Methods for wave analysis and application to the perturbed wave equation in Schwarzschild background

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Abstract

In this work we present a brief set of methods used to analyse wave equation solutions. After introducing the reader to some elementary results, we show how one can use a commuted vector field approach to establish ILED estimates without first showing boundedness of energy. We present its use in the study of solution to the wave equation in Schawrzschild spacetime and derive known results for the first order perturbed wave equation in that same spacetime. Our results elaborate on and clarify results by Holzegel and Kauffman.

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1 Introduction

The study of wave mechanics is undoubtedly omnipresent in Physics and has been for a long time. An extensive study of solutions to the wave equation in flat space has led to many results [1, 2, 16]. Many have set their focus on perturbed wave equations or non-linear wave equations [4, 21] as these represent interesting physical systems or interesting mathematical studies. In the field of differential equations, one is very often interested by the notion of existence or non-existence of solutions as well as the behaviour of such solutions. When considering a solution to the non-homogeneous wave equation in flat space, one might wonder whether it decays as time progresses and how.
All these questions have their equivalent in a different setting. For instance, many results for the wave equation in Schwarzschild spacetime have been produced [11, 3, 4, 5]. A similar treatment was also done for Kerr solutions [10, 11, 24]. It goes without saying that due to the increased complexity of the wave equation in such settings, many elementary methods could not be ported from flat space. The more recent development in energy estimates and alike have proven more robust [7, 12, 13].

In this work we will try and give a brief presentation of the current methods used in the aforementioned contexts. The aim being to provide suitable bounds on smooth solutions of the perturbed wave equation found in [17]. In Section 2, we recall analytical techniques and function spaces often used in wave equations. This section will present the basics of distribution and their use in differential equation analysis, notably how one can study C^0 solutions of a differential equation. We will also provide the reader with a set of results in Fourier analysis that will become useful when studying the decomposition of solutions to the wave equation. Section 3 will present a couple of elementary results we mentioned earlier. We will derive explicit solutions for the 1+1-dimensional wave equation (d'Alembert's formula) and the 1+3-dimensional wave equation (Kirchoff's formula). We will see how the Fourier analysis can be used to derive an energy conservation in Section 3.3 and compare our results to the more general case in Section 3.4. The majority of Section 3.4 is dedicated to the introduction to the newer methods that are more easily transferable to variable coefficient wave equations. In Section 4 we will briefly present the reader how to use those new methods in the context of Schwarzschild background using the energy-momentum tensor formalism. Finally, we will follow a paper by Holzegel and Kauffman in which a derivation of an energy estimate is done for a perturbed wave equation in a Schwarzschild spacetime in Section 5. Every calculation from [17] was reproduced and detailed in that section. These results all follow from the methods introduced in previous chapters. Their aim is to show stability for smooth solutions of the perturbed wave equation.

2 Background material

Before tackling any detailed analysis of the wave equation we must make sure we agree on some concepts and definitions. This section will give a brief summary of some properties of Sobolev spaces, Schwartz spaces and Fourier transforms. For more details on the former the reader is invited to consult [19], from which most results were taken. These concepts will become useful mainly in Section 3.3 where we will derive explicit solutions to the wave equation.

2.1 Distributions

In this section we let $\Omega \subseteq \mathbb{R}^n$ be an open subset.
Used extensively in Physics, distributions have become an essential tool in the study of solutions to partial differential equations. By the omnipresence of the latter in every field of physics, it is natural to start by defining what one means by distribution. Following [19], we define

Definition 2.1. The space of test functions in Ω, denoted $\mathcal{D}(\Omega)$ is the space of all $C^\infty(\Omega, \mathbb{R})$ which have compact support.

From this, we now define the space of interest in this section.

Definition 2.2. The dual of $\mathcal{D}(\Omega)$ is called the space of distributions on Ω, denoted $\mathcal{D}'(\Omega)$. Its elements are called distributions on Ω.

One quickly sees that any function $f \in L^p(\Omega, \mathbb{R})$ naturally defines a distribution as

$$\forall \varphi \in \mathcal{D}(\Omega), \quad f(\varphi) = \int_\Omega f\varphi < \infty$$

holds. This can be seen by applying Hölder's inequality to $f(\varphi)$ and using the definition of $L^p(\Omega, \mathbb{R})$ as well as the fact that φ has compact support. See Proposition A.3 for a detailed proof.
One important distribution, which appears everywhere in physics is the Dirac delta. One defines it, as usual, by its action on test functions

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \delta(\varphi) = \varphi(0) \tag{2.1}$$

More importantly, we can define derivatives of distributions by the use of integration by parts. Making use of the multi-index notation for $\alpha \in \mathbb{N}^n$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}$$

we define

Definition 2.3. Let $f \in \mathcal{D}'(\Omega)$ be a distribution on Ω. For any $\alpha \in \mathbb{N}^n$ we define $D^\alpha f$ as

$$\forall \varphi \in \mathcal{D}(\Omega), \quad D^\alpha f(\varphi) = (-1)^{|\alpha|} f(D^\alpha \varphi) \tag{2.2}$$

By definition of $\mathcal{D}(\Omega)$ one can easily see that if f is a distribution then so is $D^\alpha f$. See Proposition A.4.

We can use this definition of the derivative of a distribution to extend our usual definition of derivative. This extension is called the derivative in the sense of distributions. One possible example of such a derivative would be for the absolute value function $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}$

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & x < 0 \end{cases}$$

In the usual sense of derivative, the absolute value function is not $C^1(\mathbb{R})$ as the derivative at the origin is not well defined. However, being a locally integrable function it is also a distribution on $\mathbb{R}$ and it this new sense of derivatives it is $C^1(\mathbb{R})$ as we have

$$\begin{aligned} \left(\frac{d|x|}{dx}\right)(\varphi) &= \int_{\mathbb{R}} \left(\frac{d|x|}{dx}\right)(x)\varphi(x)dx \\ &= -\int_{\mathbb{R}} |x|\varphi'(x)dx \\ &= \int_{-\infty}^{0} x\varphi'(x)dx - \int_{0}^{\infty} x\varphi'(x)dx \\ &= -\int_{-\infty}^{0} \varphi(x)dx + \int_{0}^{\infty} \varphi(x)dx \end{aligned}$$

Finally, we conclude that for any $a \in \mathbb{R}$

$$\left(\frac{d|x|}{dx}\right)(x) = \begin{cases} 1, & \text{if } x > 0 \\ a, & x = 0 \\ -1, & x < 0 \end{cases} \tag{2.3}$$

We immediately see that in order to retain uniqueness of derivatives we must identify distributions if they are equal nearly everywhere. It, of course, falls immediately from the integral present in the definition of distributions.
One may also prove that the derivative in the sense of distributions conserves all the usual properties and identities of regular derivatives [19].

As a final note on distributions, this new definition of derivative allow us to study C^0 solutions to partial differential equations and even allow for the study of singularities in general solutions (and their propagation) [16, 18]. This, would, however require the introduction of the concept of Sobolev spaces [6, 19].

2.2 Schwartz space and Fourier transform

To conclude this chapter on background material, let us recall the definition of the Schwartz space of $\mathbb{R}^n$.

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \,\big|\, \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty\} \tag{2.4}$$

In other words, it is the space of infinitely differentiable functions whose derivatives decay faster than any polynomial power. On this space, we have a countable family of semi-norms

$$\|f\|_{\alpha\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \tag{2.5}$$

And in fact one can define a metric on such a vector space and show that it is a complete space with respect to that metric [15].
The Schwartz space is both Mathematically and Physically interesting as it is natural to define on it the Fourier transform.

Definition 2.4. Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform as

$$\mathcal{F}(t)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \tag{2.6}$$

We also define the inverse Fourier transforms as

$$\mathcal{F}^{-1}(t)(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx \tag{2.7}$$

The interesting property of the Fourier transform on Schwartz space is the fact that it maps it back to itself. Additionally, given that $f \in C^\infty$ we also have the property

$$\mathcal{F}(D^\alpha f)(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \mathcal{F}(f)(\xi) \tag{2.8}$$

as well as the acclaimed Plancherel theorem

Theorem 2.1 (Plancherel)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. If $\mathcal{F}(f)$ denotes the Fourier transform of f then its L^2 norm is equal to that of f. In other words*

$$\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \tag{2.9}$$

3 The wave equation in Minkowski space

In this section we will be interested in the analysis of the wave equation in Minkowski space. This will be a good place to remind the reader of the elementary formulae. We will give a brief overview of different methods that exist to find an explicit solution. Finally, we will present newer methods that are applicable to general wave equations.

$$\Box_\eta u = -\partial_t^2 u + \Delta u = 0 \tag{3.1}$$

See [16] for more details.

3.1 The 1+1-dimensional case and d'Alembert's formula

Let us first take a look at the wave equation on $\mathbb{R}^{1+1}$. It turns out that this restriction to a smaller dimension decreases greatly the difficulty to solve the equation but also allows us to forge some results that will become primordial in our higher dimensional analysis.

We follow here Holzegel [15]. Let us consider the Cauchy problem

$$-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = 0 \quad u(0,x) = f(x) \quad \partial_t u(0,x) = g(x), \tag{3.2}$$

and look for soluions $u \in C^2(\mathbb{R}^{1+1}, \mathbb{R})$. The reader will find useful to know that the following derivation is analogous to the $u \in C^2(\mathbb{R}^{1+1}, \mathbb{C}^n)$ case.

Note that, contrary to the usual convention, we are writing the wave equation with the negative sign on the time derivative. This choice, undoubtedly, won't change our analysis but is there to keep a level of consistency with our later redefinition of the wave operator on Lorentzian manifolds in section 4. We are also suppressing the constant speed of light factor in front of the spacial derivative. This can easily be restored after performing a rescaling of the time axis resulting in no loss of generality (See Proposition A.1).

We easily see that $u(t,x) = \phi(x+t) + \psi(x-t)$ solves (3.2) for any $\phi, \psi \in C^2(\mathbb{R})$ satisfying

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ \phi'(x) - \psi'(x) &= g(x) \end{aligned}$$

It turns out that this form of u encompasses all solutions (see Proposition A.2).
We can then differentiate the first equation and solve the system of equations to get

$$\begin{aligned} \phi'(x) &= \frac{1}{2}\left(f'(x) + g(x)\right) \\ \psi'(x) &= \frac{1}{2}\left(f'(x) - g(x)\right) \end{aligned}$$

We can put these back into the expression for $u(t,x)$ to get

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left(f(x+t) + \int_{x_0}^{x+t} g(s)ds\right) + \frac{1}{2}\left(f(x-t) - \int_{x_0}^{x-t} g(s)ds\right) \\ &= \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds \end{aligned} \tag{3.3}$$

This explicit form is known as d'Alembert's formula. From it we can immediately deduce the uniqueness property

Proposition 3.1. For any $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ there is a unique solution $u \in C^2(\mathbb{R}^{1+1}, \mathbb{R})$ to the Cauchy problem (3.2).

3.2 The 1+3-dimensional case and Kirchoff's formula

It goes without saying that the case of physical interest is that of the wave equation in 3-dimensional space. Following [15], we will construct an explicit solution to the Cauchy problem

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0 \quad u(0,x) = f(x) \quad \partial_t u(0,x) = g(x) \tag{3.4}$$

for which $u \in C^2(\mathbb{R}^{1+3}, \mathbb{R})$.

For a more general approach and until further notice, we will denote the number of spacial dimensions by d. This will allow us to derive results valid for any dimensionality.
To do so we will make use of a continuous function $h : \mathbb{R}^d \longrightarrow \mathbb{R}$ with which we define

$$M_h(x,r) = \frac{1}{\mathrm{Vol}(S^d)r^{d-1}} \int_{|y-x|=r} h(y)dS_y \tag{3.5}$$

We immediately see the importance of M_h for h a solution of (3.4). In that case, the former is an average over a d-sphere of the solution h. That average turns out to also be a solution of (3.4) and allow a reduction in dimensionality. More on this in Section 5.1 of [2].
What follows will simply be a set a manipulations on M_h to make the previous statement more clear and allow for an explicit solution to be written.

Performing a change of variable $y = x + r\xi$ with $|\xi| = 1$, we see that $dS_y = r^{d-1}dS_\xi$ and

$$M_h(x,r) = \frac{1}{\mathrm{Vol}(S^d)} \int_{|\xi|=1} h(x+r\xi)dS_\xi$$

By this explicit form of M_h we can extends its definition to all values of $r \in \mathbb{R}$. In other words, we define $M_h(x,-r) = \frac{1}{\mathrm{Vol}(S^d)} \int_{|\xi|=1} h(x-r\xi)dS_\xi = M_h(x,r)$.

Now, restricting ourselves to $h \in C^2(\mathbb{R}^d, \mathbb{R})$, we have

$$\begin{aligned}\partial_r M_h(x,r) &= \frac{1}{\mathrm{Vol}(S^d)}\int_{|\xi|=1}\xi\cdot\nabla_y h(x+r\xi)dS_\xi = \frac{1}{\mathrm{Vol}(S^d)r}\int_{|\xi|=1}\xi\cdot\nabla_\xi h(x+r\xi)dS_\xi \\ &= \frac{1}{\mathrm{Vol}(S^d)r}\int_{|\xi|\le 1}\nabla_\xi\cdot\nabla_\xi h(x+r\xi)d\xi = \frac{1}{\mathrm{Vol}(S^d)r}\int_{|\xi|\le 1}\Delta_\xi h(x+r\xi)d\xi\end{aligned}$$

using $\int_\Omega \nabla\cdot A d\xi = \int_{\partial\Omega} A\cdot dS$ where $A=\nabla_\xi h(x+r\xi)$, $dS=\xi dS_\xi$. Further rearranging gives us,

$$\begin{aligned}\partial_r M_h(x,r) &= \frac{r}{\mathrm{Vol}(S^d)}\int_{|\xi|\le 1}\Delta_y h(x+r\xi)d\xi = \frac{r}{\mathrm{Vol}(S^d)}\int_{|\xi|\le 1}\Delta_x h(x+r\xi)d\xi \\ &= \frac{r}{\mathrm{Vol}(S^d)}\int_0^1 d\rho\rho^{d-1}\Delta_x\int_{|\xi|=1}h(x+r\rho\xi)dS_\xi = r\int_0^1 d\rho\rho^{d-1}\Delta_x M_h(x,r\rho) \\ &= r^{1-d}\int_0^r d\rho\rho^{d-1}\Delta_x M_h(x,\rho)\end{aligned}$$

We can deduce of this result that M_h obeys the Darboux equation

$$\begin{aligned}\partial_r\left(r^{d-1}\partial_r M_h(x,r)\right) &= \partial_r\int_0^r d\rho\rho^{d-1}\Delta_x M_h(x,\rho) \\ &= r^{d-1}\Delta_x M_h(x,r) \\ \iff \Delta_x M_h(x,r) &= \left(\partial_r^2+\frac{d-1}{r}\partial_r\right)M_h(x,r)\end{aligned}\tag{3.6}$$

Now that we have established this result and given a solution $u\in C^2(\mathbb{R}^{1+d},\mathbb{R})$ to the wave equation (3.4), we define

$$M_u(t,x,r) = \frac{1}{\mathrm{Vol}(S^d)}\int_{|\xi|=1}u(t,x+r\xi)dS_\xi \tag{3.7}$$

We immediately see that we can recover our solution u from M_u by the relation

$$M_u(t,x,0) = \frac{1}{\mathrm{Vol}(S^d)}\int_{|\xi|=1}u(t,x)dS_\xi = \frac{\mathrm{Vol}(S^d)}{\mathrm{Vol}(S^d)}u(t,x) = u(t,x)$$

and that by (3.4) and (3.6) M_u satisfies

$$\partial_t^2 M_u(t,x,r) = \Delta_x M_u(t,x,r) = \left(\partial_r^2+\frac{d-1}{r}\partial_r\right)M_u(t,x,r)$$

We can now restrict ourselves back to 3-dimensional space in order to extract an explicit solution for u. Given our previous equation, we see that $rM_u(t,x,r)$ satisfies the 1-dimensional wave equation

$$\begin{aligned}\partial_t^2 M_u(t,x,r) &= \left(\partial_r^2+\frac{2}{r}\partial_r\right)M_u(t,x,r) = \frac{1}{r}\partial_r^2(rM_u(t,x,r)) \\ \iff -\partial_t^2(rM_u)(t,x,r) &+ \partial_r^2(rM_u)(t,x,r) = 0\end{aligned}$$

with initial values

$$rM_u(0,x,r) = rM_f(x,r) \qquad \partial_t(rM_u)(0,x,r) = rM_g(x,r)$$

Thus, by Proposition 3.1, we know there is a unique solution $(rM_u)(\cdot,x,\cdot) \in C^2(\mathbb{R}^{1+1},\mathbb{R})$ given by d'Alembert's formula

$$rM_u(t,x,r) = \frac{1}{2}\big((r+t)M_f(x,r+t) + (r-t)M_f(x,r-t)\big) + \frac{1}{2}\int_{r-t}^{r+t} \rho M_g(x,\rho)d\rho \tag{3.8}$$

We can rearrange the previous equation and take the limit $r \longrightarrow 0$ to get

$$\begin{aligned}
M_u(0,x,r) &= \lim_{r\longrightarrow 0}\left[\frac{1}{2}(M_f(x,r+t) + M_f(x,r-t)) + \frac{t}{2r}(M_f(x,r+t) - M_f(x,t-r)) + \frac{1}{2}\int_{r-t}^{r+t}\frac{\rho}{r}M_g(x,\rho)d\rho\right] \\
&= M_f(x,t) + t\partial_t M_f(x,t) + \frac{1}{2}\lim_{r\longrightarrow 0}\int_{-t}^{t}\left(\frac{\rho}{r}+1\right)M_g(x,\rho+r)d\rho
\end{aligned}$$

Now by the fact that $M_g \in C^1(\mathbb{R},\mathbb{R})$ we have, in a neighbourhood of $r = 0$, the Taylor expansion

$$M_g(x,\rho+r) = M_g(x,\rho) + r\partial_\rho M_g(x,\rho) + o(r)$$

and thus, by parity of $M_g(x,\cdot)$, we see that

$$\begin{aligned}
M_u(0,x,r) &= M_f(x,t) + t\partial_t M_f(x,t) + \frac{1}{2}\lim_{r\longrightarrow 0}\int_{-t}^{t}\left(\frac{\rho}{r}+1\right)\big(M_g(x,\rho) + r\partial_\rho M_g(x,\rho) + o(r)\big)\,d\rho \\
&= M_f(x,t) + t\partial_t M_f(x,t) \\
&\quad + \frac{1}{2}\lim_{r\longrightarrow 0}\int_{-t}^{t}\left(\frac{\rho}{r}M_g(x,\rho) + M_g(x,\rho) + \rho\partial_\rho M_g(x,\rho) + r\partial_\rho M_g(x,\rho) + o\left(r^0\right)\right)d\rho \\
&= M_f(x,t) + t\partial_t M_f(x,t) + \frac{1}{2}\int_{-t}^{t}\big(M_g(x,\rho) + \rho\partial_\rho M_g(x,\rho)\big)\,d\rho \\
&= M_f(x,t) + t\partial_t M_f(x,t) + \frac{1}{2}\int_{-t}^{t}\partial_\rho\left(\rho M_g(x,\rho)\right)d\rho \\
&= \partial_t\left(tM_f(x,t)\right) + tM_g(x,t)
\end{aligned}$$

Finally, we conclude that a solution to our Cauchy problem (3.4) has the explicit form

$$u(t,x) = \partial_t\left(\frac{1}{4\pi t}\int_{|y-x|=|t|} f(y)dS_y\right) + \frac{1}{4\pi t}\int_{|y-x|=|t|} g(y)dS_y \tag{3.9}$$

known as Kirchoff's formula.
However, as pointed out by Holzegel in [15], the solution explicitly shows some loss in regularity. Indeed, having performed the limit on the expression for M_u, in order to get a C^2 solution u we will require $f \in C^3$ and $g \in C^2$. Due to this, it is more common to deal with finite energy solutions such as solutions in certain Sobolev spaces

3.3 The frequency decomposition

Another way of deriving an explicit solution to (3.4) would be through the use of Fourier transforms. This method is in some sense easier to use but forces us to reduce ourselves to a smaller set of solution u: those that are smooth and behave suitably at infinity. We can formally construct this restriction on u by requiring it to be in $\mathcal{S}(\mathbb{R}^d)$, the Schwartz space. Then any such u obeys

$$\forall \alpha, \beta \in \mathbb{N}^n, \quad \|u\|_{\alpha\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty \tag{3.10}$$

as detailed in Section 2.2.
In turns out that any solution in $\mathcal{S}(\mathbb{R}^d)$ is of physical interest as such behaviours at infinity are desired in physics. However, one must keep in mind that not all physically interesting solutions are in $\mathcal{S}(\mathbb{R}^d)$ as finite energy solutions exist outside of C^∞. One can take, for example a C^2 solution u with compact support or with suitable decay of u, ∇u. More on this in Section 3.4. However, the Schwartz space is dense in many function spaces of physical interest, which makes our analysis relevant. By extension, it is also possible to define the Fourier transform for L^1 or L^2functions as the latter is of bigger interest to us. However, we will restrict ourselves to the introductory $\mathcal{S}(\mathbb{R}^d)$ case.

We follow here Section 5.1 of [2] and Section 2.6 of [15]. We also recall the definition of the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^d)$ from Section 2.2

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx$$

and simply extend it to functions $f \in C^2(\mathbb{R}_t) \times \mathcal{S}(\mathbb{R}^d)$ by only considering the Fourier transform of the "spacial part"

$$\mathcal{F}(f)(t,\xi) = \int_{\mathbb{R}^d} f(t,x)e^{-2\pi i x\cdot\xi}dx \tag{3.11}$$

With this definition, we can pose our Cauchy problem with restricted initial data. In other words, a restriction to $\mathcal{S}(\mathbb{R}^d)$ of (3.4)

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0 \quad u(0,x) = f(x) \quad \partial_t u(0,x) = g(x) \quad f, g \in \mathcal{S}(\mathbb{R}^d) \tag{3.12}$$

with of course $u \in C^2(\mathbb{R}_t) \times \mathcal{S}(\mathbb{R}^d)$. We will see a bit later that we can also use this method to solve the non-homogeneous wave equation $-\partial_t^2 u(t,x) + \Delta u(t,x) = f(t,x)$.

Nevertheless, we can now take the Fourier transform of the whole equation (3.12) and use (2.8) to get

$$\mathcal{F}(-\partial_t^2 u)(t,\xi) + \mathcal{F}(\Delta u)(t,\xi) = -\partial_t^2 \mathcal{F}(u)(t,\xi) + (2\pi i)^2|\xi|^2\mathcal{F}(u)(t,\xi) = 0$$

$$\iff \quad \partial_t^2 \mathcal{F}(u)(t,\xi) + 4\pi^2|\xi|^2\mathcal{F}(u)(t,\xi) = 0 \tag{3.13}$$

We immediately see the benefit of using the Fourier transform. We have turned our partial differential equation into an ordinary differential equation. The reader may be familiar with the general solution

$$\mathcal{F}(u)(t,\xi) = A(\xi)\sin(2\pi t|\xi|) + B(\xi)\cos(2\pi t|\xi|) \tag{3.14}$$

The initial conditions immediately give the conditions

$$A(\xi) = \frac{\mathcal{F}(g)(\xi)}{2\pi|\xi|} \qquad B(\xi) = \mathcal{F}(f)(\xi) \tag{3.15}$$

Finally, as shown in [15], taking the inverse Fourier transform leads to the representation formula

$$u(t,x) = \int_{\mathbb{R}^d} \left(\frac{\mathcal{F}(g)(\xi)}{2\pi|\xi|} \sin(2\pi t|\xi|) + \mathcal{F}(f)(\xi)\cos(2\pi t|\xi|) \right) e^{2\pi i x\cdot\xi} d\xi \tag{3.16}$$

As a final note on this section, from the representation formula we can derive the identity (See Proposition A.5)

$$|\partial_t \mathcal{F}(u)(t,\xi)|^2 + 4\pi^2|\xi|^2 \, |\mathcal{F}(u)(t,\xi)|^2 = |\mathcal{F}(g)(\xi)| + 4\pi^2|\xi|^2 \, |\mathcal{F}(f)(\xi)|^2 \tag{3.17}$$

which, by integration in ξ will lead to the energy conservation

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} + \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)} + \|\nabla f\|_{L^2(\mathbb{R}^d)} \tag{3.18}$$

3.4 The energy estimates

In the previous sections we looked at different ways to derive an explicit solution to the wave equation (3.4). Even though the methods were relatively straightforward, finding similar formulae for more generalised wave equations turns out to be a more complicated topic. Consequently, one has to make use of other techniques, that may not give an explicit formula but that can allow for the global behaviour of a solution to be determined (amongst other properties).
In this section, we will look at the physically motived energy inequalities, or energy estimates. Indeed, for a class of solutions to the wave equation (3.4), the energy, defined as

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^3} \left((\partial_t u)^2 + |\nabla u|^2 \right), \tag{3.19}$$

is constant in time. This is a strong statement that hints at the various symmetries of the wave operator. For a more Mathematical treatment, however, we are not limited by a single concept of energy. We indeed can define the energy of a solution to a linear first order system [22] just as we can define the energy of the general wave equation. It turns out that the energy, as defined above, for the non-homogeneous wave equation would not necessarily be constant in time.
This aforementioned property can be seen in multiple ways. One can consider the current associated to the vector field ∂_t and compute the energy momentum tensor for a solution u (more on this in Section 4) in order to derive an energy inequality. On the other, hand one can also use an equivalent method which very easily generalises to more complex cases, that of multipliers [12, 16]. Allow us to illustrate the latter by deriving an estimate for a C^2 solution to the Cauchy problem [2]

$$-\partial_t^2 u + \Delta u = f \qquad u(0,x) = u_0(x) \qquad \partial_t u(0,x) - u_1(x) \tag{3.20}$$

3.4.1 The standard estimate

Let $\mathcal{D} = \left\{(t,x) \in \mathbb{R}_t \times \mathbb{R}^d_x \,\middle|\, t \in [0,T[\right\}$. Let $u \in C^2(\mathcal{D},\mathbb{R})$ be a solution to (3.20), where $u(t,\cdot)$ is of compact support. following [2], we compute

$$\Box_\eta u\, \partial_t u = \left(-\partial_t^2 u + \Delta u\right) \partial_t u = -\frac{1}{2}\partial_t \left(\sum_{i=1}^d (\partial_i u)^2 + (\partial_t u)^2\right) + \sum_{i=1}^d \partial_i \left(\partial_i u\, \partial_t u\right)$$

Note how we have prepared the equation for an integration over the domain $\mathcal{D}$. The ∂_t will generate our energy terms and the ∂_i term will vanish by compactness of u. Indeed, integrating over $\mathcal{D}$ leads to the result

$$\int_{\mathcal{D}} \Box_\eta u\, \partial_t u\, d\tau dx = E[u](0) - E[u](t)$$

This result, for $f = 0$ in (3.20), would lead to the usual energy conservation $E[u](t) = E[u](0)$. For general f, however, we are left with no choice but to estimate the integral using the Cauchy-Schwartz inequality

$$\begin{aligned}\left|\int_{\mathcal{D}} \Box_\eta u\, \partial_t u\, d\tau dx\right| &\leq \int_0^t d\tau \|f(\tau,\cdot)\|_{L^2} \|(\partial_t u)(\tau,\cdot)\|_{L^2} \\ &\leq \int_0^t d\tau \|f(\tau,\cdot)\|_{L^2} \left(2E[u](\tau)\right)^{1/2}\end{aligned}$$

From this we can conclude that our solution u satisfies the following energy inequality (Theorem 6.3 in [2])

$$E[u](t) \leq E[u](0) + \int_0^t d\tau \|f(\tau,\cdot)\|_{L^2} \left(2E[u](\tau)\right)^{1/2} \tag{3.21}$$

Furthermore, one can further improve this estimate by considering the domain of dependence of the solution u. Under this setting, we can indeed drop the requirement that $u(t,\cdot)$ is of compact support [16].

However, one may wonder if we can apply a similar principle to a multiplier that is not $\partial_t u$. As a matter of fact, we can. In this setting we talk about general multipliers.

3.4.2 General Multipliers

To illustrate the principle behind the general multiplier, we follow [2] and set $\mathcal{D}$ to be an open subset of the closed half-plane $\{(t,x) \in \mathbb{R}_t \times \mathbb{R}^d_x \,\big|\, t \geq 0\}$. We know let $X = X_0 \partial_t + \sum_{i=1}^d X_i \partial_i$ be a smooth vector field on $\mathcal{D}$ such that $X_0 > 0$.
With this configuration, we compute

$$\begin{aligned}\Box_\eta u X[u] = &-\frac{1}{2}\partial_t \left[X_0 \left(\sum_{i=1}^d (\partial_i u)^2 + (\partial_t u)^2\right) + 2\partial_t u \sum_{i=1}^d X_i \partial_i u\right] \\ &- \frac{1}{2}\sum_{i=1}^d \partial_i \left[X_i \left(\sum_{j=1}^d (\partial_j u)^2 - (\partial_t u)^2\right) - 2X_0 \partial_t u \partial_i u - 2\partial_i u \sum_{j=1}^d X_j \partial_j u\right] + Q\end{aligned} \tag{3.22}$$

See Proposition A.6 for a more detailed calculation.
If we now integrate over the domain $\mathcal{D}_t = \{(x,\tau) \in \mathcal{D} \,\middle|\, 0 \leq \tau \leq t\}$ we find that

$$\begin{aligned}
\int_{\mathcal{D}_t} \Box_\eta u X[u] dx d\tau &= -\int_{\mathcal{D}_t} \left(\partial_t T_0 + \sum_{i=1}^{d} \partial_i T_i \right) dx d\tau + \int_{\mathcal{D}_t} Q dx d\tau \\
&= -\int_{\partial \overline{\mathcal{D}}_t} \left(N_0 T_0 + \sum_{i=1}^{d} N_i T_i \right) d\sigma + \int_{\mathcal{D}_t} Q dx d\tau \\
&= -\int_{\partial \overline{\mathcal{D}}_t} e(N,X) d\sigma + \int_{\mathcal{D}_t} Q dx d\tau
\end{aligned}$$

where T is the vector field whose components are found in (3.22) and N is the vector field normal to $\mathcal{D}_t$. Note that we have defined the energy density $e(N,X)$ such that

$$\begin{aligned}
e(N,X) = \frac{1}{2} \left(N_0 X_0 - \sum_{i=1}^{d} N_i X_i \right) (\partial_t u)^2 + \frac{1}{2} \left(N_0 X_0 + \sum_{i=1}^{d} N_i X_i \right) \sum_{j=1}^{d} (\partial_j u)^2 \\
+ \partial_t u \left(N_0 X[u] - X_0 N[u] \right) - X[u] N[u]
\end{aligned} \tag{3.23}$$

Allow us to define a few more quantities that will allow us to visualise the integrals more easily. Let $\Sigma_t = \{x \,|\, (x,t) \in \mathcal{D}\}$ and $\Lambda_t = \{(x,\tau) \in \partial\overline{\mathcal{D}} \,|\, \tau \neq 0, \tau \neq t\}$. With these definitions we indeed see that $\partial \overline{\mathcal{D}}_t = \Sigma_0 \cup \Sigma_t \cup \Lambda_t$. We can illustrate all those properties using the diagram in Figure 1.

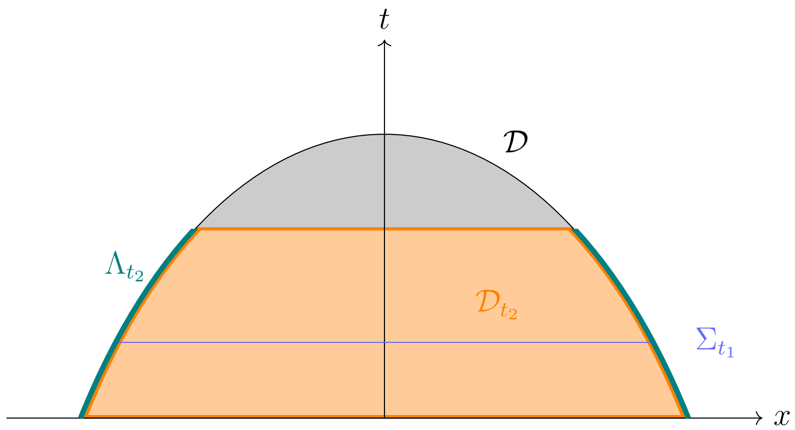


Figure 1: An illustration of the domain of integration $\mathcal{D}_t$ and its related quantities.

We then define the energy of our solution u with respect to the multiplier X and normal N as the quantity

$$E[u](t) = \int_{\Sigma_t} e(N,X) d\sigma \tag{3.24}$$

This energy can be extracted from our original integral as

$$\int_{\partial \overline{\mathcal{D}}_t} e(N,X) d\sigma = E[u](t) - E[u](0) + \int_{\Lambda_t} e(N,X) d\sigma$$

As shown in Thorem 6.7 of [2], if X and N are non-spacelike then the energy $E[u](t)$ is non negative. One can show this by considering the different cases for N and seeing that $e(N, X)$ is non negative only for the aforementioned conditions. This requirement is essential in producing an energy that has both physical sense and mathematical usefulness, as we will see in later chapters.

Using various multipliers with suitable domain $\mathcal{D}_t$ one can prove various inequalities (Morawetz inequality, KSS inequality, conformal inequality, etc). See [2] for more details.

3.5 The integrated local energy decay (ILED) estimate

Amongst the possible estimates one can build using the previous method of multiplier, there is one category called Local Energy Decay Estimates [22]. These types of estimates, for example, can help give bounds for the last term of (3.21). We will use this opportunity to introduce the equivalent formalism of energy-momentum tensors and currents. Indeed, with that formalism we will discuss how one can arrive at the following estimate [1, 2, 14].

Proposition 3.2 (Morawetz estimate). Any C^2 solution of the wave equation $\Box_\eta u = 0$ satisfies the estimate

$$\int_0^\infty \int_{\Sigma_t} \frac{1}{r} |\nabla\!\!\!/ u|^2 \leq CE[u](0) \tag{3.25}$$

for a uniform constant C.

Let us introduce the formalism described above. For a solution u to the wave equation (3.4), we define its energy-momentum tensor

$$T_{\mu\nu}[u] = \partial_\mu \partial_\nu u - \frac{1}{2}\eta_{\mu\nu}(\partial u)^2 \tag{3.26}$$

which satisfies

$$\partial_\mu T^{\mu\nu}[u] = 0 \tag{3.27}$$

For a smooth vector field $X = X^\mu \partial_\mu$ we define its deformation tensor

$${}^{(X)}\pi^{\mu\nu} = \frac{1}{2}\left(\partial^\mu X^\nu + \partial^\nu X^\mu\right) = (\mathcal{L}_X \eta)^{\mu\nu} \tag{3.28}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X. We immediately see that if X is a Killing vector field then ${}^{(X)}\pi^{\mu\nu} = 0$.
Consequently the current $J^{(X)}_\mu[u]$ will be divergence free and $K^{(X)}[u]$ will vanish when defining them as

$$J^{(X)}_\mu[u] = T_{\mu\nu}[u]X^\nu \qquad K^{(X)}[u] = T_{\mu\nu}[u]^{(X)}\pi^{\mu\nu} \tag{3.29}$$

To conclude this overview of the formalism, note that our condition on N and X at the end of Section 3.4.2 that produced positive energy does translate here as Proposition 3.2 of [14]

$$T_{\mu\nu}X^\mu Y^\nu \geq 0 \qquad \text{for X, Y both future directed causal} \tag{3.30}$$

The proof falls quite naturally when using null frames [14, 2, 1].

If one followed Holzegel [14] in their proof of Proposition 3.2 one would use the domain $\mathcal{D}_t = \left([0,t] \times \mathbb{R}^d_x\right) \setminus (]0,t[\times B_\epsilon)$.

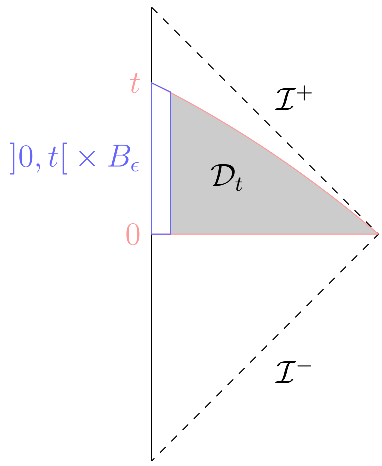
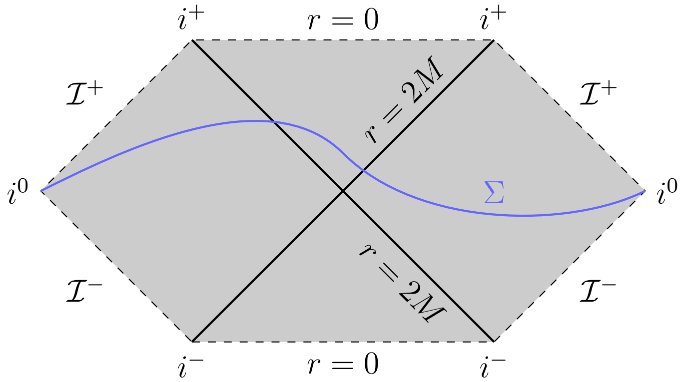


Figure 2: Representation of the domain $\mathcal{D}_t$ (in gray) and its related quantities.

On it one must consider the vector field $X = \partial_r$ and compute the corresponding current and $K^{(X)}[u]$-value. Integrating over the aforementioned domain and separating by parts would lead to an estimate. Taking the limit $\epsilon \to 0$ would give us the Morawetz inequality.
Finally, one can use these inequalities to find decay rates for solutions u such as shown in [1, 20, 21].

4 The wave equation in Schwarzschild

On a different, yet related, note, we will discuss of couple of results one can derive when considering a more complex spacetime: the Schwarzschild spacetime. Even though a lot of the methods we have introduced in earlier sections were designed to be transferable to this type of spacetime, the transfer itself is not trivial. As we will see, not only will the domain of interest change, along with its corresponding normal vector fields, but the wave equation itself will gain in complexity.
Let us recall the Carter-Penrose diagram for the maximally extended Schwazschild solution



On such a spacetime, the metric can no longer be represented globally by the Minkowski metric η but is instead given by a Lorentzian metric g, whose representation will depend on the choice of coordinates. In the so-called Schwarzschild coordinates (t, r, θ, φ) the metric can by written as

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta) d\varphi^2\right) \tag{4.1}$$

In order extend our analysis from Minkowski space to Schwarzschild in a meaningful way, we will need to define properly the wave equation operator.

It turns out that the most appropriate way to extend it comes from Differential Geometry. Indeed, the appropriate generalisation of the wave operation is the so-called geometric wave operator

$$\Box_g u = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu u\right) \tag{4.2}$$

which conserves a lot of the desired properties of the regular wave equation in flat spaceime. As one develops the expression above, they will notice that it can be written like a general wave equation which variable coefficients [2, 14].
One, however, does not look at Cauchy problems on the maximally extended Schwarzschild spacetime if they are looking for a physically interesting analysis. Indeed, given the non-causal separation of that spacetime and given the fact that we, as observers, do not live below the horizon $\mathcal{H}^+ = \{r = 2M\}$, it is much more interesting to look at the region at the far right of the Carter-Penrose diagram. In this case, we define our Cauchy surface of interest as $\Sigma \cap \{r \geq 2M\}$ (as given in the diagram above). As pointed out in [11], the hypersurface Σ may not be spherically symmetric, in which case its projection to the diagram will not be continuous. We nevertheless, keep it as such as we will later choose it to be spherically symmetric anyway (See Section 5).

4.1 An energy estimate and the Kay-Wald theorem

Allow us to put to use our formalism described in Section 3.5 as it is the easiest to use while keeping all quantities coordinate system independent.
We will aim to describe how to approach the Kay-Wald theorem which states that for certain requirements on the solution u, it must obey

$$|u| \leq C \tag{4.3}$$

for some constant C depending only on the Cauchy initial data on Σ. See Section 3.2 of [11] for a more complete description of the conditions.

In this new setting, the different quantities defined earlier are generalised to [14]

$$T_{\mu\nu}[u] = \nabla_\mu u \nabla_\nu u - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \nabla_\alpha u \nabla_\beta u \right) \qquad {}^{(X)}\pi_{\mu\nu} = \frac{1}{2} \left(\nabla_\mu X_\nu + \nabla_\nu X_\mu \right) \tag{4.4}$$

$$\nabla^\mu J^{(X)}_\mu [u] = K^{(X)}[u] \tag{4.5}$$

Allow us now to follow Section 3 of [11] and define our domain of interest $\mathcal{D}_\tau$. Remembering that the field ∂_t is Killing on this spacetime, we can define a foliation of the $r \geq 2M$ region by considering the 1-parameter group of diffeomorphisms of ∂_t, denoted φ_t. Indeed the set of all $\Sigma_\tau = \varphi_t \left(\Sigma \cap \{ r \geq 2M \} \right)$ defines such a foliation and we have

$$\mathcal{D}_\tau = \bigcup_{0 \leq t \leq \tau} \Sigma_t \tag{4.6}$$

We then also define

$$\mathcal{H}^+(0,\tau) = \mathcal{H}^+ \cap J^+(\Sigma_0) \cap J^-(\Sigma_\tau) \tag{4.7}$$

Our Carter-Penrose diagram of interest is then given by Figure 3.

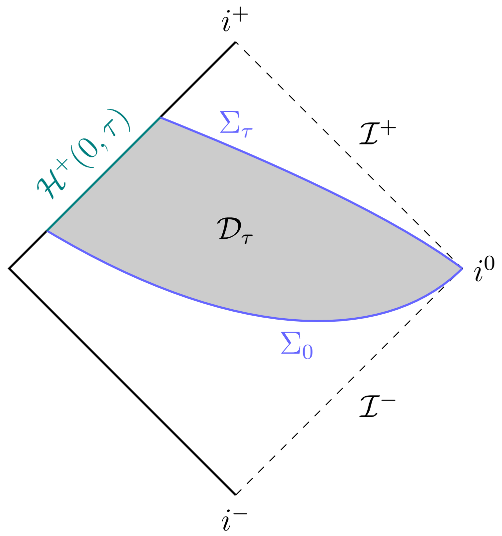


Figure 3: Carter-Penrose diagram of the region of interest $\mathcal{D}_\tau$ (in grey) and its associated quantities.

Let n_Σ denote the future directed unit normal of Σ and let $n_\mathcal{H}$ be a null generator of $\mathcal{H}^+$. The current associated to the Killing vector field ∂_t is denoted $J^T_\mu[u]$ for a solution u of $\Box_g u = 0$. Then integrating with respect to the usual volume form we find

$$\int_{\mathcal{D}_\tau} \nabla^\mu J^T_\mu[u] = \int_{\Sigma_0} J^T_\mu[u] n^\mu_{\Sigma_0} - \int_{\Sigma_\tau} J^T_\mu[u] n^\mu_{\Sigma_\tau} - \int_{\mathcal{H}^+(0,\tau)} J^T_\mu[u] n^\mu_\mathcal{H} = 0 \tag{4.8}$$

By extension to our discussion on (3.30), since ∂_t is future-directed causal in $\mathcal{D}_\tau$, we have

$$J^T_\mu[u] n^\mu_\Sigma \leq 0 \qquad J^T_\mu[u] n^\mu_\mathcal{H} \leq 0 \tag{4.9}$$

and as long as $-g(\partial_t, n_{\Sigma_0}) \leq C$ for some constant C, we can write

$$\int_{\Sigma_\tau \cap \{r \geq r_0\}} \left((\partial_t u)^2 + (\partial_r u)^2 + |\nabla\!\!\!/ u|^2 \right) \leq C(r_0, \Sigma) \int_{\Sigma_0} J^T_\mu[u] n^\mu_{\Sigma_0} \tag{4.10}$$

Given the commutation property $\left[\Box_g, \partial_t\right] = 0$ then we also have

$$\int_{\Sigma_\tau \cap \{r \geq r_0\}} \left((\partial_t^2 u)^2 + (\partial_r \partial_t u)^2 + |\nabla\!\!\!/ \partial_t u|^2 \right) \leq C(r_0, \Sigma) \int_{\Sigma_0} J^T_\mu[\partial_t u] n^\mu_{\Sigma_0} \tag{4.11}$$

Then as noted by [11], given a certain requirement on the decay of the Cauchy initial data at spacial infinity, our solution u obeys

$$|u|^2 \leq C(r_0, \Sigma) \left(\int_{\Sigma_0} J^T_\mu[u] n^\mu_{\Sigma_0} + \int_{\Sigma_0} J^T_\mu[\partial_t u] n^\mu_{\Sigma_0} \right) \tag{4.12}$$

As rejoicing as it may seem, this inequality is not quite the Kay-Wald theorem. Indeed, as discussed in [11], the condition $-g(\partial_t, n_{\Sigma_0}) \leq C$ can no longer be satisfied at the horizon as ∂_t becomes null. This does limit the estimate to the exterior only. However, there are methods around this problem. Dafermos and Rodnianski use a different multiplier to prove the Kay-Wald theorem. This allows for the control of solutions to the geometric wave equation. Notably, if the solution has suitable asymptotics, as is desired in physical systems, then it can be shown to be bounded. In other words, from a physical stand point, the solution never blows up and the study of such solution holds everywhere in $\mathcal{D}_\tau$.

Finally note that we can also apply the same ideas as in previous chapters to this problem. In other words, we can choose a different multiplier with which to commute the operator. For example, the choice of vector field $N = a(r)\partial_t - b(r)\partial_r$ can lead to a suitable result on the redshift effect [14, 11]. One can also find an ILED estimate in the presence of a Schwarzschild background. For example,

$$\int_{\mathcal{D}_\tau} \left(1 - \frac{3M}{r}\right)^2 \frac{1}{r^2} |Du|^2 \leq C \int_{\Sigma_0} |Du|^2 \tag{4.13}$$

In that case the result is interesting as it reminiscent of the phenomena of trapped photon geodesics [11, 24] at $r = 3M$. This can be made clear from the degeneracy in the estimate itself at that value of r.

5 Application to a perturbed wave equation

Now that we have looked at some of the basic properties and definitions regarding the wave function in Minkowski space and in Schwarzschild background, we are able to have a look at the process that led to the paper by Holzegel and Kauffman [17]. The reason behind the long introduction to the previous methods (for example: energy estimates) was due to their effectiveness in larger situations. Indeed we can use these energy estimates and ILEDs to control various derivatives of our solution to the wave equation in multiple background metrics. Furthermore, we can also apply them to modified wave equations and study the stability of such solutions. This is, in essence, what we will aim to do in this section, by following [17].
As described in that same paper, many results regarding the linear wave equation $\Box\phi = 0$ have already been produced [8, 11]. Would it be in exterior Schwarzschild or Kerr geometries, decay estimates are quite well understood [10]. One can also find results for more general spacetimes [23]. Nevertheless, given the number of results found for the aforementioned wave equation, Holzegel and Kauffman were interested in knowing which of these results hold for the perturbed wave equation

$$\Box_g\phi = \epsilon\beta^\mu\partial_\mu\phi \tag{5.1}$$

where β is a smooth vector field on $\mathcal{M}$ that obeys some spacial asymptotics we will specify later (See equation (5.9)). In order to achieve such a result they derived a non-degenerate integrated energy estimate, see Theorem 4.1 of [17]. In a similar fashion to what was shown in Section 3.5, the ILED can be used to show how the solution decays. In this case, from their results follows Corollary 4.1 of [17] which predicts an inverse polynomial decay in time.

5.1 Setting the Cauchy problem

Similarly to Section 4, we are not necessarily interested in the entirety of the maximally extended Schwarzschild spacetime but rather by the exterior Schwarzschild region, including the horizon $\mathcal{H}^+$. To cover this region with one set of coordinates we can extend the Schwarzschild coordinates (t, r, θ, φ) to a new set of coordinates defined by [11]

$$t^\star = t + 2M\ln(r - 2M) \tag{5.2}$$

Under these regular coordinates $(t^\star, r, \theta, \varphi)$ one can easily see that the metric takes the form

$$g = -a_-(r)(dt^\star)^2 + \frac{4M}{r}dtdt^\star + a_+(r)dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right) \tag{5.3}$$

where we defined $a_\pm = \left(1 \pm \frac{2M}{r}\right)$. This can also easily been seen to extend up to the horizon $\mathcal{H}^+$. For conciseness, from now on we will use the symbol t instead of $t^\star$.

We can also note that the manifold of interest $\mathcal{M} = \mathbb{R}_t \times [2M, \infty[\times S^2$, can be foliated by spacelike hypersurfaces Σ_τ defined as surfaces of constant $t = \tau$. Notably, we will chose Σ_0 as our Chauchy surface when setting up the Cauchy problem later. Also note that the vector field $T = \partial_t$ is Killing in the regular coordinates. Finally, for further readability of the Carter-Penrose diagram shown Figure 4, we define a subset of the horizon $\mathcal{H}^+(0,\tau) = [0,\tau] \times \{r = 2M\} \times S^2$.

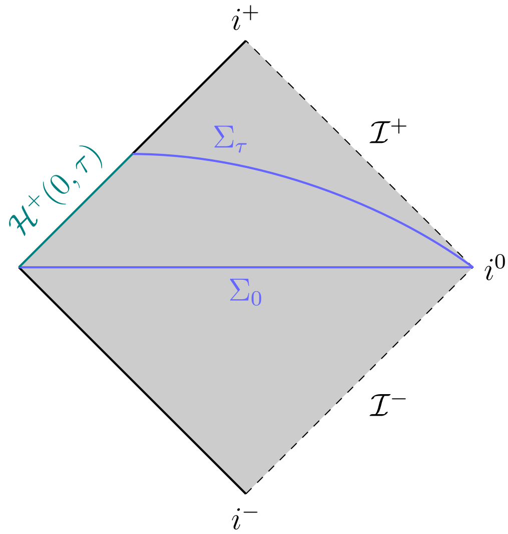


Figure 4: Carter-Penrose diagram of the exterior Schwarzschild manifold, including the horizon at $r = 2M$.

In order to to greatly simplify the analysis Holzegel and Kauffman defined a new field $\psi = r\phi$, where ϕ is a solution ϕ of (5.1). This rescaling of the solution will allow us to take advantage of the separability of the wave equation.

5.1.1 Deriving an expression for $\mathcal{P}_s\psi$

As a preliminary step, we will expand out the left hand side of (5.1) given our metric (5.3). This step is done using our results from Section 4, and more specifically, using (4.2): $\Box_g u = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu u\right)$.

We recall that the Schwarzschild metric and its inverse, in regular coordinates, have the matrix representation

$$G = \begin{pmatrix} -a_-(r) & \frac{2M}{r} & & \\ \frac{2M}{r} & a_+(r) & & \\ & & r^2 & \\ & & & r^2\sin^2(\theta) \end{pmatrix} \quad G^{-1} = \begin{pmatrix} -a_+(r) & \frac{2M}{r} & & \\ \frac{2M}{r} & a_-(r) & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2(\theta)} \end{pmatrix}$$

From that follows the expressions for the determinant and the square root of it

$$g = \det(G) = -r^4\sin^2(\theta)$$
$$\sqrt{|g|} = r^2\sin(\theta)$$

Substituting those values in our expression for $\Box_g \phi$ (i.e. inside (4.2)), we find

$$\begin{aligned}
\Box_g \phi = \frac{1}{r^2 \sin(\theta)} \Bigg[& -\partial_t \left(r^2 \sin(\theta) \left(a_+(r)\partial_t \phi - \frac{2M}{r}\partial_r \phi \right) \right) \\
& + \partial_r \left(r^2 \sin(\theta) \left(\frac{2M}{r}\partial_t \phi + a_-(r)\partial_r \phi \right) \right) \\
& + \partial_\theta \left(\sin(\theta)\partial_\theta \phi \right) + \partial_\varphi \left(\frac{1}{\sin(\theta)} \partial_\varphi \phi \right) \Bigg] \\
= -a_+(r)\partial_t^2 \phi + & \frac{2M}{r}\partial_t \partial_r \phi + \frac{1}{r^2}\left(2M\partial_t \phi + 2Mr\partial_t\partial_r\phi + 2ra_-(r)\partial_r \phi \right) \\
+ \partial_r \left(a_-(r)\partial_r \phi \right) + & \frac{\Delta_{S^2}\phi}{r^2} \\
= -a_+(r)\partial_t^2 \phi + & \frac{4M}{r}\partial_t\partial_r \phi + \frac{2M}{r^2}\partial_t \phi + \frac{2}{r}a_-(r)\partial_r \phi + \partial_r \left(a_-(r)\partial_r \phi \right) \\
+ \frac{\Delta_{S^2}\phi}{r^2} &
\end{aligned}$$

where

$$\Delta_{S^2}\phi = \frac{1}{\sin(\theta)} \left(\partial_\theta \left(\sin(\theta)\partial_\theta \phi \right) + \partial_\varphi \left(\frac{1}{\sin(\theta)}\partial_\varphi \phi \right) \right)$$

is the Laplacian on the 2-sphere. It must not be confused with that of the induced metric on a sphere of radius r denoted $\not\Delta$ and for which $\not\Delta = \frac{\Delta_{S^2}}{r^2}$.

We can now use our definition of ψ to find the explicit expression for $\Box_g \psi$ by permuting through the factor of r to get

$$\begin{aligned}
\Box_g \psi &= -a_+(r)\partial_t^2 \psi + \frac{4M}{r}\partial_t \partial_r (r\phi) + \frac{2M}{r^2}\partial_t \psi + \frac{2}{r}a_-(r)\partial_r (r\phi) + \partial_r \left(a_-(r)\partial_r (r\phi) \right) \\
&\quad + \frac{\Delta_{S^2}\psi}{r^2} \\
&= r\Box_g \phi + \frac{4M}{r}\partial_t \phi + \frac{2}{r}a_-(r)\phi + \frac{2M}{r^2}\phi + 2a_-(r)\partial_r \phi \\
&= r\Box_g \phi + \frac{4M}{r}\partial_t \phi + \frac{1}{r}\left(2 - \frac{2M}{r} \right)\phi + 2a_-(r)\partial_r \phi
\end{aligned}$$

We can now use (5.1) to substitute $\Box_g \phi$ in order to find

$$\begin{aligned}
\Box_g \psi &= \epsilon r \beta^\mu \partial_\mu \left(\frac{\psi}{r} \right) + \frac{4M}{r^2}\partial_t \psi + \frac{1}{r^2}\left(2 - \frac{2M}{r} \right)\psi + 2a_-(r)\partial_r \left(\frac{\psi}{r} \right) \\
&= \epsilon \beta^\mu \partial_\mu \psi - \epsilon \frac{\beta^r}{r}\psi + \frac{4M}{r^2}\partial_t \psi + \frac{2M}{r^3}\psi + \frac{2}{r}a_-(r)\partial_r \psi
\end{aligned}$$

In accordance with the definition of $\mathcal{P}_s$ in [17], we see that in order to recover the same expression, we must have

$$\begin{aligned}
\mathcal{P}_s \psi &= -\Box_g \psi + \frac{4M}{r^2}\partial_t \psi + (1-s^2)\frac{2M}{r^3}\psi + \frac{2}{r}a_-(r)\partial_r \psi \\
&= a_+(r)\partial_t^2 \psi - \frac{4M}{r}\partial_t \partial_r \psi + \frac{2M}{r^2}\partial_t \psi - \partial_r \left(a_-(r)\partial_r \psi \right) - \frac{\Delta_{S^2}\psi}{r^2} + (1-s^2)\frac{2M}{r^3}\psi \qquad (5.4)
\end{aligned}$$

As specified by Holzegel and Kauffman, s can take the values $0, 1, 2$ and was introduced to set similarities with the Regge-Wheeler operator. We nevertheless, recover for $s = 0$ our modified perturbed wave equation

$$\mathcal{P}_0\psi = \epsilon\left(\frac{\beta^r}{r}\psi - \beta^\mu\partial_\mu\psi\right) \tag{5.5}$$

5.1.2 Asymptotics of β and the Cauchy problem

In order to derive a suitable estimate for ψ it will turn out that β must satisfy certain asymptotic conditions. In particular, the components β^μ as well as their derivatives $\partial_t\beta_\mu$ and $\partial_r\beta^\mu$ must obey certain decays.

In order to write down a proper condition on β rather than vague statements, we must first define two new vector fields which will become essential in our analysis

$$R^\star = \frac{2M}{r}\partial_t + a_-(r)\partial_r \tag{5.6}$$

$$W = \frac{r}{\sqrt{1-\frac{2M}{r}}}\left(R^\star + f(r)\partial_t\right) \qquad f(r) = \left(1 - \frac{3M}{r}\right)\sqrt{1+\frac{6M}{r}} \tag{5.7}$$

The choice of $f(r)$ and $R^\star$ are not arbitrary. Indeed, the former satisfies certain asymptotic behaviours that make this choice of W particularly well suited for the problem at hand. Firstly

$$f(r) \xrightarrow{r\to 2M} -1 \qquad f(r) \xrightarrow{r\to\infty} 1 \tag{5.8}$$

Of course other properties of $f(r)$ and $R^\star$ are essential for the derivations in this chapter. However, we will make them explicit when they are needed.

We can now state that we require the components of β to decay inverse polynomially in r and $W(\beta^\mu)$ to do the same. Overall, it turns out to be most convenient to formulate the proper asymptotic conditions as the following inequality valid for all $2M \leq r < \infty$

$$r^2|\beta^t| + r^2|\beta^r| + r^{3/2}\sqrt{\not{g}_{AB}\not{\beta}^A\not{\beta}^B} + r^{3/2}|W(\beta^t)| + r^{3/2}|W(\beta^r)| + r\sqrt{\not{g}_{AB}W(\not{\beta}^A)W(\not{\beta}^B)} \leq C \tag{5.9}$$

where $\not{\beta}$ is the projection of the vector field β onto the hypersurface of constant t and r and $\not{g}$ is the induced metric of such a projection.

As a matter of fact, we could settle down and set our Cauchy problem using equation (5.5) and Σ_0 as the Cauchy surface of initial data. However, without compromising our analysis we can simplify things by getting rid of the zero$^{\text{th}}$ order term in (5.5). This is justified by the nature of that term which, being a zero$^{\text{th}}$ order term, can be dealt with in the various estimates we will derive. Hence, following [17] and compensating for sign differences, we define our Cauchy problem as

$$\mathcal{P}_s\psi = -\epsilon\beta^\mu\partial_\mu\psi \qquad \psi\big|_{\Sigma_0} = \psi_0 \qquad n_{\Sigma_0}\psi\big|_{\Sigma_0} = \psi_1 \tag{5.10}$$

and look for finite energy smooth solutions ψ.

5.2 The energies

Undoubtedly, one must first define their energies before requiring finite energy solutions of the Cauchy problem. Let us, nevertheless, do this a posteriori in this section. Contrary to the standard spacial volume integral $\int_{\Sigma_\tau} d\Omega = \int_{2M}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi r^2 \sin(\theta)\big|_{t=\tau}$ we will consider energies whose volume form gives

$$\int_{\Sigma_\tau} = \int_{\Sigma_\tau} \sin(\theta) dr d\theta d\varphi = \int_{2M}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sin(\theta)\big|_{t=\tau} \tag{5.11}$$

As seen in Section 3.4, this still allows for valid energies to be constructed.

We first and foremost define the degenerate energies

$$\mathbb{E}[\psi](\tau) = \int_{\Sigma_\tau} \left((\partial_t\psi)^2 + a_-(r)(\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2 + \frac{\psi^2}{r^2} \right)$$

$$\mathbb{I}_{\text{deg}}[\psi](\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \left(\left(1 - \frac{3M}{r}\right)^2 \left(\frac{1}{r^2}(\partial_t\psi)^2 + \frac{1-\frac{2M}{r}}{r^2}(\partial_r\psi)^2 + \frac{1}{r}|\nabla\!\!\!/\psi|^2 \right) + \frac{1}{r^2}|R^\star\psi|^2 + \frac{\psi^2}{r^3} \right)$$

$$\mathbb{I}_{\text{ndeg}}[\psi](\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \left(\frac{1}{r^2}(\partial_t\psi)^2 + \frac{1-\frac{2M}{r}}{r^2}(\partial_r\psi)^2 + \frac{1}{r}|\nabla\!\!\!/\psi|^2 + \frac{\psi^2}{r^3} \right)$$

which see their ∂_r terms degenerate near the horizon. As we will see, we wish to bypass this degeneracy as it doesn't allow us to place any bound on $\partial_r\psi$ near the horizon. To do so, we also define the non-degenerate energies

$$\overline{\mathbb{E}}[\psi](\tau) = \int_{\Sigma_\tau} \left((\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2 + \frac{\psi^2}{r^2} \right)$$

$$\overline{\mathbb{I}}_{\text{deg}}[\psi](\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \left(\left(1 - \frac{3M}{r}\right)^2 \left(\frac{1}{r^2}(\partial_t\psi)^2 + \frac{1}{r^2}(\partial_r\psi)^2 + \frac{1}{r}|\nabla\!\!\!/\psi|^2 \right) + \frac{1}{r^2}|R^\star\psi|^2 + \frac{\psi^2}{r^3} \right)$$

$$\overline{\mathbb{I}}_{\text{ndeg}}[\psi](\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \left(\frac{1}{r^2}(\partial_t\psi)^2 + \frac{1}{r^2}(\partial_r\psi)^2 + \frac{1}{r}|\nabla\!\!\!/\psi|^2 + \frac{\psi^2}{r^3} \right)$$

The keen eye will have noticed the degenerate terms at $r = 3M$ in the above ILEDs. As discussed at the end of Section 4.1, this property is tightly tied to the geometric wave operator on this background spacetime. Nevertheless, we will still need to find ways around this degeneracy if we wish to properly bound our solutions.

5.3 Commuting the vector field

Sticking to what we learned in Section 3.4, after multiplying equation (5.10) by $\partial_t\psi$ we would like to commute through the operators in order to be left with a total divergence. However, it turns out that the results we seek can more easily be found by considering $\mathcal{P}_s\left(W[\psi]\right)$ to which $\partial_t W[\psi]$ will be multiplied. In order to make this derivation more apparent, we first compute a few useful quantities.

5.3.1 Some useful commutation identities

Let $h \in C^1([2M, \infty[\,; \mathbb{R})$, then

$$\begin{aligned}\left[W, h(r)\right] \psi &= W\left[h(r)\psi\right] - h(r)W\left[\psi\right]\\ &= W[h(r)]\psi\\ &= r\left(a_-(r)\right)^{1/2} h'(r)\psi\end{aligned}$$

From this identity we can write

$$\begin{aligned}\left[W, a_+(r)\right]\psi &= -\left(a_-(r)\right)^{1/2}\tfrac{2M}{r}\psi & \left[W, \tfrac{2M}{r^3}\right]\psi &= -\left(a_-(r)\right)^{1/2}\tfrac{6M}{r^3}\psi\\ \left[W, \tfrac{4M}{r}\right]\psi &= -\left(a_-(r)\right)^{1/2}\tfrac{4M}{r}\psi & \left[W, \tfrac{2M}{r^2}\right]\psi &= -\left(a_-(r)\right)^{1/2}\tfrac{4M}{r^2}\psi\end{aligned}$$

Furthermore, we also have

$$\begin{aligned}\left[W, \partial_r\right]\psi &= W\left[\partial_r\psi\right] - \partial_r W\left[\psi\right]\\ &= \frac{M}{r^2}\left(a_-(r)\right)^{-3/2}\Big(\left(2M + rf(r)\right)\partial_t\psi + (r-2M)\partial_r\psi\Big)\\ &\quad - \left(a_-(r)\right)^{-1/2}\Big(\left(f(r) + rf'(r)\right)\partial_t\psi + \partial_r\psi\Big)\\ &= \frac{M}{r^2}\left(a_-(r)\right)^{-1} W[\psi] - \left(a_-(r)\right)^{-1/2}\Big(\left(f(r) + rf'(r)\right)\partial_t\psi + \partial_r\psi\Big)\end{aligned}$$

and

$$\begin{aligned}\left[W, \partial_t\right]\psi &= 0\\ \left[W, \Delta_{S^2}\right]\psi &= 0\end{aligned}$$

5.3.2 Finding an expression for $\mathcal{P}_s\left(W[\psi]\right)$

We can focus on rearranging our expression for $\mathcal{P}_s\left(W[\psi]\right)$ in order to derive equation (12) of [17]. We use our definition of $\mathcal{P}_s\psi$ in equation(5.4) to write down

$$\begin{aligned}\mathcal{P}_s\left(W[\psi]\right) &= a_+(r)\partial_t^2 W[\psi] - \frac{4M}{r}\partial_t\partial_r W[\psi] + \frac{2M}{r^2}\partial_t W[\psi] - \partial_r\left(a_-(r)\partial_r W[\psi]\right)\\ &\quad - \frac{\Delta_{S^2}W[\psi]}{r^2} + (1-s^2)\frac{2M}{r^3}W[\psi]\\ &= a_+(r)W[\partial_t^2\psi] - \frac{4M}{r}\partial_r W[\partial_t\psi] + \frac{2M}{r^2}W[\partial_t\psi] - \partial_r\left(a_-(r)\partial_r W[\psi]\right)\\ &\quad - \frac{W[\Delta_{S^2}\psi]}{r^2} + (1-s^2)\frac{2M}{r^3}W[\psi]\end{aligned} \tag{5.12}$$

And commute the derivatives through. For a detailed derivation the reader is invited to jump to Section B.1. We thus have

Proposition 5.1. Given the definition of $\mathcal{P}_s\left(W[\psi]\right)$ (5.4) it follows that

$$\mathcal{P}_s\left(W[\psi]\right) = -\frac{2}{r}\left(a_-(r)\right)^{-1}\left(1 + \frac{6M}{r}\right)^{-1/2}\partial_t W[\psi] + \mathcal{F}_1 + \mathcal{F}_2 \tag{5.13}$$

$$\mathcal{F}_1 = -\frac{\epsilon}{r^2}W[r^2\beta^\mu\partial_\mu\psi] \qquad \mathcal{F}_2 = \left(a_-(r)\right)^{-1/2}\left(h_2(r)\partial_t\psi + h_2(r)\partial_r\psi + h_3(r)\psi\right)$$

This proposition corresponds to equation (12) of [17], up to a sign difference. Note how the only terms that contain second order derivatives are either in $\partial_t W[\psi]$ or contain an ϵ. We also note the boundedness property of the $h_n(r)$ function

$$\sup_{[2M,\infty[} |r^n h_n(r)| \le \hat{C}(M) \tag{5.14}$$

5.3.3 Our first estimate

We are now ready to apply what we have learned in Section 3.4. Indeed, given our previous result (5.13) and the original definition of $\mathcal{P}_s\left(W[\psi]\right)$ (5.12), we can construct an energy-type inequality by multiplying by $\partial_t W[\psi]$ and integrating over a spacetime region.
Let us first rearrange the following expression into a total divergence.

$$\begin{aligned}
\partial_t W[\psi]\mathcal{P}_s\left(W[\psi]\right) &= a_+(r)\partial_t^2 W[\psi]\partial_t W[\psi] - \frac{4M}{r}\partial_t\partial_r W[\psi]\partial_t W[\psi] + \frac{2M}{r^2}\left(\partial_t W[\psi]\right)^2 \\
&\quad - \partial_r\left(a_-(r)\partial_r W[\psi]\right)\partial_t W[\psi] - \frac{\Delta_{S^2} W[\psi]}{r^2}\partial_t W[\psi] + (1-s^2)\frac{2M}{r^3} W[\psi]\partial_t W[\psi] \\
&= \frac{1}{2}\partial_t\left(a_+(r)\left(\partial_t W[\psi)\right)^2\right) - \frac{1}{2}\partial_r\left(\frac{4M}{r}\left(\partial_t W[\psi]\right)^2\right) - \partial_r\left(a_-(r)\partial_r W[\psi]\right)\partial_t W[\psi] \\
&\quad - \nabla\!\!\!\!/\cdot\left(\partial_t W[\psi]\nabla\!\!\!\!/ W[\psi]\right) + \frac{1}{2}\partial_t|\nabla\!\!\!\!/ W[\psi]|^2 + \partial_t\left((1-s^2)\frac{M}{r^3}\left(W[\psi]\right)^2\right) \\
&= \frac{1}{2}\partial_t\left(a_+(r)\left(\partial_t W[\psi]\right)^2 + |\nabla\!\!\!\!/ W[\psi]|^2 + (1-s^2)\frac{2M}{r^3}\left(W[\psi]\right)^2\right) - \frac{1}{2}\partial_r\left(\frac{4M}{r}\left(\partial_t W[\psi]\right)^2\right) \\
&\quad - \partial_r\left(a_-(r)\partial_r W[\psi]\partial_t W[\psi]\right) + a_-(r)\partial_r W[\psi]\partial_t\partial_r W[\psi] \\
&\quad - \nabla\!\!\!\!/\cdot\left(\partial_t W[\psi]\nabla\!\!\!\!/ W[\psi]\right) \\
&= \frac{1}{2}\partial_t\left(a_+(r)\left(\partial_t W[\psi]\right)^2 + a_-(r)\left(\partial_r W[\psi]\right)^2 + |\nabla\!\!\!\!/ W[\psi]|^2 + (1-s^2)\frac{2M}{r^3}\left(W[\psi]\right)^2\right) \\
&\quad - \partial_r\left(\frac{2M}{r}\left(\partial_t W[\psi]\right)^2 + a_-(r)\partial_r W[\psi]\partial_t W[\psi]\right) - \nabla\!\!\!\!/\cdot\left(\partial_t W[\psi]\nabla\!\!\!\!/ W[\psi]\right)
\end{aligned}$$

We can immediately see on of the advantages of having made the choice of $W[\psi]$ as the field of interest. Indeed, we are left with a total divergence and no polynomial terms (as opposed to what we have seen in Section 3.4). This fact will allow us to apply our standard results of positivity to the energy estimate we derive from this formula. We note, however, that we wish to build an energy estimate using $\mathbb{E}[\psi]$ which contains some terms that are not in our ∂_t expression above, and vice versa. This being said, we can simply construct the desired energy estimate and leave the extra terms on the side. Let us now integrate the previous expression over the spacetime region $[0,\tau]\times[2M,\infty[\times S^2$. We, again, leave the details to Section B.3 and jump to the conclusion of that calculation. We indeed find the result

Proposition 5.2. Let $\epsilon_0 > 0$ be small. Then, for all smooth solutions ψ of the Cauchy problem (5.10) and for all $\epsilon_0 \ge \epsilon \ge 0$ we have

$$\begin{aligned}
\mathbb{E}[W[\psi]](\tau) + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2 &\lesssim \mathbb{E}[W[\psi]](0) + \bar{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \\
&\quad + \epsilon\,\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)
\end{aligned} \tag{5.15}$$

where we use $\lesssim$ to denote $\le C$.

The reader might notice this is exactly equation (14) in [17].
This estimate is very powerful as it allows us to see the redshift in effect at the energy level. Indeed, thanks to the favourable non-degenerate terms on the right hand side any finite energy solution must display a finite $\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2$ term. Given the divergent term, this condition forces a decay as $r \to 2M$ of the term $\left(\partial_t W[\psi]\right)^2$ at least of the form $a_-(r)$.

5.4 A Lagrangian estimate

Following what we have done previously with equation (5.13), we can derive a second estimate using a different multiplier. Recall that we used $\partial_t W[\psi]$ as a multiplier of (5.13) to get our energy-type estimate (5.15). Just as we did in Section 3.4, we can choose a different multiplier in order to derive a different estimate. Indeed, in this section we will devote ourselves to the derivation of a Lagrangian estimate by using the multiplier $\frac{1}{r}W[\psi]$.
Let us first rewrite the following expression as a total divergence

$$
\begin{aligned}
\frac{1}{r}W[\psi]\mathcal{P}_s\left(W[\psi]\right) &= a_+(r)\partial_t^2 W[\psi]\frac{1}{r}W[\psi] - \frac{4M}{r}\partial_t\partial_r W[\psi]\frac{1}{r}W[\psi] + \frac{2M}{r^2}\partial_t W[\psi]\frac{1}{r}W[\psi] \\
&\quad - \partial_r\left(a_-(r)\partial_r W[\psi]\right)\frac{1}{r}W[\psi] - \frac{\Delta_{S^2}W[\psi]}{r^2}\frac{1}{r}W[\psi] + (1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2 \\
&= -\frac{1}{r}a_+(r)\left(\partial_t W[\psi]\right)^2 + \frac{1}{r}a_+(r)\partial_t\left(\partial_t W[\psi]W[\psi]\right) - \partial_t\left(\frac{4M}{r^2}\partial_r W[\psi]W[\psi]\right) \\
&\quad + \frac{4M}{r^2}\partial_t W[\psi]\partial_r W[\psi] + \frac{M}{r^3}\partial_t\left(W[\psi]\right)^2 - \partial_r\left(\frac{1}{r}a_-(r)\partial_r W[\psi]W[\psi]\right) \\
&\quad - \frac{1}{r^2}a_-(r)\partial_r W[\psi]W[\psi] + \frac{1}{r}a_-(r)\left(\partial_r W[\psi]\right)^2 - \frac{1}{r}\nabla\!\!\!/\cdot\left(\nabla\!\!\!/ W[\psi]W[\psi]\right) \\
&\quad + \frac{1}{r}|\nabla\!\!\!/ W[\psi]|^2 + (1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2
\end{aligned}
$$

As opposed to what we found in the previous section but also as expected from Section 3.4, we cannot turn the expression into a total divergence only. Some polynomial terms remain. This is not of any concern, however, as we will still be able to estimate them.
Leaving all the detailed to Section B.2 we see that we recover the estimate

Proposition 5.3. For all smooth solutions ψ of the Cauchy problem the following estimate holds.

$$
\begin{aligned}
\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \lesssim \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 + \mathbb{E}[W[\psi]](0) + \mathbb{E}[W[\psi]](\tau) \\
+ \overline{\mathbb{E}}[\psi](0) + \overline{\mathbb{E}}[\psi](\tau) \qquad (5.16)
\end{aligned}
$$

As illustrated in equation (15) of [17].

5.5 Combining the estimates

Our previous results are encouraging as the estimates are close to giving us full control on our solutions. We would, however, like to remove the factor of $a_-(r)$ from these estimates, as it is degenerate on the horizon $\mathcal{H}_+$. To do so we can combine both our estimates (5.15) and (5.16) with a standard result [17, 11, 9]

$$
\overline{\mathbb{E}}[\psi](\tau) \lesssim \overline{\mathbb{E}}[\psi](0) + \epsilon\,\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \qquad (5.17)
$$

For now, let C_1 denote the constant in (5.15) and C_2 denote the constant in (5.16). We first add $C_2\times$(5.15) to (5.16) in order to get

$$\begin{aligned}
C_2\mathbb{E}[W[\psi]](\tau) + C_2\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2 + \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)\\
\leq C_1C_2\mathbb{E}[W[\psi]](0) + C_1C_2\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C_1C_2\epsilon\,\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)\\
+ C_2\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2 + C_2\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C_2\mathbb{E}[W[\psi]](0) + C_2\mathbb{E}[W[\psi]](\tau)\\
+ C_2\overline{\mathbb{E}}[\psi](0) + C_2\overline{\mathbb{E}}[\psi](\tau)
\end{aligned}$$

Which, for a certain constant C, and after using (5.15) again to bound the $\mathbb{E}[W[\psi]](\tau)$, leads to

$$\begin{aligned}
\mathbb{E}[W[\psi]](\tau) + \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) &\leq C\epsilon\,\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) + C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C\mathbb{E}[W[\psi]](0) + C\mathbb{E}[W[\psi]](\tau)\\
&\quad + C\overline{\mathbb{E}}[\psi](0) + C\overline{\mathbb{E}}[\psi](\tau)\\
&\leq C\epsilon\,\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) + C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C\mathbb{E}[W[\psi]](0) + C\overline{\mathbb{E}}[\psi](0)\\
&\quad + C\overline{\mathbb{E}}[\psi](\tau)
\end{aligned}$$

Applying our, now familiar, trick of choosing ϵ smaller than $\frac{1}{c}$ we can remove the first term on the right hand side by subtraction to the left hand side. Note that we are not *setting* ϵ to any specific value. We are only restricting its range closer to 0. Then using (5.17) we get

$$\begin{aligned}
\mathbb{E}[W[\psi]](\tau) + \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) &\leq C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C\mathbb{E}[W[\psi]](0) + C\overline{\mathbb{E}}[\psi](0) + C\overline{\mathbb{E}}[\psi](\tau)\\
&\lesssim \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \mathbb{E}[W[\psi]](0) + \overline{\mathbb{E}}[\psi](0)
\end{aligned} \tag{5.18}$$

which is simply equation (16) in [17]. We start to notice the pattern in suppression degeneracies. We have, so far, successfully removed the degenerate $\partial_t W[\psi]$ term. We would now like to have a similar, non-degenerate, expression for ψ (as opposed to $W[\psi]$).

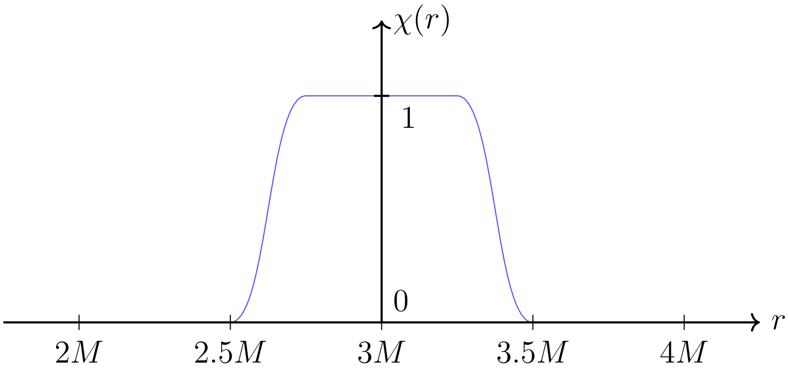
5.6 Tackling the $r = 3M$ degeneracy

To provide the appropriate estimate for ψ we will provide a bound on $\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)$. As shown with equation (17) of [17], this bound can be partly provided by $\overline{\mathbb{I}}_{\text{deg}}[\psi](0,\tau)$. Let us follow their steps and derive the aforementioned estimate.
We first recall the definition of $\overline{\mathbb{I}}_{\text{deg}}[\psi](0,\tau)$ for which the terms in $(\partial_t\psi)^2$, $(\partial_r\psi)^2$ and $|\nabla\!\!\!/\psi|^2$ are degenerate at $r = 3M$.

$$\overline{\mathbb{I}}_{\text{deg}}[\psi](0,\tau) = \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\left(\left(1-\frac{3M}{r}\right)^2\left(\frac{1}{r^2}(\partial_t\psi)^2 + \frac{1}{r^2}(\partial_r\psi)^2 + \frac{1}{r}|\nabla\!\!\!/\psi|^2\right) + \frac{1}{r^2}|R^\star\psi|^2 + \frac{\psi^2}{r^3}\right)$$

If we can remove this degeneracy at $r = 3M$ then we can easily bound $\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)$. In such an attempt, let us define $\chi : [2M,\infty[\to \mathbb{R}$, a smooth radial cut-off function which is equal to 1 in the region $\left[\frac{11M}{4},\frac{13M}{4}\right]$ and 0 in the region $\left]\frac{5M}{2},\frac{7M}{2}\right[$. Such a function would look like
Thus, adding the quantity

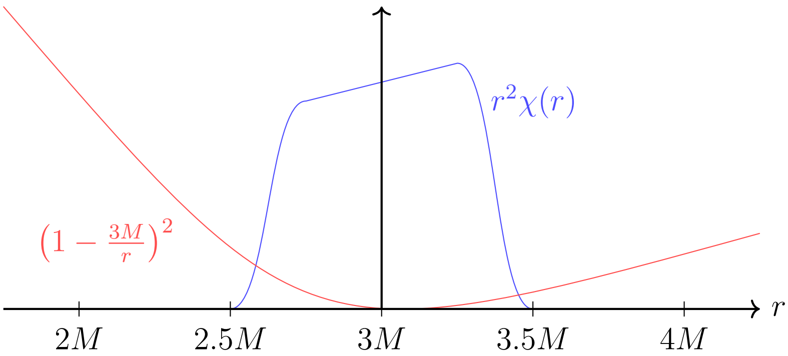
$$\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\chi^2\left((\partial_t\psi)^2 + (\partial_t\psi)^2 + |\nabla\!\!\!/\psi|^2\right)$$



to $\bar{\mathbb{I}}_{\text{deg}}[\psi](0,\tau)$ we can easily see that around $r = 3M$ the integral doesn't vanish (unless the various derivatives of ψ do). By scaling up by a suitable constant, we thus get

$$\bar{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \leq C\bar{\mathbb{I}}_{\text{deg}}[\psi](0,\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \chi^2\left((\partial_t\psi)^2 + (\partial_t\psi)^2 + |\nabla\!\!\!/\psi|^2\right) \tag{5.19}$$

This can further be illustrated by looking locally at the factors in front of $\frac{1}{r^2}(\partial_t\psi)^2$ on both sides. Indeed, around $r = 3M$ we can sketch the rough behaviour of these factors as



From it we can clearly see that we have

$$1 \leq C\left(1 - \frac{3M}{r}\right)^2 + r^2\chi^2$$

A similar argument can be made for all other factors, thus, proving (5.19).

Furthermore, we can use the fact that

$$\frac{1}{r}\left(a_-(r)\right)^{-1/2} W[r-3M] = \left(\partial_r + \frac{\frac{2M}{r} + f(r)}{a_-(r)}\partial_t\right)(r - 3M) = 1$$

to rewrite the second term on the right in (5.19). Indeed, we can then integrate by parts to find a suitable bound to that term.

Let us recall that for $X = X^\mu \partial_\mu$ a smooth vector field and f,g two smooth functions we have the integration by parts formula

$$\int f X[g] = \int \partial_\mu \left(X^\mu f g \right) - \int \partial_\mu X^\mu f g - \int X[f] g$$

Using this property for $X = \frac{1}{r} \left(a_-(r) \right)^{-1/2} W$, which is smooth on the black hole exterior, we find

$$\begin{aligned}
&\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \chi^2 \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \frac{1}{r} \left(a_-(r) \right)^{-1/2} W[r - 3M] \\
&\quad = \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[\partial_t \left(\frac{\frac{2M}{r} + f(r)}{a_-(r)} (r - 3M) \chi^2 \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \right) \\
&\qquad + \partial_r \left((r - 3M) \chi^2 \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \right) \Bigg] \\
&\qquad - \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(1 - \frac{3M}{r} \right) \left(a_-(r) \right)^{-1/2} W \left[\chi^2 \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \right]
\end{aligned}$$

By compact support of χ we see that the ∂_r term must vanish. Additionally, defining $\eta(r) = \big(2M + f(r)\big) \left(1 - \frac{3M}{r} \right) \left(a_-(r) \right)^{-1} \chi^2(r)$ we find that $|\eta(r)| \leq C a_-(r)$. Using these, we can write

$$\begin{aligned}
&\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \chi^2 \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \frac{1}{r} \left(a_-(r) \right)^{-1/2} W[r - 3M] \\
&\quad = \int_{\Sigma_\tau} \eta(r) \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) - \int_{\Sigma_0} \eta(r) \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \\
&\qquad - \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(1 - \frac{3M}{r} \right) \left(a_-(r) \right)^{-1/2} W[\chi^2] \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \\
&\qquad - \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(1 - \frac{3M}{r} \right) \left(a_-(r) \right)^{-1/2} \chi^2 W \left[\left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \right] \\
&\quad \leq C \mathbb{E}[\psi](\tau) + C \mathbb{E}[\psi](0) + C \overline{\mathbb{I}}_{\text{deg}}[\psi](0, \tau) \\
&\qquad - 2 \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(1 - \frac{3M}{r} \right) \left(a_-(r) \right)^{-1/2} \chi^2 \left(\partial_t \psi \partial_t W[\psi] + \partial_t \psi W[\partial_r \psi] + \nabla\!\!\!/ \psi \nabla\!\!\!/ W[\psi] \right) \\
&\quad \leq C \mathbb{E}[\psi](\tau) + C \mathbb{E}[\psi](0) + C \overline{\mathbb{I}}_{\text{deg}}[\psi](0, \tau) \\
&\qquad + \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(1 - \frac{3M}{r} \right) \chi^2 \Bigg(\left(a_-(r) \right)^{-1} \frac{1}{\delta} \left((\partial_t \psi)^2 + (\partial_t \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right) \\
&\qquad + \delta \left(\left(\partial_t W[\psi] \right)^2 + \left(\partial_r W[\psi] \right)^2 + |\nabla\!\!\!/ W[\psi]|^2 \right) \Bigg) \\
&\quad \leq C \mathbb{E}[\psi](\tau) + C \mathbb{E}[\psi](0) + \left(C + \frac{1}{\delta} \right) \overline{\mathbb{I}}_{\text{deg}}[\psi](0, \tau) + C \delta \mathbb{I}_{\text{ndeg}}[W[\psi]](0, \tau)
\end{aligned}$$

We have, again, used the fact that $W[\partial_r \psi]$ breaks down into terms that can be bounded by energies of ψ and $\partial_r W[\psi]$. The above expression together with (5.19), for sufficiently small δ, reduces to

$$\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0, \tau) \lesssim \mathbb{E}[\psi](\tau) + \mathbb{E}[\psi](0) + \frac{1}{\delta} \overline{\mathbb{I}}_{\text{deg}}[\psi](0, \tau) + \delta \mathbb{I}_{\text{ndeg}}[W[\psi]](0, \tau) \tag{5.20}$$

as illustrated in [17].

5.7 Towards our final estimate

We are now ready to combine the various estimates we have derived into what will become Theorem 4.1 of [17]. Let us first recall the estimates

$$\overline{\mathbb{E}}[\psi](\tau) \le C_1 \left(\overline{\mathbb{E}}[\psi](0) + \epsilon\, \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \right) \tag{5.17}$$

$$\mathbb{E}[W[\psi]](\tau) + \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \le C_2 \left(\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \mathbb{E}[W[\psi]](0) + \overline{\mathbb{E}}[\psi](0) \right) \tag{5.18}$$

$$\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \le C_3 \left(\mathbb{E}[\psi](\tau) + \mathbb{E}[\psi](0) + \frac{1}{\delta} \overline{\mathbb{I}}_{\text{deg}}[\psi](0,\tau) + \delta \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \right) \tag{5.20}$$

We will also make use of a standard estimate [9, 11, 17]

$$\overline{\mathbb{I}}_{\text{deg}}[\psi](0,\tau) \le C_4 \left(\overline{\mathbb{E}}[\psi](0) + \epsilon\, \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \right) \tag{5.21}$$

and notice that by definition we have

$$\mathbb{E}[\psi](\tau) \le \overline{\mathbb{E}}[\psi](\tau) \tag{5.22}$$

Note that we have used different symbols for each constant involved in the above estimates. We start by inserting (5.21) and (5.18) into (5.20) to get

$$\begin{aligned}\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \le{}& C_2 C_4 \frac{1}{\delta} \left(\overline{\mathbb{E}}[\psi](0) + \epsilon\, \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \right) + C_3 C_4 \delta \left(\mathbb{E}[W[\psi]](0) + \overline{\mathbb{E}}[\psi](0) + \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \right) \\ & - C_4 \delta \mathbb{E}[W[\psi]](\tau) + C_4 \mathbb{E}[\psi](\tau) + C_4 \mathbb{E}[\psi](0)\end{aligned}$$

This expression is equivalent to

$$\begin{aligned}\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \left(1 - C_2 C_4 \frac{\epsilon}{\delta} - C_3 C_4 \delta \right) + C_4 \delta \mathbb{E}[W[\psi]](\tau) \le{}& \overline{\mathbb{E}}[\psi](0) \left(C_2 C_4 \frac{1}{\delta} + C_3 C_4 \delta \right) + C_3 C_4 \delta \mathbb{E}[W[\psi]](0) \\ & + C_4 \mathbb{E}[\psi](\tau) + C_4 \mathbb{E}[\psi](0)\end{aligned}$$

We can now choose δ such that $C_3 C_4 \delta < 1$. If required, we can then further restrict the range of ϵ in order to make the first term on the left hand side positive. Together with (5.17) and a further restriction in ϵ we get

$$\overline{\mathbb{E}}[\psi](\tau) + \mathbb{E}[W[\psi]](\tau) + \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) \lesssim \overline{\mathbb{E}}[\psi](0) + \mathbb{E}[W[\psi]](0)$$

Finally we can add to this inequality the estimate (5.18) and use the previous estimate on the right hand side in order to obtain

$$\begin{aligned}\overline{\mathbb{E}}[\psi](\tau) + \mathbb{E}[W[\psi]](\tau) + \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) &\lesssim \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \overline{\mathbb{E}}[\psi](0) + \mathbb{E}[W[\psi]](0) \\ &\lesssim \overline{\mathbb{E}}[\psi](0) + \mathbb{E}[W[\psi]](0)\end{aligned}$$

This concludes the derivation as we have successfully recovered Theorem 4.1 of [17]. Indeed, from our derivation we can express the following theorem.

Theorem 5.1 (Holzegel and Kauffman). *Given the Cauchy problem (5.10) with β satisfying (5.9) in the region $\{r \ge 2M\}$, then there exists $\epsilon_0 > 0$ such that for all $\epsilon_0 \ge \epsilon \ge 0$ all smooth solutions to (5.10) satisfy for all $\tau > 0$*

$$\overline{\mathbb{E}}[\psi](\tau) + \mathbb{E}[W[\psi]](\tau) + \overline{\mathbb{I}}_{ndeg}[\psi](0,\tau) + \mathbb{I}_{ndeg}[W[\psi]](0,\tau) \lesssim \overline{\mathbb{E}}[\psi](0) + \mathbb{E}[W[\psi]](0) \tag{5.23}$$

6 Conclusion

To conclude, this work presented multiple techniques to study the wave equation. We started by presenting some elementary results one can easily derive for flat space homogeneous wave equation. In a later treatment we looked at how the energy estimates can be derived and discussed how they can be used to show the boundedness of solutions both in flat spacetime and in Schwarzschild spacetime. In a final note we have applied these ideas to the paper by Holzegel and Kauffman [17]. We have indeed, successfully derived many results claimed in that work. We note, however, the sudden increase in difficulty the curved background and perturbation term brought to the calculations.

Additionally, as discussed by Holzegel and Kauffman this treatment of the wave equation can be extended to Kerr solutions by considering the Teukolsky equation [17]. It is without a doubt the a direction on could take if they wished to continue on the work by Holzegel and Kauffman.

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A Some proofs

Proposition A.1. There is a one-to-one correspondence between C^2 solutions of (3.2) and C^2 solutions of the same Cauchy problem with the speed of light constant inserted, i.e. of

$$-\partial_t^2 u(t,x) + c^2\partial_x^2 u(t,x) = 0 \quad u(0,x) = f(x) \quad \partial_t u(0,x) = g(x)$$

Proof. Assuming we have such a solution to (3.2), denoted $u_0(t,x)$, we can performing a rescaling of the time axis

$$t' = \frac{t}{c}$$

This leads to the equation

$$\begin{aligned} -\partial_t^2 u_0(t,x) + \partial_x^2 u_0(t,x) &= -\frac{1}{c^2}\partial_{t'}^2 u_0(t'c,x) + \partial_x^2 u_0(t'c,x) = 0 \\ \iff \quad & -\partial_{t'}^2 u_0(t'c,x) + c^2\partial_x^2 u_0(t'c,x) = 0 \end{aligned}$$

We see that $u(t',x) = u_0(t'c,x)$ is a C^2 solution of the new Cauchy problem. The one-to-one correspondence is also made clear by the previous statement.

□

Proposition A.2. The general solution of $-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = 0$ is of the form $u(t,x) = \phi(x+t) + \psi(x-t)$.

Proof. Let us make the change of variables

$$\xi = x + t\,, \quad \eta = x - t$$

The wave equation then becomes

$$\begin{aligned} -\partial_t^2 u(t,x) + \partial_x^2 u(t,x) &= -\left(\partial_\xi - \partial_\eta\right)\left(\partial_\xi - \partial_\eta\right)u(t,x) + \left(\partial_\xi + \partial_\eta\right)\left(\partial_\xi + \partial_\eta\right)u(t,x) \\ &= 4\partial_\xi\partial_\eta u(t,x) \\ &= 0 \end{aligned}$$

$$\iff \quad \partial_\xi\partial_\eta u(t,x) = 0$$

It is well known that this equation has general solution

$$\begin{aligned} u(t,x) &= \phi(\xi) + \psi(\eta) \\ &= \phi(x+t) + \psi(x-t) \end{aligned}$$

□

Proposition A.3. Let Ω be an open subset of $\mathbb{R}^n$. For all $f \in L^p(\Omega, \mathbb{R})$ and all $\varphi \in \mathcal{D}(\Omega)$ we have

$$f(\varphi) = \int_\Omega f\varphi < \infty$$

Proof. By definition of $L^p(\Omega, \mathbb{R})$, we have

$$\left(\int_\Omega |f|^p\right)^{1/p} < \infty \quad \Rightarrow \quad \int_\Omega |f| < \infty$$

Since φ has compact support, we have

$$\int_\Omega |\varphi| < \infty$$

Finally, using Hölder's inequality we get

$$\begin{aligned} |f(\varphi)| = |\int_\Omega f\varphi| &\leq \int_\Omega |f\varphi| \\ &\leq \int_\Omega |f| \int_\Omega |\varphi| \\ &< \infty \end{aligned}$$

□

Proposition A.4. Let $f \in \mathcal{D}'(\Omega)$ be a distribution on Ω, an open subset of $\mathbb{R}^n$. Also let $\alpha \in \mathbb{N}^n$, then $D^\alpha f$ is well defined and is also a distribution on Ω.

Proof. As usual $D^\alpha f$ is defined through its action on test functions $\varphi \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega) \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ then $D^\alpha \varphi$ is well defined. By compact support of φ and after integrating by parts $|\alpha|$ times, we have

$$\begin{aligned} D^\alpha f(\varphi) &= \int_\Omega D^\alpha f\varphi \\ &= (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi \end{aligned}$$

By compact support of φ we have that $D^\alpha \varphi$ is also of compact support and so $D^\alpha f$ is well defined and is also a distribution on Ω.

□

Proposition A.5. Given the representation formula

$$\mathcal{F}(u)(t,\xi) = \frac{\mathcal{F}(g)(\xi)}{2\pi|\xi|} \sin(2\pi t|\xi|) + \mathcal{F}(f)(\xi)\cos(2\pi t|\xi|)$$

then $|\partial_t \mathcal{F}(u)(t,\xi)|^2 + 4\pi^2|\xi|^2 \, |\mathcal{F}(u)(t,\xi)|^2 = |\mathcal{F}(g)(\xi)| + 4\pi^2|\xi|^2 \, |\mathcal{F}(f)(\xi)|^2$ holds for all $\xi \in \mathbb{R}^d$.

Proof. We start by computing

$$\partial_t \mathcal{F}(u)(t,\xi) = \mathcal{F}(g)(\xi)\cos(2\pi t|\xi|) - 2\pi|\xi|\mathcal{F}(f)(\xi)\sin(2\pi t|\xi|)$$

Then it follows that

$$\begin{aligned}
|\partial_t \mathcal{F}(u)(t,\xi)|^2 + 4\pi^2|\xi|^2|\mathcal{F}(u)(t,\xi)|^2 &= |\mathcal{F}(g)(\xi)|^2\cos^2(2\pi t|\xi|) + 4\pi^2|\xi|^2|\mathcal{F}(f)(\xi)|^2\sin^2(2\pi t|\xi|) \\
&\quad - 2\pi|\xi|\mathcal{F}(g)(\xi)\overline{\mathcal{F}(f)(\xi)}\cos(2\pi t|\xi|)\sin(2\pi t|\xi|) \\
&\quad - 2\pi|\xi|\overline{\mathcal{F}(g)(\xi)}\mathcal{F}(f)(\xi)\cos(2\pi t|\xi|)\sin(2\pi t|\xi|) \\
&\quad + 4\pi^2|\xi|^2\Big(\frac{|\mathcal{F}(g)(\xi)|^2}{4\pi^2|\xi|^2}\sin^2(2\pi t|\xi|) + |\mathcal{F}(f)(\xi)|^2\cos^2(2\pi t|\xi|) \\
&\quad + \frac{\mathcal{F}(g)(\xi)}{2\pi|\xi|}\overline{\mathcal{F}(f)(\xi)}\cos(2\pi t|\xi|)\sin(2\pi t|\xi|) \\
&\quad + \frac{\overline{\mathcal{F}(g)(\xi)}}{2\pi|\xi|}\mathcal{F}(f)(\xi)\cos(2\pi t|\xi|)\sin(2\pi t|\xi|)\Big) \\
&= |\mathcal{F}(g)(\xi)|^2 + 4\pi^2|\xi|^2|\mathcal{F}(g)(\xi)|^2
\end{aligned}$$

Then integrating by parts, and using Plancherel theroem, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} d\xi\, \Big(|\partial_t \mathcal{F}(u)(t,\xi)|^2 + 4\pi^2|\xi|^2|\mathcal{F}(u)(t,\xi)|^2\Big) &= \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^d)} + \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \\
&= \|g\|_{L^2(\mathbb{R}^d)} + \|\nabla g\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

$\square$

Proposition A.6. For Q a quadratic form in the derivatives of u, whose coefficients are the derivatives of the components of X, we have

$$\begin{aligned}
\Box_\eta u X[u] = &-\frac{1}{2}\partial_t\left[X_0\left(\sum_{i=1}^d(\partial_i u)^2 + (\partial_t u)^2\right) + 2\partial_t u\sum_{i=1}^d X_i\partial_i u\right] \\
&-\frac{1}{2}\sum_{i=1}^d\partial_i\left[X_i\left(\sum_{j=1}^d(\partial_j u)^2 - (\partial_t u)^2\right) - 2X_0\partial_t u\partial_i u - 2\partial_i u\sum_{j=1}^d X_j\partial_j u\right] + Q
\end{aligned}$$

Proof. The proof follows from a lengthy but simple calculation

$$
\begin{aligned}
\Box_\eta u X[u] &= \left(-\partial_t^2 u - \sum_{i=1}^d \partial_i^2 u\right)\left(X_0\partial_t u + \sum_{j=1}^d X_j\partial_j u\right) \\
&= -\partial_t^2 u X_0 \partial_t u + \sum_{i=1}^d \partial_i^2 u X_0\partial_t u - \partial_t^2 u \sum_{i=1}^d X_i \partial_i u + \sum_{i=1}^d \partial_i^2 u \sum_{j=1}^d X_j \partial_j u \\
&= -\frac{1}{2}\partial_t\left(X_0(\partial_t u)^2\right) + \frac{1}{2}\partial_t X_0 (\partial_t u)^2 - \partial_t\left(\sum_{i=1}^d X_i\partial_i u\partial_t u\right) + \sum_{i=1}^d \partial_t X_i \partial_i u \partial_t u + \sum_{i=1}^d X_i \partial_t\partial_i u\partial_t u \\
&\quad + \sum_{i=1}^d \partial_i\left(\partial_i u X_0 \partial_t u\right) - \sum_{i=1}^d \partial_i u \partial_i X_0 \partial_t u - \sum_{i=1}^d \partial_i u X_0 \partial_i\partial_t u + \sum_{i=1}^d \partial_i\left(\partial_i u \sum_{j=1}^d X_j\partial_j u\right) \\
&\quad - \sum_{i=1}^d \partial_i u \sum_{j=1}^d \partial_i X_j \partial_j u - \sum_{i=1}^d \partial_i u \sum_{j=1}^d X_j \partial_i\partial_j u \\
&= -\frac{1}{2}\partial_t\left(X_0(\partial_t u)^2 + X_0\sum_{i=1}^d (\partial_i u)^2 + \partial_t u \sum_{i=1}^d X_i\partial_i u\right) + \frac{1}{2}\partial_t X_0 \sum_{i=1}^d (\partial_i u)^2 + \frac{1}{2}\sum_{i=1}^d \partial_i\left(X_i(\partial_t u)^2\right) \\
&\quad + \sum_{i=1}^d \partial_i\left(\partial_i u X_0\partial_t u\right) + \sum_{i=1}^d \partial_i\left(\partial_i u\sum_{j=1}^d X_j\partial_j u\right) - \sum_{i=1}^d \partial_i u \sum_{j=1}^d X_j\partial_i\partial_j u + \frac{1}{2}\partial_t X_0(\partial_t u)^2 \\
&\quad + \sum_{i=1}^d \partial_t X_i \partial_i u\partial_t u - \frac{1}{2}\sum_{i=1}^d \partial_i X_i (\partial_t u)^2 - \sum_{i=1}^d \partial_i u\partial_i X_0 \partial_t u - \sum_{i=1}^d \partial_i u \sum_{j=1}^d \partial_i X_j\partial_j u \\
&= -\frac{1}{2}\partial_t\left[X_0\left(\sum_{i=1}^d (\partial_i u)^2 + (\partial_t u)^2\right] + 2\partial_t u\sum_{i=1}^d X_i\partial_i u\right] \\
&\quad - \frac{1}{2}\sum_{i=1}^d \partial_i\left[X_i\left(\sum_{j=1}^d(\partial_j u)^2 - (\partial_t u)^2\right) - 2X_0\partial_t u\partial_i u - 2\partial_i u\sum_{j=1}^d X_j\partial_j u\right] + \frac{1}{2}\sum_{i=1}^d \partial_i X_i \sum_{j=1}^d (\partial_j u)^2 \\
&\quad + \frac{1}{2}\partial_t X_0 \sum_{i=1}^d (\partial_i u)^2 + \frac{1}{2}\partial_t X_0(\partial_t u)^2 + \sum_{i=1}^d \partial_t X_i\partial_i u\partial_t u - \frac{1}{2}\sum_{i=1}^d \partial_i X_i(\partial_t u)^2 - \sum_{i=1}^d \partial_i u\partial_i X_0\partial_t u \\
&\quad - \sum_{i=1}^d \partial_i u \sum_{j=1}^d \partial_i X_j\partial_j u \\
&= -\frac{1}{2}\partial_t\left[X_0\left(\sum_{i=1}^d (\partial_i u)^2 + (\partial_t u)^2\right) + 2\partial_t u\sum_{i=1}^d X_i\partial_i u\right] \\
&\quad - \frac{1}{2}\sum_{i=1}^d \partial_i\left[X_i\left(\sum_{j=1}^d(\partial_j u)^2 - (\partial_t u)^2\right) - 2X_0\partial_t u\partial_i u - 2\partial_i u\sum_{j=1}^d X_j\partial_j u\right] + Q
\end{aligned}
$$

□

B Derivation from Section 5

B.1 Details from Section 5.3.2

Proposition B.1. 8 Given the definition of $\mathcal{P}_s\left(W[\psi]\right)$ (5.4) it follows that

$$\mathcal{P}_s\left(W[\psi]\right) = -\frac{2}{r}\left(a_-(r)\right)^{-1}\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t W[\psi] + \mathcal{F}_1 + \mathcal{F}_2 \tag{5.13}$$

$$\mathcal{F}_1 = -\frac{\epsilon}{r^2}W[r^2\beta^\mu\partial_\mu\psi] \qquad \mathcal{F}_2 = \left(a_-(r)\right)^{-1/2}\left(h_2(r)\partial_t\psi + h_2(r)\partial_r\psi + h_3(r)\psi\right)$$

Proof. We recall

$$\begin{aligned}
\mathcal{P}_s\left(W[\psi]\right) &= a_+(r)\partial_t^2 W[\psi] - \frac{4M}{r}\partial_t\partial_r W[\psi] + \frac{2M}{r^2}\partial_t W[\psi] - \partial_r\left(a_-(r)\partial_r W[\psi]\right) \\
&\quad - \frac{\Delta_{S^2}W[\psi]}{r^2} + (1-s^2)\frac{2M}{r^3}W[\psi] \\
&= a_+(r)W[\partial_t^2\psi] - \frac{4M}{r}\partial_r W[\partial_t\psi] + \frac{2M}{r^2}W[\partial_t\psi] - \partial_r\left(a_-(r)\partial_r W[\psi]\right) \\
&\quad - \frac{W[\Delta_{S^2}\psi]}{r^2} + (1-s^2)\frac{2M}{r^3}W[\psi]
\end{aligned} \tag{5.12}$$

With the help of our results from Section 5.3.1 we can, therefore, determine the following commutations of W with all the terms in the above expression.

$$a_+(r)W[\partial_t^2\psi] = W\left[a_+(r)\partial_t^2\psi\right] + \left(a_-(r)\right)^{1/2}\frac{2M}{r}\partial_t^2\psi$$

$$\begin{aligned}
-\frac{4M}{r}\partial_r W[\partial_t\psi] &= -\frac{4M}{r}W[\partial_t\partial_r\psi] + \frac{4M}{r}[W,\partial_r]\partial_t\psi \\
&= -W\left[\frac{4M}{r}\partial_t\partial_r\psi\right] - \left(a_-(r)\right)^{1/2}\frac{4M}{r}\partial_r\partial_r\psi + \frac{4M}{r}[W,\partial_r]\partial_t\psi
\end{aligned}$$

$$\frac{2M}{r^2}W[\partial_t\psi] = W\left[\frac{2M}{r^2}\partial_t\psi\right] + \frac{4M}{r^2}\left(a_-(r)\right)^{1/2}\partial_t\psi$$

$$\begin{aligned}
-\partial_r\left(a_-(r)\partial_r W[\psi]\right) &= -\partial_r\left(a_-(r)W[\partial_r\psi]\right) + \partial_r\left(a_-(r)[W,\partial_r]\psi\right)\\
&= -\partial_r W\left[a_-(r)\partial_r\psi\right] + \partial_r\left(\frac{2M}{r}\left(a_-(r)\right)^{1/2}\partial_r\psi\right)\\
&\quad + \partial_r\left(a_-(r)[W,\partial_r]\psi\right)\\
&= -W\left[\partial_r\left(a_-(r)\partial_r\psi\right)\right] + [W,\partial_r]\left(a_-(r)\partial_r\psi\right)\\
&\quad + \partial_r\left(\frac{2M}{r}\left(a_-(r)\right)^{1/2}\partial_r\psi\right) + \partial_r\left(a_-(r)[W,\partial_r]\psi\right)
\end{aligned}$$

$$-\frac{W[\Delta_{S^2}\psi]}{r^2} = -W\left[\frac{1}{r^2}\Delta_{S^2}\psi\right] - \frac{2}{r^2}\left(a_-(r)\right)^{1/2}\Delta_{S^2}\psi$$

$$\frac{2M}{r^3}(1-s^2)W[\psi] = W\left[\frac{2M}{r^3}(1-s^2)\psi\right] + \left(a_-(r)\right)^{1/2}\frac{6M}{r^3}(1-s^2)\psi$$

As lengthy as it might seem, we must now place those identities back into equation (5.12) regroup terms that are contained in $\mathcal{P}_s\psi$.

$$\begin{aligned}
\mathcal{P}_s\left(W[\psi]\right) &= W\left[a_+(r)\partial_t^2\psi\right] + \left(a_-(r)\right)^{1/2}\frac{2M}{r}\partial_t^2\psi - W\left[\frac{4M}{r}\partial_t\partial_r\psi\right] - \left(a_-(r)\right)^{1/2}\frac{4M}{r}\partial_t\partial_r\psi\\
&\quad + \frac{4M}{r}[W,\partial_r]\partial_t\psi + W\left[\frac{2M}{r^2}\partial_t\psi\right] + \frac{4M}{r^2}\left(a_-(r)\right)^{1/2}\partial_t\psi - W\left[\partial_r\left(a_-(r)\partial_r\psi\right)\right]\\
&\quad + [W,\partial_r]\left(a_-(r)\partial_r\psi\right) + \partial_r\left(\frac{2M}{r}\left(a_-(r)\right)^{1/2}\partial_r\psi\right) + \partial_r\left(a_-(r)[W,\partial_r]\psi\right)\\
&\quad - W\left[\frac{1}{r^2}\Delta_{S^2}\psi\right] - \frac{2}{r^2}\left(a_-(r)\right)^{1/2}\Delta_{S^2}\psi + W\left[\frac{2M}{r^3}(1-s^2)\psi\right] + \left(a_-(r)\right)^{1/2}\frac{6M}{r^3}(1-s^2)\psi\\
&= W[\mathcal{P}_s\psi] + \left(a_-(r)\right)^{1/2}\frac{2M}{r}\partial_t^2\psi - \left(a_-(r)\right)^{1/2}\frac{4M}{r}\partial_t\partial_r\psi + \frac{4M}{r}[W,\partial_r]\partial_t\psi\\
&\quad + \frac{4M}{r^2}\left(a_-(r)\right)^{1/2}\partial_t\psi + [W,\partial_r]\left(a_-(r)\partial_r\psi\right) + \partial_r\left(\frac{2M}{r}\left(a_-(r)\right)^{1/2}\partial_r\psi\right)\\
&\quad + \partial_r\left(a_-(r)[W,\partial_r]\psi\right) - \frac{2}{r^2}\left(a_-(r)\right)^{1/2}\Delta_{S^2}\psi + \left(a_-(r)\right)^{1/2}\frac{6M}{r^3}(1-s^2)\psi
\end{aligned}$$

We can now expand out all terms in $[W,\partial_r]\psi$ and all derivatives. We will also use our commutation identities to rewrite $W[\mathcal{P}_s\psi] = \frac{r^2}{r^2}W[\mathcal{P}_s\psi] = \frac{1}{r^2}W[r^2\mathcal{P}_s\psi] - 2\left(a_-(r)\right)^{1/2}\mathcal{P}_s\psi$. Finally, we will also use the definition of our Cauchy problem (5.10) to replace all instances of $\mathcal{P}_s\psi$ by $-\epsilon\beta^\mu\partial_\mu\psi$.

$$\begin{aligned}
\mathcal{P}_s\left(W[\psi]\right) &= \frac{1}{r^2}W[r^2\mathcal{P}_s\psi] - 2\left(a_-(r)\right)^{1/2}\mathcal{P}_s\psi + \left(a_-(r)\right)^{1/2}\frac{2M}{r}\partial_t^2\psi - \left(a_-(r)\right)^{1/2}\frac{4M}{r}\partial_t\partial_r\psi\\
&\quad + \frac{4M^2}{r^3}\left(a_-(r)\right)^{-1}W[\partial_t\psi]\\
&\quad - \frac{4M}{r}\left(a_-(r)\right)^{-1/2}\left(\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t^2\psi + \partial_t\partial_r\psi\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{4M}{r^2}\left(a_-(r)\right)^{1/2}\partial_t\psi + \frac{M}{r^2}\left(a_-(r)\right)^{-1} W\left[a_-(r)\partial_r\psi\right] \\
&- \left(a_-(r)\right)^{-1/2}\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2} a_-(r)\partial_t\partial_r\psi \\
&- \left(a_-(r)\right)^{1/2}\partial_r\left(a_-(r)\partial_r\psi\right) + \partial_r\left(\frac{2M}{r}\left(a_-(r)\right)^{1/2}\partial_r\psi\right) + \partial_r\left(\frac{M}{r^2}W[\psi]\right) \\
&- \partial_r\left(\left(a_-(r)\right)^{1/2}\left(\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t\psi + \partial_r\psi\right)\right) \\
&- \frac{2}{r^2}\left(a_-(r)\right)^{1/2}\Delta_{S^2}\psi + \left(a_-(r)\right)^{1/2}\frac{6M}{r^3}(1-s^2)\psi \\
= &-\frac{\epsilon}{r^2}W[r^2\beta^\mu\partial_\mu\psi] - s^2\frac{2M}{r^2}\left(a_-(r)\right)^{-1/2}\left(\frac{2M}{r}+f(r)\right)\partial_t\psi + s^2\frac{2M}{r^3}\left(a_-(r)\right)^{1/2}\psi \\
&- s^2\frac{2M}{r^2}\left(a_-(r)\right)^{1/2}\partial_r\psi - 2\left(a_-(r)\right)^{1/2}a_+(r)\partial_t^2\psi + \frac{8M}{r}\left(a_-(r)\right)^{1/2}\partial_t\partial_t\psi \\
&- \frac{4M}{r}\left(a_-(r)\right)^{1/2}\partial_t\psi + 2\left(a_-(r)\right)^{1/2}\partial_t\left(a_-(r)\partial_r\psi\right) + \frac{2}{r^2}\left(a_-(r)\right)^{1/2}\Delta_{S^2}\psi \\
&- \frac{4M}{r^3}(1-s^2)\left(a_-(r)\right)^{1/2}\psi + \left(a_-(r)\right)^{1/2}\frac{2M}{r}\partial_t^2\psi - \left(a_-(r)\right)^{1/2}\frac{4M}{r}\partial_t\partial_r\psi \\
&+ \frac{4M^2}{r^3}\left(a_-(r)\right)^{-1}\partial_t(W[\psi]) - \frac{4M}{r}\left(a_-(r)\right)^{-1/2}\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t^2\psi \\
&- \frac{4M}{r}\left(a_-(r)\right)^{-1/2}\partial_t\partial_r\psi + \frac{4M}{r^2}\left(a_-(r)\right)^{1/2}\partial_t\psi \\
&+ \frac{M}{r}\left(a_-(r)\right)^{-1/2}\left(\frac{2M}{r}+f(r)\right)\partial_t\partial_r\psi + \frac{M}{r}\left(a_-(r)\right)^{-1/2}\partial_r\left(a_-(r)\partial_r\psi\right) \\
&- \left(a_-(r)\right)^{1/2}\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t\partial_r\psi \\
&- \left(a_-(r)\right)^{-1/2}\partial_r\left(a_-(r)\partial_r\psi\right) - \frac{2M}{r^2}\left(a_-(r)\right)^{1/2}\partial_r\psi \\
&+ \frac{2M}{r^3}\left(a_-(r)\right)^{-1/2}\partial_r\psi + \frac{2M}{r}\left(a_-(r)\right)^{1/2}\partial_r^2\psi - \frac{2M}{r^2}\left(a_-(r)\right)^{-1/2}\left(\frac{2M}{r}+f(r)\right)\partial_t\psi \\
&- \frac{2M}{r^2}\left(a_-(r)\right)^{1/2}\partial_r\psi - \frac{M^2}{r^3}\left(a_-(r)\right)^{-3/2}\left(\frac{2M}{r}+f(r)\right)\partial_t\psi - \frac{M^2}{r^3}\left(a_-(r)\right)^{-1/2}\partial_r\psi \\
&+ \frac{M}{r^2}\left(a_-(r)\right)^{-1/2}\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t\psi + \frac{M}{r^2}\left(a_-(r)\right)^{-1/2}\partial_r\psi \\
&+ \frac{M}{r}\left(a_-(r)\right)^{-1/2}\left(\frac{2M}{r}+f(r)\right)\partial_t\partial_r\psi + \frac{M}{r}\left(a_-(r)\right)^{1/2}\partial_r^2\psi \\
&- \frac{M}{r^2}\left(a_-(r)\right)^{-1/2}\left(1+\frac{3M}{r}+\frac{9M^2}{r^2}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t\psi - \frac{M}{r^2}\left(a_-(r)\right)^{-1/2}\partial_r\psi \\
&+ \left(a_-(r)\right)^{1/2}\left(\frac{3M}{r^2}+\frac{18M^2}{r^3}\right)\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t\psi
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3M}{r^2} \left(a_-(r)\right)^{1/2} \left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) \left(1 + \frac{6M}{r}\right)^{-3/2} \partial_t \psi \\
&- \left(a_-(r)\right)^{1/2} \left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) \left(1 + \frac{6M}{r}\right)^{-1/2} \partial_t \partial_r \psi - \left(a_-(r)\right)^{1/2} \partial_r^2 \psi \\
&- \frac{2}{r} \left(a_-(r)\right)^{1/2} \Delta_{S^2} \psi + \left(a_-(r)\right)^{1/2} \frac{6M}{r^3} (1 - s^2) \psi
\end{aligned}
$$

Given the length of the previous equation, we can simply pick out every term with common factors $\partial_t^2 \psi$, $\partial_t \partial_r \psi$, $\partial_r^2 \psi$, $\partial_r \psi$, $\partial_t \psi$ and ψ. This choice leads to the simplifications

$$
\begin{aligned}
\partial_t^2 \psi :& - 2 \left(a_-(r)\right)^{1/2} a_+(r) + \left(a_-(r)\right)^{1/2} \frac{2M}{r} \\
&- \frac{4M}{r} \left(a_-(r)\right)^{-1/2} \left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) \left(1 + \frac{6M}{r}\right)^{-1/2} + \frac{4M}{r^2} \left(a_-(r)\right)^{-3/2} \left(\frac{2M}{r} + f(r)\right) \\
&= \left(a_-(r)\right)^{-3/2} \left[-2 \left(a_-(r)\right)^2 a_+(r) + \left(a_-(r)\right)^2 \frac{2M}{r} + \frac{8M^3}{r^3}\right] \\
&\quad + \left(1 + \frac{6M}{r}\right)^{-1/2} \left(a_-(r)\right)^{-3/2} \left[- \frac{4M}{r} a_-(r) \left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) \right. \\
&\qquad \left. + \frac{4M^2}{r^2} \left(1 - \frac{3M}{r}\right) \left(1 + \frac{6M}{r}\right) \right] \\
&= \left(a_-(r)\right)^{-3/2} \left[-2 + \frac{6M}{r}\right] + \left(1 + \frac{6M}{r}\right)^{-1/2} \left(a_-(r)\right)^{-3/2} \left[-\frac{4M}{r}\right] \\
&= -\frac{2}{r} \left(a_-(r)\right)^{-3/2} \left(1 + \frac{6M}{r}\right)^{-1/2} r \left[\frac{2M}{r} + f(r)\right]
\end{aligned}
$$

which is the $\partial_t^2 \psi$ term in $-\frac{2}{r} \left(a_-(r)\right)^{-1} \left(1 + \frac{6M}{r}\right)^{-1/2} \partial_t W[\psi]$. On the other hand we also have

$$
\begin{aligned}
\partial_t \partial_r \psi :& \frac{4M}{r} \left(a_-(r)\right)^{1/2} + \frac{4M^2}{r^2} \left(a_-(r)\right)^{-1/2} - \frac{4M}{r} \left(a_-(r)\right)^{-1/2} \\
&+ 2 \frac{M}{r} \left(a_-(r)\right)^{-1/2} \left(\frac{2M}{r} + f(r)\right) - 2 \left(a_-(r)\right)^{1/2} \left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) \left(1 + \frac{6M}{r}\right)^{-1/2} \\
&= \left(a_-(r)\right)^{-3/2} \left(1 + \frac{6M}{r}\right)^{-1/2} \left[\frac{2M}{r} a_-(r) \left(1 - \frac{3M}{r}\right) \left(1 + \frac{6M}{r}\right) \right. \\
&\qquad \left. - 2 \left(a_-(r)\right)^2 \left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) \right] \\
&= \left(a_-(r)\right)^{-3/2} \left(1 + \frac{6M}{r}\right)^{-1/2} \left[2 \left(\frac{2M}{r} - 1\right)\right] \\
&= -\frac{2}{r} \left(a_-(r)\right)^{-3/2} \left(1 + \frac{6M}{r}\right)^{-1/2} r a_-(r)
\end{aligned}
$$

which is the $\partial_t\partial_r\psi$ term in $-\frac{2}{r}\left(a_-(r)\right)^{-1}\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t W[\psi]$. Furthermore, we also have

$$\begin{aligned}
\partial_r^2\psi : &2\left(a_-(r)\right)^{3/2} + \frac{M}{r}\left(a_-(r)\right)^{1/2} - \left(a_-(r)\right)^{1/2} + \frac{2M}{r}\left(a_-(r)\right)^{1/2} + \frac{M}{r}\left(a_-(r)\right)^{1/2} \\
&\quad - \left(a_-(r)\right)^{1/2} \\
&= \left(a_-(r)\right)^{1/2}\left[2a_-(r) + \frac{4M}{r} - 2\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\partial_r\psi : &- s^2\frac{2M}{r^2}\left(a_-(r)\right)^{1/2} + \frac{4M}{r^2}\left(a_-(r)\right)^{1/2} - \frac{2M}{r^2}\left(a_-(r)\right)^{-1/2} - \frac{2M}{r^2}\left(a_-(r)\right)^{1/2} \\
&\quad + \frac{2M^2}{r^3}\left(a_-(r)\right)^{-1/2} - \frac{2M}{r^2}\left(a_-(r)\right)^{1/2} - \frac{M^2}{r^3}\left(a_-(r)\right)^{-1/2} + \frac{2M^2}{r^3}\left(a_-(r)\right)^{-1/2} \\
&= \left(a_-(r)\right)^{-1/2}\left[-s^2\frac{2M}{r^2}a_-(r) + \frac{4M}{r^2}a_-(r) - \frac{2M}{r^2} - \frac{2M}{r^2}a_-(r) + \frac{2M^2}{r^3} + \frac{2M^2}{r^3}\right] \\
&= \left(a_-(r)\right)^{-1/2}\left[-s^2\frac{2M}{r^2}a_-(r)\right] \\
&= \left(a_-(r)\right)^{-1/2}h_2(r)
\end{aligned}$$

Note that we have defined $h_2(r)$ such that $\sup_{[2M,\infty[}|r^2h_2(r)| = 2Ms^2$. Pursuing our endeavour,

$$\begin{aligned}
\partial_t\psi : &- \frac{4M}{r^2}\left(a_-(r)\right)^{1/2} + \frac{4M}{r^2}\left(a_-(r)\right)^{1/2} - \frac{2M}{r^2}\left(a_-(r)\right)^{-1/2}\left(\frac{2M}{r} + f(r)\right) \\
&\quad - \frac{M^2}{r^3}\left(a_-(r)\right)^{-3/2}\left(\frac{2M}{r} + f(r)\right) + \left(a_-(r)\right)^{1/2}\left(\frac{3M}{r^2} + \frac{18M^2}{r^3}\right)\left(1 + \frac{6M}{r}\right)^{-1/2} \\
&\quad - \frac{3M}{r^2}\left(a_-(r)\right)^{1/2}\left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right)\left(1 + \frac{6M}{r}\right)^{-3/2} \\
&= \left(a_-(r)\right)^{-3/2}\left[-\frac{4M^2}{r^3}a_-(r) - \frac{2M^3}{r^4}\right] \\
&\quad + \left(a_-(r)\right)^{-3/2}\left(1 + \frac{6M}{r}\right)^{-3/2}\left[\frac{3M}{r^2}\left(1 + \frac{6M}{r}\right)^2\left(a_-(r)\right)^2\right. \\
&\quad - \frac{3M}{r^2}\left(a_-(r)\right)^2\left(1 + \frac{3M}{r} + \frac{9M^2}{r^2}\right) - \frac{2M}{r^2}\left(1 - \frac{3M}{r}\right)\left(1 + \frac{6M}{r}\right)^2 a_-(r) \\
&\quad \left. - \frac{M^2}{r^3}\left(1 - \frac{3M}{r}\right)\left(1 + \frac{6M}{r}\right)^2\right] \\
&= \left(a_-(r)\right)^{-3/2}\left[-\frac{4M^2}{r^3} + \frac{6M^3}{r^4}\right] + \left(a_-(r)\right)^{-3/2}\left(1 + \frac{6M}{r}\right)^{-3/2}\left[\frac{2M}{r^2}\left(\frac{6M}{r} - 1\right)\right] \\
&= \left(a_-(r)\right)^{-1/2}h_2(r)
\end{aligned}$$

Note that we have, this time, defined $h_2(r)$ such that $\sup_{[2M,\infty[}|r^2h_2(r)| = 2M$. We do not mind defining twice the function $h_2(r)$ as we will only interested in their boundedness. One can indeed

see that they both obey $\sup_{[2M,\infty[}|r^2 h_2(r)| \leq \hat{C}(M)$.
Finally, the last terms give us

$$\begin{aligned}\psi : s^2 \frac{2M}{r^3}\left(a_-(r)\right)^{1/2} - \frac{4M}{r^3}(1-s^2)\left(a_-(r)\right)^{1/2} + \left(a_-(r)\right)^{1/2}\frac{6M}{r^3}(1-s^2) \\ = \left(a_-(r)\right)^{-1/2}\left[\frac{2M}{r^3}a_-(r)\right] \\ = \left(a_-(r)\right)^{-1/2} h_3(r)\end{aligned}$$

Similarly to the previous cases, we have defined $h_3(r)$ such that $\sup_{[2M,\infty[}|r^3 h_3(r)| = 2M$.
We are now able to write out our expression for $\mathcal{P}_s\left(W[\psi]\right)$ using our definitions above.

$$\begin{aligned}\mathcal{P}_s\left(W[\psi]\right) &= -\frac{\epsilon}{r^2}W[r^2\beta^\mu\partial_\mu\psi] - \frac{2}{r}\left(a_-(r)\right)^{-1}\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t W[\psi] \\ &\quad + \left(a_-(r)\right)^{-1/2}\left(h_2(r)\partial_t\psi + h_2(r)\partial_r\psi + h_3(r)\psi\right) \\ &= -\frac{2}{r}\left(a_-(r)\right)^{-1}\left(1+\frac{6M}{r}\right)^{-1/2}\partial_t W[\psi] + \mathcal{F}_1 + \mathcal{F}_2\end{aligned} \tag{5.13}$$

□

B.2 Details from Section 5.4

Proposition B.2. For all smooth solutions ψ of the Cauchy problem the following estimate holds.

$$\mathbb{I}_{\mathrm{ndeg}}[W[\psi]](0,\tau) \lesssim \overline{\mathbb{I}}_{\mathrm{ndeg}}[\psi](0,\tau) + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 + \mathbb{E}[W[\psi]](0) + \mathbb{E}[W[\psi]](\tau) + \overline{\mathbb{E}}[\psi](0) + \overline{\mathbb{E}}[\psi](\tau) \tag{5.16}$$

Proof. Recall that

$$\begin{aligned}
\frac{1}{r}W[\psi]\mathcal{P}_s\left(W[\psi]\right) &= a_+(r)\partial_t^2 W[\psi]\frac{1}{r}W[\psi] - \frac{4M}{r}\partial_t\partial_r W[\psi]\frac{1}{r}W[\psi] + \frac{2M}{r^2}\partial_t W[\psi]\frac{1}{r}W[\psi] \\
&\quad - \partial_r\left(a_-(r)\partial_r W[\psi]\right)\frac{1}{r}W[\psi] - \frac{\Delta_{S^2}W[\psi]}{r^2}\frac{1}{r}W[\psi] + (1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2 \\
&= -\frac{1}{r}a_+(r)\left(\partial_t W[\psi]\right)^2 + \frac{1}{r}a_+(r)\partial_t\left(\partial_t W[\psi]W[\psi]\right) - \partial_t\left(\frac{4M}{r^2}\partial_r W[\psi]W[\psi]\right) \\
&\quad + \frac{4M}{r^2}\partial_t W[\psi]\partial_r W[\psi] + \frac{M}{r^3}\partial_t\left(W[\psi]\right)^2 - \partial_r\left(\frac{1}{r}a_-(r)\partial_r W[\psi]W[\psi]\right) \\
&\quad - \frac{1}{r^2}a_-(r)\partial_r W[\psi]W[\psi] + \frac{1}{r}a_-(r)\left(\partial_r W[\psi]\right)^2 - \frac{1}{r}\nabla\!\!\!/\cdot\left(\nabla\!\!\!/ W[\psi]W[\psi]\right) \\
&\quad + \frac{1}{r}|\nabla\!\!\!/ W[\psi]|^2 + (1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2
\end{aligned}$$

Integrating over the spacetime region $\{r \geq 2M \,|\, \tau \geq t \geq 0\}$ leads to

$$\begin{aligned}
\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}W[\psi]\mathcal{P}_s\left(W[\psi]\right) &= \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\Bigg[-\frac{1}{r}a_+(r)\left(\partial_t W[\psi]\right)^2 + \frac{4M}{r^2}\partial_t W[\psi]\partial_r W[\psi] \\
&\quad - \frac{1}{r^2}a_-(r)\partial_r W[\psi]W[\psi] + \frac{1}{r}a_-(r)\left(\partial_r W[\psi]\right)^2 + \frac{1}{r}|\nabla\!\!\!/ W[\psi]|^2 \\
&\quad + (1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2\Bigg] \\
&\quad + \left[\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_+(r)\partial_t W[\psi]W[\psi] - \frac{4M}{r^2}\partial_r W[\psi]W[\psi] + \frac{M}{r^3}\left(W[\psi]\right)^2\right]_0^\tau \\
&\quad - \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\left[\partial_r\left(\frac{1}{r}a_-(r)\partial_r W[\psi]W[\psi]\right) + \nabla\!\!\!/\cdot\left(\frac{1}{r}\nabla\!\!\!/ W[\psi]W[\psi]\right)\right]
\end{aligned}$$

By analogy to the previous estimate and by using our expression for $\mathcal{P}_s\left(W[\psi]\right)$ we write

$$\begin{aligned}
&\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\Bigg[-\frac{1}{r}a_+(r)\left(\partial_t W[\psi]\right)^2+\frac{4M}{r^2}\partial_t W[\psi]\partial_r W[\psi]-\frac{1}{r^2}a_-(r)\partial_r W[\psi]W[\psi]\\
&\qquad+\frac{1}{r}a_-(r)\left(\partial_r W[\psi]\right)^2+\frac{1}{r}|\nabla\!\!\!\!/ W[\psi]|^2+(1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2\Bigg]\\
&\leq C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)+\int_{\Sigma_0}\left[\frac{1}{r}a_+(r)\partial_t W[\psi]W[\psi]-\frac{4M}{r^2}\partial_r W[\psi]W[\psi]+\frac{M}{r^3}\left(W[\psi]\right)^2\right]\\
&\qquad-\int_{\Sigma_\tau}\left[\frac{1}{r}a_+(r)\partial_t W[\psi]W[\psi]-\frac{4M}{r^2}\partial_r W[\psi]W[\psi]+\frac{M}{r^3}\left(W[\psi]\right)^2\right]\\
&\qquad-\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{2}{r^2}\left(1+\frac{6M}{r}\right)^{-1/2}\left(a_-(r)\right)^{-1}\partial_t W[\psi]W[\psi]+\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\left(\frac{1}{r}\mathcal{F}_1 W[\psi]+\frac{1}{r}\mathcal{F}_2 W[\psi]\right)
\end{aligned}$$

Since we are interested in constructing an estimate of $\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)$ we add and remove terms on both sides to make that energy more apparent.

$$\begin{aligned}
\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)\leq C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)&+\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_+(r)\left(\partial_t W[\psi]\right)^2+\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\Bigg[-\frac{4M}{r^2}\partial_t W[\psi]\partial_r W[\psi]\\
&+\frac{1}{r^2}\left(\partial_t W[\psi]\right)^2+\left(\frac{1}{r^2}-\frac{1}{r}\right)a_-(r)\left(\partial_r W[\psi]\right)^2+\frac{1}{r^2}a_-(r)\partial_r W[\psi]W[\psi]\\
&-(1-s^2)\frac{2M}{r^4}\left(W[\psi]\right)^2+\frac{1}{r}\mathcal{F}_1 W[\psi]+\frac{1}{r}\mathcal{F}_2 W[\psi]\Bigg]\\
&+\int_{\Sigma_0}\left[\frac{1}{r}a_+(r)\partial_t W[\psi]W[\psi]-\frac{4M}{r^2}\partial_r W[\psi]W[\psi]+\frac{M}{r^3}\left(W[\psi]\right)^2\right]\\
&-\int_{\Sigma_\tau}\left[\frac{1}{r}a_+(r)\partial_t W[\psi]W[\psi]-\frac{4M}{r^2}\partial_r W[\psi]W[\psi]+\frac{M}{r^3}\left(W[\psi]\right)^2\right]\\
&-\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{2}{r^2}\left(1+\frac{6M}{r}\right)^{-1/2}\left(a_-(r)\right)^{-1}\partial_t W[\psi]W[\psi]
\end{aligned}$$

Now, we can further simplify this estimate by using similar elementary estimates to last time, namely $\frac{1}{r^2} \leq \frac{1}{2Mr}$, $a_+(r) \leq \left(a_-(r)\right)^{-1}$, $\left(1+\frac{6M}{r}\right)^{-1/2} \leq 1$ and $1 \leq \left(a_-(r)\right)^{-1}$.

$$
\begin{aligned}
\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \leq C\bar{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} a_-(r) \left(\partial_t W[\psi]\right)^2 \\
+ \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{2Mr} a_-(r) \left(\partial_t W[\psi]\right)^2 + \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[\frac{2}{r^2} \left(a_-(r)\right)^{-1} \left|\partial_t W[\psi] W[\psi]\right| \\
- \frac{4M}{r^2} \partial_t W[\psi] \partial_r W[\psi] + \left(\frac{1}{r^2} - \frac{1}{r}\right) a_-(r) \left(\partial_r W[\psi]\right)^2 + \frac{1}{r^2} a_-(r) \partial_r W[\psi] W[\psi] \\
- (1-s^2) \frac{2M}{r^4} \left(W[\psi]\right)^2 + \frac{1}{r} \mathcal{F}_1 W[\psi] + \frac{1}{r} \mathcal{F}_2 W[\psi] \Bigg] \\
+ \int_{\Sigma_0} \left[\frac{1}{r} a_+(r) \partial_t W[\psi] W[\psi] - \frac{4M}{r^2} \partial_r W[\psi] W[\psi] + \frac{M}{r^3} \left(W[\psi]\right)^2 \right] \\
- \int_{\Sigma_\tau} \left[\frac{1}{r} a_+(r) \partial_t W[\psi] W[\psi] - \frac{4M}{r^2} \partial_r W[\psi] W[\psi] + \frac{M}{r^3} \left(W[\psi]\right)^2 \right]
\end{aligned}
$$

We also apply the familiar Cauchy-Schwartz inequality to get

$$
\begin{aligned}
\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \leq C\bar{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} a_-(r) \left(\partial_t W[\psi]\right)^2 \\
+ \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[\left(a_-(r)\right)^{-1} \left(\frac{1}{r} \left(\partial_t W[\psi]\right)^2 + \frac{1}{r^3} \left(W[\psi]\right)^2 \right) \\
+ \frac{2M}{r^2} \left(\delta_1 \left(\partial_r W[\psi]\right)^2 + \frac{1}{\delta_1} \left(\partial_t W[\psi]\right)^2 \right) + C a_-(r) \left(\partial_r W[\psi]\right)^2 \\
+ \frac{1}{r^2} a_-(r) \left(\frac{\delta_2}{2} \left(\partial_r W[\psi]\right)^2 + \frac{1}{2\delta_2} \left(W[\psi]\right)^2 \right) + C \frac{2M}{r^4} \left(W[\psi]\right)^2 \\
+ \frac{\epsilon}{r^3} W[r^2 \beta^\mu \partial_\mu \psi] W[\psi] + \frac{1}{r} \left(a_-(r)\right)^{-1/2} \left(h_2 \partial_t \psi + h_2 \partial_r \psi + h_3 \psi\right) W[\psi] \Bigg] \\
+ \int_{\Sigma_0} \Bigg[2M \left(\delta_3 \left(\partial_t W[\psi]\right)^2 + \frac{1}{r^4 \delta_3} \left(W[\psi]\right)^2 \right) + \frac{2M}{r^2} \left(\delta_4 \left(\partial_r W[\psi]\right)^2 + \frac{1}{\delta_4} \left(W[\psi]\right)^2 \right) \\
+ \frac{M}{r^3} \left(W[\psi]\right)^2 \Bigg] + \int_{\Sigma_\tau} \Bigg[2M \left(\delta_5 \left(\partial_t W[\psi]\right)^2 + \frac{1}{r^4 \delta_5} \left(W[\psi]\right)^2 \right) \\
+ \frac{2M}{r^2} \left(\delta_6 \left(\partial_r W[\psi]\right)^2 + \frac{1}{\delta_6} \left(W[\psi]\right)^2 \right) + \frac{M}{r^3} \left(W[\psi]\right)^2 \Bigg]
\end{aligned}
$$

With the perspective of recovering our energy formulae in the process, we set the various δ coefficients to

$$
\delta_1 = \frac{1}{r} a_-(r) \qquad \delta_2 = r \left(a_-(r)\right)^2 \qquad \delta_3 = \delta_5 = \frac{1}{r^2} \qquad \delta_4 = \delta_6 = a_-(r)
$$

Leading to

$$
\begin{aligned}
\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \leq C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 \\
+ \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \Bigg[\frac{1}{r^3}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \frac{2M}{r^2}a_-(r)\left(\partial_r W[\psi]\right)^2 \\
+ \frac{2M}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2 + Ca_-(r)\left(\partial_r W[\psi]\right)^2 \\
+ \frac{1}{2r}\left(a_-(r)\right)^3\left(\partial_r W[\psi]\right) + \frac{1}{2r^3}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + C\frac{2M}{r^4}\left(W[\psi]\right)^2 \\
+ \frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi] + \frac{\hat{C}}{r}\left(a_-(r)\right)^{-1/2}\left(\frac{1}{r^2}\partial_t\psi + \frac{1}{r^2}\partial_r\psi + \frac{1}{r^3}\psi\right)W[\psi]\Bigg] \\
+ \int_{\Sigma_0}\Bigg[\frac{2M}{r^2}\left(\partial_t W[\psi]\right)^2 + \frac{2M}{r^2}\left(W[\psi]\right)^2 + \frac{2M}{r^2}a_-(r)\left(\partial_r W[\psi]\right)^2 \\
+ \frac{2M}{r^2}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \frac{M}{r^3}\left(W[\psi]\right)^2\Bigg] + \int_{\Sigma_\tau}\Bigg[\frac{2M}{r^2}\left(\partial_t W[\psi]\right)^2 + \frac{2M}{r^2}\left(W[\psi]\right)^2 \\
+ \frac{2M}{r^2}a_-(r)\left(\partial_r W[\psi]\right)^2 + \frac{2M}{r^2}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \frac{M}{r^3}\left(W[\psi]\right)^2\Bigg]
\end{aligned}
$$

We can now use another identity that is straightforward to prove and is provided by [17]

$$
\int_{\Sigma_\tau}\frac{\left(W[\psi]\right)^2}{1-\frac{2M}{r}}\frac{1}{r^2} \lesssim \overline{\mathbb{E}}[\psi](\tau)
$$

We also recall the identity

$$
\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{\left(W[\psi]\right)^2}{1-\frac{2M}{r}}\frac{1}{r^3} \lesssim \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)
$$

With them, we find

$$
\begin{aligned}
\mathbb{I}_{\rm ndeg}[W[\psi]](0,\tau) \leq{}& C\overline{\mathbb{I}}_{\rm ndeg}[\psi](0,\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 + C\mathbb{E}[W[\psi]](0) + C\mathbb{E}[W[\psi]](\tau)\\
& C\overline{\mathbb{E}}[\psi](0) + C\overline{\mathbb{E}}[\psi](\tau) + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\Bigg[Ca_-(r)\left(\partial_r W[\psi]\right)^2 + C\frac{2M}{r^4}\left(W[\psi]\right)^2\\
& + \frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi] + \frac{1}{2}\hat{C}\left(\frac{1}{r^3}\left(\partial_t\psi\right)^2 + \frac{1}{r^3}\left(\partial_r\psi\right)^2 + \frac{1}{r^5}\psi^2\right)\\
& + \frac{3\hat{C}}{2r^3}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2\Bigg]\\
\leq{}& C\overline{\mathbb{I}}_{\rm ndeg}[\psi](0,\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 + C\mathbb{E}[W[\psi]](0) + C\mathbb{E}[W[\psi]](\tau)\\
& C\overline{\mathbb{E}}[\psi](0) + C\overline{\mathbb{E}}[\psi](\tau)\\
& + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\left[Ca_-(r)\left(\partial_r W[\psi]\right)^2 + C\frac{2M}{r^4}\left(W[\psi]\right)^2 + \frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi]\right]\\
\leq{}& C\overline{\mathbb{I}}_{\rm ndeg}[\psi](0,\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 + C\mathbb{E}[W[\psi]](0) + C\mathbb{E}[W[\psi]](\tau)\\
& C\overline{\mathbb{E}}[\psi](0) + C\overline{\mathbb{E}}[\psi](\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}a_-(r)\left(\partial_r W[\psi]\right)^2\\
& + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{2M}{r^4}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi]\\
\leq{}& C\overline{\mathbb{I}}_{\rm ndeg}[\psi](0,\tau) + C\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r}a_-(r)\left(\partial_t W[\psi]\right)^2 + C\mathbb{E}[W[\psi]](0) + C\mathbb{E}[W[\psi]](\tau)\\
& C\overline{\mathbb{E}}[\psi](0) + C\overline{\mathbb{E}}[\psi](\tau) + \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi]
\end{aligned}
$$

As a final step towards a satisfying estimate, we must determine a bound to the last term above.

Expanding it out and applying our β estimates, we get

$$\begin{aligned}
\frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi] &= \frac{\epsilon}{r}W[\beta^\mu]\partial_\mu\psi W[\psi] + \frac{\epsilon}{r}W[\partial_\mu\psi]\beta^\mu W[\psi] + \frac{\epsilon}{r}\left(a_-(r)\right)^{1/2}\beta^\mu\partial_\mu\psi W[\psi] \\
&= \frac{\epsilon}{r}W[\psi]\left(W[\beta^t]\partial_t\psi + W[\beta^r]\partial_r\psi + W[\beta^\theta]\partial_\theta\psi + W[\beta^\varphi]\partial_\varphi\psi\right) \\
&\quad + \frac{\epsilon}{r}W[\psi]\left(\beta^t\partial_r W[\psi] + \beta_r W[\partial_r\psi] + \beta^\theta\partial_\theta W[\psi] + \beta^\varphi\partial_\varphi W[\psi]\right) \\
&\quad + \frac{\epsilon}{r}\left(a_-(r)\right)^{1/2}W[\psi]\left(\beta^t\partial_t\psi + \beta^r\partial_r\psi + \beta^\theta\partial_\theta\psi + \beta^\varphi\partial_\varphi\psi\right) \\
&\le C\frac{\epsilon}{r}W[\psi]\Bigg(\frac{1}{r^{3/2}}\partial_t\psi + \frac{1}{r^{3/2}}\partial_r\psi + \frac{1}{r^2}\partial_\theta\psi + \frac{1}{r^2|\sin(\theta)|}\partial_\varphi\psi + \frac{1}{r^2}\partial_t W[\psi] + \frac{1}{r^2}W[\partial_r\psi] \\
&\quad + \frac{1}{r^{5/2}}\partial_\theta W[\psi] + \frac{1}{r^{5/2}|\sin(\theta)|}\partial_\varphi W[\psi] + \frac{1}{r^2}\partial_t\psi + \frac{1}{r^2}\partial_r\psi + \frac{1}{r^{5/2}}\partial_r\psi + \frac{1}{r^{5/2}}\partial_\theta\psi \\
&\quad + \frac{1}{r^{5/2}|\sin(\theta)|}\partial_\varphi\psi\Bigg) \\
&\le C\frac{\epsilon}{2}\Bigg(\frac{\delta_7}{r^5}\left(W[\psi]\right)^2 + \frac{1}{\delta_7}\left(\partial_t\psi\right)^2 + \frac{\delta_8}{r^5}\left(W[\psi]\right)^2 + \frac{1}{\delta_8}\left(\partial_r\psi\right)^2 + \frac{\delta_9}{r^6}\left(W[\psi]\right)^2 + \frac{1}{\delta_9}\left(\partial_\theta\psi\right)^2 \\
&\quad + \frac{\delta_{10}}{r^6}\left(W[\psi]\right)^2 + \frac{1}{\delta_{10}\sin^2(\theta)}\left(\partial_\varphi\psi\right)^2 + \frac{\delta_{11}}{r^6}\left(W[\psi]\right)^2 + \frac{1}{\delta_{11}}\left(\partial_t W[\psi]\right)^2 + \frac{\delta_{12}}{r^7}\left(W[\psi]\right)^2 \\
&\quad + \frac{1}{\delta_{12}}\left(\partial_\theta W[\psi]\right)^2 + \frac{\delta_{13}}{r^7}\left(W[\psi]\right)^2 + \frac{1}{\delta_{13}\sin^2(\theta)}\left(\partial_\varphi W[\psi]\right)^2 + \frac{\delta_{14}}{r^6}\left(W[\psi]\right)^2 + \frac{1}{\delta_{14}}\left(\partial_t\psi\right)^2 \\
&\quad + \frac{\delta_{15}}{r^6}\left(W[\psi]\right)^2 + \frac{1}{\delta_{15}}\left(\partial_r\psi\right)^2 + \frac{\delta_{16}}{r^7}\left(W[\psi]\right)^2 + \frac{1}{\delta_{16}}\left(\partial_\theta\psi\right)^2 + \frac{\delta_{17}}{r^7}\left(W[\psi]\right)^2 \\
&\quad + \frac{1}{\delta_{17}\sin^2(\theta)}\left(\partial_\varphi\psi\right)^2\Bigg) + C\frac{\epsilon}{r^3}W[\psi]W[\partial_r\psi] \\
&\le C\frac{11\epsilon}{2}\frac{\delta}{r^3}\left(W[\psi]\right)^2 + C\frac{\epsilon}{2}\left(\partial_t\psi\right)^2\frac{1}{\delta r^2}a_+(r) + C\frac{\epsilon}{2}\left(\partial_r\psi\right)^2\frac{1}{\delta r^2}a_+(r) + C\frac{\epsilon}{2}\left(\partial_\theta\psi\right)^2\frac{1}{\delta r^3}a_+(r) \\
&\quad + C\frac{\epsilon}{2}\left(\partial_\varphi\psi\right)^2\frac{1}{\delta r^3\sin^2(\theta)}a_+(r) + C\frac{\epsilon}{2}\left(\partial_t W[\psi]\right)^2\frac{1}{\delta r^3} + C\frac{\epsilon}{2}\left(\partial_\theta W[\psi]\right)^2\frac{1}{\delta r^4} \\
&\quad + C\frac{\epsilon}{2}\left(\partial_\varphi W[\psi]\right)^2\frac{1}{\delta r^4\sin^2(\theta)} + C\frac{\epsilon}{r^3}W[\psi]W[\partial_r\psi]
\end{aligned}$$

We can now integrate this term over the spacetime region used previously. Using the fact that $W[\partial_r\psi]$ breaks down into $\partial_r W[\psi]$ and lower order terms, we can easily see that the whole expression is bounded by

$$\begin{aligned}
\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{\epsilon}{r^3}W[r^2\beta^\mu\partial_\mu\psi]W[\psi] &\le C\epsilon\delta\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\frac{1}{r^3}\left(W[\psi]\right)^2 + C\epsilon\frac{1}{\delta}\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C\epsilon\frac{1}{\delta}\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \\
&\le C\epsilon\left(1+\frac{1}{\delta}\right)\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + C\epsilon\frac{1}{\delta}\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)
\end{aligned}$$

We can now choose δ such that $C\epsilon\frac{1}{\delta}$ is less than 1. In that case, we can then subtract the last term from the left of our inequality with $\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)$. Using this, we thus conclude that

$$\begin{aligned}\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \lesssim \bar{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} a_-(r) \left(\partial_t W[\psi]\right)^2 + \mathbb{E}[W[\psi]](0) + \mathbb{E}[W[\psi]](\tau) \\ + \overline{\mathbb{E}}[\psi](0) + \overline{\mathbb{E}}[\psi](\tau)\end{aligned} \tag{5.16}$$

□

B.3 Details from Section 5.3.3

Proposition B.3. Let $\epsilon_0 > 0$ be small. Then, for all smooth solutions ψ of the Cauchy problem (5.10) and for all $\epsilon_0 \geq \epsilon \geq 0$ we have

$$\mathbb{E}[W[\psi]](\tau) + \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 \lesssim \mathbb{E}[W[\psi]](0) + \overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \epsilon\, \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) \tag{5.15}$$

where we use $\lesssim$ to denote $\leq C$.

Proof. Starting from the expression

$$\begin{aligned}\partial_t W[\psi] \mathcal{P}_s\left(W[\psi]\right) = \frac{1}{2}\partial_t \left(a_+(r)\left(\partial_t W[\psi]\right)^2 + a_-(r)\left(\partial_r W[\psi]\right)^2 + |\nabla\!\!\!/ W[\psi]|^2 + (1-s^2)\frac{2M}{r^3}\left(W[\psi]\right)^2\right) \\ - \partial_r\left(\frac{2M}{r}\left(\partial_t W[\psi]\right)^2 + a_-(r)\partial_r W[\psi]\partial_t W[\psi]\right) - \nabla\!\!\!/ \cdot \left(\partial_t W[\psi]\nabla\!\!\!/ W[\psi]\right)\end{aligned}$$

and integrating over the region $\{r \geq 2M \,\big|\, \tau \geq t \geq 0\}$, we get

$$\begin{aligned}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{P}_s\left(W[\psi]\right) = &\frac{1}{2}\left(\mathbb{E}[W[\psi]](\tau) - \mathbb{E}[W[\psi]](0)\right) \\ &+ \frac{1}{2}\left[\int_{\Sigma_{\tilde{\tau}}}\left(\frac{2M}{r}\left(\partial_t W[\psi]\right)^2 + (1-s^2)\frac{2M}{r^3}\left(W[\psi]\right)^2 - \frac{1}{r^2}\left(W[\psi]\right)^2\right)\right]_0^\tau \\ &- \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \partial_r\left(\frac{2M}{r}\left(\partial_t W[\psi]\right)^2 + a_-(r)\partial_r W[\psi]\partial_t W[\psi]\right) \\ &- \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \nabla\!\!\!/ \cdot \left(\partial_t W[\psi]\nabla\!\!\!/ W[\psi]\right)\end{aligned}$$

This then leads to the estimate

$$\begin{aligned}\mathbb{E}[W[\psi]](\tau) \leq C\mathbb{E}[W[\psi]](0) - \left[\int_{\Sigma_{\tilde{\tau}}}\left(\frac{2M}{r}\left(\partial_t W[\psi]\right)^2 + (1-s^2)\frac{2M}{r^3}\left(W[\psi]\right)^2 - \frac{1}{r^2}\left(W[\psi]\right)^2\right)\right]_0^\tau \\ + 2\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{P}_s\left(W[\psi]\right)\end{aligned}$$

Note that he second term on the right, either has the correct sign (in other words, is negative) or can be estimated by $C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)$. Also note that from now on we will use the symbol C to represent any strictly positive constant. This, of course, doesn't affect the validity of our derivations as for $\alpha \leq A\beta + B\gamma$, defining $C = \max(A,B)$ leads to $\alpha \leq C\beta + C\gamma$, where all quantities have suitable signs.
Applying our formula (5.13) and further estimating we get

$$\begin{aligned}\mathbb{E}[W[\psi]](\tau) \leq C\mathbb{E}[W[\psi]](0) + C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) - 4\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}\left(a_-(r)\right)^{-1}\left(1+\frac{6M}{r}\right)^{-1/2}\left(\partial_t W[\psi]\right)^2 \\ + 2\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\left(\partial_t W[\psi]\mathcal{F}_1 + \partial_t W[\psi]\mathcal{F}_2\right)\end{aligned}$$

Taking the third term to the other side of the inequality and using the following bounds $2 \leq 4\left(1+\frac{6M}{r}\right)^{-1/2} \leq 4$ we can write

$$\mathbb{E}[W[\psi]](\tau) + 2\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2 \leq C\mathbb{E}[W[\psi]](0) + C\overline{\mathbb{I}}_{\mathrm{ndeg}}[\psi](0,\tau) \\ + 2\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(\partial_t W[\psi]\mathcal{F}_1 + \partial_t W[\psi]\mathcal{F}_2\right) \tag{B.1}$$

To obtain estimate (14) from [17] all that is left for us to do is to estimate the two $\mathcal{F}$-terms. Starting with the $\mathcal{F}_1$ term, we expand

$$\begin{aligned} W[r^2\beta^\mu\partial_\mu\psi] &= \frac{r}{\sqrt{1-\frac{2M}{r}}}\left(\left(\frac{2M}{r}+f(r)\right)\partial_t\left(r^2\beta^\mu\partial_\mu\psi\right) + a_-(r)\partial_r\left(r^2\beta^\mu\partial_\mu\psi\right)\right) \\ &= r^3\left(a_-(r)\right)^{-1/2}\left(\left(\frac{2M}{r}+f(r)\right)\partial_t\beta^\mu + a_-(r)\partial_r\beta^\mu\right)\partial_\mu\psi \\ &\quad + r^3\left(a_-(r)\right)^{-1/2}\left(\left(\frac{2M}{r}+f(r)\right)\partial_t\partial_\mu\psi + a_-(r)\partial_r\partial_\mu\psi\right)\beta^\mu \\ &\quad + 2r^2\left(a_-(r)\right)^{1/2}\beta^\mu\partial_\mu\psi \\ &= r^2W[\beta^\mu]\partial_\mu\psi + r^2W[\partial_\mu\psi]\beta^\mu + 2r^2\left(a_-(r)\right)^{1/2}\beta^\mu\partial_\mu\psi \end{aligned}$$

and use it in order to write

$$\begin{aligned} \int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{F}_1 &= \epsilon\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}} \frac{1}{r^2}\partial_t W[\psi]W[r^2\beta^\mu\partial_\mu\psi] \\ &= \epsilon\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\left(\partial_t W[\psi]W[\beta^\mu]\partial_\mu\psi + \partial_t W[\psi]W[\partial_\mu\psi]\beta^\mu + 2\left(a_-(r)\right)^{1/2}\partial_t W[\psi]\beta^\mu\partial_\mu\psi\right) \\ &= \epsilon\int_0^\tau d\tilde{\tau}\int_{\Sigma_{\tilde{\tau}}}\Big(W[\beta^t]\partial_t W[\psi]\partial_t\psi + W[\beta^r]\partial_t W[\psi]\partial_r\psi + W[\beta^\theta]\partial_t W[\psi]\partial_\theta\psi \\ &\quad + W[\beta^\varphi]\partial_t W[\psi]\partial_\varphi\psi + \beta^t\left(\partial_t W[\psi]\right)^2 + \beta^r\partial_t W[\psi]W[\partial_r\psi] + \beta^\theta\partial_t W[\psi]W[\partial_\theta\psi] \\ &\quad + \beta^\varphi\partial_t W[\psi]W[\partial_\varphi\psi] + \beta^t 2\left(a_-(r)\right)^{1/2}\partial_t W[\psi]\partial_t\psi \\ &\quad + \beta^r 2\left(a_-(r)\right)^{1/2}\partial_t W[\psi]\partial_r\psi + \beta^\theta 2\left(a_-(r)\right)^{1/2}\partial_t W[\psi]\partial_\theta\psi \\ &\quad + \beta^\varphi 2\left(a_-(r)\right)^{1/2}\partial_t W[\psi]\partial_\varphi\psi\Big) \end{aligned}$$

We now would like to separate every term that contains derivatives of $W[\psi]$ in such a way that we are left with their squares. To make this more apparent and to illustrate how to perform such an estimate, consider $a, b \in \mathbb{R}$ and $\delta > 0$. In this case we can derive a Cauchy-Schwartz inequality

$$\begin{aligned} \left(\frac{a}{\sqrt{\delta}} - \sqrt{\delta}b\right)^2 \geq 0 \qquad &\iff \qquad \frac{a}{\delta} - 2ab + \delta b \geq 0 \\ &\iff \qquad ab \leq \frac{1}{2\delta}a + \frac{\delta}{2}b \end{aligned}$$

A similar computation can be done with the opposite sign

$$-ab \leq \frac{1}{2\delta}a + \frac{\delta}{2}b$$

Keeping this method in mind, we can now expand all the terms in our $\mathcal{F}_1$ expression above, each time using different δ coefficients. This will allow us to set those values at a later stage and get the correct estimate. Note that we can also commute trough all derivatives with W, adding the corresponding commutator value afterwards.

$$\begin{aligned}
\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{F}_1 \leq \epsilon \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[& \left(\frac{1}{2\delta_1} \left(W[\beta^t]\right)^2 \left(\partial_t W[\psi]\right)^2 + \frac{\delta_1}{2} \left(\partial_t \psi\right)^2 \right) \\
& + \left(\frac{1}{2\delta_2} \left(W[\beta^r]\right)^2 \left(\partial_t W[\psi]\right)^2 + \frac{\delta_2}{2} \left(\partial_r \psi\right)^2 \right) + \left(\frac{1}{2\delta_3} \left(W[\beta^\theta]\right)^2 \left(\partial_t W[\psi]\right)^2 + \frac{\delta_3}{2} \left(\partial_\theta \psi\right)^2 \right) \\
& + \left(\frac{1}{2\delta_4} \left(W[\beta^\varphi]\right)^2 \left(\partial_t W[\psi]\right)^2 + \frac{\delta_4}{2} \left(\partial_\varphi \psi\right)^2 \right) + \beta^t \left(W[\psi]\right)^2 \\
& + \beta^r \frac{M}{r^2} \left(a_-(r)\right)^{-1} \partial_t W[\psi] W[\psi] - \beta^r \left(a_-(r)\right)^{-1/2} \left(f(r) + r f'(r)\right) \partial_t W[\psi] \partial_t \psi \\
& - \beta^r \left(a_-(r)\right)^{-1/2} \partial_t W[\psi] \partial_r \psi + \beta^r \partial_t W[\psi] \partial_r W[\psi] \\
& + |\beta^\theta| \left(\frac{1}{\delta_5} \left(\partial_t W[\psi]\right)^2 + \frac{\delta_5}{2} \left(\partial_\theta W[\psi]\right)^2 \right) + |\beta^\varphi| \left(\frac{1}{\delta_6} \left(\partial_t W[\psi]\right)^2 + \frac{\delta_6}{2} \left(\partial_\varphi W[\psi]\right)^2 \right) \\
& + |\beta^t| \left(a_-(r) \frac{1}{\delta_7} \left(\partial_t W[\psi]\right)^2 + \delta_7 \left(\partial_t \psi\right)^2 \right) \\
& + |\beta^r| \left(a_-(r) \frac{1}{\delta_8} \left(\partial_t W[\psi]\right)^2 + \delta_8 \left(\partial_r \psi\right)^2 \right) \\
& + |\beta^\theta| \left(a_-(r) \frac{1}{\delta_9} \left(\partial_t W[\psi]\right)^2 + \delta_9 \left(\partial_\theta \psi\right)^2 \right) \\
& + |\beta^\varphi| \left(a_-(r) \frac{1}{\delta_{10}} \left(\partial_t W[\psi]\right)^2 + \delta_{10} \left(\partial_\varphi \psi\right)^2 \right) \Bigg]
\end{aligned}$$

One can now see the motivation behind the asymptotic behaviour we enforced on the vector field β. Indeed, from equation (5.9) we can deduce

$$\beta^t, \beta^r \leq \frac{C}{r^2} \qquad \beta^\theta \leq \frac{C}{r^{5/2}} \qquad \beta^\varphi \leq \frac{C}{r^{5/2}|\sin(\theta)|}$$

$$W[\beta^t], W[\beta^r] \leq \frac{C}{r^{3/2}} \qquad W[\beta^\theta] \leq \frac{C}{r^2} \qquad W[\beta^\varphi] \leq \frac{C}{r^2|\sin(\theta)|}$$

We can use these to further improve our estimate and get rid of all β terms. We also regroup all term by their common factors and shift some δ coefficients by a $\sin(\theta)$ factor, i.e. $\delta_i' = \delta_i|\sin(\theta)|$.

This is allowed in the limit $\theta \to 0$ as divergences cancel for those terms.

$$\begin{aligned}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi] \mathcal{F}_1 \leq \epsilon C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[& \frac{1}{2} \left(\partial_t W[\psi]\right)^2 \left(\frac{1}{r^2} \left(2 + \frac{2}{\delta_7} a_-(r) + \frac{2}{\delta_8} a_-(r) \right) \right. \\ & + \frac{1}{r^{5/2}} \left(\frac{1}{\delta_5} + \frac{1}{\delta_6'} + \frac{2}{\delta_9} a_-(r) + \frac{2}{\delta_{10}'} a_-(r) \right) \\ & \left. + \frac{1}{r^3} \left(\frac{1}{\delta_1} + \frac{1}{\delta_2} \right) + \frac{1}{r^4} \left(\frac{1}{\delta_3} + \frac{1}{\delta_4'} \right) \right) \\ & + \left(\partial_\theta W[\psi]\right)^2 \frac{\delta_5}{2r^{5/2}} + \left(\partial_\varphi W[\psi]\right)^2 \frac{\delta_6'}{2r^{5/2}\sin^2(\theta)} + (\partial_t \psi)^2 \left(\frac{\delta_7}{r^2} + \frac{\delta_1}{2r^3} \right) \\ & + (\partial_r \psi)^2 \left(\frac{\delta_8}{r^2} + \frac{\delta_2}{2r^3} \right) + (\partial_\theta \psi)^2 \left(\frac{\delta_9}{r^{5/2}} + \frac{\delta_3}{2r^4} \right) + \frac{1}{\sin^2(\theta)} (\partial_\varphi \psi)^2 \left(\frac{\delta_{10}'}{r^{5/2}} + \frac{\delta_4'}{2r^4} \right) \\ & + \frac{M}{r^4} \left(a_-(r)\right)^{-1} \left(\frac{1}{\delta_{11}} \left(\partial_t W[\psi]\right)^2 + \frac{\delta_{11}}{2} \left(W[\psi]\right)^2 \right) \\ & + \frac{1}{2r^2} \left(\frac{1}{\delta_{12}} \left(\partial_t W[\psi]\right)^2 + \delta_{12} \left(\partial_r W[\psi]\right)^2 \right) \\ & + \frac{1}{r^2} \left(f(r) + r f'(r)\right) \left(\frac{1}{2\delta_{12}} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 + \frac{\delta_{13}}{2} (\partial_t \psi)^2 \right) \\ & + \frac{1}{r^2} \left(\frac{1}{2\delta_{14}} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 + \frac{\delta_{14}}{2} (\partial_r \psi)^2 \right) \Bigg]\end{aligned}$$

In order to estimate the first term, we would like it to have an $\frac{1}{r^2}$ factor. That would allow us to estimate it using $\mathbb{I}_{\mathrm{ndeg}}[W[\psi]](0,\tau)$. To enforce this condition, we must thus set the various δ coefficients to

$$\delta_1 = \delta_2 = \frac{1}{r} \qquad \delta_3 = \delta_4' = \frac{1}{r^2} \qquad \delta_5 = \delta_6' = \frac{1}{r^{1/2}} \qquad \delta_7 = \delta_8 = a_-(r) \qquad \delta_9 = \delta_{10}' = a_-(r)\frac{1}{r^{1/2}}$$

Using these values and using the fact that $f(r) + r f'(r)$ is bounded from above by $\frac{19}{8}$ we get

$$\begin{aligned}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi] \mathcal{F}_1 \leq \epsilon C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[& \frac{8}{r^2} \left(\partial_t W[\psi]\right)^2 + \frac{\delta_{12}}{2r^2} \left(\partial_r W[\psi]\right)^2 \\ & + \frac{1}{2r} \left(\frac{1}{r^2} \left(\partial_\theta W[\psi]\right)^2 + \frac{1}{r^2 \sin^2(\theta)} \left(\partial_\varphi W[\psi]\right)^2 \right) + \frac{M}{r^4} \left(a_-(r)\right)^{-1} \frac{\delta_{11}}{2} \left(W[\psi]\right)^2 \\ & + (\partial_t \psi)^2 \left(\frac{1}{r^2} a_-(r) + \frac{19\delta_{13}}{16r^2} + \frac{1}{2r^4} \right) + (\partial_r \psi)^2 \left(\frac{1}{r^2} a_-(r) + \frac{\delta_{14}}{2r^2} + \frac{1}{2r^4} \right) \\ & + (\partial_\theta \psi)^2 \left(\frac{1}{r^3} a_-(r) + \frac{1}{2r^6} \right) + \frac{1}{\sin^2(\theta)} (\partial_\varphi \psi)^2 \left(\frac{1}{r^3} a_-(r) + \frac{1}{2r^6} \right) \\ & + \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 \left(\frac{M}{2r^4 \delta_{11}} + a_-(r) \frac{1}{2r^2 \delta_{12}} + \frac{19}{16 r^2 \delta_{13}} + \frac{1}{2r^2 \delta_{14}} \right) \Bigg]\end{aligned}$$

We can now finalize this estimate by dealing with the last term above. I clearly cannot be bounded from above as it diverges at infinity. However, we can arrange for the coefficient in front to be less or equal to $\frac{1}{2}$ so that we can inject it into the left-hand-side of (B.1). This can be done by choosing

$$\delta_{11} = \frac{8M}{r^3}\epsilon C \qquad \delta_{12} = \frac{8}{r}a_-(r)\epsilon C \qquad \delta_{13} = \frac{19}{r}\epsilon C \qquad \delta_{14} = \frac{8}{r}\epsilon C$$

As done previously, we substitute these values of δ and use the elementary bounds $\frac{1}{r^2} \leq \frac{1}{2Mr}$ and $a_-(r) \leq 1$.

$$\begin{aligned}
\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{F}_1 \leq \epsilon C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[& \frac{8}{r^2}\left(\partial_t W[\psi]\right)^2 + \epsilon C \frac{4}{r^3} a_-(r) \left(\partial_r W[\psi]\right)^2 + \frac{1}{2r}|\nabla\!\!\!/ W[\psi]|^2 \\
& + \epsilon C \frac{4M^2}{r^7}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \left(\partial_t\psi\right)^2\left(\frac{1}{r^2} + \epsilon C\frac{361}{16r^3} + \frac{1}{2r^4}\right) \\
& + \left(\partial_r\psi\right)^2\left(\frac{1}{r^2} + \epsilon C\frac{4}{r^3} + \frac{1}{2r^4}\right) + \left(\partial_\theta\psi\right)^2\left(\frac{1}{r^3} + \frac{1}{2r^6}\right) + \frac{1}{\sin^2(\theta)}\left(\partial_\varphi\psi\right)^2\left(\frac{1}{r^3} + \frac{1}{2r^6}\right) \Bigg] \\
& + \frac{1}{4}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2 \\
\leq \epsilon C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[& \frac{8}{r^2}\left(\partial_t W[\psi]\right)^2 + \epsilon C \frac{2}{Mr^2} a_-(r) \left(\partial_r W[\psi]\right)^2 + \frac{1}{2r}|\nabla\!\!\!/ W[\psi]|^2 \\
& + \epsilon C \frac{1}{4M^2r^3}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \left(\partial_t\psi\right)^2\frac{1}{r^2}\left(1 + \epsilon C\frac{361}{32M} + \frac{1}{8M^2}\right) \\
& + \left(\partial_r\psi\right)^2\frac{1}{r^2}\left(1 + \epsilon C\frac{2}{M} + \frac{1}{8M^2}\right) + \left(1 + \frac{1}{16M^3}\right)\frac{1}{r}|\nabla\!\!\!/ \psi|^2 \Bigg] \\
& + \frac{1}{4}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2
\end{aligned}$$

Given that all the terms are positive we can estimate the expression by taking the biggest constant factor in front of the integral, which we call again C. Note that since ϵ is as small as we want, and more importantly $\epsilon < 1$, we have that $\epsilon^2 \leq \epsilon$. Thus, we arrive at the conclusion that

$$\begin{aligned}
\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{F}_1 \leq \epsilon C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \Bigg[& \frac{1}{r^2}\left(\partial_t W[\psi]\right)^2 + \frac{1}{r^2} a_-(r) \left(\partial_r W[\psi]\right)^2 + \frac{1}{r}|\nabla\!\!\!/ W[\psi]|^2 \\
& + \frac{1}{r^3}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 + \frac{1}{r^2}\left(\partial_t\psi\right)^2 + \frac{1}{r^2}\left(\partial_r\psi\right)^2 + \frac{1}{r}|\nabla\!\!\!/ \psi|^2 \Bigg] \\
& + \frac{1}{4}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2
\end{aligned}$$

We immediately recognise the terms from $\mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau)$ and $\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)$ which can, thus, be estimated by

$$\begin{aligned}
\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{F}_1 \leq \epsilon C \mathbb{I}_{\text{ndeg}}[W[\psi]](0,\tau) + C\overline{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau) + \epsilon C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r^3}\left(a_-(r)\right)^{-1}\left(W[\psi]\right)^2 \\
+ \frac{1}{4}\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r}\left(a_-(r)\right)^{-1}\left(\partial_t W[\psi]\right)^2
\end{aligned}$$

As a final step and as pointed out by [17], the last term on the first line can be estimated by

$$\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r^3} \left(a_-(r)\right)^{-1} \left(W[\psi]\right)^2 \leq C\bar{\mathbb{I}}_{\text{ndeg}}[\psi](0,\tau)$$

This can easily be seen by replacing $W[\psi]$ by its definition and applying the same type of estimates as we have throughout this section.

We now place our focus on the second $\mathcal{F}$-term. Applying the same method as for the first $\mathcal{F}$-term with the addition of the bounds on the h_n functions (5.14), we write

$$\begin{aligned}
\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi]\mathcal{F}_2 &= \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(a_-(r)\right)^{-1/2} \partial_t W[\psi] \left(h_2 \partial_t \psi + h_2 \partial_r \psi + h_3 \psi\right) \\
&= \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(a_-(r)\right)^{-1/2} \partial_t W[\psi] \left(r^2 h_2 \frac{1}{r^2} \partial_t \psi + r^2 h_2 \frac{1}{r^2} \partial_r \psi + r^3 h_3 \frac{1}{r^3} \psi\right) \\
&\leq \hat{C} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left(a_-(r)\right)^{-1/2} \partial_t W[\psi] \left(\frac{1}{r^2} \partial_t \psi + \frac{1}{r^2} \partial_r \psi + \frac{1}{r^3} \psi\right) \\
&\leq \hat{C} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left[\frac{1}{2r^2} \left(\frac{1}{\delta_{15}} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 + \delta_{15} \left(\partial_t \psi\right)^2\right)\right. \\
&\quad + \frac{1}{2r^2} \left(\frac{1}{\delta_{16}} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 + \delta_{16} \left(\partial_r \psi\right)^2\right) \\
&\quad \left.+ \frac{1}{2r^3} \left(\frac{1}{\delta_{17}} \left(a_-(r)\right)^{-1} \left(\partial_t W[\psi]\right)^2 + \delta_{17} \psi^2\right)\right] \\
&\leq \frac{1}{2} \hat{C} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left[\frac{\delta_{15}}{r^2} \left(\partial_t \psi\right)^2 + \frac{\delta_{16}}{r^2} \left(\partial_r \psi\right)^2 + \frac{\delta_{17}}{r^3} \psi^2\right. \\
&\quad \left.+ \left(a_-(r)\right)^{-1} \left(\frac{1}{\delta_{15} r^2} + \frac{1}{\delta_{16} r^2} + \frac{1}{\delta_{17} r^3}\right) \left(\partial_t W[\psi]\right)^2\right]
\end{aligned}$$

Again, we choose the δ coefficients such that the last term can be subtracted from the left of (B.1).

$$\delta_{15} = \delta_{16} = \hat{C}\frac{6}{r} \qquad \delta_{17} = \hat{C}\frac{6}{r^2}$$

With this choice we find

$$
\begin{aligned}
\int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \partial_t W[\psi] \mathcal{F}_2 \leq & \; 9\hat{C} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left[\frac{1}{r^3} (\partial_t \psi)^2 + \frac{1}{r^3} (\partial_r \psi)^2 + \frac{1}{r^5} \psi^2 \right] \\
& + \frac{1}{4} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \big(a_-(r) \big)^{-1} \big(\partial_t W[\psi] \big)^2 \\
\leq & \; 9\hat{C} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left[\frac{1}{2Mr^2} (\partial_t \psi)^2 + \frac{1}{2Mr^2} (\partial_r \psi)^2 + \frac{1}{4M^2 r^3} \psi^2 \right] \\
& + \frac{1}{4} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \big(a_-(r) \big)^{-1} \big(\partial_t W[\psi] \big)^2 \\
\leq & \; C \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \left[\frac{1}{r^2} (\partial_t \psi)^2 + \frac{1}{r^2} (\partial_r \psi)^2 + \frac{1}{r^3} \psi^2 \right] \\
& + \frac{1}{4} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \big(a_-(r) \big)^{-1} \big(\partial_t W[\psi] \big)^2 \\
\leq & \; C \overline{\mathbb{I}}_{\mathrm{ndeg}}[\psi](0,\tau) + \frac{1}{4} \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \big(a_-(r) \big)^{-1} \big(\partial_t W[\psi] \big)^2
\end{aligned}
$$

We are now ready to substitute back into (B.1) our results for the estimates of both $\mathcal{F}$-terms.

$$
\begin{aligned}
\mathbb{E}[W[\psi]](\tau) + \int_0^\tau d\tilde{\tau} \int_{\Sigma_{\tilde{\tau}}} \frac{1}{r} \big(a_-(r) \big)^{-1} \big(\partial_t W[\psi] \big)^2 \lesssim & \; \mathbb{E}[W[\psi]](0) + \overline{\mathbb{I}}_{\mathrm{ndeg}}[\psi](0,\tau) \\
& + \epsilon \, \mathbb{I}_{\mathrm{ndeg}}[W[\psi]](0,\tau)
\end{aligned}
\tag{5.15}
$$

□