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# An Introduction to the AdS/CFT Correspondence

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## Abstract

Conformal field theories, Anti-de Sitter spacetime and superstring theory are reviewed. The duality between open and closed strings and the ensuing interpretation of Dp-branes motivates the AdS/CFT correspondence. The conjectured map between observables in the two theories is presented. Strongly-coupled thermal plasma is explored via a 5-dimensional black hole. Results for its entropy and shear viscosity are compared with QCD and lead to a possible universality of gauge theories.

## 1 Introduction

There has been ongoing research for a theory consistent with quantum mechanics and general relativity since the early twentieth century. Dirac's quantisation method leads to field theories describing electromagnetism and the strong and weak nuclear forces. However, attempts to quantise gravity this way lead to a theory that is non-renormalisable. One of the modern approaches to a consistent framework of quantum gravity is string theory. Ideas from string theory motivated Maldacena to propose the AdS/CFT correspondence. This states that certain field theories in  $d$ -dimensions, without gravity, are dual to specific gravitational theories in  $d+1$  dimensions; the physics described by a gravitational theory is equivalent at the full quantum level to a field theory in flat spacetime on the boundary. The correspondence makes manifest ideas of holography and that gravity and quantum mechanics are inextricably linked. The aim of this dissertation is to present the AdS/CFT correspondence as originally proposed by Maldacena [17]. The conjecture is that a supersymmetric Yang-Mills ('SYM') gauge theory in four dimensions, a conformal field theory ('CFT'), is dual to a type IIB gravitational theory which has local asymptotics of a five-dimensional Anti-de Sitter ('AdS') spacetime. The correspondence is a conjecture with no rigorous mathematical proof however, strong evidence suggests the duality is valid [43]. Motivation for the conjecture and the context of theoretical developments at the time it was proposed begins with some background knowledge and history of string theory.

String theory was initially formulated to describe strong interactions. Strings, as extended objects, vibrate. Different vibrational modes give rise to different particles. Strong interactions were subsequently explained in the field theoretical context of QCD [51]. QCD is a Yang-Mills theory with gauge group  $SU(3)$ . It is well known that QCD is asymptotically free; the coupling constant runs with energy. At low energies the theory is non perturbative. 't Hooft proposed [39] that general gauge theories simplify when the gauge colour  $N$  is large. The planar and non-planar expansion demonstrated that in the limit  $N \rightarrow \infty$  there is a controllable coupling, the 't Hooft coupling,  $\lambda = g^2 N$  that is kept finite. More importantly, it was suggested that in this limit there is a dual description of a weakly-coupled string theory. This provided evidence that there exists a connection between gauge theories and string theory.

The 't Hooft argument was valid for gauge theories in general. One is then prompted to consider field theories that have desirable properties, albeit not present in QCD. One such class are

conformal field theories. CFTs have a symmetry group that includes the familiar Poincaré transformations. In addition, the theory is invariant under scale transformations (and special conformal transformations). Overall, a conformal field theory in four-dimensional Minkowski space has fifteen operators that define the Lie algebra  $SO(2,4)$  [31]. Scale invariance means that the coupling constant does not run with energy [48]. The dynamics of the theory can be further constrained if it is supersymmetric. The candidate theory for the prototypical example of the AdS/CFT correspondence is the maximally SYM gauge theory, with gauge group  $SU(N)$ . In four dimensions, the theory has four supercharges [3]. The  $\mathcal{N} = 4$  SYM also has an  $SU(4) \sim SO(6)$  R-symmetry that rotates the scalars and fermions present in the theory.

In the 1980s, string theory was recast as a theory of quantum gravity. The spectrum of closed strings naturally gives rise to massless states of spin two, the graviton. In a way, closed strings predict gravity [51]. In general relativity, gravity includes black holes. Consideration of the thermodynamic laws of black holes motivated the AdS/CFT correspondence [18, 19]. Fundamental string theory was expanded to include massive, extended, dynamical objects that curve spacetime and can lead to black holes in the higher-dimensional background that string theory predicts. The point-particle and string were generalised to  $p$ -dimensional objects called a  $p$ -branes. These couple to a  $p + 1$  gauge potential and have a  $p + 2$  field strength; similar to the Maxwell field-strength  $F = dA$ . Black holes in  $p$ -dimensions are solutions in the low-energy, semi-classical limit of closed string theory, i.e. supergravity ('SUGRA') [35]. These solutions are analogous to the extremal Reissner-Nordstrom black hole in four dimensions.

Polchinski [30] considered D-branes in open string theory. D-branes are dynamical hypersurfaces on which open strings can end i.e. open strings with Dirichlet boundary conditions. These were identified with the  $p$ -brane solutions of the supergravity limit of closed string theory and led to a duality between open and closed strings [34]. The most important property of D-branes for the purposes of this thesis is that gauge theories are realised on their world volume. Witten [45] showed that a system of  $N$  coincident D-branes will have a maximally supersymmetric  $U(N)$  gauge theory. In the low energy limit,  $N$  D3-branes will carry a  $\mathcal{N} = 4$ ,  $SU(N)$  SYM theory. On the other hand, D3-branes embedded into a closed type IIB string theory are described by a metric with a near-throat geometry that is  $AdS_5 \times S^5$  [35]. AdS spacetime in  $d + 1$  dimensions is the maximally symmetric space which is a solution to Einstein's field equations with a negative cosmological constant [49] and has a  $d$ -dimensional Minkowski conformal boundary. Symmetry considerations provide some immediate evidence in favour of the duality between a gravity theory in AdS space and a CFT. In five dimensions, the isometry group of  $AdS_5$  is  $SO(2,4)$  which acts on any theory living on the Minkowski boundary as the conformal group [49]. The symmetries of the five-sphere in the near-throat geometry also match the R-symmetry of the gauge theory.

Considerations of the duality between open and closed strings in describing interactions of D-branes lead to the conclusion that a duality between gravity and gauge theories is holographic [34]. Ideas of holography were first presented by Bekenstein and Hawking [10] who showed that, quantum mechanically, black holes emit radiation at a temperature proportional to the surface gravity and have an entropy proportional to the area of the horizon. The holographic principle was further developed by 't Hooft [40] and Susskind [38]. The AdS/CFT correspondence proposed by Maldacena [17] is a concrete manifestation of the holographic principle.

Maldacena's conjecture was that type IIB string theory with  $AdS_5 \times S^5$  geometry is dual to the  $\mathcal{N} = 4$  SYM theory. Since the original formulation of the AdS/CFT conjecture, research has expanded

the nature of theories that are dual. These include field theories with less supersymmetry that do not exhibit conformal invariance and where the bulk space is not AdS or a five-sphere. Therefore, the research area is more generally known as gauge/gravity duality [25]. It was further conjectured [13, 15, 46] that there exists a one-to-one correspondence between gauge-invariant operators on the field theory side to bulk fields in the AdS space. Calculating correlation functions for fields in AdS space provides a testing ground, but not a proof, for the validity of the correspondence.

The duality between gauge theories and gravity gives profound insights into the real physical world. Firstly, its holographic nature suggests that spacetime is an emergent property [43]. Further, it has already been mentioned that research in black holes motivated the AdS/CFT correspondence. Indeed, since the 1996 seminal paper of Andrew Strominger and Cumrun Vafa [37], string theory has been used to count the microstates of black holes as a way of providing a statistical mechanics interpretation of the entropy - an area Maldacena worked on prior to his formulation of the correspondence [16]. A long-standing problem of black hole thermodynamics is the information paradox; the loss of information as it falls past the event horizon. Since the gravity theory however, is dual to a unitary field theory on the boundary, black holes must also evolve in a unitary way and obey the laws of quantum mechanics [28]. Finally, the duality provides a relation between the two sets of parameters. The relations show that the AdS/CFT correspondence is a useful method to providing physical insights into strongly-coupled gauge theories, where QCD is non-perturbative, but where the dual description is a semi-classical supergravity regime [32].

Recent research is presented that aims to use the AdS/CFT correspondence to get a better understanding of strong interactions governed by QCD [9, 36]. In particular, the duality of a strongly-coupled gauge theory to a semi-classical description of gravity has prompted the exploration of the thermal properties of strongly-coupled quark-gluon plasma ('sQGP') observed at RHIC [7]. Finite temperature field theory with broken supersymmetry is achieved by placing a black hole in the  $AdS_5$  space [47]. The gravitational entropy of the near-extremal black hole at strong coupling is identified with the entropy of a free gauge theory and compared with results from lattice QCD. Similar results from calculations of the shear viscosity of the plasma suggests there is a universality of phenomena for gauge theories at finite temperature [25].

The structure of this dissertation is as follows. Chapter two extends the familiar Poincaré symmetries of QFTs to include scale invariance and special conformal transformations. The Lie algebra of the conformal group is presented along with restrictions on the observables, namely the form of two- and three-point correlation functions. Chapter three introduces AdS spacetime and its isometries. The geometry of the space is explored via different coordinate charts. It is shown that AdS has a conformal boundary where the metric is asymptotically flat. Chapters four and five provide a brief overview of string theory. The spectra of open and closed strings are discussed. Open strings can have their endpoints fixed on dynamical membranes, D-branes; gauge theories live on the membrane's world volume. Closed strings necessarily include gravity. Lorentz invariance and supersymmetry heavily constrain superstring theory. However, different formulations of string theory exist. For the purposes of this review, the objects supported by type IIB superstring theory are discussed, specifically D3-branes.

The duality of open and closed strings is explored via the modern interpretation of D-branes. The comparison of a stack of D3-branes and the curved backgrounds they produce motivate the AdS/CFT correspondence in chapter six. The holographic dictionary is presented along with detailed calculations of the two- and three-point correlation functions of bulk scalar fields in AdS.

Finally, chapter seven describes how a finite temperature is introduced in field theories and the laws of thermodynamics of black holes. The thermal properties of sQGP is computed from the AdS/CFT correspondence and compared to lattice QCD results.

Throughout this dissertation  $\hbar = 1 = c$ . The Minkowski metric is mostly positive,  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$  unless stated otherwise.

## 2 From Poincaré to conformal symmetry

### 2.1 Conformal transformations

Scale transformations in quantum field theory play an important and subtle role in the context of renormalization. Integrating out the Fourier modes of the field in question over a specified energy cutoff leads to an effective Lagrangian and action. The *running* of the coupling parameter with the scale is quantified by the  $\beta$  function:

$$\beta(\lambda, \epsilon) = \mu \frac{d\lambda}{d\mu} \quad (2.1)$$

where  $\epsilon$  is the cutoff and  $\mu$  is the renormalization scale [29]. Scale invariance means that the  $\beta$  function is zero. It must either be zero everywhere (Feynman diagrams cancel and no infinities) or, the  $\beta$  function is non-trivial but can have zero fixed points which corresponds to scale invariance and leads to a CFT.

Besides Lorentz transformations and translation, there are additional transformations that can be applied to spacetime. For example, it was noted early on that Maxwell's equations,  $\partial^\mu F_{\mu\nu} = 0$  are invariant under scale transformations  $x^\mu \rightarrow \lambda x^\mu$ .

The set of transformations that preserves angles between two line segments generates the conformal algebra [50]. Consider a change of coordinates  $x \rightarrow x'$  where infinitesimally this is given by  $x'^\mu = x^\mu + \epsilon^\mu(x)$ . A d-dimensional metric, under the coordinate change, transforms as a two-tensor  $g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$ . Imposing the condition that such a coordinate transformation leaves the metric unchanged up to a scale, locally:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu} \quad (2.2)$$

then the generators of the conformal group are found by looking at an infinitesimal coordinate transformation that leaves the metric invariant:  $\Lambda(x) g_{\mu\nu} = g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ .

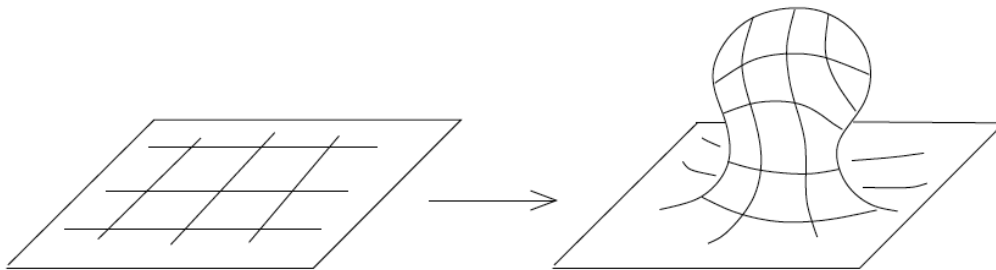


Figure 2.1: An example of two metric tensors related by a scale transformation. It preserves the angles, but not necessarily the distance, between all lines [41].

It follows that the infinitesimal conformal transformations obey the conformal Killing vector equation:  $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) g_{\mu\nu}$ . Here, setting the scale factor  $\Lambda(x) = 1$  corresponds to the Poincaré



group of translations and Lorentz transformations. For  $d > 2$ , the solution to this differential equation is given by Eq. 2.3

$$\epsilon_\mu = \alpha_\mu + \lambda x_\mu + \omega_{\nu\mu} x^\nu + b_\mu x^2 - 2(b \cdot x) x_\mu. \quad (2.3)$$

The solutions are enumerated below as follows:

1.  $\epsilon_\mu = \alpha_\mu$  describes an infinitesimal translation with corresponding generator the momentum operator  $P_\mu = \partial_\mu$ .
2.  $\epsilon_\mu = \omega_{\nu\mu} x^\nu$ .  $\omega_{\mu\nu}$  is an antisymmetric tensor and describes the infinitesimal Lorentz transformations: rotations and boosts. The generators are the angular momentum operators which, in differential form, are written as  $J_{\mu\nu} = (x_\mu \partial_\nu - x_\nu \partial_\mu)$ . Together with translations, they form the Poincaré subgroup of the conformal group.
3.  $\epsilon_\mu = \lambda x_\mu$ , where  $\lambda$  is a number, corresponds to infinitesimal scale transformations with corresponding generator the dilation,  $D = x^\mu \partial_\mu$ .
4.  $\epsilon_\mu = b_\mu x^2 - 2(b \cdot x) x_\mu$  are called special conformal transformations and describe an infinitesimal inversion followed by a translation and then another inversion [50]. The generator is  $K_\mu = (\eta_{\mu\nu} x^2 - 2x_\mu x_\nu) \partial^\nu$ .

In summary, the generators of the conformal group are:

$$P_\mu = \partial_\mu \quad (2.4)$$

$$J_{\mu\nu} = (x_\mu \partial_\nu - x_\nu \partial_\mu) \quad (2.5)$$

$$D = x^\mu \partial_\mu \quad (2.6)$$

$$K_\mu = (\eta_{\mu\nu} x^2 - 2x_\mu x_\nu) \partial^\nu \quad (2.7)$$

and the associated algebra of the group is:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [K_\mu, K_\nu] &= 0 \\ [D, P_\mu] &= -P_\mu, & [D, K_\mu] &= K_\mu & [D, J_{\mu\nu}] &= 0 \\ [J_{\mu\nu}, K_\rho] &= -\eta_{\mu\rho} K_\nu + \eta_{\nu\rho} K_\mu, & [J_{\mu\nu}, P_\rho] &= -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu \\ [J_{\mu\nu}, J_{\lambda\rho}] &= -\eta_{\mu\lambda} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\lambda} + \eta_{\nu\lambda} J_{\mu\rho} + \eta_{\mu\rho} J_{\nu\lambda} \\ [K_\mu, P_\nu] &= 2(\eta_{\mu\nu} D + J_{\mu\nu}) \end{aligned}$$

The translation and special conformal transformation operators on the first line commute with themselves. The dilation can be considered as "counting" the length dimension [50] and gives the

second line of the algebra while the third line states  $K$  and  $P$  transform as vectors. The commutation of the angular momentum operators gives the usual Lorentz algebra. Further, the fact that the translation operator does not commute with the dilation operator means that  $m^2 = -P^\mu P_\mu$  is no longer a Casimir operator as it was in the Poincaré group. Hence, a field with a mass is not a good representation [31].

Overall, there are  $\frac{(d+2)(d+1)}{2}$  generators in the conformal group. The conformal algebra is isomorphic to  $SO(2,d)$  [50]. This can be made more manifest by redefining the generators and packaging them in a group defined by a  $(d+2) \times (d+2)$  antisymmetric matrix. Denote the  $SO(2,d)$  generators by  $J^{MN}$ , ( $M, N = 0, 1, \dots, d+1$  and  $\mu, \nu = 0, 1, \dots, d-1$ ) satisfying the algebra

$$[J_{MN}, J_{PQ}] = -\eta_{MP}J_{NQ} - \eta_{NQ}J_{MP} + \eta_{NP}J_{MQ} + \eta_{MQ}J_{NP}.$$

The dilation generator,  $D$ , is a scalar and transforms as such while the momentum and special conformal operators transform as vectors, as shown in the algebra. Redefining the vectors as:

$$J_{\mu,d+1} = \frac{K_\mu - P_\mu}{2}, \quad J_{\mu,d} = \frac{K_\mu + P_\mu}{2},$$

then:

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & J_{\mu,d+1} & J_{\mu,d} \\ -J_{\nu,d+1} & 0 & D \\ -J_{\nu,d} & -D & 0 \end{pmatrix}$$

Therefore, conformal invariance in flat Minkowski space with symmetry group  $SO(1,d-1)$  is isomorphic to  $SO(2,d)$ . It will be shown that the symmetry group  $SO(2,d)$  is the same as that for a  $d+1$ -dimensional Anti-de Sitter space,  $AdS_{d+1}$ . The fact that the symmetry group of a  $d$ -dimensional CFT and a  $d+1$  AdS space is the same provides a strong argument - and necessary condition - in favour of the duality of a gravity theory in the bulk space and a CFT at its boundary [25].

It's briefly stated that in two-dimensions, the conformal algebra is isomorphic to the Virasoro algebra, where the Virasoro operators are the operators for the transverse oscillations of a quantum relativistic string and generate reparametrizations of the worldsheet of the string [51].

## 2.2 Representations of the conformal group

One defines representations of the conformal group by the action of the scaling operator  $D$  on a scalar field with eigenvalue  $-\Delta$ , where  $\Delta$  is the scaling dimension [25], i.e. as  $x^\mu \rightarrow \lambda x^\mu$ ,  $\Phi(x) \rightarrow \Phi'(x) = \lambda^\Delta \Phi(\lambda x)$ . Acting with the momentum and dilation operators on  $\Phi$  and using the conformal algebra:

$$D(P_\mu \Phi) = [D, P_\mu] \Phi + P_\mu D \Phi = -(\Delta + 1)(P_\mu \Phi)$$

and

$$D(K_\mu\Phi) = [D, K_\mu]\Phi + K_\mu D\Phi = -(\Delta - 1)(K_\mu\Phi).$$

In as sense,  $P_\mu$  and  $K_\mu$  act as raising and lowering operators, respectively. The representation is then obtained by acting with these operators. The lowest dimension is at the origin  $x = 0$ ,  $\Phi(0)$ , called the conformal primary field. It follows that  $K_\mu\Phi(0) = 0$  and the spectrum is built by successive action of the "raising operator" and by the action of  $J^{\mu\nu}$  which fills the spin states [25].

## 2.3 Conformal symmetry in field theory

### 2.3.1 Correlation functions and constraints

Quantities of interest in any field theory are vacuum correlation functions which, in a way, describe how the theory behaves. For real scattering, one can relate correlation functions to S-matrices via the LSZ reduction formalism [29]. Further, correlation functions are important within the context of evidence for the AdS/CFT conjecture when calculated from both sides of the duality.

In the path integral formalism of QFTs, the n-point function for a scalar field is defined as in Eq. 2.8

$$G_n(x_1, \dots, x_n) = \langle 0 | T \hat{\Phi}(x_1) \dots \hat{\Phi}(x_n) | 0 \rangle \quad (2.8)$$

with a similar expression for objects with tensor indices. By defining a partition function

$$Z[J] = \int D\phi \exp^{iS[\phi] + i \int d^d x J(x)\phi(x)} \quad (2.9)$$

the correlation functions are obtained by appropriate differentiation of the generating function [29]:

$$G_n(x_1, \dots, x_n) = (-i)^n \frac{\delta^n \ln[Z[J]]}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0}. \quad (2.10)$$

The required symmetry of a theory imposes constraints on correlation functions. Already Poincaré invariance constraints the one-point function,  $\langle \hat{\phi}(x) \rangle$  to be a constant due to translation invariance of the vacuum. The two-point function, which describes how a field propagates from one point in spacetime to another, must be a function of the norm of the difference in spacetime positions:  $\langle \hat{\phi}(x_1)\hat{\phi}(x_2) \rangle = f(|x_1 - x_2|)$  [29].

Scale invariance imposes additional constraints. A scale-invariant theory requires the action to be scale invariant. Therefore, a field (considering only a scalar field, suppressing any tensor indices) must transform as follows:  $\Phi'(x) = \lambda^\Delta \Phi(\lambda x)$ . That is, the field transforms as a scalar quantity up to an overall scaling where  $\Delta$  is the scale dimension, as aforementioned. Therefore, in a CFT, the one-point function must vanish because if the field has a scale, the constant obtained from Poincaré

invariance will also have a scale and would break scale invariance. The vacuum expectation value of the two-point function of two scalar fields of scaling dimensions  $\Delta_1$  and  $\Delta_2$  transforms as [35]:

$$\langle 0 | \hat{\Phi}(x_1) \hat{\Phi}(x_2) | 0 \rangle \rightarrow \lambda^{\Delta_1} \lambda^{\Delta_2} \langle 0 | \hat{\Phi}(\lambda x_1) \hat{\Phi}(\lambda x_2) | 0 \rangle \quad (2.11)$$

As mentioned, invariance under translations and rotations implies the two-point function only depends on the absolute difference  $|x_1 - x_2|$ . Scale invariance further implies:

$$\langle \hat{\Phi}(x_1) \hat{\Phi}(x_2) \rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \quad (2.12)$$

where  $c_{12}$  is a constant [25]. Similar analysis leads to the three-point function,

$$\langle \hat{\Phi}(x_1) \hat{\Phi}(x_2) \hat{\Phi}(x_3) \rangle = \frac{c_{123}}{\left(r_{12}^{\Delta_1 + \Delta_2 - \Delta_3}\right) \left(r_{23}^{\Delta_2 + \Delta_3 - \Delta_1}\right) \left(r_{13}^{\Delta_1 + \Delta_3 - \Delta_2}\right)}, \quad (2.13)$$

where  $c_{123}$  is another constant and  $r_{ij}$  denotes the absolute distances. Therefore, conformal invariance restricts the two- and three-point functions but not higher order functions.

### 2.3.2 Stress-energy tensor

Finally, one can define the stress-energy tensor to be [41]:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial S}{\partial g^{\mu\nu}}. \quad (2.14)$$

This is obtained by varying the action with respect to the metric tensor,

$$\delta S = \frac{1}{2} \int d^d x \sqrt{-g} (\delta g^{\mu\nu}) T_{\mu\nu}.$$

In a scale-invariant theory the stress-energy tensor is traceless, at least classically, or for a flat spacetime [35]. From the variation of the action, it is clear that the stress-energy tensor must have dimension  $d = \Delta$  since the action is dimensionless. The two-point correlation function of the stress-energy tensor will also depend on a constant but will further depend on an object built from scale invariant tensors:

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle = \frac{c}{|x - y|^{2\Delta}} I_{\mu\nu, \alpha\beta}. \quad (2.15)$$

Here  $c$  is called the central charge and counts the degrees of freedom of the CFT. In string theory (discussed in a later chapter), and in particular bosonic string theory, the central charge is  $c = 26$ . For a special class of CFTs, the central charge can be considered as a spacetime dimension. This means that string theory is constrained to be Lorentz invariant only if the string propagates in a

background spacetime of  $d=26$  dimensions [35, 51].

## 2.4 QCD, supersymmetry and $\mathcal{N} = 4$ SYM

The theory describing strong interactions is quantum chromodynamics (QCD). It is a non-abelian theory of gauge group  $SU(3)$ . The gauge fields are in the adjoint representation. On-shell, fermions obey the Dirac equation and are in the fundamental representation. Unlike the mediator of electromagnetism, the photon, gluons carry colour charge; the non-abelian nature of  $SU(3)$  means QCD describes self-interacting gauge fields as opposed to the abelian  $U(1)$  theory of quantum electrodynamics. Famously, the beta function is negative and so QCD is asymptotically free. At low energy levels the coupling constant is high and the theory is strongly coupled and non-perturbative.

Generalising the above description, one considers an  $SU(N)$  theory with the gauge fields in the adjoint representation given by an  $N \times N$  matrix and four supercharges. This is the ( $\mathcal{N} = 4$ ) super Yang-Mills theory. The Lagrangian is then completely determined by supersymmetry [20]. In schematic form the action is given by Eq. 2.16

$$S = -\frac{1}{4g_{YM}^2} \int d^4x \text{Tr} \left[ F^2 + 2(D_\mu \Phi)^2 + \chi \not{D} \chi + \chi [\Phi, \chi] - \sum_{I,J} [\Phi^I, \Phi^J]^2 \right] + \frac{\theta}{8\pi^2} \int \text{Tr} [F \wedge F]. \quad (2.16)$$

The coupling constants are  $g_{YM}^2$  and  $\theta$ ; the latter couples to a topological factor. It can be shown that the coupling constant of this theory doesn't run with the energy [48]. The beta function is given by

$$\beta = -\frac{g^3}{16\pi^2} \left( \frac{11}{3} c(A) - \frac{2}{3} \sum c(\text{weyl}) - \frac{1}{6} \sum c(\text{scalar}) \right)$$

where  $c$  denotes the second Casimir of the representations of the gauge fields, fermions and scalars, respectively. In  $\mathcal{N} = 4$  SYM, all the fields are in the adjoint. The theory has four fermions,  $\chi_\alpha$  and six scalars  $\Phi^I$ . Hence the  $\beta$  function is zero and finite at all orders. This means the coupling constant does not run with the energy scale. Since the coupling does not run it can be fixed to a value. The effective coupling will turn out to be the 't Hooft coupling  $\lambda = g_{YM}^2 N$  where  $N$  arises because there are  $N$  mediators that can be exchanged between two fields.

In contrast to QCD, the fermions and scalars in SYM are also in the adjoint – a supersymmetric theory relates bosons to fermions and vice versa - and the two types of matter fields must be in the same representation. The theory also has a global  $SO(6)$  or  $SU(4)$  R-symmetry that rotates the six scalars into each other, and also rotates the fermions. This R-symmetry is related to the symmetry of  $S^5$  in  $AdS^5 \times S^5$  [20]. A detailed formulation of supersymmetry, superspace and the  $\mathcal{N} = 4$  SYM Lagrangian is found in [3].

The efficacy of the AdS/CFT correspondence is that the duality between the gravity theory and gauge theory in question is defined in regimes where the coupling constants are of different magnitude. That is it relates a perturbative gravity theory where calculations are easier to do to a non-

perturbative gauge theory. Since QCD has a strong coupling and is non-perturbative, one would like to extend the conjecture to make a dual description of strong interactions. However, super Yang-Mills theory is a conformal field theory with a  $SU(N)$  gauge group with maximal supersymmetry. Its physics is different from that of QCD, a gauge theory with no supersymmetry. Further, the AdS/CFT correspondence is best described when the gauge theory has large degrees of freedom. Nevertheless the literature on the correspondence has been extended since the original work by Maldacena. A discussion on applications to QCD is included in the last chapter of this thesis.

### 3 Anti-de Sitter spacetime

Anti-de Sitter spacetime is best understood by comparing and contrasting with the more familiar (and arguably more relevant to our observed universe) de Sitter spacetime.

#### 3.1 de Sitter space

Analogous to the embedding of a  $d$ -dimensional sphere,  $S^d$ , of radius  $L$  in  $d+1$  Euclidean space,  $E^{d+1}$ , de Sitter spacetime,  $dS^d$ , with length  $L$  can be defined as the embedding in  $d+1$  Minkowski spacetime,  $M^{(1,d)}$ . The metric is

$$ds^2 = \eta_{MN} dX^M dX^N; \quad M, N = 0, 1, \dots, d \quad (3.1)$$

and satisfies

$$\eta_{MN} X^M X^N = L^2; \quad M, N = 0, 1, \dots, d. \quad (3.2)$$

The  $X^0$  coordinate is endowed with a minus sign as opposed to the Euclidean metric for the embedding of  $S^d$ . With the  $X^0$  coordinate (taken to be the time coordinate) running from  $-\infty$  to  $+\infty$ , the de Sitter spacetime is a hyperbola where every point on the hyperbola is a  $S^{d-1}$  sphere. A familiar schematic of de Sitter spacetime is shown in figure 3.1.

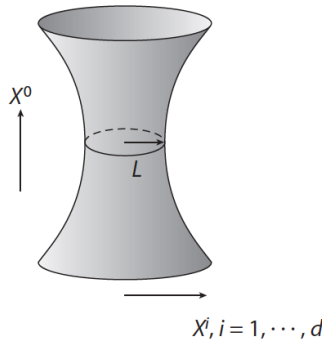


Figure 3.1: de Sitter spacetime embeds as a hyperboloid in Minkowski spacetime [49].

The manifest isometry group is  $SO(1,d)$  i.e. Lorentz transformations leave the metric invariant. As mentioned above, a constant time coordinate in Eq. 3.2 results in spheres of ever-expanding radii hence, the topology of de Sitter is  $\mathbb{R} \times S^{d-1}$  [49], namely a point in  $dS^d$  is left invariant by  $SO(1, d-1)$ , the Lorentz subgroup of  $SO(1, d)$ . One concludes that de Sitter spacetime is a maximally symmetric space. This constraints the Riemann curvature tensor to the form in Eq. 3.3

$$R_{\mu\nu\lambda\sigma} = \frac{\kappa}{d(d-1)} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}). \quad (3.3)$$

Here,  $\kappa$  is the curvature constant [5]. For de Sitter space, the curvature constant is given by  $\kappa = \frac{d(d-1)}{L^2}$ , where  $L$  is the radius. Hence, the Ricci tensor, Ricci scalar and Einstein tensor are fixed:

$$R_{\mu\nu} = \frac{(d-1)}{L^2}g_{\mu\nu}, \quad \mathcal{R} = \frac{d(d-1)}{L^2}, \quad G_{\mu\nu} = -\frac{(d-1)(d-2)}{2L^2}g_{\mu\nu}. \quad (3.4)$$

The Einstein tensor in Eq. 3.4 states that de Sitter spacetime is a solution to Einstein's field equation  $R^{\mu\nu} = 8\pi G\Lambda g^{\mu\nu}$  with a positive cosmological constant. The field equations with a positive cosmological constant are described as an exponentially expanding universe that is homogeneous and isotropic in space.

## 3.2 Anti de Sitter space

The metric of  $d$ -dimensional AdS spacetime,  $AdS^d$ , of length  $L$  is

$$ds^2 = (-f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_{d-2}^2) \quad (3.5)$$

where  $f(r) = 1 + \frac{r^2}{L^2}$  and  $d\Omega_{d-2}^2$  is the round metric on the  $(d-2)$ -sphere. The metric in these **static coordinates** is written similar to the Schwarzschild metric but the asymptotic behaviour differs. At  $r = 0$  the spacetime is flat. In the limit  $r \rightarrow \infty$  the metric looks like:

$$ds^2 = L^2(\exp^{2r}(-dt^2 + d\Omega^2) + dr^2).$$

The metric on the sphere and  $g_{00}$  grow. One can analyse this space by tracing geodesics along the spacetime. Massless and massive geodesics are best analysed in **Poincaré coordinates** introduced later in the chapter.

The manifest isometry group of AdS in static coordinates is  $R \times SO(d)$ . However, a larger symmetry group is found by recasting the metric - in an analogous way to de Sitter - as the embedding in a  $d+1$  dimensional Minkowski space with two time coordinates,  $M^{2,d-1}$  i.e. a spacetime metric

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{d-1} (dX^i)^2 + (dX^d)^2 \quad (3.6)$$

satisfying

$$(X^0)^2 - \sum_{i=1}^{d-1} (X^i)^2 + (X^d)^2 = L^2. \quad (3.7)$$

The full isometry group of  $AdS^d$  is then  $SO(2,d-1)$  as opposed to  $SO(1,d)$  for de Sitter space. Like de Sitter spacetime, AdS is maximally symmetric and has a Riemann tensor of the form



$$R_{\mu\nu\lambda\sigma} = \frac{-1}{L^2}(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}). \quad (3.8)$$

The Ricci tensor, Ricci scalar and Einstein tensor have the same form as de Sitter and differ only by a sign:

$$R_{\mu\nu} = -\frac{(d-1)}{L^2}g_{\mu\nu}, \quad \mathcal{R} = -\frac{d(d-1)}{L^2}, \quad G_{\mu\nu} = \frac{(d-1)(d-2)}{2L^2}g_{\mu\nu}. \quad (3.9)$$

Hence, AdS spacetime is a solution to Einstein's field equation  $R^{\mu\nu} = 8\pi G\Lambda g^{\mu\nu}$  with a negative cosmological constant,  $\Lambda = -\frac{(d-1)(d-2)}{16L^2}$ . Substituting for the length in Eq. 3.5 and suppressing some constants, the metric in terms of the cosmological constant in four spacetime dimensions is:

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\Lambda r^2}{3}r^2\right)} + r^2 d\Omega_{d-2}^2. \quad (3.10)$$

This form of the metric will be useful later when considering charged black holes. Another coordinate transformation for  $AdS^d$  is in **conformal coordinates** [49]:

$$ds^2 = \frac{L^2}{\cos^2\theta}(-dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2) = \frac{L^2}{\cos^2\theta}(-dt^2 + d\Omega_{d-1}^2) \quad (3.11)$$

where  $d\Omega_{d-i}^2$  is the round metric on the  $d-i$ -sphere. Note the time coordinate is arbitrary while  $0 \leq \theta \leq \pi/2$ .

#### The boundary of AdS

The fact that  $\theta$  is between 0 and  $\pi/2$  and not  $\pi$  means that only half the  $S^{d-2}$  sphere is covered i.e. the spatial component of  $AdS_d$  is bounded by  $S^{d-2}$ . With a time dimension, the critical statement is that  $AdS_d$  is bounded by  $R \times S^{d-2}$ . Specifically, in a four-dimensional universe,  $d = 4$ ,  $AdS_5$  has the isometry group  $SO(4,2)$  with a boundary  $M^{1,3}$ . As mentioned in the previous chapter, the conformal group in flat Minkowski space is also  $SO(4,2)$ . [49].

The asymptotic behaviour at the boundary is best understood in **Poincaré coordinates**. The  $AdS^d$  metric is then

$$ds^2 = \frac{L^2}{z^2}(-\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \quad \mu, \nu = 0, 1, \dots, d-1. \quad (3.12)$$

Here the conformal boundary is at  $z = 0$ . Up to a conformal factor, Eq. 3.12 is the flat Minkowski space. The Penrose diagram is then given by a triangle. Note that the Poincaré coordinates do not cover the whole of the AdSs space but slices that display the Poincaré symmetry [20]. The conformal boundary is manifestly invariant under spacetime translations and Lorentz transformations. Further, the metric in Eq. 3.12 is invariant under rescaling  $(t, \vec{x}, z) \rightarrow \lambda(t, \vec{x}, z)$ . As a result, at the conformal boundary the metric is Minkowski and the action of the isometries on the boundary is the conformal group.

### 3.2.1 Geodesics in AdS

In these coordinates photons travelling from a fixed point along the radial  $z$ -axis towards the boundary at  $z = 0$  return back at a finite time. A massive particle in the  $(t, z)$  plane is described by the differential  $\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = \frac{z^2}{L^2}$ . The Killing vector  $\partial t$  gives the conserved quantity and in particular  $g_{00}\frac{dt}{d\tau}$  is proportional to a constant,  $E$  [5]. The motion of the particle is then described in an analogous way to experiencing a Newtonian potential:

$$\left(\frac{dz}{dt}\right)^2 + \left(\frac{L}{Ez}\right)^2 = 1.$$

Therefore, a massive particle with finite energy will not reach the boundary but will turn back. Thus, the Penrose diagram of  $AdS^d$  is a cylinder with time at the vertical and a conformal boundary in the limit  $r \rightarrow \infty$ . The boundary contains the time direction and a  $S^{d-2}$  as shown in figure 3.2. The Poincaré patch is a slice of the full AdS space with manifest  $SO(1, d)$  and scale symmetry.

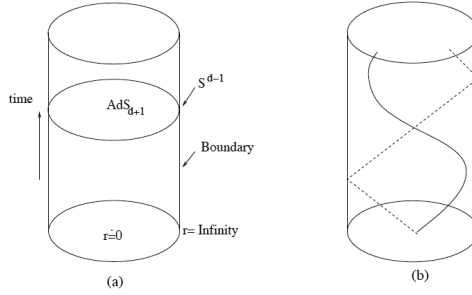


Figure 3.2: a) The  $AdS^d$  spacetime is a cylinder. Time runs in the vertical. The boundary at  $r = \infty$ . The boundary contains the time direction and  $S^{d-2}$ . b) Massless geodesics reach the boundary and back in a finite time. Massive geodesics don't reach the boundary [20].

## 4 String Theory

String theory describes the dynamics of relativistic open and closed strings and their interactions. The AdS/CFT conjecture, a gauge/gravity duality, is in some sense a duality between open and closed string theories.

Open strings have endpoints that are either free to move in the background space or are fixed to higher-dimensional hyperplanes called D-branes. The endpoints carry non-abelian charges, analogous to the  $SU(3)$  colour charges of Yang-Mills theory [35]. Polchinski's work on D-branes showed they are dynamical objects and can be thought of as the source of these charges [30]. On the other hand, closed strings propagate freely in space. Their excitations includes a state that contains the graviton, together with a scalar and an antisymmetric tensor, the Kalb-Ramond field. At low energy, supersymmetric string theories are approximated by supersymmetric gravity theories (SUGRA) [35]. These theories include the familiar Einstein-Hilbert action as well as additional matter fields, bosonic and fermionic, that can transform into each other according to supersymmetry. The equations of motion resulting from the SUGRA action describe p-branes which are a generalisation of a point particle and a string. These turn out to have black hole solutions in higher dimensions and can have the geometry of AdS spacetime [8].

A quantum relativistic string is a one-dimensional object which sweeps out a two dimensional worldsheet as it propagates through the target space [41]. The worldsheet is parametrized by a time-like coordinate  $\tau$  and a space-like coordinate  $\sigma$ , where  $\sigma$  is periodic. The fields  $X^\mu$  are parametrized by the worldsheet coordinates and describe how the string propagates in the  $D$ -dimensional background spacetime,  $\mu = 0, 1, \dots, D - 1$ . For now, in the classical theory, the dimension of the background spacetime  $D$  is arbitrary and unconstrained. Strings have tension hence the action must minimise the worldsheet area. It is given by the Polyakov action in Eq. 4.1

$$S_{POL} = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \alpha} \frac{\partial X^\nu}{\partial \beta} \eta_{\mu\nu}. \quad (4.1)$$

Here,  $T$  is the string tension, sometimes written as  $T = \frac{1}{2\pi\alpha'}$ .  $h^{\alpha\beta}$  is the induced metric on the worldsheet surface and  $\eta_{\mu\nu}$  is the metric in the background spacetime taken to be Minkowski. The fields  $X^\mu$  are bosonic and the Polyakov action describes the motion of a bosonic string.

The symmetries of the Polyakov action are:

- Global Poincaré invariance on the world sheet;
- Local reparametrisation invariance, a gauge symmetry on the worldsheet; and
- Weyl invariance - local rescaling of the metric.

As aforementioned in the section on CFTs, the stress energy tensor of a theory that is invariant under Weyl transformations is traceless. Using the redundancies of the degrees of freedom and Weyl invariance, one can fix a gauge so that the intrinsic metric on the worldsheet is the two-dimensional flat metric [35]. By varying the action with respect to the fields,

$$\delta S_{POL} = T \int d\tau \int d\sigma \sqrt{-h} \delta X^\mu \partial_\alpha \partial^\alpha X_\mu - T \int d\tau \sqrt{-h} \delta X^\mu \partial_\sigma X_\mu \Big|_0^\pi \quad (4.2)$$

the equations of motions describe a free wave. Setting the boundary condition in Eq. 4.2 to zero defines the type of strings in the theory. For closed strings, one requires that  $X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau)$ . For open strings, the boundary condition is either Neumann,

$$\partial_\sigma X^\mu(\tau, \sigma) = \partial_\sigma X^\mu(\tau, \sigma + \pi) = 0 \quad (4.3)$$

which means the endpoints of the string are free to propagate at the speed of light, or Dirichlet,

$$\delta X^\mu(\tau, \sigma) = \delta X^\mu(\tau, \sigma + \pi) = 0. \quad (4.4)$$

The string endpoints with Dirichlet boundary conditions are constrained on fixed points. One is free to specify a string with both Neumann and Dirichlet boundary conditions so that some endpoints are constrained on hypersurfaces and some are free to propagate. This type of objects are called Dp-branes and will be discussed in the next section as they play a crucial role in the AdS/CFT correspondence.

The field equations and their constraints are more easily solved by defining light-cone coordinates on the worldsheet. Following [35],  $\sigma^\pm = \tau \pm \sigma$ . In these coordinates the field equations of free waves are written as  $\partial_+ \partial_- X^\mu = 0$ . Varying the action with respect to the induced metric in the manner of Eq.2.15 one finds two constraints to the free wave equations:

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0.$$

The field equations are linear in  $\sigma^+$  and  $\sigma^-$  hence, the general solution to the field equations for  $X^\mu(\sigma^+, \sigma^-)$  is obtained by a linear combination of left and right movers defined as:

$$X^\mu = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+).$$

Expanding the solution for a closed string i.e. no boundary conditions, in Fourier modes, the mode expansion of a closed string is as follows:

$$X^\mu = x^\mu + \frac{1}{2} \tau l_s^2 p^\mu + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{2in\sigma} + \bar{\alpha}_n^\mu e^{-2in\sigma}, \quad (4.5)$$

Here  $x^\mu$  and  $p^\mu$  are constants called the centre of mass and total momentum of the string, respectively. These describe the motion of a string analogous to a point particle. The infinite summed series describes the vibrational modes of the string; strings possess an infinite number of vibrational modes. In an analogous way to the harmonic oscillator's modes, strings support an infinite stack of massive particles and some massless particle excitations. The string length is also a constant and is given by the relations  $T = \frac{1}{2\pi\alpha'}$ ,  $\alpha' = \frac{1}{2} l_s^2$  [35]. One notes that closed strings include left and right movers with distinct Fourier modes with coefficients  $\alpha$  and  $\bar{\alpha}$ . The distinction between the two becomes important upon quantisation. A similar expression is obtained for the mode expansion of an open string satisfying appropriate boundary conditions. Open strings however are furnished with only one type of Fourier mode coefficient.

The Hamiltonian and canonical momentum are found as the components of the conserved current corresponding to a translational symmetry. One starts with the classical Poisson brackets for the field  $X^\mu$  and its canonical momentum  $P^\mu$  and derives the Poisson brackets for the modes  $\alpha_m^\mu$  and  $\bar{\alpha}_m^\mu$ .

The constraints are also expanded in Fourier modes, defined as:

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}, \quad L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad (4.6)$$

and

$$T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \bar{L}_m e^{-2im(\tau+\sigma)}, \quad \bar{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n. \quad (4.7)$$

where  $L_m$  and  $\bar{L}_m$  are called the Virasoro operators. The zero modes of the constraints give the classical mass formula for a bosonic string

$$L_0 + \bar{L}_0 = -p^\mu p_\mu = M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} \left( \alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \right). \quad (4.8)$$

The constants in the Fourier mode expansion of the field,  $\alpha$  and  $\bar{\alpha}$ , have a real physical effect in determining the spectrum of the theory. For a quantum string, these turn out to be analogous to the creation/annihilation operators of the harmonic oscillator and the summation of terms like  $\alpha_{-n} \cdot \alpha_n$  analogous to the number operator which counts the excited modes. Further, since left and right movers are furnished with different modes and since the zero modes are equal, there is a level matching condition that relates the number of right moving and left moving excitations.

## 4.1 Quantization of the relativistic string

The easiest way to quantize the relativistic string is in light-cone gauge with spacetime coordinates  $X^-, X^+, X^I, I = 1, \dots, D-2$ . Picking a gauge makes the theory no-longer manifestly Lorentz invariant. The fields and conjugate momenta are promoted to operators and their Poisson brackets into commutation relations. These then give the commutation relations of the Fourier modes in light-cone gauge. The oscillatory modes define an infinite set of creation/annihilation operators. The transverse Virasoro operators correspond to oscillators in the  $X^-$  direction [51]. There is an ambiguity in quantizing the Virasoro operators from the usual Normal ordering of a product of operators, the oscillatory excitation modes. Requiring the theory to be Lorentz invariant fixes the spacetime dimension to be 26 and shifts the mass formula from the classical one in Eq. 4.8. The mass-shell condition for open strings is:

$$M^2 = \frac{2}{l_s^2} \sum_{I=1}^{D-2} \sum_{n=-\infty}^{\infty} ( : \alpha_{-n}^I \alpha_n^I : -1 ) = \frac{2}{l_s^2} (N - 1) \quad (4.9)$$

where  $N$  is the number operator that counts the number of excitations. The ground state that corresponds to  $N = 0$  admits a tachyon state. The first excited state,  $N = 1$ , is a vector boson  $\alpha_{-1}^I |0; k^\mu\rangle$  which must be massless due to Lorentz invariance [35]. Higher states will describe massive excitations. Hence, at low energies open strings admit massless photon states.

A closed string has both left and right moving modes with a constraint that matches their number operators. The mass-shell condition for closed strings is:

$$\alpha' M^2 = 4(N - 1) = 4(\bar{N} - 1). \quad (4.10)$$

The ground state of the closed string is again a Tachyon. The first excited state for  $N = 1$  is massless and is obtained by tensoring the right and left moving states  $\sum_{I,J} R_{IJ} \alpha_{-1}^I \bar{\alpha}_{-1}^J |0; 0; k^\mu\rangle$  [51].  $R_{IJ}$  is a  $(D-2) \times (D-2)$  matrix and can be decomposed into a symmetric traceless tensor, the graviton, an antisymmetric part called the Kalb-Ramond field and the trace called the dilaton. Hence closed strings naturally admit graviton states.

## 4.2 Superstring theory

The above brief introduction to bosonic string theory indicates that open strings can describe gauge theories while closed strings include gravity. Both open and closed strings were shown to have tachyonic ground states. Furthermore, the Polyakov action did not include fermions. To include fermions, string theory is extended to supersymmetric string theory which also projects out the tachyons [35]. One adds a Majorana field for each bosonic field. The full action is the bosonic Polyakov action and the action describing Majorana fermions [35]:

$$S = -\frac{1}{2\pi} \int d\tau d\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu - \frac{1}{2\pi} \int d\tau d\sigma \bar{\psi}^\mu \not{\partial} \Psi_\mu. \quad (4.11)$$

The two worldsheet fields are the bosonic,  $X^\mu$ , and fermionic,  $\Psi^\mu$  which are related by supersymmetry. The fermions are two component spinors  $\Psi_\pm^\mu$  where the subscript  $\pm$  indicates the chirality. Both bosons and fermions transform under the vector representation [8] of the Lorentz group. The theory has two sectors, the Neveu-Schwarz (NS) and Ramond (R) sectors [8]. A closed string state was obtained by tensoring left and right movers as in the mode expansion Eq. 4.5 therefore, one gets a total of four different sectors the NS-NS, the NS-R, R-NS and R-R sectors. Lorentz invariance constrained the bosonic string to propagate in  $D=26$  spacetime dimensions; in supersymmetric string theory the critical dimension becomes  $D=10$ . In [35] this is shown as a requirement of removing unwanted ghost states in the spectrum. The mass spectrum is obtained in the usual way of acting with creation operators on the vacuum. To remove the tachyonic states and define a consistent supersymmetric theory, one projects out in a way that eliminates the tachyon and leads to a supersymmetric theory in  $D=10$  spacetime [35]. This is called GSO projection. It turns out imposing the GSO projection leads to a constraint on the number of oscillators in the NS and R

sectors. These are referred to as having positive or negative G-parity. The number of oscillators in the NS sector must be odd. In the R sector one projects states with either odd or even excitations depending on the chirality. These possibilities lead to two types of superstring theories, types IIA and IIB which can be obtained depending on whether the G-parity of the left and right moving R sectors is the same or opposite. Table 4.1 below summarises the sectors found in each theory:

IIA	NS $\otimes$ NS	R $_-$ $\otimes$ R $_+$	NS $\otimes$ R $_+$	R $_-$ $\otimes$ NS
IIB	NS $\otimes$ NS	R $_+$ $\otimes$ R $_+$	NS $\otimes$ R $_+$	R $_+$ $\otimes$ NS

Table 4.1: The two types of superstring theory indicating the chirality of the R-sector [35].

The massless spectrum of IIA and IIB string theories is obtained by decomposing the tensor product of right and left moving string into representations of the Lorentz group [35]. The NS-NS and R-R sectors are bosonic. The former is made up of the tensor product of two vectors and decomposes into a symmetric traceless tensor (the graviton) the trace (dilaton) and the antisymmetric tensor (the Kalb-Ramond field). This decomposition was encountered in the spectrum of the closed bosonic string. The NS-R sector is the tensor product of a vector with a spinor and decomposes into the gravitino and the dilatino and similarly for the R-NS sector. The R-R sector is the tensor product of two spinors. The field strength tensors of the two theories are identified with spinor bilinears:

$$IIA : F^{\mu_1 \dots \mu_{d+1}} = \bar{\Psi}_-^L \gamma^{\mu_1 \dots \mu_{d+1}} \Psi_+^R, \quad IIB : F^{\mu_1 \dots \mu_{d+1}} = \bar{\Psi}_+^L \gamma^{\mu_1 \dots \mu_{d+1}} \Psi_+^R$$

where the superscripts  $L, R$  denote the the left and right movers and  $\pm$  the chiralities [35]. The field strength tensors are  $(d + 1)$ -forms and are the exterior derivatives of the  $d$ -form R-R potentials,  $F_{d+1} = dC_d$ . This is analogous to the electromagnetic field strength  $F_{\mu\nu}$  as the components of the two form,  $F$  where  $F = dA$ ,  $A = A_\mu dx^\mu$ .

Using chirality arguments, the field strengths for the R-R sectors of type IIA string theory are  $F_2$  and  $F_4$  with R-R potentials  $C_1$  and  $C_3$ , respectively. Type IIB superstring theory has  $F_1$ ,  $F_3$  and  $F_5$  field strengths with potentials  $C_0$ ,  $C_2$  and  $C_4$ , respectively. For the specific example of gauge/gravity duality examined in the dissertation where the bulk space is  $AdS_5 \times S^5$  the  $C_4$  potential is of particular interest. This is examined in more detail in the next section on p-branes.

## 5 D-branes and p-branes

### 5.1 Black holes

Black holes naturally arise in general relativity. They are defined by a causal boundary beyond which signals cannot escape. The familiar example of a black hole in four dimensions is the Schwarzschild solution which is a spherically symmetric solution in vacuum and has  $SO(1,3)$  symmetry. The horizon is a coordinate singularity. By a coordinate transformation, one extends the description to the entire spacetime. On the other hand, the Kretschmann invariant,  $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim 1/r^6$  diverges at the singularity  $r = 0$ , an "honest singularity" where the classical description of gravity cannot accurately describe the physics [42].

Black holes can have mass, charge and angular momentum. Rotating black holes are outside the scope of this thesis however, describing the physics of charged black holes plays an important role in the original formulation of AdS/CFT by Maldacena. In four dimensions one starts with the Einstein-Hilbert action that includes Maxwell fields [6]:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - F_2^2). \quad (5.1)$$

The Einstein-Maxwell equations have the spherically-symmetric Reissner-Nordström (RN) solution:

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega^2 \quad (5.2)$$

where

$$\Delta = r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-), \quad r_{\pm} = M \pm \sqrt{M^2 - Q^2}.$$

Evidently the RN solution has two horizons. However, solutions with  $M < Q$  lead to a "naked singularity" at  $r = 0$  which is excluded according to Penrose's cosmic censorship hypothesis [6]. An extremal black hole is the special situation where the charge is maximal given its mass i.e.  $M = Q$ , ( $r_{\pm} = M$ ) [1]. The metric in Eq.5.2 for an extremal black hole is asymptotically Minkowski when  $r \rightarrow \infty$  and singular at  $r \rightarrow M$ . To determine the asymptotic behaviour at this limit, one defines a new coordinate  $r = M + V$  which leaves the metric in a convenient form.

$$ds^2 = -H^{-2} dt^2 + H^2 (dV^2 + V^2 d\Omega_2^2), \quad H = 1 + \frac{M}{V}, \quad (5.3)$$

Here  $H$  is a harmonic function in three dimensions. The horizon is located at  $V = 0$ . Maldacena [17–19] investigated the geometry in the near-horizon limit  $V \rightarrow 0$  of extremal black holes (in higher dimensions) which paves the way to the gauge/gravity duality. Taking the approximation  $V \approx 0$  and after a coordinate transformation  $V = \frac{M^2}{\rho}$  one finds the Bertotti-Robinson metric [1]



$$ds^2 = \underbrace{\frac{M^2}{\rho^2}(-dt^2 + d\rho^2)}_{AdS_2} + \underbrace{M^2 d\Omega_2^2}_{2\text{-sphere}}. \quad (5.4)$$

The geometry is a two-dimensional  $AdS_2$  space with a negative cosmological constant and with two dimensions effectively compactified i.e.  $AdS_2 \times S^2$  [20]. It will be shown that the geometry near the horizon of a D3-branes is also AdS. The conclusion reached is that charged black holes are solutions to the Einstein-Maxwell field equations. Extremal black holes arise when the horizon bound is saturated and the mass equals the charge. In four dimensions the solution is a two-dimensional AdS space with the other two dimensions compactified on a two-sphere defined by a harmonic function in three dimensions.

## 5.2 p-branes

In the previous section on string theory, the relativistic string action was introduced as a generalisation of the point particle's worldline; the string is one dimensional and sweeps a two-dimensional world volume. This concept can be further generalised to a  $(p+1)$ -dimensional object that sweeps a certain world volume. These extended objects are called p-branes. If the solutions to their equations of motion include a horizon, they represent a higher-dimensional analog of black holes.

Just as a point particle, a 0-brane, couples to an electromagnetic U(1) gauge field, a 1-brane (a string) couples to a two-form gauge potential [35]

$$S_B = \int_{\mathcal{M}} d\tau d\sigma B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta},$$

where  $\mathcal{M}$  is the worldsheet mapped by the string as it propagates. The Kalb-Ramond two-form,  $B_{\mu\nu}$  has an associated three form field strength,  $H_3 = dB_2$ .

Generalising to p-branes as extended objects in higher dimensions, Polchinski [30] showed that a p-brane couples to a  $p + 1$ -form. In Type IIA superstring theory,  $A_\mu$  couples to D0-branes and  $A_{\mu\nu\rho}$  couples to D2-branes. In type IIB superstring theory,  $A_{\mu\nu}$  couples to D1-branes and  $A_{\mu\nu\rho\sigma}$  couples to D3-branes. D3-branes of type IIB theory are particularly important in the context of the gauge/gravity duality. The world volume of D3 branes is four-dimensional spacetime; it couples to the four-form with an associated five-form field strength, which, in ten dimensions is self-dual [20]. This is analogous to the two-form electromagnetic field strength in four dimensions that gives rise to both electric and magnetic fields.

A low energy limit can be thought of as fixing the energy level to be much lower than the string mass. The mass-shell spectrum for strings was given as:

$$M^2 \sim \frac{(N - 1)}{\alpha'},$$

hence one takes the limit  $\alpha \rightarrow 0$ . At low energies the massive modes decouple and only the massless modes remain. For a closed string, these modes are the graviton, the Kalb-Ramond field

and the dilaton. In this low energy regime, Einstein's field equations with matter are described by ten-dimensional supergravity actions. These theories include the familiar Einstein-Hilbert action coupled to some matter fields that make the theory supersymmetric.

As extended massive objects which move through spacetime, p-branes provide sources of curvature. Beginning with the low energy effective action of a type IIB SUGRA action relevant for the purposes of this thesis, [20]

$$S = \frac{1}{(2\pi)^7 l_p^8} \int d^{10}x \sqrt{g} (\mathcal{R} - F_5^2), \quad F_5 = \star F_5. \quad (5.5)$$

where the string length is  $l_s = g_s^{(-1/4)} l_p$ . The field strength five-form  $F_5$  is self-dual. The self duality of the five-form gives rise to both electric and magnetic fields. The solutions to the equations of motions are expressed by Harmonic functions as was the case for charged black hole solutions in four dimensions. In a spacetime with ten dimensions, the D3-brane of type IIB SUGRA theory is the solution to [35]:

$$\begin{aligned} ds^2 &= H(r)^{-1/2} (-dt^2 + \dots + d^2x_3) + H(r)^{1/2} (dx_4^2 + \dots + d^2x_9) \\ e^{2\phi} &= 1, \quad \phi = 0 \end{aligned} \quad (5.6)$$

$$A_{0,1,3,4} = H^{-1} - 1$$

$$H(r) = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s N (\alpha')^2, \quad Q = g_s N$$

where  $H$  is the harmonic function with  $R$  the radius of the D3-brane,  $\Phi$  is the dilaton scalar field related to the string coupling by  $g_s = e^{(-\phi)}$ , which is a constant in this case.

It is important to note that solutions to SUGRA can be extremal p-branes that preserve the maximal supersymmetry and are known as BPS states. In 1995, Witten [44] connected the various types of ten-dimensional string theories via a series of dualities. Underlying these was an 11-dimensional supergravity theory, called *M-theory* [35]. An important argument in Witten's theory concerned a class of BPS states. From supersymmetry arguments the solutions to the equations of motion had a horizon. They correspond to charged, extremal black holes that are stable. As was the case in the charged Reissner-Nordström black hole, the extremal solution was at  $Q = M$ . This equality saturates the physical bound of the horizon  $Q \leq M$ . In supersymmetry, the BPS bound implies the solution preserves the maximal supersymmetry, which is 1/2 [25].

### 5.3 D-branes

Closed strings have no boundary condition (except the condition relating the left and right movers) and are free to propagate in spacetime. For open strings however, one needs to specify boundary conditions. As mentioned these are either Neumann or Dirichlet. In the case of Neumann conditions, the string endpoints move at the speed of light. Open strings with Dirichlet boundary

conditions have string endpoints that are fixed in space i.e. they end on hypersurfaces called Dp-branes where  $p$  denotes the the spatial dimensions of the branes. A string can have  $p + 1$  Neumann boundary conditions and  $d - p - 1$  Dirichlet boundary conditions. The mode expansion for the coordinates with Neumann boundary conditions  $X^\mu$ ,  $\mu = 0, 1, \dots, p$  is given by [35]

$$X^\mu = x^\mu + \tau l_s^2 p^\mu + i l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{(-in\tau)} \cos(n\sigma). \quad (5.7)$$

The fields with Dirichlet boundary conditions are  $X^I$ ,  $I = p + 1, \dots, D$ . Considering a configuration where there are  $N$  number of D-branes the mode expansion will is:

$$X^I = x_i^I + \frac{\sigma}{\pi} (x_j^I - x_i^I) + l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{(-in\tau)} \sin(n\sigma). \quad (5.8)$$

Here  $i, j$  are the Chan-Patton factors that label the different Dp-branes a string's endpoints can be fixed on. The states are also labeled appropriately [1]. The figure below gives a schematic of a configuration of two parallel Dp branes. The strings can begin and end on the same D-brane or they can stretch between the two. The orientation of the stretched strings must be specified. The orientation plays an important role when D-branes are seen as carriers of electric charge of Ramond-Ramond fields [51].

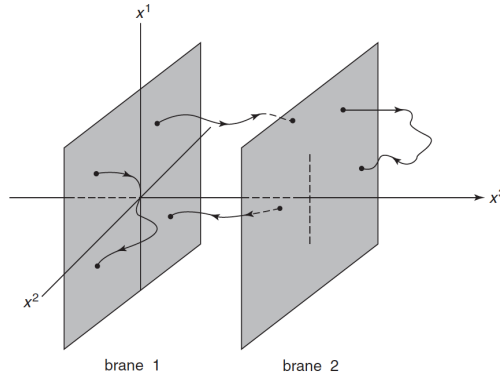


Figure 5.1: A two D-brane configuration where string endpoints are fixed. Strings can have their endpoints fixed on the same brane or stretch between the two [51].

The schematic in fig. 5.1 also shows another important feature of D-branes in string theory. If one considers a spacetime with many D-branes present then one must "count" the number of different configurations of strings stretched and fixed on them. The presence of different states in such a system contributes to a statistical mechanics interpretation of black hole entropy within string theory. The mass squared of states is:

$$M^2 = \frac{N - 1}{\alpha'} + T^2 (x_i^I - x_j^I)^2.$$

The bracketed term is then thought as the tension of the string stretched between then  $i^{th}$  and  $j^{th}$

brane. The massless states are obtained following [1]. In a system with only one D-brane, the massless state admits only a single oscillation. There are excitations,

$$\alpha_{-1}^I |0, k\rangle, \quad (5.9)$$

which correspond to  $D - p - 1$  massless scalar particles and states

$$\alpha_{-1}^\mu |0, k\rangle, \quad (5.10)$$

which corresponds to a  $p + 1$ -dimensional vector,  $A_\mu$ , the gauge field. In this system with one D-brane there exists a U(1) Maxwell gauge field as well as a massless scalar field along each normal direction. Witten [45] showed that in a system of  $N$  coincident branes the gauge symmetry is U(N);  $N$  coincident D-branes carry a U(N) Yang-Mills theory. It can be shown that in the low energy limit the non-abelian gauge group is SU(N). A theory with  $N$  D3-branes will also have  $\mathcal{N} = 4$  supersymmetry and will carry a  $\mathcal{N} = 4$  SU(N) supersymmetric Yang-Mills theory, a conformal field theory.

### 5.3.1 Open/closed string duality

Since p-branes are massive and carry a charge, in a closed-string background they interact via the exchange of closed strings (lhs of figure 5.2) [34]. In 1995, Polchinski [30] showed that the one-loop amplitude for the interaction of open strings stretched between two parallel D-branes is equivalent to a closed string at tree level. Hence it is possible to reinterpret the p-brane interaction to include open strings. On the rhs of figure 5.2, an in-falling closed string forms an excited unstable state by splitting to form an open string with endpoints attached to the brane. The state decays by emitting a closed string [34].

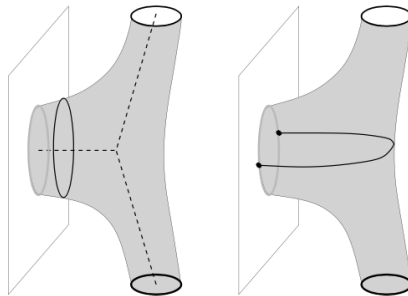


Figure 5.2: Different interpretations of a p-brane interaction in closed string theory [34].

Polchinski's set up consisted of two branes in a ten-dimensional background separated by some distance  $\Delta y$ . Branes are dynamical extended objects; they are massive and carry Ramond-Ramond charges hence they will interact with each other. In quantum field theories, interactions are mediated by gauge bosons; string theory includes an infinite series of vibrational modes. In a low energy regime however, the physics simplifies. Closed string theory includes the graviton while

open string theory contains Yang-Mills gauge fields. A tree level exchange is shown in figure 5.3 below. The diagram can be interpreted in two different but equivalent ways. On the lhs. a closed string is exchanged between the two D-branes while the rhs. is interpreted as a one-loop open string.

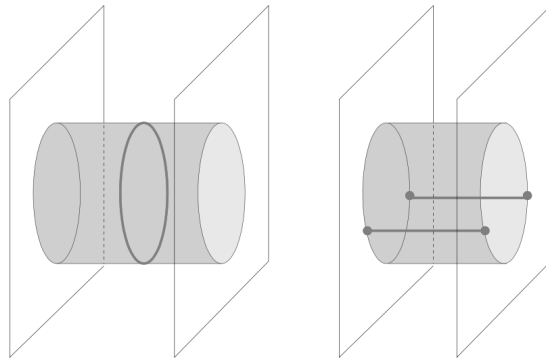


Figure 5.3: Lhs: A closed string exchanged between two branes. Rhs. one-loop open string [34].

The equivalence between open and closed string theories (presented schematically) reveals important aspects that are relevant for the conjecture. Namely, closed string theory encodes information of a gauge theory but at different loop-expansions [34]. Further, the degrees of freedom of the two theories are bound to different dimensional backgrounds and is thus holographic in the manner of t' Hooft [40] and Susskind [38]. One also notes that the decoupling limit, one for closed strings and the other in terms of open strings, is non-trivial. If the distance between the two branes is large - the distance the closed string propagates - the closed string massive modes are neglected and the low-energy effective action of string theory is SUGRA. If on the other hand the distance  $\Delta y$  is small in relation to the string length, a low energy effective action is not appropriate and one must consider the full string theory spectrum. The reverse is the case from the equivalent view of open string theory. A small separation means that pair creation is restricted to a low loop order; for  $\Delta y \sim l_s$  only massless modes are considered [34].

One concludes that string theory requires the existence of D-branes, the hyperplanes where open strings end and that these correspond to p-branes, solutions to supergravity, the low energy limit of string theory. Extremal p-branes have a non-zero horizon hence they have entropy. Dp-branes then carry the Ramond-Ramond charges and different configurations arise from strings stretching between the D-branes [51]. At first glance, it seems improbable that the abstract description of the open closed string duality described would lead to a duality between gauge theories and gravity. The duality between  $\mathcal{N} = 4$  SYM and type IIB string theory follows from the the decoupling argument.

## 6 AdS/CFT Correspondence

### 6.1 Symmetry arguments

A necessary but not sufficient argument for the correspondence is based on the symmetry group present in the gauge theory on the boundary and bulk gravity theory. For the specific example originally proposed by Maldacena, the gauge theory is maximally supersymmetric with four spinor supercharges. It has the conformal group  $SO(2,4)$  as well as a global R-symmetry  $SU(4) \sim SO(6)$ . The five-dimensional  $AdS_5$  space was shown to also have a local  $SO(2,4)$  symmetry. Since strings propagate in ten dimensions, the extra five dimensions are compactified [20] on the five dimensional sphere  $S^5$ . Therefore, from symmetry arguments alone the  $\mathcal{N} = 4$  SU(N) SYM theory is the appropriate candidate gauge theory dual to the ten-dimensional superstring theory on  $AdS_5 \times S^5$  [27].

### 6.2 The decoupling limit

In field theory one adds the amplitudes of all Feynman diagrams in a perturbative expansion for a particular interaction. Alternatively, one considers an effective action with a background field and an effective potential. In a similar manner the decoupling limit involves examining a string theory with D-branes in two different ways.

Maldacena's argument arises by considering  $N$  coincident D3-branes in ten-dimensional spacetime and type IIB closed strings propagating in the spacetime. One point of view is to consider the scattering of closed strings and D-branes perturbatively. In this regime, a closed string can split into many closed strings or it can split to form open strings whose endpoints are on the D-branes. Open strings on the branes can further split into more open strings. They can also form closed strings by joining their endpoints. These then are not bound to the branes and are emitted as Hawking radiation [25]. This system is depicted in fig. 6.1

In the low energy limit the dynamics of the open strings on the D-branes are described by  $\mathcal{N} = 4$  SYM theory. The closed strings in the bulk spacetime are described by a ten-dimensional type IIB supergravity. The gravitational interactions between the two - resulting in Hawking radiation - can be switched off since the ten-dimensional Newton constant  $G^{(10)} \propto g_s^2 (\alpha')^4$  goes to zero when the energy is bounded while  $\alpha' \rightarrow 0$ . From this point of view, the system describes a decoupled type IIB supergravity in ten dimensions and a  $\mathcal{N} = 4$  SYM living on the four-dimensional brane.

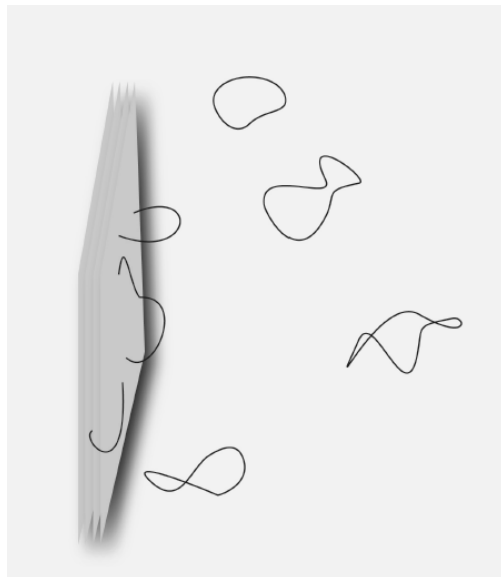


Figure 6.1: Scattering of closed strings with  $N$  D-branes [2].

An alternative description of the system is to replace the D3-branes with their SUGRA solution; the D-branes act as sources for closed strings [25]. The space-time metric of  $N$  coincident D3-branes is

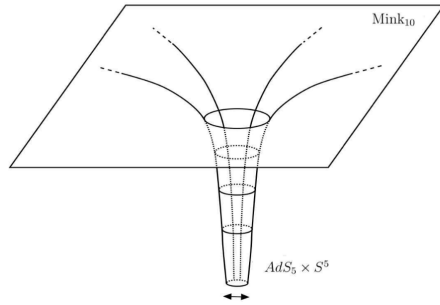
$$ds^2 = H(r)^{-1/2}(-dt^2 + d\vec{x}^2) + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \quad (6.1)$$

$$H(r) = 1 + \frac{L^4}{r^4}, \quad L^4 = g_s N 4\pi (\alpha')^2,$$

where  $L$  is the radius of the D3-brane and  $\vec{x}$  is a three dimensional vector along which the branes stretch; they appear as a point along the transverse six dimensions. Comparing with Eq. 5.3 this is an extremal black hole. For  $r \gg L$ , the metric approaches ten-dimensional flat spacetime. A near-horizon geometry appears in the limit  $r \ll L$ . By a similar redefinition as the RN black hole,  $w = L^2/r$ , one finds the asymptotic form of the metric to be:

$$ds^2 = \frac{L^2}{w^2}(\eta_{ij} dx^i dx^j + dw^2) + L^2 d\Omega_5^2. \quad (6.2)$$

This is the  $AdS_5 \times S^5$  metric where the radius of the five-sphere is  $L$ .



The conformal boundary is far away from the branes at  $r \rightarrow \infty, w \rightarrow 0$ . The spacetime at the boundary is ten-dimensional Minkowski. Near the horizon a highly symmetrical throat geometry survives [8], shown in fig. 6.2. The origin where the branes live is an infinite distance down the throat. Asymptotically, the throat becomes an infinite cylinder. The horizon at the end of the throat is an infinite distance away from any point on the plane.

Figure 6.2: The boundary is flat spacetime and is infinitely far away from the horizon. The horizon geometry is  $AdS_5 \times S^5$  [22].

The energy measured by an observer at the boundary,  $E$ , and proper energy,  $E_p$  is given by  $E = \sqrt{g_{00}} E_p$ . Hence, excitations of finite energy near the horizon  $w \rightarrow \infty$  appear infinitely red-shifted to an observer at the boundary. In addition, all excitations far from the boundary are of large wavelength in the low energy limit. As a result, this configuration is again decoupled into a system of ten-dimensional supergravity at the boundary and type IIB superstring theory in the near-horizon in  $AdS_5 \times S^5$ .

In this low energy limit, the two configurations match one of the decoupled systems. The candidates for the second decoupled system are, as aforementioned, supersymmetric Yang-Mills in four dimensions and type IIB superstrings in  $AdS_5 \times S^5$ . Therefore, Maldacena [20] argued that the two are in fact the same.

### 6.2.1 Parameters in AdS/CFT

The AdS/CFT dictionary relates the two sets of parameters for the two dual theories. The gauge theory on one side of the duality has two dimensionless parameters. The coupling constant in the SYM action Eq. 2.16 was given by  $g_{YM}^2$ . In addition, the gauge theory depends on the number of coincident D-branes,  $N$ . The 't Hooft limit  $N \rightarrow \infty$  gives planar diagrams and is controlled by the effective coupling  $\lambda = g_{YM}^2 N$ , which is fixed. The gravity theory also contains two parameters, the radius of the brane in units of string length  $\frac{L}{l_s}$  and the string coupling constant  $g_s$  which determines the strength of interactions between strings. The two sets are related as follows:

$$\lambda \equiv g_{YM}^2 N = \left(\frac{L}{l_s}\right)^4, \quad g_{YM}^2 = \frac{\lambda}{N} = 4\pi g_s. \quad (6.3)$$

The strong version of the duality states that this equivalence of parameters holds true for all values of the coupling constant  $g_s$  and for any  $N$ . In 1973 't Hooft [39] showed that field theories simplify when the number of "colours" is large i.e. in the limit  $N \rightarrow \infty$  while the 't Hooft parameter  $\lambda$  remains fixed. In this limit, 't Hooft showed that only planar diagrams contribute since non-planar diagrams have a perturbative expansion in powers of  $1/N$ . Following Aharony et al. [27] one takes the 't Hooft limit and varies the 't Hooft parameter,  $\lambda$ , from weak to strong by dialing the string coupling  $g_s$  from small to large. From Eq. 6.3 what changes is the AdS radius in units of string length which determines the magnitude of gravitational effects.

Perturbative computations on the gauge theory side are possible at small values of the 't Hooft coupling and these are mapped to a regime where the radius of AdS is of the order of the string length i.e. supergravity is not a good approximation to superstring theory. On the other hand, if the radius is large in units of string length then the curvature is small and supergravity is a good approximation to superstring theory but gauge theory computations are not tractable [20]. That the regimes in which calculations are more easily computed do not match makes the AdS/CFT duality powerful and useful in physics; Yang-Mills theories with large coupling can be matched to the other side of the duality which is computationally easier and vice versa.

### 6.3 The holographic dictionary

The exact gauge/gravity duality described above means that the symmetries, degrees of freedom and dynamics of the theories are in one-to-one correspondence; one must be able to define a map between the two sides of the duality. When dealing with gauge theories, one is interested in gauge invariant operators such as the trace of the field strength tensor  $Tr[F_{\mu\nu}F^{\mu\nu}]$ , the stress-energy tensor etc. According to the AdS/CFT dictionary [15, 46], each gauge-invariant operator on the field theory side corresponds to a bulk field defined on the interior space i.e. to closed string modes moving in the  $AdS_{d+1}$  space. The bulk fields with boundary conditions are then related to the generating function of correlation functions for the corresponding operator in the field theory where the field evaluated at the boundary acts as the source:

$$Z_O[\Phi_0]_{CFT} = \langle e^{\int d^d x \Phi_0 \mathcal{O}(x)} \rangle = Z_\Phi[\Phi_0]_{grav} = e^{-S_{grav}[\Phi]}, \quad (6.4)$$



where  $\Phi_0$  is the bulk field at the boundary and  $\mathcal{O}$  is the relevant gauge invariant operator. From Eq. 2.10, correlation functions of operators are then given by [20]

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \dots \rangle = \frac{\delta}{\delta \Phi_0(x_1)} \dots \frac{\delta}{\delta \Phi_0(x_n)} Z_{grav}[\Phi_0] \Big|_{\Phi_0=0}. \quad (6.5)$$

The bulk fields and operators must have the same structure (Lorentz indices and quantum numbers). For example, a bulk scalar field is associated with a scalar operator at the boundary. Further, Gubser et al. [15] showed it is possible to determine anomalous dimensions from string theory. The relevant quantity is the mass of the associated string modes; massless string states have vanishing anomalous dimensions. It is then natural, for example, to match the rank-two stress-energy tensor, with vanishing anomalous dimension, to the graviton mode of the closed string. The relations between the masses of bulk fields and the conformal dimension of the dual operators [8] are summarised in table 6.1

Fields	Relations
Scalars or massive spin 2 fields	$m^2 L^2 = \Delta(\Delta - d)$
Massless spin 2 fields	$m^2 L^2 = 0, \Delta = d$
Spin 1/2, 3/2	$ m L = \Delta - \frac{d}{2}$
p-form	$m^2 L^2 = (\Delta - p)(\Delta + p - d)$

Table 6.1: The relations between the mass of bulk fields and conformal dimension of the dual operators [8].

### 6.3.1 Bulk scalar field

In order to explain how these relations arise, the simplest but physical example of a bulk scalar is considered. The important step is solving the equations of motion in  $AdS_{d+1}$ . The metric in Poincaré coordinates was given in Eq. 3.12. The scalar action in AdS space is:

$$S_\Phi = -\frac{1}{2} \int_{AdS} dz d^d x \sqrt{-g} (g^{MN} \partial_M \Phi \partial_N \Phi + m^2 \Phi^2). \quad (6.6)$$

The bulk scalar obeys the Klein-Gordon equation,

$$(-\square + m^2)\Phi = \frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N \Phi) - m^2 \Phi = 0. \quad (6.7)$$

The field is decomposed into a superposition of Fourier modes:

$$\Phi(x, z) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{\Phi}(z, k), \quad |k| = \sqrt{\eta_{\mu\nu} k^\mu k^\nu} \quad (6.8)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric in  $d$ -dimensions. The equations of motion in terms of the Fourier modes follow [32],

$$z^{d+1}\partial_z(z^{1-d}\partial_z\Phi) - k^2z^2\Phi - m^2L^2\Phi = 0. \quad (6.9)$$

Near the conformal boundary at  $z \rightarrow 0$  the second term drops out and there are two independent solutions

$$\Phi(z, k) \propto A(k)z^{d-\Delta} + B(k)z^\Delta, \quad \Delta(\Delta - d) = m^2L^2. \quad (6.10)$$

and

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + (mL)^2}, \quad (6.11)$$

with  $\Delta$  being the larger of the two roots. For  $\Delta$  to be real (and taking  $\Delta > d/2$  for simplicity) requires the mass of the field to be bound from below by the Breitenlohner–Freedman bound [4],  $L^2m_{BF}^2 \geq \frac{-d^2}{4}$ . In principle  $m^2$  could be negative i.e. tachyonic. It is an instability in the system for large wavelengths but it's allowed as long as it is not too tachyonic [20].

Near the asymptotic boundary the leading behaviour of the bulk field is given by  $A(k)$ . Motivated by the holographic dictionary that the source of the boundary operator ( $\mathcal{O}$ ) is the bulk field at the boundary, one is prompted to set a boundary condition. Further, one imposes a condition at the interior so that there is no energy flux through the past Cauchy horizon at  $z \rightarrow \infty$  [43]. After taking the inverse Fourier transform, the general solution in real space which vanishes as  $z \rightarrow \infty$  behaves at the boundary condition as [23]:

$$\Phi(z, x) \equiv \lim_{z \rightarrow 0} z^{d-\Delta} \Phi_0(x). \quad (6.12)$$

It is worth noting that the isometry  $z \rightarrow \lambda z$  in Eq. 3.12, which generates scale transformations at the conformal boundary, induces a transformation to the bulk scalar field in Eq. 6.12 which is dependent on the  $z$ -coordinate. The transformation is as follows  $\Phi_0(x) \rightarrow \lambda^{d-\Delta} \Phi_0(\lambda x)$ .

One proceeds to compute the on-shell action which will connect to the discussion on matching observables in the dual theories. After integrating by parts, introducing a cut-off surface at  $z = \epsilon$  i.e. in the limit  $\epsilon \rightarrow 0$  the surface is the AdS boundary, and evaluating on the equations of motion, one obtains a boundary term [43]

$$S_{on-shell} = -\frac{1}{2} \int_{\partial M} d^d x \left( \frac{L}{z} \right)^{d-1} \Phi \partial_z \Phi \Big|_\epsilon. \quad (6.13)$$

Given a boundary condition, the bulk field can be constructed using the bulk-to-boundary propagator [13, 46]. The propagator,  $K_\Delta(z, x, y)$ , is defined such that it satisfies

$$(-\square + m^2)K_\Delta(z, x, y) = 0, \quad \lim_{z \rightarrow 0} K_\Delta(z, x, y) \rightarrow \delta^d(x - y)z^{d-\Delta}. \quad (6.14)$$

For a field that obeys the equations of motion and with specified boundary condition, Eq. 6.12 can be written in terms of the field at the boundary and the propagator:

$$\Phi(x, z) = \int d^d y K_\Delta(z, x, y) \Phi_0(y). \quad (6.15)$$

The correct expression for the propagator is the form: [13]

$$K_\Delta(z, x, y) = \mathcal{B}_\Delta \frac{z^\Delta}{(z^2 + (x - y)^2)^\Delta}, \quad \mathcal{B}_\Delta \equiv \pi^{-d/2} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \quad (6.16)$$

which can be shown to obey the equations of motion and have the appropriate asymptotic behaviour. The bulk field  $\Phi(z, x)$  is expressed in terms of the source on the boundary  $\Phi_0(x)$  in Eq. 6.15 which can be inserted into the on-shell action:

$$S_{on-shell} = -\frac{1}{2} \int_{\partial M} d^d x \left( \frac{L}{z} \right)^{d-1} \Phi \partial_z \Phi \Big|_\epsilon \quad (6.17)$$

$$= \lim_{z \rightarrow 0} -\frac{1}{2} \left( \frac{L}{z} \right)^{d-1} \int d^d x d^d y d^d w K_\Delta(z, x, y) \Phi_0(y) \partial_z (K_\Delta(z, x, w) \Phi_0(w)). \quad (6.18)$$

The derivative of the propagator is

$$\partial_z (K_\Delta(z, x, w)) = \partial_z \frac{\mathcal{B}_\Delta z^\Delta}{(z^2 + (x - y)^2)^\Delta} \quad (6.19)$$

$$= \Delta \mathcal{B}_\Delta \left( \frac{z}{(z^2 + (x - y)^2)} \right)^{\Delta-1} \left[ \frac{1}{z^2 + (x - y)^2} - 2z^2 \right]. \quad (6.20)$$

Taking the limit  $z \rightarrow 0$ , the resulting expression for the on-shell action simplifies considerably as all the terms in  $z$  cancel:

$$\lim_{z \rightarrow 0} S_{on-shell} \propto -\frac{1}{2} z^{1-d} \int d^d x d^d y d^d w z^{d-\Delta} \delta^d(x - y) \frac{z^{\Delta-1}}{(z^2 + (x - w)^2)^\Delta} \Phi_0(y) \Phi_0(w) \quad (6.21)$$

$$= -\frac{1}{2} \Delta \mathcal{B}_\Delta L^{d-1} \int d^d y d^d w \frac{\Phi_0(y) \Phi_0(w)}{|y - w|^{2\Delta}}. \quad (6.22)$$

The observables in the theory correspond to correlation functions of boundary operators. These can be calculated via appropriate Witten diagrams in the bulk [46]; akin to summing over Feynman diagrams in quantum field theory. In Witten diagrams, the boundary of AdS is represented by a circle where the sources  $\Phi_0$  live. The bulk space is the interior of the circle. There exist boundary-to-boundary propagators that connect two points on the boundary. The more important objects are

the bulk-to-boundary propagators, already discussed, which connect points on the boundary of the circle with an interaction vertex in the bulk.

Using Eq. 6.4, the two-point function is given by:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = -\frac{\delta^2 S_{grav}[\Phi_0]}{\delta\Phi_0(x_1)\delta\Phi_0(x_2)}\Big|_{\Phi_0=0} \quad (6.23)$$

$$= \mathcal{B}_\Delta L^{d-1} \frac{1}{|x_1 - x_2|^{2\Delta}}. \quad (6.24)$$

The two-point function gives the correct behavior for a scalar field of conformal dimension  $\Delta$  as was shown in Eq. 2.12. This is the simplest but a valid check of the AdS/CFT correspondence. The two-point function was fixed by conformal invariance and the numerical constant was determined from normalisation.

Dynamics are encoded in three and higher-point functions. The form of the three-point function is also fixed by conformal invariance but the overall coefficient is not [25]. The three-point functions were calculated in [23, 24, 33]. For the three-point function, a cubic vertex interaction is added to the action [24],

$$S_{int\Phi} = \frac{1}{2} \int_M d^{d+1}x \sqrt{-g} ((\nabla\Phi)^2 + m^2\Phi^2 + \lambda - 3\Phi^3). \quad (6.25)$$

A tree-level Witten diagram for the three-point function is depicted in fig. 6.3,

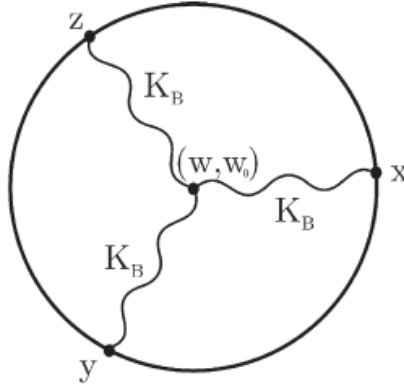


Figure 6.3: A tree-level Witten diagram for the three-point function. Bulk-to-boundary propagators intersect at a cubic vertex [25].

The three-point correlator is obtained by integrating over all possible positions of the bulk vertex [24]:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3) \rangle = \frac{\delta^3 S_{int\Phi}[\Phi_0]}{\delta\Phi_{0,1}(x_1)\delta\Phi_{0,2}(x_2)\Phi_{0,3}(x_2)} \Big|_{\Phi_0=0} \quad (6.26)$$

$$= -\lambda_3 \int d^d x dz \sqrt{-g} K_{\Delta_1}(z, x; x_1) K_{\Delta_2}(z, x; x_2) K_{\Delta_3}(z, x; x_3) \quad (6.27)$$

$$= \frac{\lambda_3 \alpha_1}{(r_{12})^{\Delta_1+\Delta_2-\Delta_3} (r_{13})^{\Delta_1+\Delta_3-\Delta_2} (r_{23})^{\Delta_2+\Delta_3-\Delta_1}}, \quad (6.28)$$

where  $\alpha_1$  is a constant. The agreement of the form with Eq. 2.13 is apparent.

## 7 AdS/CFT at finite temperature

Quantum chromodynamics is fundamentally different from the supersymmetric Yang-Mills theory which appears as the dual gauge theory in the original formulation of the AdS/CFT correspondence. As aforementioned, in SYM all quarks and fields transform in the adjoint representation of the gauge group whilst for QCD the quarks transform in the fundamental representation. Further, the conformal invariance of SYM theories implies that their  $\beta$ -function vanishes; the coupling is scale independent. However, QCD is asymptotically free and thus non-perturbative which makes calculations very difficult.

In order to make use of the AdS/CFT correspondence for real-world physics it must be extended to non-conformal and non-supersymmetric quantum field theories. The correspondence is then more generally known as gauge/gravity duality. One way to arrive at a theory closer to QCD is to consider systems with a finite temperature. A non-zero temperature introduces an energy scale which breaks conformal invariance. It will be shown that supersymmetry is also broken. It will then be possible to describe heavy ion collisions in QCD at finite temperature [25] using the duality formalism.

### 7.1 Finite temperature in field theory

A finite temperature can be introduced in field theory by switching to the Euclidean action following the transformation  $t = i\tau$ . The Euclidean functional integral is  $z = \int \mathcal{D}\Phi e^{-S_E[\Phi]}$ , where  $S_E$  is the Euclidean action. The functional integral can then be associated with the partition function for a canonical ensemble given by  $Z = \text{Tr}[e^{-\beta H}]$ . Euclidean time is therefore periodic with period  $\beta = (k_B T)^{-1}$  and  $k_B$  is the Boltzmann constant. It follows that bosonic fields in the functional integral are periodic while fermionic fields are anti-periodic [42].

### 7.2 Thermodynamics of Black Holes

Black holes play an important role in string theory and a key motivation for the AdS/CFT correspondence. Indeed, the bulk theory of AdS/CFT is a gravity theory which can give rise to black holes. The Euclidean Schwarzschild metric is obtained by Wick-rotating the time coordinate:

$$ds_E^2 = A(r)d\tau^2 + A(r)^{-1}dr^2 + r^2d\Omega^2, \quad A(r) = 1 - \frac{2GM}{r}. \quad (7.1)$$

At  $r \rightarrow \infty$  the metric topology is  $R^3 \times S^1$ , from the periodicity of the Euclidean time. The topology at the singularity,  $r = 2GM$ , is  $R^2 \times S^2$ . The singularity is a smooth coordinate singularity only if the Euclidean time is identified as periodic  $\tau \sim \frac{2\pi}{\kappa}$ , where  $\kappa$  is the surface gravity ( $1/4GM$  for Schwarzschild). One then identifies the Hawking temperature of a black hole:

$$T_H = \frac{\kappa}{2\pi}. \quad (7.2)$$

The fact that fermions are anti-periodic around the circle  $S^1$  at infinity implies they acquire a mass while bosons, which are periodic, do not [25]. Since the masses of bosons and fermions are not the same means that, in the presence of a black hole, supersymmetry is broken. It can be shown that finite temperature always breaks supersymmetry. However, a Schwarzschild black hole has a negative specific heat,  $C = \frac{\partial M}{\partial T}$  and is therefore thermodynamically unstable. This means that the QFT for this system does not have a finite temperature.

The thermodynamics of black holes in AdS space were studied before the AdS/CFT correspondence by Hawking and Page [10], subsequently re-examined by Witten [47] and Gubser et al. [14]. Witten showed that a finite temperature QFT is achieved by adding a black hole in AdS space where the system is thermodynamically stable if the black hole has high enough mass  $M$ . The metric for a five-dimensional Schwarzschild black hole in  $AdS_5$  is:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_3^2, \quad f(r) = \frac{r^2}{L^2} + 1 - \frac{r_0}{r^2}, \quad (7.3)$$

where  $L$  is the radius of curvature of the AdS space,  $d\Omega_3^2$  is the round metric of the three-sphere and  $r_0$  is a constant

$$r_0 = \omega_4 M, \quad \omega_4 \equiv \frac{16\pi G_{N,5}}{3Vol(S^3)}. \quad (7.4)$$

The black hole horizon is at  $r = r_+$  where  $r_+$  is the larger, positive real root of the polynomial  $f(r)$ . One obtains the surface gravity using the formula [6]

$$\kappa^2 = \lim_{r \rightarrow r_+} g^{\mu\nu} \nabla_\mu V \nabla_\nu V, \quad V^2 = -g_{\mu\nu} K^\mu K^\nu, \quad (7.5)$$

where  $K = \partial_t$  is a Killing vector at the Killing horizon at  $r = r_+$ . The surface gravity of this black hole in  $AdS_5$  is:

$$\kappa = \frac{2r_+^2 + L^2}{L^2 r_+}. \quad (7.6)$$

The Hawking temperature from Eq. 7.2 is  $T(M) = \kappa/2\pi$ . Unlike the Schwarzschild black hole, there is a region where the specific heat is positive. However, the topology at  $r \rightarrow \infty$ , where the  $\mathcal{N} = 4$  SYM gauge theory lives is  $S^3 \times S^1$  since the Euclidean time is periodic with radius  $\propto 1/T$  [25]. Requiring the topology to be flat Minkowski in four dimensions, Witten obtained a finite temperature AdS/CFT in the limit the mass of the black hole  $M \rightarrow \infty$  [25]. Following re-scaling and redefinition of coordinates, one obtains the  $AdS_5 \times S^5$  metric at finite temperature in Poncaré coordinates:

$$ds = \frac{L^2}{z^2} \left( -f(z)dt^2 + d\vec{y}^2 + \frac{dz^2}{f(z)} \right) + L^2 d\Omega_5^2 \quad (7.7)$$

$$f(z) = 1 - \frac{z^4}{z_0^4}, \quad (7.8)$$

with Hawking temperature:

$$T = \frac{1}{\pi z_0}. \quad (7.9)$$

The entropy of the five-dimensional black hole is given by the Bekenstein–Hawking formula,

$$S = \frac{A}{4G_{N,5}}, \quad (7.10)$$

where  $A$  is the area of the horizon. This is infinite therefore, one defines an entropy density given by:

$$S = \frac{L^3}{4G_{N,5}z_0^3}. \quad (7.11)$$

The five-dimensional Newton constant  $G_{N,5}$  is found from string theory by dimensional reduction from ten dimensions. Using the relations of parameters from the AdS/CFT correspondence in Eq. 6.3, the constant is written explicitly in terms of the parameters of the dual field theory [25]:

$$G_{N,5} = \frac{\pi L^3}{2N^2}. \quad (7.12)$$

With  $z_0$  given in terms of the Hawking temperature, the entropy density of the AdS black hole at infinite coupling, written in terms of the entropy of the  $\mathcal{N} = 4$  SYM parameters is,

$$S_{BH} = \frac{\pi^2}{2} N^2 T^3, \quad (7.13)$$

which has the expected form  $S \sim T^3$  for blackbody radiation. The AdS/CFT correspondence connects the entropy of a black hole with the entropy of a gauge field theory. From a conceptual point of view, this provides a statistical mechanics interpretation for black hole entropy as per the Strominger and Vafa entropy calculation. In addition, black holes are seen as thermal states consistent with quantum mechanics and unitary evolution [20]. Consequently, information is not lost inside a black hole [43]. Finally, the duality allows for the computation of thermodynamic properties from the field theory side and compare with QCD physics.

### 7.3 Thermal properties of sQGP

As aforementioned, the gauge theory on the boundary and QCD are very different theories. However, at a non-zero temperature where the temperature is high enough to form a quark-gluon plasma, many of the distinctions between the two theories disappear [32]. In particular, confinement of quarks into hadrons is no longer present at high temperatures and the plasma behaviour is similar to scale invariance. Further, the Witten prescription of having a black hole in the bulk space breaks supersymmetry. Therefore, it is of interest to consider whether certain quantities are universally described across gauge theories at finite temperature. In particular, one compares the thermal



properties for strongly-coupled quark-gluon plasma from the  $\mathcal{N} = 4$  SYM side at finite temperature to results from QCD obtained from heavy ion collisions in Brookhaven's RHIC experiment and the LHC's ALICE experiment [7].

The entropy in Eq. 7.13 was at infinite coupling and was calculated using the entropy of a five-dimensional black hole. One calculates the pressure and energy density of the sQGP fluid from  $s = \partial P / \partial T$  and  $E = -P + TS$ , respectively. At infinite coupling these are [32]:

$$P_{\lambda=\infty} = \frac{\pi^2}{8} N^2 T^4 \quad (7.14)$$

$$E_{\lambda=\infty} = \frac{3\pi^2}{8} N^2 T^4. \quad (7.15)$$

The same quantities can be obtained at weak coupling,  $\lambda = 0$  from the gauge theory side. The fields in  $\mathcal{N} = 4$  SYM are in the adjoint, hence each has  $N^2 - 1$  components. The theory has eight fermionic and eight bosonic degrees of freedom. In total the entropy is [32]:

$$s_{\lambda=0} = \left(8 + 8 \times \frac{7}{8}\right) \frac{2\pi^2}{45} (N^2 - 1) T^3 \approx \frac{2\pi^2}{3} N^2 T^3. \quad (7.16)$$

Comparing the thermal quantities between strong and weak coupling, one finds:

$$\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{P_{\lambda=\infty}}{P_{\lambda=0}} = \frac{E_{\lambda=\infty}}{E_{\lambda=0}} = \frac{3}{4}. \quad (7.17)$$

An interesting observation about the result in Eq. 7.17 is that these thermodynamic properties do not change much as the coupling goes from weak to strong. On the other hand, it is known that the transport properties and physical composition of the plasma *do* vary dramatically with the coupling [32]. This phenomenon was shown that is not unique to the  $\mathcal{N} = 4$  case; Nishioka and Takayanagi [26] computed the ratio of the entropy of the free Yang-Mills to the entropy of the strongly-coupled Yang-Mills for infinite many cases and found that it does not change drastically around this value. The relative factor of 3/4 in eq.7.17 can be viewed not as a discrepancy of the theory but as a prediction [12]; the two calculations were taken at different coupling strengths hence, the deficit is a prediction of the gauge theory behaviour at strong coupling.

Further, in lattice calculations of QCD thermodynamics, the numerical value for the ratio of the pressure of strongly-coupled Yang-Mills to free Yang-Mills was found to be around 80%, which is of the same order as the calculation done via the gauge/gravity duality [9].

### 7.3.1 Shear viscosity

The AdS/CFT correspondence can also be used to relate the shear viscosity at finite temperature in the strong coupling regime of the gauge theory with the low-energy background of D3-branes [36]. Treating the sQGP as a fluid, its viscosity is obtained from the stress tensor:

$$T_{ij} = \delta_{ij}p - \eta \left( \partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u_k \right) - \zeta \delta_{ij} \partial_k u_k, \quad (7.18)$$

where  $u_i$  is the flow velocity,  $p$  is the pressure and  $\eta$  and  $\zeta$  are the shear and bulk viscosity, respectively. The shear viscosity can be expressed via the Kubo relations to the correlation functions of the stress tensor,

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d^3x e^{i\omega t} \langle [T_{xy}(t, \vec{x}), T_{xy}(0, 0)] \rangle. \quad (7.19)$$

For a D3-brane, the result at strong coupling is [36]

$$\eta = \frac{\pi}{8} N^2 T^3, \quad (7.20)$$

which is equal to the entropy density, eq.7.13 up to a factor:

$$\frac{\eta}{S_{BH}} = \frac{1}{4\pi}. \quad (7.21)$$

The constant of proportionality,  $1/4\pi$  is close to results obtained from QCD [11].

## 8 Conclusions

This dissertation presented the prototypical example of the AdS/CFT correspondence as conjectured by Maldacena; a  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory with gauge group  $SU(N)$  is dual to a background spacetime with  $AdS_5 \times S^5$  geometry. Preliminary material was presented on the constituent parts of the duality, namely conformal field theories and AdS spacetime.

The conformal group was shown as the extension of the Poincaré group that includes scale transformations (and special conformal transformations). The Lie algebra is  $SO(2,d)$ . Scale invariance in a field theory restricts its observables. In particular, the form of the two- and three-point functions for scalar fields were shown to be functions of their relative distance and their scale dimension. The candidate CFT in four dimensions is the maximally supersymmetric Yang-Mills gauge theory.

In general relativity, a  $d+1$ -dimensional Anti-de Sitter spacetime is as a solution to Einstein's field equations with a negative cosmological constant. Its embedding in a Minkoswian spacetime makes manifest the isometries of the metric; these were shown to also be  $SO(2,d)$ . AdS spacetime has a conformal boundary - for  $AdS_5$  the boundary is 4-d Minkowski where the conformal group acts on it. From the perspective of the AdS/CFT correspondence, the boundary of the bulk space is where the gauge theory lives. The correspondence is therefore holographic in the sense that it relates two theories of different dimensions.

The bulk space is a solution to a specific superstring theory. Open and closed superstrings propagate in a ten-dimensional background. The spectrum of closed strings includes gravitons while open strings have vibrational modes that one identifies as massless vector bosons. Open strings can be described with Dirichlet boundary conditions. The strings' endpoints are fixed on dynamical membranes, D-branes, that carry charges. The key idea is that interactions between D-branes can be described either as closed strings at tree level or mediated by open strings at a one-loop amplitude. Closed strings interacting with the D-branes can open up and turn into open strings whose ends are free to move along the D-brane. Conversely, open strings bound on the D-brane can join their endpoints and form closed strings. These in turn are not bound to the membrane and can be emitted as Hawking radiation.

D-branes realise gauge theories on their world volume and preserve half of the bulk supersymmetry. The spectrum of open strings living on a stack of  $N$  coincident D-branes will correspond to a maximally supersymmetric  $SU(N)$ , Yang-Mills gauge theory. Further, it was argued that D-branes are massive objects that curve spacetime and have a dual description when embedded into a background of closed strings.

Consideration of low-energy processes for the D3-branes in type IIB string theory indicates that the solution has a near-throat geometry that is  $AdS_5 \times S^5$ . In this low energy limit, the system decouples from the asymptotically flat region at the boundary. Similarly, in the dual description of the system of  $N$  coincident D3-branes, the  $\mathcal{N} = 4$   $SU(N)$  gauge theory decouples from the bulk supergravity in ten dimensions in the low energy limit. Maldacena argued that the dual description of this configuration, which results in decoupled systems, leads to the assertion that computations in  $\mathcal{N} = 4$  SYM theory and in string theory on  $AdS_5 \times S^5$  give the same results.

The AdS/CFT correspondence is a conjecture with no rigorous mathematical proof. However, there is strong evidence to support it. An immediate requirement comes from symmetry arguments. The

fact that the symmetries of the conformal group in  $d$  dimensions are equivalent to the isometries of AdS in  $d+1$  dimensions is a necessary condition and motivates the correspondence. In addition, the near-throat geometry includes a compactified space of a five-sphere. The isometries of this space form the group  $SU(4) \sim SO(6)$  and are identical to the R-symmetry of the gauge theory.

From the duality of the two theories, the thesis presented the conjectured map between the gauge-invariant operators in the CFT and the bulk fields in  $AdS_5$ . The dimension of the operators is determined by their mass; the simplest example of a bulk scalar was presented where the relation was found to be  $\Delta(\Delta - d) = m^2 L^2$ . It was shown that correlation functions can be computed via Witten diagrams and bulk-to-boundary propagators. The two- and three-point functions of bulk scalars in AdS were computed and shown to match the form expected from the conformal theory side.

The supersymmetric Yang-Mills theory differs greatly from QCD. To make use of the duality, some of the features of the dual gauge theory must be relaxed. It was shown that it is possible to break supersymmetry and conformal invariance by placing a black hole into the five-dimensional geometry.

The holographic dictionary relates the set of parameters in the gauge theory to those of the bulk spacetime. Perturbative gauge theory computations are mapped onto the stringy regime of the full type IIB string theory. On the other hand, the power and usefulness of the gauge/gravity duality is that it relates quantities in strongly-coupled gauge theories to quantities that can be calculated more easily in the dual classical supergravity regime. The thesis presented an example from recent research on strongly-coupled quark-gluon plasma observed at the RHIC experiment. The entropy of the fluid was computed by exploring a five-dimensional black hole using the Bekenstein-Hawking formula. This was compared to the weak coupling entropy from the gauge side. The numerical factor of the ratio of the entropies yields a result which closely matches those obtained from lattice QCD. Calculations of the fluid's shear viscosity yield similar findings. This suggests that some thermal properties are universal for gauge theories.

There has been a flurry of research activity in gauge/gravity dualities since Maldacena's original formulation of the correspondence. Despite their great theoretical value, the ultimate interest lies in furthering an understanding of QCD. As aforementioned, QCD is non-perturbative and exhibits asymptotic freedom, confinement and has a mass gap. There are two approaches to obtaining a dual gravity theory to QCD: the so-called "top-down" and "bottom-up". The former starts from a well-defined string theory system and arrives at a duality with a well-defined gauge theory whose properties are more akin to QCD. With no conformal invariance and less (or no) supersymmetry, the background spacetime will differ from  $AdS_5$  and the compact space is not necessarily an  $n$ -Sphere. These string theories are related to the 11-dimensional M-theory but are not of type IIB string theory as the original formulation. The bottom-up method is a phenomenological approach whereby one starts with QCD and looks for its gravity dual. It would be interesting to study these dualities from a theoretical point of view, compare calculations with data and explore new insights into the physical world.

As a final note, two recently-published papers are mentioned that have been developed within the AdS/CFT duality paradigm and have the potential of resolving a long standing problem in theoretical physics - black hole information paradox. The duality of the gravity theory to the unitarity at the boundary where the CFT lives means that information is not lost. However, there remained the issue of showing the mechanism by which the information is encoded in the radiation

evaporating from the black hole. In [21, 28] it was shown that the entropy of the black hole was consistent with information conservation. It would also be interesting to extend the scope of this dissertation to cover the information paradox history and recent resolution.

## 9 References

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