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Department of Physics

Canonical Quantum Gravity

Author:
Michael Schick

Supervisor:
Prof. Joao Magueijo

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Abstract

In this work we are going to discuss some of the key ideas in Canonical Quantum Gravity. We will focus on the classical theory, giving a background in constrained Hamiltonian systems and then applying this to General Relativity. We will also discuss the Ashtekar variables, matter and an introduction to quantisation.

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Chapter 1

Introduction

In this dissertation we will be discussing a number of topics in Canonical Quantum Gravity with the aim of giving a reader with only a background in Quantum Field Theory (QFT) and General Relativity (GR) an introduction into the field. We focus mainly on the classical aspects of the theory with a small discussion on quantisation at the end. The aim is to arm the reader with the knowledge required to continue on to the full theory. Quantum gravity is wide field with many different approaches including String/M Theory, Loop Quantum Gravity, Twistor theory and many more, for a recent review see [14]. We will be focused on the so called "Canonical Quantum Gravity" theory described by Bergmann [9] in 1966 and more substantially by DeWitt a year later [15]. Since the introduction of the Ashtekar variables [3] in 1986, this area has seen an increase in interest.

We will begin in section 2.2 by giving an overview on constrained Hamiltonian systems. This is mainly based on work done by Dirac and Bergmann. We will describe the different types of constraints that arise, the consistency conditions that must be considered as well as the equations of motion. We will then apply this to the basic case of electromagnetism to show how we can find the Hamiltonian constraints from the Lagrangian. The hope is that we can then follow a similar procedure for GR so that we can arrive at a Hamiltonian form of GR which can be quantised using the canonical approaches.

In section 3.1 we will deriving the "Arnowitt-Deser-Misner" (ADM) action first introduced in 1959 [1]. The ADM action is an action for GR that we can through the use of Legendre transformations convert into Hamiltonian form. This is performed in section 3.1.1. Once we have arrived at our Hamiltonian we will evaluate the equations of motion in section 3.1.2 as well as the boundary terms which were ignored before in section 3.1.3.

While the ADM formalism is a good start a more useful set of coordinates is the Ashtekar variables which we introduce in section 3.2. This reformulation simplifies the Hamiltonian constraint at the expense of introducing complex variables. We end the section with a brief discussion on the Immirzi parameter. We next provide a small discussion on the issue of time in section 3.3. While we do not go into much technical detail, we aim to provide an overview on the different approaches used in the field for introducing time.

We end our classical analysis with a discussion of matter in section 3.4. We will discuss how the introduction of fermions creates torsion as well as the new terms that must be added to the Hamiltonian constraint from different matter sources. While a full derivation of these constraints is not provided due to time constraints we will provide references for further study in this area.

In section 3.5 we will give an overview of the steps needed to quantise the theory and introduce the Master constraint. The Master constraint is a more modern attempt at quantisation first proposed by Thiemann in 2006 [53], it attempts to reduce the infinite set of Hamiltonian constraints into a single constraint which then has to be quantised. Unfortunately due to time constraints a thorough discussion on quantisation was not possible but we have attempted to provide at least an indication of what is required. We end with a summary discussion, with reference to some recent review for further study in 4.

Chapter 2

Background

2.1 Motivation

We begin with a small section asking the question: why? Over the twentieth century fundamental physics was refined down mostly to two major ideas, Quantum Mechanics and General Relativity. It is natural therefore to wish to find a way to combine both theories and find a true quantum gravity theory. Any new theory must therefore hold with the key parts we of each respective theory that came before which have been verified experimentally. In the case of Quantum mechanics the feature we most want to preserve is the probabilistic nature, and for General Relativity our metric no longer is an observer but instead a dynamical part of the theory. In other words our physics must now be background independent. This background independence presents a major challenge as we simply do not know how to approach QFT on spaces without a fixed background metric as we find in theories such as the Standard model.

While there are numerous attempts at finding new quantum gravity theories we can broadly categorise them into two camps. The first takes a perturbation approach. This approach preserves the parts of QFT that can be saved while dropping the restriction of background independence, and trying to recover this restriction by perturbing the background and in essence performing a sum over all possible backgrounds - this is what one might call the string theory approach [25]. The other camp takes the opposite view and tries to preserve background independence from the outset but is now forced to create new mathematical approaches and tools beyond what is used in QFT. This second approach will be the one we are concerned with. Among the non-perturbative approaches the canonical approach is the oldest, originating with work done by Dirac in the 1940s and 1950s [18, 20, 21] and then carried on by Wheeler and DeWitt in the 1960s [15, 31]. The main idea behind the approach is to apply a Legendre transformation to the Einstein-Hilbert action to cast into Hamiltonian form, and then to quantise using Dirac's theory of quantisation of constraints.

2.2 Constrained Hamiltonian Systems

In this section we are going to introduce some of the key ideas behind constrained Hamiltonian systems. The motivation for this will be twofold. First, we note that GR is a constrained Hamiltonian system and therefore it is best to introduce this idea now before we apply it to the Hamiltonian formalism of GR. Secondly Gauge theories in general are examples of constrained Hamiltonian systems and so a study of this topic can be widely applied to many areas of physics. The main body of work in this area was done by Dirac [41] with a more modern treatment given in [42].

We start by considering a system with a finite number of degrees of freedom and dynamical coordinate $q_n, n = 1, ..., N$ where N is the number of degrees of freedom. From this we can derive the velocities $\frac{dq_n}{dt} = \dot{q}_n$ and we have a Lagrangian which is a function $L = L(q, \dot{q})$ of our coordinates and velocities. We already can note the importance of the time parameter in our system as we not only use it to calculate our velocities but by necessity require it for our Lagrangian. Following the standard procedure of varying the action we can derive the Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n} \tag{2.1}$$

As usual to reach our Hamiltonian form we introduce the momentum $p_n = \frac{\partial L}{\partial \dot{q}_n}$. Usually here one would make the assumption that our momenta are independent functions of the velocities, however we will take a more general approach and instead will define the relationships connecting the momentum variables as $\phi_m(q, p) = 0$ where $m = 1, ..., M$ runs over all the possible independent relations. We will call ϕ_m our primary constraints. We now wish to consider the Legendre transformation $H = p_n\dot{q}_n - L$ and take the variation of it.

$$\begin{aligned}
\delta H &= \delta p_n \dot{q}_n + p_n \delta\dot{q}_n - \left(\frac{\partial L}{\partial q_n}\right)\delta q_n - \left(\frac{\partial L}{\partial \dot{q}_n}\right)\delta\dot{q}_n \\
&= \delta p_n \dot{q}_n - \left(\frac{\partial L}{\partial q_n}\right)\delta q_n
\end{aligned} \tag{2.2}$$

Where we have used our definition for the momenta to go from the first to the second line. We can note that the variation of H only involves q and p and does not contain any terms involving variation of the velocities. By writing out an equation for the general variation of H and comparing with 2.2, using our primary constraint above it is easy to find that

$$\begin{aligned} \dot{q}_n &= \frac{\partial H}{\partial p_n} + \lambda_m \frac{\partial \phi_m}{\partial p_n} \\ \dot{p}_n &= -\frac{\partial L}{\partial q_n} = -\frac{\partial H}{\partial q_n} - \lambda_m \frac{\partial \phi_m}{\partial q_n} \end{aligned} \tag{2.3}$$

where our λ_m are unknown coefficients.

2.2.1 The Poisson Bracket

It is now convenient to introduce the Poisson bracket $\{f, g\}$ for functions $f(q, p), g(q, p)$ which we will define as

$$\{f, g\} = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} \tag{2.4}$$

The Poisson bracket is anti-symmetric in f and g, linear in both, satisfies the product law

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\}f_2 \tag{2.5}$$

and satisfies the *Jacobi* identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{2.6}$$

For any of the functions g we defined above we can write

$$\dot{g} = \frac{\partial g}{\partial q_n}\dot{q}_n + \frac{\partial g}{\partial p_n}\dot{p}_n \tag{2.7}$$

and with the use of 2.3 we can write this as

$$\dot{g} = \{g, H + \lambda_m \phi_m\} \tag{2.8}$$

This motivates us to define a new Hamiltonian $H_{tot} = H + \lambda_m \phi_m$ which gives us a very nice equation of motion

$$\dot{g} = \{g, H_{tot}\} \tag{2.9}$$

Let us now consider what happens when g is one of our primary constraints, that is to say $g = \phi_m$ and for consistency we must have $\dot{g} = 0$. Putting these into 2.9 we find

$$\{\phi_m, H\} + \lambda_{m'}\{\phi_m, \phi_{m'}\} \approx 0 \tag{2.10}$$

where here the use of $\approx$ is to remind us that our primary constraint $\phi_m = 0$ should be applied at the end of our calculations.

2.2.2 Consistency Conditions

We now have a number of consistency conditions that we need to consider. The first condition we must apply is that the Lagrangian equations of motion are consistent, this is important as can easily be seen by considering a Lagrangian $L = q$, which when applied to 2.1 directly leads to the contradiction $1 = 0$. After applying this condition we can then split our equation 2.10 into three further types. The first and most simple is if our equation reduces to $0 = 0$, in other words it is already satisfied, in this case there is nothing further for us to do. The second kind is when our equation reduces to a new equation which is independent of λ, it therefore only contains q and p. We can write this as

$$\chi(q,p) = 0 \tag{2.11}$$

which we call "secondary" constraints. When this occurs following the same procedure as before we find new consistency conditions

$$\{\chi, H\} + \lambda_m \{\chi, \phi_m\} \approx 0 \tag{2.12}$$

which we need to treat in the same way as before. This can lead to more secondary constraints (sometimes referred to as "tertiary"). We may have to apply this procedure many times until we have exhausted all of our consistency conditions and at the end we will be left with a set of secondary constraints $\phi_k = 0$ where $k = M+1, ..., M+K$ for K secondary constraints. We can then combine our constraints together as

$$\phi_j = 0, j = 1, ..., M + K \equiv J \tag{2.13}$$

The final type is when 2.10 does not reduce and we are therefore required to impose a condition on λ. We will look for solutions of the form $\lambda_m = \Lambda_m(q, p)$, and we know that solutions must exist because no solution implies that Lagrangian's equations of motion are inconsistent but consistency was the first step we required. The solution we have found is not unique and a more general solution can be written as

$$\lambda_m = \Lambda_m(q, p) + v_a V_{am} \tag{2.14}$$

where v_a is an arbitrary constant and V_{am} are solutions of the homogeneous equation associated with 2.10. We can then substitute this into our total Hamiltonian to find the new total Hamiltonian

$$\begin{aligned} H_T &= H + \Lambda_m \phi_m + v_a V_{am} \phi_m \\ \Rightarrow H_T &= H' + v_a \phi_a \end{aligned} \tag{2.15}$$

Where $H' = H + \Lambda_m \phi_m, \phi_a = V_{am} \phi_m$. Our new total Hamiltonian still has the same equations of motion found in 2.9 and this completes our consistency requirement. The arbitrariness of the constant v_a can be understood by comparison with things such as the gauge in electrodynamics, or the arbitrary choice of coordinates which we are often allowed to make. It is not a problem to have such arbitrariness in our system, it instead suggests that the mathematical framework we are using simply contains arbitrary features. We also wish to introduce the idea of "classes" of constraints. We will call a constraint "first class" if for any dynamical variable R it has zero Poisson bracket with all the $\phi' s$

$$\{R, \phi_j\} \approx 0 \tag{2.16}$$

and we call a constraint "second class" otherwise. First class constraints are important as it is these that generate our gauge transformations [36].

2.2.3 The Hamiltonian Form of Electromagnetism

Let us now briefly consider electromagnetism which has the Lagrangian

$$L = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^3x \tag{2.17}$$

where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{2.18}$$

Following the formalism above we now wish to go to the Hamiltonian form. Firstly we must introduce our momenta by varying the velocities in the Lagrangian

$$\begin{aligned}\delta L &= -\frac{1}{2}\int F_{\mu\nu}\delta F_{\mu\nu} d^3x \\ &= \int F_{\mu 0}\delta A_{\mu,0} d^3x \end{aligned} \tag{2.19}$$

and we define the momenta B^μ

$$\delta L = \int B^\mu \delta A_{\mu,0} d^3x \tag{2.20}$$

Our momenta obey the Poisson bracket

$$\{A_{\mu x}, B^\nu_{x'}\} = g^\nu_\mu \delta^3(x - x') \tag{2.21}$$

By comparing 2.19 and 2.20 we can easily see that $B^\mu = F^{\mu 0}$ and since $F^{\mu\nu}$ is antisymmetric we can see that $B^0_x = 0$ and we find our first primary constraint. It should be noted that this is not a single primary constraint as x represents a point in space so in fact it is an infinite number of primary constraints. We now define the Hamiltonian in the usual way as

$$\begin{aligned} H &= \int B^\mu A_{\mu,0} d^x - L \\ &= \int \left(F^{r0}A_{r,0} + \frac{1}{4}F_{rs}F_{rs} + \frac{1}{2}F^{r0}F_{r0} \right) d^3x \\ &= \int \left(\frac{1}{4}F^{rs}F_{rs} - \frac{1}{2}F^{r0}F_{r0} + F^{r0}A_{0,r} \right) d^x \\ &= \int \left(\frac{1}{4}F^{rs}F_{rs} + \frac{1}{2}B^r B^r - A_0 B^r_{,r} \right) d^3x \end{aligned} \tag{2.22}$$

where we used partial integration on the last term. We can see that our Hamiltonian now does not contain any velocities and only involves dynamical coordinates and momenta. One might think that since our F_{rs} terms contain partial derivatives inside them we may have velocities however our partial derivative acts only over the spacial terms and therefore does not result in any velocities. Now that we have our Hamiltonian we are next required to work out the consistency conditions starting from our primary constraint. Since our primary constraint has to be satisfied at all times we require that $\left\{B^0, H\right\} = 0$, from this we can calculate that

$$B^r_{,r} = 0 \tag{2.23}$$

which we recognise as a secondary constraint. Again as part of our consistency check we must now check $\{B^r_{r0}, H\} = 0$ which happens to reduce to the first type we considered $0 = 0$ and we are now finished with our consistency check. After doing this we are now left with our set of primary and secondary constraints. We can now check as to whether they are first or second class constraints and it is easy to see that since they are all momenta variables they all have zero Poisson bracket with each other and are therefore first class constraints. We can now write the total Hamiltonian

$$H_T = \int \left(\frac{1}{4}F^{rs}F_{rs} + \frac{1}{2}B_rB_r\right) d^3x - \int A_0 B^r_{,r} d^3x + \int v_x B^0 d^3x \tag{2.24}$$

and equations of motion

$$\dot{g} \approx \{g, H_T\} \tag{2.25}$$

To summarise, we have shown that we can write electromagnetism in a Hamiltonian form where we are then concerned with calculating all the constraints, both primary and secondary, and first and second class. We can then find the equations of motion for our theory and our *Gauge* transformations are generated by our first class constraints. At this stage one would then continue on to quantise the system to get a full quantum mechanical description but that is beyond the scope of this discussion. We will simply state here that in general one attempts to replace the constraints with operators, and the Poisson brackets with commutators scaled with $i\hbar$. For more information on the quantisation see [19].

Chapter 3

Canonical Gravity

3.1 ADM Action

Our aim in this section will be to begin to develop the Hamiltonian formalism of General Relativity first proposed by Arnowitt, Deser and Misner [1] with more modern treatment found in [10, 60]. We will mainly be following the procedure as set out by Thiemann [58] and begin by discussing the vacuum case only. To do this we will have to split our spacetime manifold into spacial and temporal sections. Our hope is that we will then be able to use Hamiltons equations to generate time evolution of our system. We begin with the Einstein-Hilbert action.

$$S = \frac{1}{\kappa}\int_M d^{D+1}X\sqrt{|\det(g)|}R^{(D+1)} \tag{3.1}$$

Where from here on we will mostly be concerned with D=3. We will use the signature convention $(-,+,+,+)$ and our indices $\mu,\nu,\rho,... = 0,1,2,3$ correspond to components of spacetime tensors. As usual $\kappa = 16\pi G$ for Newton's constant G. Here we make our assumption that our manifold M has topology $M \cong \mathbb{R} \times \sigma$ where σ is a fixed three-dimensional manifold of arbitrary topology. If our spacetime is globally hyperbolic then due to a theorom by Geroch [23] it is necessarily the topology we have assumed. It is clear that M foliates into hypersurfaces $\Sigma_t := X_t(\sigma)$ where σ are embedded in M. We can interpret $t \in \mathbb{R}$ as a time coordinate however it is important to not specify any of our coordinates so as to preserve the diffeomorphism invariance of the theory. Another way to view this is to say that a theory with a preferred foliation and thus a prefered time will break diffeomorphism invariance. It is useful now to define the time flow vector

$$T^\mu(X) := \left(\frac{\partial X^\mu(t,x)}{\partial t}\right) = N(X)n^\mu(X) + N^\mu(X) \tag{3.2}$$

where we have further decomposed it into its normal and tangential components. The coefficient of proportionality N here corresponds to the *lapse* function and N^μ the $shift$ vector. This leads us to our first constraint; as we are only dealing with spacelike embeddings the T is required to be timelike. Thus we get the constraint $-N^2 = g_{\mu\nu}N^\mu N^\nu < 0$. From this it can be inferred that the lapse function is nowhere vanishing and since we want our foliation to be future directed we also require that N be positive everywhere. A useful parametrisation is to use $n^\mu = \left(\frac{1}{N}, \frac{-N^a}{N}\right)$ and $N^\mu = (0, N^a)$ where our indices $a,b,c,...$ run over space. The intrinsic metric on Σ_t is now given by

$$q_{\mu\nu} := g_{\mu\nu} - sn_\mu n_\nu \tag{3.3}$$

and the extrinsic curvature by

$$K_{\mu\nu} \coloneqq q_\mu^\rho q_\nu^\sigma \boldsymbol{\nabla}_\rho n_\sigma \tag{3.4}$$

where $s = -1$ for Lorentzian and $s = +1$ for Euclidian. Finally it is necessary to define the Riemann tensor as

$$R^{(D+1)}_{\mu\nu\rho\sigma} u^\sigma = [\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu] u_\rho \tag{3.5}$$

which can be written in an equivalent form

$$R^{(D)}_{\mu\nu\rho\sigma} = 2sK_{\rho[\mu}K_{\nu]\sigma} + q_\mu^{\mu'} q_\nu^{\nu'} q_\rho^{\rho'} q_\sigma^{\sigma'} R^{(D+1)}_{\mu'\nu'\rho'\sigma'} \tag{3.6}$$

which is often called the *Gauss* equation. We need now to derive the Riemann curvature scalar so that we can insert it into our Einstein-Hilbert action. Using the abbreviations $K \coloneqq K_{\mu\nu} q^{\mu\nu}, K^{\mu\nu} = q^{\mu\rho} q^{\nu\sigma} K_{\rho\sigma}$ we can write the curvature scalar as

$$R^{(D)} = R^{(D)}_{\mu\nu\rho\sigma} q^{\mu\rho} q^{\nu\sigma} = s\left[K^2 - k_{\mu\nu}K^{\mu\nu}\right] + q^{\mu\rho} q^{\nu\sigma} R^{(D+1)}_{\mu\nu\rho\sigma} \tag{3.7}$$

We wish to arrive at an expression relating $R^{(D)}$ to $R^{(D+1)}$ and we can see we are almost there. The next step is to use our definition for the intrinsic metric 3.3 along with our definition for the Riemann curvature tensor 3.5 to derive $R^{(D+1)}$ as

$$\begin{aligned} R^{(D+1)} &= R^{(D+1)}_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \\ &= q^{\mu\rho} q^{\nu\sigma} R^{(D+1)}_{\mu\nu\rho\sigma} + 2 s q^{\rho\mu} n^{\nu} [\boldsymbol{\nabla}_{\mu}, \boldsymbol{\nabla}_{\nu}] n_{\rho} \\ &= q^{\mu\rho} q^{\nu\sigma} R^{(D+1)}_{\mu\nu\rho\sigma} + 2 s n^{\nu} [\boldsymbol{\nabla}_{\mu}, \boldsymbol{\nabla}_{\nu}] n^{\nu} \end{aligned} \tag{3.8}$$

Where we have eliminated the term quartic in n using the antisymmetry of the Riemann tensor. Next we wish to evaluate the final term which can be written as

$$n^{\nu}[\boldsymbol{\nabla}_{\mu}, \boldsymbol{\nabla}_{\nu}] n^{\mu} = -(\boldsymbol{\nabla}_{\mu} n^{\nu})(\boldsymbol{\nabla}_{\nu} n^{\mu}) + (\boldsymbol{\nabla}_{\mu} n^{\mu})(\boldsymbol{\nabla}_{\nu} n^{\nu}) + \boldsymbol{\nabla}_{\mu} \left(n^{\nu} \boldsymbol{\nabla}_{\nu} n^{\mu} - n^{\mu} \boldsymbol{\nabla}_{\nu} n^{\nu} \right) \tag{3.9}$$

and we make the identifications

$$\begin{aligned} K &= \boldsymbol{\nabla}_{\mu} n^{\nu} \\ K_{\mu\nu} K^{\mu\nu} &= (\boldsymbol{\nabla}_{\mu} n^{\nu})(\boldsymbol{\nabla}_{\nu} n^{\mu}) \end{aligned} \tag{3.10}$$

We now can combine 3.7, 3.8 and 3.10 to arrive at the *Codacci* equation

$$R^{(D+1)} = R^{(D)} - s \left[K_{\mu\nu} K^{\mu\nu} - K^2 \right] + 2 s \boldsymbol{\nabla}_{\mu} \left(n^{\nu} \boldsymbol{\nabla}_{\nu} n^{\mu} - n^{\mu} \boldsymbol{\nabla}_{\nu} n^{\nu} \right) \tag{3.11}$$

and now finally we have our relationship between $R^{(D+1)}$ and $R^{(D)}$. We will for now drop the last term as it is a surface term, we will recover it when we discuss boundaries. Using these we can now rewrite the action 3.1 as

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\sigma} d^D x \sqrt{\det(q)}|N|(R - s[K_{ab}K^{ab} - (K^a_a)^2]) \tag{3.12}$$

which is the ADM action. We should note that here that the Latin indices represent three-dimensional spacial indices not the full spacetime coordinates we were using before. For a proof that this is allowed see [58].

3.1.1 Legendre transformation

Now that we have the ADM action our next step is to perform a Legendre transformation on our Lagrangian density to find our corresponding Hamiltonian density. We can see that time derivatives of both N and N^a do not appear in 3.12 hence they are Lagrange multipliers. The action does depend on both $\dot{q}_{ab}$ and q_{ab}. Therefore our conjugate momenta are

$$P^{ab}(t,x) := \frac{\delta S}{\delta \dot{q}_{ab}(t,x)} = -s\frac{|N|}{N\kappa}\sqrt{det(q)}[K^{ab} - q^{ab}(K^c_c)] \tag{3.13}$$

$$\Pi(t,x) := \frac{\delta S}{\delta \dot{N}(t,x)} = 0 \tag{3.14}$$

$$\Pi_a(t,x) := \frac{\delta S}{\delta \dot{N}^a(t,x)} = 0 \tag{3.15}$$

where we have also used the fact that R does not contain any time derivatives. From these we can see that 3.14 and 3.15 are primary constraints which we will denote as

$$C(t,x) \coloneqq \Pi(t,x) = 0 \qquad \text{and} \qquad C^a(t,x) \coloneqq \Pi^a(t,x) = 0 \tag{3.16}$$

Following now the procedure set out by Dirac [41] we introduce lagrange multiplier fields $\lambda(t,x), \lambda_a(t,x)$ for the primary constraints and perform a usual Legendre transformation on the remaining velocities. Using these we can recast our action 3.12 as

$$S = \int_{\mathbb{R}} \int_{\sigma} d^D x \left(\dot{q}_{ab} P^{ab} + \dot{N}\Pi + \dot{N}^a \Pi_a - [\lambda C + \lambda^a C_a + N^a H_a + |N| H] \right) \tag{3.17}$$

where

$$\begin{aligned} H_a &\coloneqq -2 q_{ac} D_b P^{bc} \\ H &\coloneqq -\left(\frac{s\kappa}{\sqrt{det(q}} \left[q_{ac} q_{bd} - \frac{1}{D-1} q_{ac} q_{bd} \right] P^{ab} P^{cd} + \sqrt{det(q)} \frac{R}{\kappa} \right) \end{aligned} \tag{3.18}$$

where D_a is a covariant derivative such that $D_\mu f \coloneqq q_\mu^\nu \boldsymbol{\nabla}_\nu \tilde{f}$. H_a is called the Diffeomorphism constraint and H the Hamiltonian constraint. We now need to define the Poisson bracket

$$\left\{ P^{ab}(t,x), q_{cd}(t,x') \right\} = \kappa \delta^a_{(c} \delta^b_{d)} \delta^{(D)}(x,x') \tag{3.19}$$

where as usual our motivation is to study the evolution of our system. Next we want to study the square bracket term in 3.17 which we will call the "Hamiltonian"

$$\kappa \boldsymbol{H} := \int_\sigma d^D x[\lambda C + \lambda^a C_a + N^a H_a + |N| H] = C(\lambda) + \vec{C}(\vec{\lambda}) + \vec{H}(\vec{N}) + H(|N|) \tag{3.20}$$

We expect that variation of this action with respect to our Lagrange multiplier fields λ and $\vec{\lambda}$ should reproduce the primary constraints found in 3.16. For the dynamics of this system to be consistent we must have the constraints preserved under evolution of the system, or in other words $\dot{C}(t,x) := \{\boldsymbol{H}, C(T,x)\} = 0$ $\forall x \in \sigma$ and similar for $\dot{\vec{C}}$. However on calculation we instead find that

$$\left\{\vec{C}(\vec{f}), \boldsymbol{H}\right\} = \vec{H}(\vec{f}) \qquad \text{and} \qquad \{C(f), \boldsymbol{H}\} = H\left(\frac{N}{|N|} f\right) \tag{3.21}$$

which should vanish for all $f, \vec{f}$. Therefore we are forced to introduce the secondary constraints

$$H(x,t) = 0 \qquad \text{and} \qquad H_a(x,t) = 0 \tag{3.22}$$

Looking at 3.20 we see that in General Relativity the "Hamiltonian" is constrained to be 0.

3.1.2 Equations of Motion

We now wish to study the equations of motion of our phase space. We have already seen that $C = \Pi, C_a = \Pi_a$ however we are yet to study N, N^a, q_{ab} and P^{ab}. For the shift and lapse it is easy to see that $\dot{N}^a = \lambda^a, \dot{N} = \lambda$. Since the equations of motion for q_{ab}, P^{ab} are not affected by the terms with $\vec{C}, C$ in the "Hamiltonian" it is straightforward to solve for the equations of motion. We simply have to treat N, N^a as Lagrange multipliers as stated above and drop all terms proportional to C, C_a from the action 3.17. This leads us to what is known as the canonical ADM action

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\sigma} d^D x \left(\dot{q}_{ab} P^{ab} - [N^a H_a + |N| H] \right) \tag{3.23}$$

from which we can obtain our new Hamiltonian

$$\boldsymbol{H} = \frac{1}{\kappa} \int_{\sigma} d^D x \, [N^a H_a + |N| H] \tag{3.24}$$

After much lengthy calculation we can see that our system evolves as

$$\left\{ H(\vec{N}), q_{\mu\nu} \right\} = \mathcal{L}_{\vec{N}} q_{\mu\nu} \qquad \left\{ H(\vec{N}), P^{\mu\nu} \right\} = \mathcal{L}_{\vec{N}} P^{\mu\nu} \tag{3.25}$$

where we have used the Lie derivative, and we can see that the vector constraint generates space diffeomorphisms on Σ. For the Hamiltonian constraint we see that

$$\{H(N), q_{\mu\nu}\} = \mathcal{L}_{n\vec{N}} q_{\mu\nu} \tag{3.26}$$

$$\{H(N), P^{\mu\nu}\} = \frac{q^{\mu\nu} NH}{2} - 2N\sqrt{\det(q)}\left[q^{\mu[\rho} q^{\nu]\sigma}\right] R_{\rho\sigma} + \mathcal{L}_{Nn} P^{\mu\nu} \tag{3.27}$$

where the first expression is simply the action of time diffeomorphism on $q_{\mu\nu}$. The second expression gives the time diffeomorphism on $P_{\mu\nu}$ only when we have two extra conditions. Namely that the Hamiltonian is 0 and we are on the constrained surface, and that $R_{\mu\nu} = 0$, which is the statement we are considering physical solutions of the vacuum Einstein equations.

3.1.3 Boundary Conditions

We have so far ignored all boundary terms but would now like to do a preliminary analysis. Indeed this is an issue that goes back to our introduction of the Einstein-Hilbert action 3.1 as strictly speaking we should include the Gibbons-Hawking boundary term [24, 61] such that our action becomes

$$S = S_{EH} + \frac{1}{8\pi G}\int_{\partial M} d^3x \sqrt{-det(q)} K \tag{3.28}$$

This is required such that our action is zero upon variation at a boundary. However we are not done yet as 3.12 contains another boundary term which we will now derive. First we need to consider two separate 3-dimensional submanifolds of our original spacetime manifold. The first of these is the boundary submanifold which we will assume to be timelike with unit normal r^a, and induced metric $h^{ab} = g^{ab} - r^a r^b$. The second is a spacial slice Σ with unit normal n^a and induced metric $q^{ab} = g^{ab} + n^a n^b$ as found before. At the boundary $\partial\Sigma$ these two submanifolds intersect giving a 2-dimensional surface on which we will use the metric $\sigma^{ab} = q^{ab} - r^a r^b = h^{ab} + n^a n^b$. For simplicity we will make the assumption that $n^a r_n = 0$ so that r^a is tangent to Σ and provied the normal to $\partial\Sigma$ in Σ, similar for n^a and M. Our definition for the Ricci tensor 3.11 is

$$R = {}^{(3)}R + K^{ab}K_{ab} - K^2 - 2\boldsymbol{\nabla}_a \left(n^b \boldsymbol{\nabla}_b n^a - n^a \boldsymbol{\nabla}_b n^b \right) \tag{3.29}$$

where in the original derivation of our ADM action we ignored the total divergence term. We now wish to examine this term as we will now receive a boundary term precisely from this in addition to the Gibbons-Hawking term we know about. If we use the normal r^a to the boundary ∂M and our induced metric h^{ab} we can see that

$$\begin{aligned} -2\int_M d^4x\sqrt{-\det(g)}\boldsymbol{\nabla}_a v^a &= -2\int_{\partial M} d^3y\sqrt{-\det(h)}r_a \left(n^b \boldsymbol{\nabla}_b n^a - n^a \boldsymbol{\nabla}_b n^b \right) \\ &= -2\int_{\partial M} d^3y\sqrt{-\det(h)}r_a n^b \boldsymbol{\nabla}_b n^a \end{aligned} \tag{3.30}$$

where we used our assumption $r_a n^a = 0$ to remove the second term in the bracket. Now considering the intersection with our submanifold Σ we get the relation $N\sqrt{\det(\sigma)} = \sqrt{-\det(h)}$. We also will split $d^3t = d^2zdt$ so that we can consider purely the spacial boundary condition given by

$$-2\int_{\partial\Sigma} d^2zN\sqrt{\det(\sigma)}r_a n^b \boldsymbol{\nabla}_b n^a = 2\int_{\partial\Sigma} d^2zN\sqrt{\det(\sigma)}n^a n^b \boldsymbol{\nabla}_b r_a \tag{3.31}$$

where we simplified using our assumption $n^a r_a = 0$. Combining this term with the Gibbons-Hawking term gives

$$
\begin{aligned}
2\int_{\partial\Sigma} d^2 z N\sqrt{\det(\sigma)}\left(K + n^a n^b \boldsymbol{\nabla}_b r_a\right) &= 2\int_{\partial\Sigma} d^2 z N\sqrt{\det(\sigma)}\left(h^{ab} + n^a n^b\right)\boldsymbol{\nabla}_b r_a \\
&= 2\int_{\partial\Sigma} d^2 z N\sqrt{\det(\sigma)}\sigma^{ab}\boldsymbol{\nabla}_b r_a \\
&= 2\int_{\partial\Sigma} d^2 z N\sqrt{\det(\sigma)}k
\end{aligned}
\tag{3.32}
$$

There is one final term we have to consider coming from our diffeomorphism constraint 3.18, as strictly speaking there was a surface term we should have included but dropped earlier. This term gives

$$
2\int d^2 z\sqrt{\det(\sigma)}r_a\left(\frac{P^{ab}N_b}{\sqrt{\det(q)}}\right) \tag{3.33}
$$

We can finally write down our full Hamiltonian including boundary terms as

$$
H_{tot} = H + H_{\partial\Sigma} \tag{3.34}
$$

Where here H refers to the Hamiltonian given in 3.24 and

$$
H_{\partial\Sigma} = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2 z\sqrt{\det(\sigma)}\left(Nk + \frac{r_a P^{ab}N_b}{N}\right) \tag{3.35}
$$

first found by Brown and York [13]. Strictly speaking, while the timelike boundary of our spacetime does not appear here only the spacial slice, due to our assumption this boundary condition only applies when we have foliations of spacetime where $\partial\Sigma_t$ forms a timelike boundary orthogonal to Σ_t. For a more general treatment one should read [12, 27, 28]. We wish to give some kind of interpretation to this boundary term and it turns out it can be interpreted as quasilocal quantities of energy and momentum. To avoid our Minkowski space from having non zero energy we must normalise, which we do by subtracting our boundary terms from its value obtained in a known reference space such that the Minkowski space will have zero energy. The normalised boundary term is thus

$$H_{\partial\Sigma}^{norm} = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2z\sqrt{\det(\sigma)}\left(N(k-k_0) + N_b\frac{(r_a P^{ab} - \bar{r}_a\bar{P}^{ab})}{N}\right) \quad (3.36)$$

where in particular we interpret the energy as

$$E = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2z\sqrt{\det(\sigma)}N(k-k_0) \quad (3.37)$$

and the momentum as

$$J = \int_{\partial\Sigma} d^2z\sqrt{\det}N_b\frac{(r_a P^{ab} - \bar{r}_a\bar{P}^{ab})}{N} \quad (3.38)$$

and when considering spaces away from Minkowski space, non zero values are related to energies.

This concludes our section on the ADM action, it is worth taking a moment to consider what we have done. We started with the Einstein-Hilbert action 3.1, transformed it into the ADM action 3.12, performed a Legendre transformation to derive our Hamiltonian 3.20 and then studied the constraints and equations of motion. Finally we discussed the boundary terms and showed how they lead to new terms in our Hamiltonian.

3.2 Ashtekar Variables

Attempts at quantising the Hamiltonian constraint have been made before with little success, most notably by Wheeler and DeWitt [15, 16, 17] who arrived at the "Wheeler-DeWitt equation" $\hat{H}\,|\psi\rangle = 0$ for the Hamiltonian constraint. The issue is clear when one considers that 3.18 does not even depend polynomially on the field variables, which when quantised are therefore divergent. This is where the field was stuck until Ashtekar introduced a new set of variables [3, 4]. These new variables cast the theory into one more like a gauge theory, and usefully the Hamiltonian constraint when written in these new variables is polynomial. The idea will be to extend our phase space from the ADM phase space, we will consider the ADM phase space to be a reduction of our new phase space, which we can arrive back at with the introduction of constraints on our new phase space. Let us now introduce a new set of vector fields e^i_a the so-called *triads* where $i, j, k = 1, 2, 3$. We wish for these fields to be orthogonal which we do by requiring

$$q_{ab} \coloneqq \delta_{jk} e^j_a e^k_b \tag{3.39}$$

We also wish to introduce another field K^i_a from which we an derive the extrinsic curvature as

$$-sK_{ab} \coloneqq K^i_{(a}e^i_{b)} \tag{3.40}$$

From this we can see that our new field must satisfy the constraint

$$G_{ab} \coloneqq K^i_{[a}e^j_{b]} = 0 \tag{3.41}$$

which comes from the fact that K_{ab} was symmetric. Finally we wish to introduce

$$E^a_i \coloneqq \sqrt{\det(q)}e^a_i \tag{3.42}$$

which we can use to rewrite 3.41 in equivalent form

$$G_{ij} = K_{a[i}E^a_{j]} = 0 \tag{3.43}$$

We now wish to consider the following functions on our extended phase space

$$q_{ab} = E^i_a E^i_b \left|\det(E^c_j)\right|^{\frac{2}{(D-1)}}, \quad P^{ab} = 2|\det(E^c_i)|^{\frac{-2}{(D-1)}} E^a_k E^d_k K^J_{[d}\delta^b_{c]}E^c_j \tag{3.44}$$

With the constraint $G_{ij} = 0$ we can see that 3.44 reduces to the ADM coordinates we found above. We can also check that our Poisson bracket in our extended phase space is equal to the Poisson bracket of the ADM space at least under our constraint.

$$
\begin{aligned}
\left\{P^{ab}(x), q_{cd}(x')\right\} &= \left(\left[q^{a(e}q^{b)f} - q^{ab}q^{ef}\right]E^j_f\right)(x)\left\{K^j_e(x), \left(|\det(E)|^{\frac{2}{D-1}}E^k_cE^k_d\right)(x')\right\} \\
&= \left(\left[q^{a(e}q^{b)f} - q^{ab}q^{ef}\right]E^j_f\right)(x)\left[\frac{2}{D-1}q_{cd}(x')\frac{\left\{K^j_e(x), |\det(E)|(x')\right\}}{|\det(E)|(x)}\right. \\
&\left. + 2\det(q)E^k_{(c}(x)\left\{K^j_e(x), E^k_{d)}(x')\right\}\right] \\
&= \kappa\left(\left[q^{a(e}q^{b)f} - q^{ab}q^{ef}\right]\left[\frac{-1}{D-1}q_{cd}q_{ef} + q_{e(c}q_{d)f}\right]\right)(x)\delta(x,x') \\
&= \kappa\delta^a_{(c}\delta^b_{d)}\delta(x,x') \qquad (3.45)
\end{aligned}
$$

and we can see that we arrive at the same answer we got in 3.19 as long as $G_{ab} = 0$. We have used the identities $\delta E^{-1} = -E^{-1}\delta E E^{-1}$, $\frac{[\delta|\det E|]}{|\det E|} = \frac{\delta det(E)}{\det(E)}$. Now that we have seen that our extended phase space and our ADM phase space are equivalent as far as the physics is concerned we can choose to work with the extended one. Therefore our action will now become

$$
S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\sigma} d^Dx\left(2\dot{K}^j_aE^a_j - \left[-\Lambda^{jk}G_{jk} + N^aH_a + NH\right]\right) \qquad (3.46)
$$

which reduces to the ADM action again with our constraint on G_{jk} applied. Here Λ is an arbitrary antisymmetric matrix such that $\Lambda^T = -\Lambda$. We next are going to introduce what is called the *spin connection* and attempt to redefine the constraint G_{jk} such that it takes the form of a *Gauss* constraint for a SO(D) Gauge theory. We need to extend our condition of metric compatibility which is to say that $D_aq_{bc} = 0$. This leads us to

$$
D_ae^j_b = 0 \Rightarrow \Gamma_{ajk} = -e^b_k\left[\partial_ae^j_b - \Gamma^c_{ab}e^j_c\right] \qquad (3.47)
$$

From this we can see that $D_a \delta_{jk} = D_a e^b_j e^k_b = 0$ and thus $D_a v^j = \partial_a v^j + \Gamma_{ajk} v^k$. Returning to the constraint we would like to write $G_{jk} = (\partial_a E^a + [A_a, E^a])_{jk}$. We said before that we would be working with D = 3 and now we can see the reason why, we see that our object E^a_j transforms in the defining representation of SO(D) and Γ^a_{jk} transforms in the adjoint representation. For D=3 these are equivalent hence why we restrict ourselves. We now will look at some of the possible transformations that we can do, namely (1) constant Weyl or rescaling transformations and (2) affine transformations. We begin here with the Weyl transformation where it is easy to see that under the transformation $(K^j_a, E^a_j) \mapsto ({}^{(\beta)}K^j_a := \beta K^j_a, {}^{(\beta)}E^a_j := \frac{E^a_j}{\beta})$ the Poisson bracket 3.45 is invariant. Here β is the *Immirzi parameter* which will be discussed in section 3.2.1. We can note that in particular a rotational constraint

$$G_j = \epsilon_{jkl} K^k_a E^a_l = \epsilon_{jkl} \left({}^{(\beta)}K^k_a\right) \left({}^{(\beta)}E^a_l\right) \tag{3.48}$$

is invariant under this rescaling. We now consider an affine transformation. It is clear from 3.47 that $D_a E^b_j = 0$, a special case of this we want to consider is

$$D_a E^a_j = D_a E^a_j + \Gamma^k_{aj} E^a_k =_a E^a_j + \epsilon_{jkl} \Gamma^k_a E^a_l = 0 \tag{3.49}$$

where we are able to change between a partial and covariant derivative as the covariant derivative is only acting on our tensorial indices. Next we want to solve explicitly for our spin connection in terms of E^a_j for which we find

$$
\begin{aligned}
\Gamma_a^i &= \frac{1}{2}\epsilon^{ijk}e_k^b\left[e_{a,b}^j - e_{b,a}^j + e_j^c e_a^l e_{c,b}^l\right] \\
&= \frac{1}{2}\epsilon^{ijk}E_k^b\left[E_{a,b}^j - E_{b,a}^j + E_j^c E_a^l E_{c,b}^l\right] \\
&+ \frac{1}{4}\epsilon^{ijk}E_k^b\left[2E_a^j\frac{(\det(E)),b}{\det(E)} - E_b^j\frac{(\det(E)),a}{\det(E)}\right]
\end{aligned}
\tag{3.50}
$$

From this we can see that our spin connection is invariant under our rescaling transformation, or

$$
\left({}^{(\beta}\Gamma_a^j\right) = \Gamma_a^j\left({}^{(\beta)}E\right) = \Gamma_a^j = \Gamma_a^j\,(E) \tag{3.51}
$$

The same is true for the Christoffel symbol Γ_{bc}^a. From these we can also see that our derivative D_a is independent of β and from 3.47 we can therefore conclude that $D_a({}^{(\beta)}E_j^a) = 0$. We can use this result in 3.48 to rewrite our constraint as

$$
\begin{aligned}
G_j &= \epsilon_{jkl}\left({}^{(\beta)}K_a^k\right)\left({}^{(\beta)}E_l^a\right) = \partial_a\left({}^{(\beta)}E_j^a\right) + \epsilon_{jkl}\left[\Gamma_a^k + \left({}^{(\beta)}K_a^k\right)\right]\left({}^{(\beta)}E_l^a\right) \\
&= {}^{(\beta)}\mathcal{D}_a{}^{(\beta)}E_j^a
\end{aligned}
\tag{3.52}
$$

Looking at the term in the square brackets this suggests that we define a new connection

$$
\left({}^{(\beta)}A_a^j\right) := \Gamma_a^k + \left({}^{(\beta)}K_a^k\right) \tag{3.53}
$$

which we will call the Sen-Ashtekar-Immirzi-Barbero connection (SAIB) and from here will replace our spin connection. The Sen connection is for $\beta = \pm i, G_j = 0$, the Ashtekar connection for $\beta = \pm i$, the Immirizi connetion when β is complex and the Barbero connetion for real β. It is worth taking a second to discuss why we bother introducing more complexity into our system as one might naively think that this will make any results long and complicated. However upon calculation we infact find that

$$\left\{ {}^{(\beta)}A_a^j(x), {}^{(\beta)}A_b^k(y) \right\} = \left\{ {}^{(\beta)}E_j^a(x), {}^{(\beta)}E_k^b(y) \right\} = 0, \left\{ {}^{(\beta)}E_j^a(x), {}^{(\beta)}A_b^k(y) \right\} = \frac{\kappa}{2}\delta_b^a\delta_j^k\delta(x,y) \tag{3.54}$$

It is this simplicity that motivates us following this procedure and is what makes introducing these new variables useful when trying to find the Hilbert space representation that will allow us to transform our Poisson brackets into commutators and thus obtain a quantum theory.

So far in our discussion we have not talked about the constraints of our original ADM system found in 3.18. Our first task will be to transform to our new variables, first by going to the extended phase space

$$\begin{aligned} H_a &= 2sD_b\left[K_a^j E_j^b - \delta_a^b K_c^j E_j^c\right] \\ H &= -\frac{s}{\sqrt{\det(q)}}\left(K_a^l K_b^j - K_a^j K_b^l\right) E_j^a E_l^b - \sqrt{\det(q)}R \end{aligned} \tag{3.55}$$

and then by introducing new curvature variables as

$$
\begin{aligned}
R_{ab}^{j} &\coloneqq 2\partial_{[a}\Gamma_{b]}^{j} + \epsilon_{jkl}\Gamma_{a}^{k}\Gamma_{b}^{l} \\
{}^{(\beta)}F_{ab}^{j} &\coloneqq 2\partial_{[a}{}^{(\beta)}A_{b]}^{j} + \epsilon_{jkl}{}^{(\beta)}A_{a}^{k}{}^{(\beta)}A_{b}^{l}
\end{aligned}
\tag{3.56}
$$

which we can relate to the covariant derivative as $[D_a, D_b]v_j = R_{abjl}v^l = \epsilon_{jkl}R_{ab}^{k}v^l$ and $\left[{}^{(\beta)}D_a, {}^{(\beta)}D_b\right]v_j = {}^{(\beta)}F_{abjl}v^l = \epsilon_{jkl}{}^{(\beta)}F_{ab}^{k}v^l$. We now wish to expand ${}^{(\beta)}F$ in terms of Γ and ${}^{(\beta)}K$ and contract with ${}^{(\beta)}E$ which results in

$$
{}^{(\beta)}F_{ab}^{j}{}^{(\beta)}E_{j}^{b} = \frac{R_{ab}^{j}E_{j}^{b}}{\beta} + 2D_{[a}\left(K_{b]}^{j}E_{j}^{b}\right) + \beta K_{a}^{j}G_{j}
\tag{3.57}
$$

where we have used our *Gauss* constraint 3.48. We next will use a version of the *Bianchi identity* to show that the first term vanishes.

$$
\begin{aligned}
& dx^a \wedge dx^b D_a e_b^j = de^j + \Gamma_k^j \wedge e^k = 0 \\
& \Rightarrow 0 = -d^2 e^j = d\Gamma_k^j \wedge e^k - \Gamma_l^j \wedge de^l = \left[d\Gamma_k^j + \Gamma_l^j \wedge \Gamma_k^l\right] \wedge e^k = \Omega_k^j \wedge e^k \\
& \Rightarrow \Omega = d\Gamma + \Gamma \wedge \Gamma = d\Gamma^i T_i + \frac{1}{2}[T_j, T_k]\Gamma^j \wedge \Gamma^k = \frac{1}{2}dx^a \wedge dx^b R_{ab}^i T_i
\end{aligned}
\tag{3.58}
$$

Which we can write in equivelant form as

$$
\epsilon_{ijk}\epsilon^{efc}R_{ef}^{j}e_c^k = 0 \Rightarrow \frac{1}{2}\epsilon_{ijk}\epsilon^{efc}R_{ef}^{j}e_c^k e_a^i = \frac{1}{2}E_j^b\epsilon_{cab}\epsilon^{efc}R_{ae}^{j} = R_{ab}^{j}E_j^b = 0
\tag{3.59}
$$

as required. Finally by using 3.55 we can write 3.57 as

$$ {}^{(\beta)}F^j_{ab}{}^{(\beta)}E^b_j = -sH_a +^{(\beta)} K^j_a G_j \tag{3.60}$$

It is important here to note that 3.60 still contains our *Gauss* constraint since our transformations were canonical. We now wish to return to 3.56 and again expanding ${}^{(\beta)}F$ in terms of Γ and ${}^{(\beta)}K$ but this time not contracting with ${}^{(\beta)}E$ to get

$$ {}^{(\beta)}F^j_{ab} = R^j_{ab} + 2\beta D_{[a}K^j_{b]} + \beta^2 \epsilon_{jkl} K^k_a K^l_b \tag{3.61}$$

and now contracting instead with $\epsilon^{(\beta)}_{jkl} E^a_k {}^{(\beta)}E^b_l$ we find

$$\begin{aligned} {}^{(\beta)}F^j_{ab}\epsilon_{jkl}{}^{(\beta)}E^a_K{}^{(\beta)}E^b_l =& -det(q)\frac{R_{abkl}e^a_k e^b_l}{\beta^2} - 2\frac{E^a_j D_a G_j}{\beta} \\ &+ \left(K^j_a E^a_j\right)^2 - \left(K^j_b E^a_j\right)\left(K^k_a E^b_k\right) \end{aligned} \tag{3.62}$$

by expanding $v_j = e^a_j v_a, v_a = e^j_a v_j$, using the fact that $D_a e^j_b = 0$ and comparing to our identities above for $[D_a, D_b]v_j$ we see that we can write $R_{abcd}e^c_i e^d_j = R_{abij}$ which we can insert into 3.62 to find

$$\begin{aligned} {}^{(\beta)}F^j_{ab}\epsilon_{jkl}{}^{(\beta)}E^a_K{}^{(\beta)}E^b_l =& -det(q)\frac{R}{\beta^2} - 2^{(\beta)}E^a_j D_a G_j \\ &+ \left(K^j_a E^a_j\right)^2 - \left(K^j_b E^a_j\right)\left(K^k_a E^b_k\right) \end{aligned} \tag{3.63}$$

Finally we again use 3.55 to write 3.63 as

$$
\begin{aligned}
&{}^{(\beta)}F^j_{ab}\epsilon_{jkl}{}^{(\beta)}E^a_k{}^{(\beta)}E^b_l + 2^{(\beta)}E^a_j D_a G_j \\
&= \sqrt{\det(q)}\left[-\sqrt{\det(q)}\frac{R}{\beta^2} - \frac{\left(K^j_b E^a_j\right)\left(K^k_a E^b_k\right) - \left(K^j_a E^a_j\right)^2}{\sqrt{\det(q)}}\right] \\
&= \frac{\sqrt{\det(q)}}{\beta^2}\left[-\sqrt{\det(q)}R - \beta^2\frac{\left(K^j_b E^a_j\right)\left(K^k_a E^b_k\right) - \left(K^j_a E^a_j\right)^2}{\sqrt{\det(q)}}\right] \\
&= \frac{\sqrt{\det(q)}}{\beta^2}\left[H + \left(s-\beta^2\right)\frac{\left(K^j_b E^a_j\right)\left(K^k_a E^b_k\right) - \left(K^j_a E^a_j\right)^2}{\sqrt{\det(q)}}\right] \\
&= s\sqrt{\det(q)}\left[-\frac{s}{\sqrt{\det(q)}}\left[\left(K^j_b E^a_j\right)\left(K^k_a E^b_k\right) - \left(K^j_a E^a_j\right)^2\right] - \frac{s}{\beta^2}\sqrt{\det(q)}R\right] \\
&= s\sqrt{\det(q)}\left[H + \left(1 - \frac{s}{\beta^2}\right)\sqrt{\det(q)}R\right]
\end{aligned}
\tag{3.64}
$$

From 3.64 we can see that the left hand side is proportional to H when $\beta = \pm\sqrt{s}$. Again we also note that 3.64 contains our *Gauss* constraint as we saw before with 3.60. For convenience we will solve 3.64 for H as

$$
\begin{aligned}
H =& \frac{\beta^2}{\sqrt{\left|det({}^{(\beta)}E\beta)\right|}}\left[{}^{(\beta)}F^j_{ab}\epsilon_{jkl}{}^{(\beta)}E^a_k{}^{(\beta)}E^b_l + 2^{(\beta)}E^a_j D_a G_j\right] \\
&+ \left(\beta^2 - s\right)\frac{\left({}^{(\beta)}K^j_b{}^{(\beta)}E^a_j\right)\left({}^{(\beta)}K^j_a{}^{(\beta)}E^b_j\right) - \left({}^{(\beta)}K^k_c{}^{(\beta)}E^c_k\right)^2}{\sqrt{\left|det({}^{(\beta)}E\beta)\right|}}
\end{aligned}
\tag{3.65}
$$

Since the transformations we have used are canonical the Poisson brackets among the constraints G_j, H_a, H are unchanged. We can therefore write $H_a = H'_a + f^j_a G_j, H = H' + f^j G_j$ where we have separated out the parts of H, H_a

proportional to G_j and know that it is completely equivalent to work with set of constraint H', H'_a, G_j as it is with H, H_a, G_j. We will therefore use these new constraints and dropping the primes we can write these as

$$
\begin{aligned}
G_j &=^{(\beta)} D_a{}^{(\beta)}E^a_j = \partial_a{}^{(\beta)}E^a_j + \epsilon_{jkl}{}^{(\beta)}A^{j}_{a}{}^{(\beta)}E^a_j \\
H_a &= -s^{(\beta)}F^j_{ab}{}^{(\beta)}E^b_j \\
H &= \left[\beta^{2(\beta)}F^j_{ab} - \left(\beta^2 - s\right)\epsilon_{jmn}{}^{(\beta)}K^{m}_{a}{}^{(\beta)}K^n_b\right]\frac{\epsilon_{jkl}E^a_kE^b_l}{\sqrt{det(q)}}
\end{aligned}
\tag{3.66}
$$

Finally we can now write the Einstein-Hilbert action 3.1 in the equivalent form

$$
S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_\sigma d^3x \left(2^{(\beta)}\dot{A}^{i}_{a}{}^{(\beta)}E^a_i - \left[\Lambda^j G_j + N_a H_a + NH\right]\right) \tag{3.67}
$$

where now our constraints are the ones found in 3.66. We can here see the motivation to introduce these variables in action as for $s = -1, \beta = \pm i$ the second term in 3.66 is dropped leading to a polynomial constraint. This makes the quantisation of the Hamiltonian constraint considerably easier and has led to much further study. We should note that we have not been working with any of the boundary conditions we found in 3.1.3, for a discussion on this see [52].

3.2.1 The Immirzi parameter

The Immirzi parameter was first introduced by Barbero [8] in 1995 and later in more depth by Immirzi [29] in 1997. When Ashtekar introduced his variables he used $s = -1, \beta = i$ this has the advantage of reducing 3.65 to a simpler form by eliminating the term with $\left(\beta^2 - s\right)$ this however has the problem of making our variable complex which leads to a lot of technicalities especially when dealing with our Poisson brackets. An alternate choice introduced by Barbero is to use $\beta = 1$ where the choice of the value 1 was ultimately arbitrary and we could have used any real positive. For $s = +1$ the Euclidean case this has the same advantage with respect to our constraints as Ashtekars choice however without the ability to "Wick rotate" our theory this does not help us much [54, 55] and when we take $s = -1$ we end up with a complicated Hamiltonian constraint that we have to deal with. The Immirzi parameter is important physically and could be viewed as a renormalisation of Newton's constant. It also appears in calculations of black hole entropy [2].

3.3 A Brief Story of Time

In this section we want to mention very briefly some of the ideas around the issue of time. We will not perform any detailed analysis as that is not the aim of this work but instead wish to leave the reader with at least a basic understanding of some of the complications that arise and must be dealt with. For a more comprehensive study of these issues one should read [30, 34]. One can already see how we must put some careful thought into these issues by considering that the Hamiltonian we found in 3.20 was constrained to vanish. This at first glance would suggest that in this framework time does not exist as usually we generate time evolution of our system via the Poisson bracket between our Hamiltonian and our observables. We also should be motivated to consider these ideas due to the difference in how time is treated in Quantum Mechanics and General Relativity.

Most approaches to dealing with time attempt to treat time as an internal structure inherent in the system rather than an external parameter. The main differences lie more in whether to identify time before or after quantisation as well as well as the final interpretation we use. We will here briefly discuss the three broad categories that we can organise our approaches into as set out in [34].

The first approach we will discuss is where we identify time as a functional of the canonical variables. In this scheme we solve for the constraints before quantising our system and hope to reproduce something similar to the Schrodinger equation. This is the most conservative approach of the three as we are assuming that time is already part of the background structure. There are many different approaches here but we will just discuss one. The basic steps are as follows

1. Impose a suitable gauge condition. This should result in the shift and lapse function dropping out.
2. Construct a new action which should reproduce the dynamics of our remaining physical variables.
3. Impose the canonical commutation relations and proceed as normal for a quantum system by forming a Schrödinger equation

This however has some downsides, namely that by choosing a gauge in step one we are breaking the invariance in our theory and for it to be a true quantum system we expect that our results should be independent of our choice of gauge. In this approach one also has to remove parts of the metric tensor which makes it hard to explore the geometry of these theories. There is also the problem that it is hard to solve for the Hamiltonian constraints however the hope is that by using the Ashtekar variables we may be able to solve these issues.

The second approach essentially is the inverse of the first. We only apply our constraints on the quantum system, by applying restrictions on our allowed state vectors and our time identification appears after quantisation. The main operator constraint that is considered in this approach is the so called Wheeler-DeWitt equation. Solving the Wheeler-DeWitt equation has been a difficult task and is at the heart of the canonical theory of quantum gravity. Again this approach can be further subdivided, for example one approach attempts to consider the Wheeler-DeWitt equation to be analogous to the Klein-Gordon equation. The main problems encountered are that our constraints are often highly non-linear which makes them difficult to work with. We also have to spend a lot of thought on the ideas of what is an observable here.

The final approach covers a wide variety of ideas, the similarity between them all is the attempt to not introduce a concept of time at all in the quantum theory. Many begin by following the same techniques as the second approach but they differ in the final step by not making a time identification at the end. It is common in these approaches to introduce some kind of internal clock to the system to measure time, however these are considered to be purely phenomenological and therefore of little physical significance. In particular one might want to read about the consistent histories approach which we will not disccuss here but can be found at [26, 38, 39, 40].

We will not discuss more of the issues in this area here but hope to have at least shown some of the technicalities that must be considered.

3.4 Matter

So far in our formalism we have been working purely for a vacuum case. We now would like to attempt to add some matter in to the picture to see what modifications we need to make. If we consider the two constraints we found in 3.18 it is natural to want to define new constraints

$$
\begin{aligned}
C &= H + H_{matter} \\
C_a &= H_a + H_a^{matter}
\end{aligned} \tag{3.68}
$$

and it is now convenient to smear the constraints as

$$
\begin{aligned}
H[N] &= \int d^3x N(x) C(x) \\
\vec{D}[\vec{N}] &= \int d^3x N^a(x) C_a(x)
\end{aligned} \tag{3.69}
$$

In the case of two Hamiltonian constraints we then have

$$
\begin{aligned}
\{C[N], C[M]\} &= \{H[N] + H_{matter}[N], H[M] + H_{matter}[M]\} \\
&= \{H[N], H[M]\} + \{H[N], H_{matter}[M]\} + \{H_{matter}[N], H[M]\} \\
&+ \{H_{matter}[N], H_{matter}[M]\}
\end{aligned} \tag{3.70}
$$

Since the non matter part of the Hamiltonian constraint does not contain any

terms with spatial derivatives of momenta, if we choose a matter contribution that only couples with the spacial metric - as in a minimal coupled scalar field - the combination $\{H[N], H_{matter}[M]\} + \{H_{matter}[N], H[M]\}$
is equivalent to $NM - MN = 0$ and thus we have

$$\{C[N], C[M]\} = \{HN, H[M]\} + \{H_{matter}[N], H_{matter}[M]\} \tag{3.71}$$

3.4.1 Torsion

Let us now consider fermions following the procedure in [10]. We introduce the bi-spinor $\Psi = (\psi, \eta)^T$, where ψ and η are 2-spinors transforming in $SL(2,\mathbb{C})$. As shown in [37] minimal coupling of fermions to gravity is given by the action

$$S^{\beta}_{Dirac} = \frac{1}{2}\int_M d^4x|e|\left(\bar{\Psi}\gamma^I e^a_I(1 - \frac{i}{\beta}\gamma^5)D_a\Psi - \bar{D}_a\bar{\Psi}(1 - \frac{i}{\beta}\gamma^5)\gamma^I e^a_i\Psi\right) \tag{3.72}$$

where D_a is the covariant derivative on a spinor field given by

$$D_a\Psi = \partial_a\Psi + \frac{1}{2}w^{IJ}_a\sigma_{IJ}\Psi = \partial_a\Psi + \frac{1}{4}w^{IJ}_a\gamma_{[I}\gamma_{J]}\Psi \tag{3.73}$$

with Lorentz generator σ_{IJ} and Lorentz connection w^{IJ}_a. γ^I are the Dirac matrices which satisfy the Clifford algebra $\gamma_I\gamma_J + \gamma_J\gamma_I = 2\eta_{IJ}\mathbb{I}$ and $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$.

Considering the terms which contain time derivatives of the fermions we have

$$-\frac{i}{2}\int_\sigma d^3x\sqrt{\det q}\left((1+\frac{i}{\beta})\psi^\dagger\dot{\psi}-(1-\frac{i}{\beta})\dot{\psi}^\dagger\psi\right)$$

$$=\int_\sigma d^3x\left(p_\psi\dot{\psi}-2\pi iG(1-\frac{i}{\beta})\beta\psi^\dagger\psi e^i_c\dot{P}^c_i\right)-\int_\sigma d^3x\frac{1-i/\beta}{2}\mathcal{L}_t(p_\psi\psi) \quad (3.74)$$

where $p_\psi = -i\sqrt{\det(q)}\psi^\dagger$. By performing a similar analysis to the vacuum case we can show that the Ashtekar-Barbero connection 3.53 receives a torsion contribution from the spin connection and extrinsic curvature. The new connection can be written as

$$A^i_a = \tilde{\Gamma}^i_a + \beta K^i_a - 2\pi G e^i_a J^0 \quad (3.75)$$

where $J^0 = \psi^\dagger\psi - \eta^\dagger\eta$. For more information about fermionic fields see [11, 32]

3.4.2 Standard Model Fields

We now wish to add in the matter contents in the standard model to see how our constraints are changed as shown in [45, 50]. We will consider a new total action

$$S_{tot} = S_{EH} + S_{cosmo} + S_{YM} + S_{Higgs} + S_{Dirac} \quad (3.76)$$

where

$$
\begin{aligned}
S_{EH} &= \frac{1}{\kappa}\int_M d^4X\sqrt{|\det(g)|}R \\
S_{cosmo} &= \frac{\Lambda}{\kappa}\int_M d^4X\sqrt{|\det(g)|} \\
S_{YM} &= -\frac{1}{4Q^2}\int_M d^4X\sqrt{|\det(g)|}g^{\mu\nu}g^{\rho\sigma}\underline{F}^I_{\mu\rho}\underline{F}^J_{\nu\sigma}\delta_{IJ} \\
S_{Higgs} &= \frac{1}{2\lambda}\int_M d^4X\sqrt{|\det(g)|}\left(g^{\mu\nu}[\boldsymbol{\nabla}_\mu\phi_I][\boldsymbol{\nabla}_\nu\phi_J] + V(\phi)\right) \\
S_{Dirac} &= \frac{i}{2}\int_M d^4X\sqrt{|\det(g)|}\left(\left[\bar{\Psi}_r\gamma^\alpha\epsilon^\mu_\alpha\boldsymbol{\nabla}_\mu\Psi_s - \bar{\boldsymbol{\nabla}}_\mu\bar{\Psi}_r\gamma^\alpha\epsilon^\mu_\alpha\boldsymbol{\nabla}_\mu\Psi_s\right]\delta^{rs} - iJ(\bar{\Psi},\Psi)\right)
\end{aligned}
\tag{3.77}
$$

S_{EH} is the same Einstein-Hilbert action as in 3.1 and S_{cosmo} is a cosmological term for which it is clear that the canonical form is given by $\frac{\Lambda}{\kappa}\int_{\mathbb{R}} dt\sigma d^3xN\sqrt{\det(q)}$. S_{YM} is the Yang-Mills action for a compact gauge group G - $G = SU(3)\times SU(2)\times U(1)$ for the standard model - with $\underline{F}$ being the curvature associated to some connection $\underline{A}$ and Q being a coupling constant. S_{Higgs} is the scalar Higgs term for a potential V and coupling constant λ. S_{Dirac} is the fermionic contribution for Dirac spinor Ψ, with γ^α Dirac matrices in Minkowski space and tetrads e^μ_α. We define the conjugate spinor as usual $\bar{\Psi} = (\Psi^*)^T\gamma^0$. We note here that to discuss leptons we should insert additional chiral projectors $\frac{1}{2}(\mathbb{I}\pm\gamma_5$ where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. so that we can differentiate between the left-handed and right-handed contributions. We could also included here extra contributions from the supersymmetric extension of the standard model which one can read about here [32, 33].

The effect of each of these terms is to add to the Hamiltonian constraint. We will not here do the explicit derivations which can be found at [58], the procedure is essentially the same as what we have done before, namely one needs to rewrite the actions above using the Ashtekar variables and perform the Legendre transformations. After doing this procedure we see that we have the following Hamiltonian constraints

$$
\begin{aligned}
H_{Einstein} &= \frac{1}{\kappa\sqrt{\det(q)}} tr\left(2([K_a, K_b] - F_{ab})\left[E^a, E^b\right]\right) + \lambda\sqrt{\det(q)} \\
H_{Dirac} &= E_i^a \frac{1}{2\sqrt{\det(q)}}\left[i\pi^T \tau_i D_a \xi + D_a(\pi_i^T \xi) + \frac{1}{2} i K_a^j \pi^T \xi + cc\right] \\
H_{YM} &= \frac{q_{ab}}{2Q^2\sqrt{\det(q)}}\left[\underline{E}_I^a \underline{E} b_I + \underline{B}_I^a \underline{B}_I^b\right] \\
H_{Higgs} &= \frac{1}{2}\left(\frac{p^I p^I}{\kappa\sqrt{\det(q)}} + \sqrt{\det(q)}\left[q^{ab}(D_a\phi_I)(D_b\phi_I)/\kappa + P\frac{\phi_I\phi_I}{\bar{h}\kappa^2}\right]\right)
\end{aligned}
\tag{3.78}
$$

where $_i$ are the generators of $SU(2)$, F_{ab} the curvature of A_a, D_a the covariant derivative, $\xi = \sqrt{\det q}\eta$. We see that the addition of matter acts to add more terms to our Hamiltonian constraint.

3.5 Quantisation

We have so far discussed the classical aspects of Canonical quantum gravity, however we wish to end with a discussion on the issues of quantisation. We will aim to provide a brief overview of the quantisation procedure. We will not provide a rigorous treatment, instead we wish to give an idea of the steps that are required. For a full description see [46, 47, 48, 49, 50, 51]. We will with the Hamiltonian constraint with real connection ($\beta = 1$) which we can write as

$$
H = \frac{1}{\kappa\sqrt{\det(q)}} tr\left([F_{ab} - 2R_{ab}]\left[E^a, E^b\right]\right) \tag{3.79}
$$

We next need to introduce the identities

$$
\begin{aligned}
\frac{\left[E^a, E^b\right]^i}{\sqrt{\det(q)}} &= 2\epsilon^{abc}\left\{\frac{A_c^i}{\kappa}, V\right\} \\
K_a^i &= \frac{\delta K}{\delta E_i^a} = \left\{A_a^i, K\right\}
\end{aligned}
\tag{3.80}
$$

where

$$
\begin{aligned}
V &= \int_\sigma d^3x \sqrt{\det(q)} \\
K &= \int_\sigma d^3x \sqrt{\det(q)} K_{ab} q^{ab} = \int_\sigma d^3x K_a^i E_i^a
\end{aligned}
\tag{3.81}
$$

V has the interpretation of the total volume in σ and K is the integrated trace of the extrinsic curvature. We have made use of the identity $\left\{\Gamma_a^i, K\right\} = 0$ as well. The reason to introduce these is so we can remove the complicated curvature term R_{ab} from our constraint and write it now as

$$
\begin{aligned}
H + H^E &= \frac{2}{\sqrt{\det(q)}} tr\left([K_a, K_b]\left[E^a, E^b\right]\right) = \frac{2}{\sqrt{\det(q)}} tr\left(\left[\left\{\frac{A_a}{\kappa}, K\right\}, \left\{\frac{A_b}{\kappa}, K\right\}\right]\left[E^a, E^b\right]\right) \\
&= \frac{4}{\kappa^3}\epsilon^{abc} tr\left([\{A_a, K\}, \{A_b, K\}]\{A_c, V\}\right) = \frac{8}{\kappa^3}\epsilon^{abc} tr\left(\{A_a, K\}\{A_b, K\}\{A_c, V\}\right)
\end{aligned}
\tag{3.82}
$$

where

$$H^E = \frac{1}{\sqrt{\det(q)}} tr\left(F_{ab}\left[E^a, E^b\right]\right) = \frac{2}{\kappa}\epsilon^{abc} tr\left(F_{ab}\{A_c, V\}\right) \tag{3.83}$$

is the so-called "Euclidean Hamiltonian constraint" which would be the Hamiltonian constraint for canonical Euclidean gravity. We can now see that the obvious path towards quantisation would be to replace H^E, V, K with quantum operators and to replace our Poisson brackets with commutators as $\{.,.\} \mapsto [.,.]/i\hbar$. Indeed it is in fact possible to find operators for $\hat{V}, \hat{K}, \hat{H}^E$ as shown in [46]. We will not go through this in detail but will discuss briefly the idea of triangulation that is required in finding $\hat{H}^E$. We start from our classical expression for H^E

$$H^E[N] = \frac{2}{\kappa}\int_\sigma d^3x N(x)\epsilon^{abc} tr\left(F_{ab}\{A_c, V\}\right) \tag{3.84}$$

and we wish to split our space σ into tetrahedra denoted Δ where we will assume the edges to be analytic. Let us single out a vertex from each tetrahedra and label it $v(\Delta)$, each vertex will be made up of the intersection of three edges which we label $s_i(\Delta)$ where $i = 1, 2, 3$. We define $\alpha_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)$ to be a loop based at $v(\Delta)$ where $a_{ij}(\Delta)$ is the corresponding "base" of the tetrahedra, connecting the three sides $s_i(\Delta)$. We will denote this triangulation T, then

$$H^E_T[N] = \sum_{\Delta \in T} H^E_\Delta[N] \tag{3.85}$$

where

$$H^E_\Delta[N] = -\frac{2}{3}N_v\epsilon^{ijk}tr\left(h_{\alpha_{ij}(\Delta)}h_{s_k(\Delta)}\left\{h^{-1}_{s_k(\Delta)}, V\right\}\right) \tag{3.86}$$

noting the use of h comes from the cylindrical functions that we will not focus on here but are described in [7]. As we shrink Δ to the point $v(\Delta)$ this approaches our classical expression. We then find the quantum operator as

$$\hat{H}^E_T[N] = \sum_{\Delta \in T} \hat{H}^E_\Delta[N] \tag{3.87}$$

where of course we have to replace the Poisson bracket in $\hat{H}^E_\Delta[N]$ with a commutator and $V \mapsto \hat{V}$. This idea of triangulation is key and naturally comes with a lot of consistency checks that can be seen in [46].

3.5.1 The Master Constraint

We lastly with to discuss the Master constraint first introduced by Theimann [56, 57]. The general idea behind quantisation was to replace the Hamiltonian constraint with an operator which can then be used to solve the Wheeler-DeWitt equation. Indeed there have been proposal for Hamiltonian constraint operators before as discussed above, which was already a surprise given how the Hamiltonian constraint is a highly non-polynomial function and one would therefore expect the operator version to have many UV singularities. Fortunately this does not happen here precisely due to the choice of working in a background independent regime from the outset. Unfortunately it is not clear if these attempts at quantising the Hamiltonian constraint result in the correct classical limit. This led to the so called "spin reformulation of LQG", an attempt to avoid this issue entirely, except the problem reappeared. This led to the Master constraint which is an attempt to reformulate the Hamiltonian constraint such that it can be quantised without these issues. As found in [22, 35, 44] the main issues with the original quantisations were that

1. The Hamiltonian constraint is not a spatially diffeomorphism invariant function.
2. The Hamiltonian constraint algebra does not close.
3. The coefficient of proportionality is not a constant.

which can be summarised by the equations

$$\begin{aligned}
\left\{\vec{D}(\vec{N}), \vec{D}(\vec{N}')\right\} &= \kappa \vec{D}\left(\mathcal{L}_{\vec{N}} \vec{N}'\right) \\
\left\{\vec{D}(\vec{N}), H(N')\right\} &= \kappa H\left(\mathcal{L}_{\vec{N}} N'\right) \\
\{H(N), H(N')\} &= \kappa \int_\sigma d^3x \left(N_{,a}N' - NN'_{,a}\right)(x) q^{ab}(x) H_b(x)
\end{aligned} \tag{3.88}$$

where again we have used our smeared constraints from 3.69. The idea behind the Master constraint is to reformulate the Hamiltonian constraint such that it becomes a spatially diffeomorphism invariant function from the outset, thus removing these issues. We therefore define the Master constraint as

$$\boldsymbol{M} = \int_\sigma d^3x \frac{|H(x)|^2}{\sqrt{\det(q)}} \tag{3.89}$$

where the division by $\sqrt{\det(q)}$ ensures that the constraint is a density of weight one. Now instead of working with an infinite number of constraints we only need to consider the Master constraint which is already a great simplification. Indeed the equation $\boldsymbol{M} = 0$ corresponds to $H(x) = 0, \forall x \in \sigma$. The issue now is to simply quantise $\boldsymbol{M}$ as discussed in [57]. There is a price to pay for this approach however, to be able to detect the weak Dirac observables O, they must satisfy the master equation

$$\{O, \{O, \boldsymbol{M}\}\}_{\boldsymbol{M}=\boldsymbol{0}} = 0 \tag{3.90}$$

which is clearly non-linear.

Chapter 4

Conclusion

We would like to conclude with a brief summary reminding the reader of what we have considered. We began by discussing constrained Hamitonian systems. The key results were how we can classify our constraints, both into “primary” and “secondary” as well as into “first” and “second” class. We also discussed how we could find the equations of motion and applied our knowledge to electrodynamics.

We then discussed the classical aspects of the theory, deriving the ADM action and casting it into Hamiltonian form. We considering the constraints, equations of motion and boundary conditions that arose. We then looked at the Ashtekar variables and showed how we can use them to simplify the constraints and cast our theory into one more familiar to us. We briefly discussed the issues of time and how to introduce matter to our system.

We finished with a preliminary discussion on quantisation, discussing the idea of triangulation and how to approach quantising the Hamiltonian constraint. We also introduced the master constraint programme, an alternative approach that attempts to simplify our constraints into one single constraint, where observables obey the master equation.

Currently much of the work in the field is done on the quantisation and quantum dynamics of the theory. This is extended into what is called "Loop Quantum Gravity" where hopefully a reader can begin to see from the triangulation where we introduce loops. For two recent reviews on the field one should see [5, 59]. There has also been much work recently on applying some of these ideas to cosmology and black holes [6, 43].

Unforunately due to time constraints we were not able to describe the full mathematical details involed for quantisation which would be an interesting topic for further work.

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