi_-^i \in \bar{K} \otimes \Phi^*(TM^+) \tag{3.7.1.b}$$

$$\psi_+^{\bar{\imath}} \in \Phi^*(TM^-) \tag{3.7.1.c}$$

$$\psi_-^{\bar{\imath}} \in \Phi^*(TM^-) \tag{3.7.1.d}$$

One may find that in the B-model, the new twisted scalar fields $\psi_\pm^{\bar{\imath}}$ are both the space-time (1,0)-forms, which is slightly different from the case in the A-model. Therefore, the new scalar field can be chosen in a more convenient way, and we rename all new fields as:

$$\eta^{\bar{\imath}} = \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}} \tag{3.7.2.a}$$

$$\theta_i = g_{i\bar{\jmath}}(\psi_+^{\bar{\jmath}} - \psi_-^{\bar{\jmath}}) \tag{3.7.2.b}$$

$$\rho_z^i = \psi_+^i \tag{3.7.2.c}$$

$$\rho_{\bar{z}}^i = \psi_-^i \tag{3.7.2.d}$$

One of the reason that we introduce θ field with lower holomorphic indices is that it may lead to some simpler expressions, e.g. $\{Q_B, \theta_i\} = 0$, but $\{Q_B, \theta^{\bar{\imath}}\} = -2\Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}}\eta^{\bar{\jmath}}\theta^{\bar{k}}$. Then we can also write the action of the B-model in the rename terms.

$$S = t\int_\Sigma d^2z\left(\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\eta^{\bar{\imath}}\left(D_z\rho_{\bar{z}}^i + D_{\bar{z}}\rho_z^i\right)g_{i\bar{\imath}} + i\theta_i(D_{\bar{z}}\rho_z^i - D_z\rho_{\bar{z}}^i) + R_{i\bar{\imath}j\bar{\jmath}}\rho_z^i\rho_{\bar{z}}^j\eta^{\bar{\imath}}\theta_k g^{k\bar{\jmath}}\right) \tag{3.7.3}$$

As the scalar supercharge operator in B-model is defined as $Q_B = \bar{Q}_- + \bar{Q}_+$, we can also find the supersymmetry transformation $\delta = \bar{\epsilon}\bar{Q}_+ + \bar{\epsilon}\bar{Q}_-$ by setting $\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}$ to be constant and $\epsilon_\pm = 0$ and the following relations:

$$\delta\phi^{\bar{\imath}} = \bar{\epsilon}\eta^{\bar{\imath}} \tag{3.7.4.a}$$

$$\delta\phi^i = \delta\theta_i = \delta\eta^{\bar{\imath}} = 0 \tag{3.7.4.b}$$

$$\delta\rho_\mu^i = \pm i\bar{\epsilon}\partial_\mu\phi^i \tag{3.7.4.c}$$

The action of the B-model can also be rewritten partly in the form of $\{Q_B, V\}$ which is in the same way as what we do in the A-model section.

$$S = t\int_\Sigma \{Q_B, V\} + tW \tag{3.7.5}$$

where

$$V = g_{i\bar{j}}(\rho_z^i \partial_{\bar{z}} \phi^{\bar{j}} + \rho_{\bar{z}}^i \partial_z \phi^{\bar{j}}) \quad (3.7.6)$$

and

$$W = \int_\Sigma \left(-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right). \quad (3.7.7)$$

D in the equation (3.7.7) is the exterior derivative on the worldsheet Σ, and $\wedge$ is the wedge product. It can be found that the term W in the action is anti-symmetric in the exchange of the holomorphic and anti-holomorphic z indices, and it can be written as a differential $(1,1)$-form. Then integral of such a form over the two-dimensional worldsheet Σ is independent of the metric, so the only metric dependence of the action is in the term $\{Q_B, V\}$. Furthermore the variations of W with respect to the cohomology class of the Kähler form Ω on the target space M are Q_B-exact, so the metric dependence on the Kähler structure of the Calabi-Yau target space is also vanishing. However, by looking through the anti-commutation relations between the B-model scalar supercharge operator and the untwisted original scalar field ϕ, we can find the asymmetry in the relation,

$$\{Q_B, \phi^i\} = 0 \qquad \{Q_B, \phi^{\bar{i}}\} = -\eta^{\bar{i}}, \quad (3.7.8)$$

so that it implies the theory depends on the choice of the complex structure of the target space [24]. It shows that the B-model is also “half-topological” with respect to the target space, and the theory only topologically depends on the complex structure moduli of the Calabi-Yau target space. Furthermore for the t-dependence, we can use the same way as we did in the A-model introduction part to show that the Q_B-exact term of the action is also independent of the coupling constant t. On the other hand, one can find that the term W in (3.7.7) is linear in θ, but the other term, i.e. the Q_B-exact term has not θ-dependence. Therefore we can remove the t-dependence by redefining $\theta \to \theta/t$ so that the term tW in the action is changed to be just W. Then the theory becomes independent of the coupling constant t except for factors that come from the θ-dependence of the observables. Due to the trivial t-dependence of the B-model, the calculations can be taken for large t limit.

3.8 Coupling to Gravity

So far the A- and the B-model introduced are both still in topological field theory, in which the metric of the theory are not dynamical. To make the models be string theories, we need to couple them to the gravity, which means the worldsheet theory should not only involve the path integral over the maps Φ to the target space and their fermionic partners, but also a path integral over the metric $h_{\mu\nu}$ of the worldsheet. Then we call such theory the topological string theory. There are several things needed to achieve such coupling. Firstly, of course the Lagrangean of the theory should be rewritten in a covariant way by changing the flat metrics to the dynamical ones, and the covariant derivatives and the factor of $\sqrt{\det h}$ should be introduced as well. Furthermore an Einstein-Hilbert term also needs to be introduced to act like the kinetic term of the metric field, and the new term should still preserve the symmetries of the original theories. Finally, the theory should be able to be integrated over space of all metrics. The first two steps are relatively straightforward to achieve, because changing the Lagrangean of theory may not lead to a large change of the properties of the theory. However, integrating over the space of all metric may result in some difficulties. Even though we have a metric independence of the theory, it is not a correct way to just integrate the partition function over space of all metrics and divide the results by a volume of the topological "gauge group". Therefore an alternative way of integrating is first to do the integration over all conformally equivalent metrics and to do the integral over the remaining finite-dimensional moduli space of the worldsheet [24]. It may not be explained further as the theories treated as the cohomological field theories could give the sufficient explanation for the next chapter.

4 Mirror Symmetry

After discussing the two models of the topological string theory, we may talk about the mirror symmetry, which will make two twisted models isomorphic to each other by exchanging the two topological structures. Besides the mirror symmetry can also link the different types of superstring theories as well. This kind of duality relation will make it possible to calculate one theory by doing the calculation on the other, which is similar to the cases for T-duality and S-duality. For example some calculation on A-model will be way more difficult than the one in B-model, but the mirror symmetry conjecture state it would be possible to just obtain the result of A-model by just calculate the dual case in B-model. Such symmetry relation between two internal spaces may lead a new understanding of the string theory to the physicists nowadays.

In this chapter, we will firstly give some brief introduction to the mirror symmetry and show how this type of duality relates two string theories with totally different topological properties [33][34]. Then we will discuss the mirror symmetry in different aspects and some applications of mirror symmetry will be given as well [25[28][32].

4.1 Brief Introduction to Mirror Symmetry

Generally speaking, mirror symmetry is a conjecture that there are pairs of Calabi-Yau manifolds with different topological properties that imply the same superconformal field theory, and we call such pair of Calabi-Yau manifold a mirror pair.

$$\mathrm{SCFT}(X) \simeq \mathrm{SCFT}(\hat{X}) \tag{4.1.1}$$

where X and $\hat{X}$ are the mirror pair. In superstring theory, the mirror symmetry lead two type-II theories to have mirror duality, which make an isomorphism between the type-IIA theory in 3-dimensional Calabi-Yau manifold X and the type-IIB theory in the mirror manifold $\hat{X}$ and vice versa (while in the Calabi-Yau mirror pairs with even $\dim_{\mathbb{C}} X$ the mirror symmetry will lead a self-duality on both type-IIA and type-IIB superstring theories). Meanwhile, there is another duality relation between two type-II theories, which is called the T-duality, a duality between different target spaces, and the simplest case of such duality is to lead a equivalence between one theory in a compact internal manifold with a circle of radius R and the other with radius $1/R$ (in the natural units). Then there is a conjecture that there is connection between mirror symmetry and T-duality, which is called the SYZ conjecture proposed by Strominger, Yau and Zaslow in 1996 [4][16]. Furthermore, in the topological string theory, the mirror symmetry also implies that the A-model in target space X is isomorphic to the B-model in the mirror manifold $\hat{X}$ and vice versa.

In the aspect of Hodge numbers, the mirror symmetry implies an extra symmetry relation on the Hodge diamond for Calabi-Yau manifolds such that

$$h^{r,s}(X) = h^{n-r,s}(\hat{X}) \tag{4.1.2}$$

where n is the complex dimension of the Calabi-Yau manifolds, $0 \leq r, s \leq n$, and X and $\hat{X}$ are the mirror pairs. For the $n = 3$ case, we can find the mirror manifold through relation, but for the manifold that $h^{2,1} = 0$, there is not mirror symmetry since that the Hodge number $h^{1,1}$ of a Calabi-Yau manifold must be positive integer number as shown in (2.5.10.c). By looking at the Hodge diamond of the Calabi-Yau threefold, we can find the symmetry switches the only two independent Hodge numbers to each other, i.e. $h^{1,1}(X) = h^{2,1}(\hat{X})$ and $h^{2,1}(X) = h^{1,1}(\hat{X})$. As the mirror symmetry exchange the dimensions of two cohomology groups, it would lead the isomorphism that $H^{1,1}(X) \simeq H^{2,1}(\hat{X})$ and $H^{2,1}(X) \simeq H^{1,1}(\hat{X})$. By using the knowledge in the moduli space section, we can find that the Kähler moduli space and the complex structure moduli space are exchanged under the mirror symmetry for the Calabi-Yau manifolds. Such moduli exchange can be understood in a different direction. In the topological string theory, we have found that the A-model has the topological dependence of the only Kähler moduli on the target space (A-model may have some other topological dependence of the worldsheet of the strings), and the B-model only has the topological dependence of the complex structure moduli of the target space. Then the mirror symmetry exchanges the two moduli to lead the equivalence between two models.

$$\begin{array}{ccc} A\text{-model on } X & \leftrightarrow & \text{Kähler moduli of } X \\ & & \text{Mirror symmetry } \updownarrow \ (X, \hat{X}) \\ B\text{-model on } \hat{X} & \leftrightarrow & \text{complex moduli of } \hat{X} \end{array} \tag{4.1.3}$$

Consequently, the triple products of the coholomogical fields of each model will also have an isomorphism under the mirror symmetry, and that will lead a kind of Yukawa coupling equivalence relation between two theories.

4.2 Mirror Symmetry in Aspects of T-duality

Let's consider the simplest case for T-duality (which is also the mirror symmetry), in which the compact manifold is just a circle of radius R, i.e. $M_{10} = M_9 \times S^1$, and such way of hiding extra dimension is called the Kaluza-Klein compactification. Then for the coordinate in the 10$^{\text{th}}$

dimension X^9, we will have the following relation:

$$X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) + 2\pi R w \qquad (4.2.1)$$

where τ and σ are the coordinate on the worldsheet of the string, and w is the wind number which is the number of times that the string winds around on S^1. The number w is a quantum number of the spectrum of the physical state, and there is another quantum number for the theory, which is corresponding to the momentum of the center of mass of the string going around S^1 and is denoted as n. Then the contribution to the worldsheet energy of the state form these two quantum numbers is [6]

$$E_{n,w} = (wR)^2 + \left(\frac{n}{R}\right)^2. \qquad (4.2.2)$$

We can find the value $E_{n,w}$ here is invariant under the transformation that $R \leftrightarrow 1/R$. Taking the inverse of the radius of the compact S^1 as the new radius leads the symmetry, and consequently the wind number and the momentum quantum number are exchanged. This is the T-duality for the circle.

Then we can talk about the T-duality for a rectangular 2-torus T^2. Such compact manifold is the simplest Calabi-Yau manifold, and it is also the only case for one-dimensional Calabi-Yau manifold. The Hodge numbers of T^2 are all just one, so the mirror manifold of a 2-torus is just another 2-torus. Taking the two radius of the torus as R_1 and R_2, we can define

$$A = iR_1R_2 \qquad \tau = iR_2/R_1 \qquad (4.2.3)$$

which characterizing the Kähler structure and the complex structure of the manifold. The T-duality applying on such manifold will lead the transformation of $R_1 \to 1/R_1$ to be symmetric under exchanging $A \leftrightarrow \tau$, an exchange between the Kähler moduli and the complex structure moduli. The exchange between two moduli spaces in such mirror symmetry may naturally lead to the exchange of A-model and B-model. T^2 is a trivial Cablabi-Yau manifold with only 1 complex dimension, and it has genus 1 which can lead us to still be able to look at the one-loop free energy F_1 of the theory. The B-model at one loop computes the inverse of the determinant of the $\bar{\partial}$ operator acting on T^2, and the determinant is the Deddkind η function:

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{4.2.4}$$

where $q = e^{2\pi i \tau}$. As the mirror symmetry implies the B-model in target space of T^2 is isomorphic to the A-model in the mirror manifold, another T^2, the A-model will be able to have the calculation with η function but with an exchange $\tau \leftrightarrow A$. The factor q will be related to e^{-A} by the mirror symmetry, and the coefficient of e^{-nA} will counts maps which wrap the torus over itself n times.[25]

4.3 Yukawa Couplings in Mirror Symmetry

As mentioned in the mirror symmetry introduction section, the cohomology class of the mirror pairs can be linked by the mirror symmetry, and it leads an isomorphism between $H^{1,1}$ and $H^{2,1}$ for the triple products, which could determine a Yukawa coupling of the theories. The fields in the interaction are considered in the fundamental (248) representation of E_8 which is the gauge group of the uncompactified heterotic string theory. Under the compactification of the Calabi-Yau internal space with $SU(3)$ holonomy group, the group E_8 will be break into a E_6 group for the

4-dimensional Minkowshi spacetime and a $SU(3)$ group, i.e. $E_8 \subset E_6 \times SU(3)$. Therefore we have the decomposition $248 = (27,3) \oplus (\overline{27},\overline{3}) \oplus (1,8) \oplus (78,1)$, where the 4-dimensional matter field transform as 27 and $\overline{27}$ of E_6 (which are also the fundamental representation of E_6 and its dual) and the zero modes transform in 3 and $\overline{3}$ of the group $SU(3)$. Then we can find there are two kind of Yukawa coupling for the fields, which are $\langle 27^3 \rangle$ and $\langle \overline{27}^3 \rangle$, and the first one is in the form that

$$\kappa_{abc}^{0(27)}(X) \equiv \kappa_{abc}^{0}(X) = \int_X h_a \wedge h_b \wedge h_c, \tag{4.3.1}$$

where $h_a \in H_{\bar{\partial}}^{1,1}(X)$ and the index a, b and c are in the range from 1 to the Hodge number $h^{1,1}$. Then the second kind of Yukawa coupling is that

$$\kappa_{\alpha\beta\gamma}^{(\overline{27})}(X) \equiv \bar{\kappa}_{\alpha\beta\gamma}(X) = \int_X \Omega \wedge b_\alpha^i \wedge b_\beta^j \wedge b_\gamma^k \, \Omega_{ijk} \tag{4.3.2}$$

where Ω is the unique non-vanishing holomorphic (3,0)-form on the Calabi-Yau manifold, $b_\alpha^i = \Omega^{ijk}(b_\alpha)_{jk\bar{l}}\, dz^{\bar{l}}$ such that $b_\alpha \in H_{\bar{\partial}}^{2,1}(X)$, and the index α, β and γ are in the range from 1 to $h^{2,1}$. We can find that the coupling in (4.3.1) in purely topological and the one in (4.3.2) depends on the complex structure through Ω. There are one-to-one correspondence relations between the field and the moduli: $27 \leftrightarrow$ Kähler moduli and $\overline{27} \leftrightarrow$ complex structure moduli. Under the mirror symmetry, the full $\langle 27^3 \rangle$ couplings on the manifold X depend on the Kähler moduli in such a way that the full $\langle \overline{27}^3 \rangle$ couplings on the mirror manifold $\hat{X}$ depend on the complex structure moduli. [28][32]

In the A-model the Yukawa three-point correlation function is computed as

$$\langle h_a, h_b, h_c \rangle = \kappa^0_{abc}(X) + \sum_{\beta \neq 0} n_\beta \int_\beta h_a \int_\beta h_b \int_\beta h_c \frac{e^{2\pi i \int_\beta \Omega^{\mathbb{C}}}}{1 - e^{2\pi i \int_\beta \Omega^{\mathbb{C}}}} \tag{4.3.3}$$

where n_β is the instanton number and $\beta \in H_2(X; \mathbb{Z})$ and $\Omega^{\mathbb{C}}$ is the complexified Kähler form. Then the corresponding dual case in B-model have the three point correlation defined as

$$\langle b_\alpha, b_\beta, b_\gamma \rangle = \int_X \Omega \wedge (\nabla_{b_\alpha} \nabla_{b_\beta} \nabla_{b_\gamma} \Omega) \tag{4.3.4}$$

where Ω is still the holomorphic 3-form on X and ∇_{b_α} is the Gauss-Manin connection taking a (r, s) class to a $(r+1, s-1)$ class.

5 Conclusion

We has discussed how the compactification of string theory required the internal space to be 3-dimensional Calabi-Yau manifold and also discussed the holonomy group, cohomology class and the two moduli spaces of such manifolds. Furthermore the two twisted $N = (2,2)$ topological string theories, A-model and B-model were also introduced and we showed that they are totally determined by the Kähler structure and the complex structure of the target space respectively. Then, the mirror symmetry conjecture were illustrated and it makes the mirror pairs by exchanging the two topological structures and makes an isomorphism between two string theories, including two twisted models. Some applications of the mirror symmetry were briefly introduced in the end of the final chapter.

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