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Bringing order to consistent truncations using generalised geometry

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Abstract

In this paper we introduce a type of fibre bundle with added structure known as a Courant Algebroids. We aim to show how Courant Algebroids can be used to build the generalised geometry of type II supergravity and how their structure capture the symmetries of their bosonic fields. We also aim to show that it is possible to characterise whether a Courant Algebroid is parallelisable and how this enables us to predict consistent truncations on theories of gravity constructed using them.

Contents

1 Introduction

As string theory remains a favoured candidate for quantum gravity, tools for investigating the intimidatingly broad landscape in order to identify consistent and physically relevant theories are all the more vital [1]. Consistent truncations provide us with a means of reducing the complexity of a theory without losing key information about the symmetries. They are however rare and in better need of categorising. In this paper we shall consider one area where this is possible using spaces that are parallelisable. In particular we shall consider consistent truncations of theories represented by a type of fibre bundle known as a Courant algebroid which turn out to be closely linked to the bosonic fields of type IIA supergravity.

The structure of this paper will therefore be as follows. First we shall introduce type II supergravity (section 2) and then the mathematical objects (algebroids) that we shall use to capture their structures (section 3).

After this we aim to introduce a few key ideas of generalised geometry in the context of type II supergravity or order for us to introduce the concept of general parallelisability. We shall use parallelisability in the final section on consistent truncations (section 5) where we hope to both illustrate an example of a truncation on S^3 followed by a discussion on how we can use generalised geometry to identify a particular set of consistent truncations found on pullbacks of algebras.

2 Type II Supergravity

Theories of supergravity can be thought of as either the direct application of a supersymmetry to a theory with gravity [2] [3] or as the extension of bosonic string theory to include fermions. Type II supergravity is of dimension $d = 10$ and unlike type I supergravity has no open strings. By assessing the boundary conditions [4] [5] we can identify two sectors of relevance for a superstring theory: the Neveu-Schwarz Neveu-Schwarz sector (NSNS) which contains the bosonic fields of the theory, and the Ramond Ramond sector (RR) which introduce chiral sources and will not be considered in much detail in this paper.

Type II supergravity consists of the following fields:

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(n)}_{\mu_1,\mu_2,\ldots,\mu_n}, \psi^{\pm}_{\mu}, \lambda^{\pm}\} \tag{1}$$

As expected for a theory of gravity we have a metric $g_{\mu\nu}$ but we also have a few new objects resulting from the extended theory. $B_{\mu\nu}$ is a two form tensor analogous to the one form field strength tensor found in theories of electromagnetism. The raising of the form is to reflect the fact that within the regime of string theory we have moved from one to two dimensional sources of fields. ϕ is ta scalar field also known as the dilaton another product of string theory which appears in theories consisting of extra dimensions which are compacted down. The remaining fields are related to the RR sector: $A_{\mu_1,\mu_2,\ldots,\mu_n}$ are the potentials, $\psi^{\pm}$ are chiral gravitini and $\lambda^{\pm}$ are a pair of chiral dilatinis.

Type II supergravity can be distinguished further by the choice of type IIA and type IIB. This reflects two separate theories one that is chiral (IIB) and one that is not (IIA).

Let us just consider the Bosonic fields of type II supergravity [6]. The action of the fields have the form:

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[\mathrm{e}^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right) - \tfrac{1}{4}\sum_n \tfrac{1}{n!}(F^{(B)}_{(n)})^2\right] \tag{2}$$

where $H = dB$ and the last term depends on both $B_{\mu\nu}$ and the RR potentials $A_{\mu_1,\mu_2,\ldots,\mu_n}$.

The equations of motions resulting from the action, remembering that we have switched off the fermion contribution for the time being, are given by:

$$\begin{aligned}
\mathcal{R}_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi - \tfrac{1}{4}\mathrm{e}^{2\phi}\sum_n \tfrac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1\ldots\lambda_{n-1}}F_{\nu}^{(B)\lambda_1\ldots\lambda_{n-1}} &= 0,\\
\nabla^\mu\left(\mathrm{e}^{-2\phi}H_{\mu\nu\lambda}\right) - \tfrac{1}{2}\sum_n \tfrac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} &= 0,\\
\nabla^2\phi - (\mathcal{R}\phi)^2 + \tfrac{1}{4}\mathcal{R} - \tfrac{1}{48}H^2 &= 0,\\
\mathrm{d}F^{(B)} - H \wedge F^{(B)} &= 0,
\end{aligned} \tag{3}$$

The bosonic symmetries of this theory are given by:

$$\delta_{v+\lambda}g = \mathcal{L}_v g, \qquad \delta_{v+\lambda}\phi = \mathcal{L}_v\phi, \qquad \delta_{v+\lambda}B_{(i)} = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)}, \tag{4}$$

3 Fibre Bundles and Algebroids

The mathematical language of generalised geometry is principally based on two simpler mathematical structures: fibre bundles, a type of manifold whose local structure is equivalent to the Cartesian product of two other manifolds, and algebroids, a generalisation of an algebra which can be defined using fibre bundles.

A full introduction to fibre bundles can be found in [7], it will suffice to list the key definitions used as the basis for generalised geometry. A fibre bundle (E, π, M, F, G) also denoted $E \xrightarrow{\pi} M$ consists of three (differentiable) manifolds: M known as the *base space*; F the *fibre*, and E the *total space*. It also includes a surjective map $\pi : E \to M$ called the *projection* whose pre-image for any point $p \in M$ is called the fibre at p and is equivalent to the fibre F. Lastly the bundle contains a Lie group G known as the *structure group*, this acts on F to the left by convention. To complete the definition, it must be possible to take an open covering of the base space $\{U_i\}$ such that $U_i \times F \to \pi^{-1}(U_i)$ is a diffeomorphism. Furthermore, any transition between two overlapping open sets U_i and U_j must be consistent. Importantly, it does not necessarily follow that $E \cong M \times F$, when true the bundle is said to be trivial.

The *section* of a fibre bundle is as a smooth map $s : M \to E$ such that $\pi \circ s = id_M$, the space of all sections is denoted $\Gamma(E)$.

The *tangent bundle* is an example of a fibre bundle where for any point $p \in M$ the fibre F_p is equivalent to the tangent space $T_pM \cong \mathbb{R}^m$ where m is the dimension of M. This means the tangent bundle takes the form of a disjoint union

$$TM := \bigsqcup_{p \in M} T_pM \tag{5}$$

It is possible to define other bundles based on the properties of their fibres. The fibres of a *vector bundle* are vector spaces. The structure group of a vector bundle is the general linear group of dimension $m = \dim M$. [1]

It is also possible to introduce a right action on a fibre bundle (in addition to the left already included by definition) using the structure group G. In particular, we are interested in the case where we can define a right action $E \times G \to E$ that is regular(transitive and free), preserves the fibre of any point p, and where at least one fibre F_p is diffeomorphic to G. It follows from having this single fibre that all fibres are diffeomorphic to G. A *principal bundle* denoted in this case $E(M, G)$ is therefore a bundle whose fibre is identified as G.

Suppose our principal bundle was also a vector bundle. We are free to choose an ordered basis for the fibre (which is a vector space) at some point p. This fibre, which we shall call a *frame*, is homeomorphic to the general linear group $GL(m, \mathbb{R})$ since this is the structure group of the bundle. We may therefore interpret the action of G on the fibres as a change of basis between frames. A bundle which is identified as the disjoint union of frames is called a *frame bundle* $F(E)$.

We can also construct new fibre bundles from old ones using pullbacks. Let $E \xrightarrow{\pi} M$ be a fibre bundle with fibre F. Let N be a manifold and $f : N \to M$ be a smooth function, then the *pullback of E* is given by:

$$f^*E = \{(p, u) \in N \times E : f(p) = \pi(u)\} \tag{6}$$

If we introduce a new projection map $\pi' : f^*E \to N$, that is $(p, u) \mapsto p$, then f^*E is endowed with the structure of a fibre bundle. Furthermore, the following diagram commutes.

$$\begin{tikzcd} f^*E \arrow[r, "g"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd} \tag{7}$$

We can therefore interpret the map $g : f^*E \to E$ as yet another projection, that is $(p, u) \mapsto u$.

[1] It is worth noting that the general linear group is also the structure group of a tangent bundle. This simply follows because a tangent space is a vector space by definition.

3.1 Algebroids

Now that we have introduced fibre bundles we can use them to construct a generalised algebra over them much in the same way to how we could define a Lie algebra over a manifold. Specifically these are algebroids [2].

Definition 3.1 (Lie Algebroid) *Let* $E \xrightarrow{\pi} M$ *be a vector bundle, with the following two objects:*

- *A Lie bracket defined on the bundle's space of sections:* $[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$
- *A map from the total space of the vector bundle to the tangent bundle* $\rho : E \to TM$

The triple $(E, [\cdot,\cdot], \rho)$ *forms a Lie Algebroid if the following condition is satisfied:*

$$[u, fv] = \rho(u)f \cdot v + f[u, v]$$

where $u, v \in \Gamma(E)$, f *is smooth and* $\rho(u)f$ *denotes the derivative of* f *along the vector field* $\rho(u)$ *in the tangent bundle.*

It follows from the last condition that

$$\rho([u, v]) = [\rho(u), \rho(v)] \tag{8}$$

We can already consider how an algebroid may be used to construct an extended theory of gravity. Intuitively it should include a vector bundle which we have defined as a smooth manifold, furthermore the objects defined on it ought to be diffeomorphisms which we expect to encode the symmetries associated with a theory that includes gravity.

3.1.1 Lie Algebroids as a Theory of Gravity

Let us make a quick comparison:

Let $E \cong TM \oplus \mathbb{R}$ be a bundle and let us consider an element in the space of sections to take the form (ξ^μ, α) where $\xi \in TM$ and $\alpha \in \mathbb{R}$. Let us suppose that the bundle is not twisted, then with a slight relaxing of notation we can express a particular element as simply $\xi^\mu + \alpha$.

Let us now promote this bundle to a Lie Algebroid by defining the following bracket and anchor:

$$\begin{aligned}[\xi^\mu + \alpha, {\xi'}^\mu + \alpha'] &= [\xi^\mu, {\xi'}^\mu] + \mathcal{L}_{\xi^\mu}\alpha' - \mathcal{L}_{\xi'^\mu}\alpha \\ \rho(\xi^\mu + \alpha) &= \xi^\mu\end{aligned} \tag{9}$$

[2] In representation theory a groupoid structure is achieved by relaxing the totality condition on a group. It is indeed possible to approach these objects from the field of category theory but we shall not take this more mathematical approach as we have further generalisations to consider.

Let us compare this to the symmetry found in a simple theory of gravity with a charged scalar field given by the complex function ϕ. The action for said theory will have the form:

$$\int \sqrt{-g}\left(\tfrac{1}{2}D_\mu\varphi D^\mu\varphi^* - \tfrac{1}{2}m^2\varphi\varphi^*\right) \tag{10}$$

D is a covariant derivative since this is a local theory with gravity. Its full form is $D_\mu\varphi = \partial_\mu\varphi - iA_\mu\varphi$.

Let us now consider a variation on our scalar field:

$$\delta_{\xi+\alpha}\varphi = \mathcal{L}_\xi\varphi + i\alpha\varphi \tag{11}$$

If we take the bracket we are left with:

$$[\delta_{\xi+\alpha}, \delta_{\xi'+\alpha'}]\varphi = [\mathcal{L}_\xi, \mathcal{L}_{\xi'}]\varphi + \mathcal{L}_\xi(i\alpha'\varphi) - i\alpha'\mathcal{L}_\xi\varphi - \mathcal{L}_{\xi'}(i\alpha\varphi) + i\alpha'\mathcal{L}_\xi\varphi \tag{12}$$

noting that the lie derivative is distributive we can rearrange to get:

$$[\delta_{\xi+\alpha}, \delta_{\xi'+\alpha'}]\varphi = \mathcal{L}_{\underbrace{[\xi,\xi']}_{=\xi''}}\varphi + i\underbrace{(\mathcal{L}_\xi\alpha' - \mathcal{L}_{\xi'}\alpha)}_{=\alpha''}\varphi \tag{13}$$

or specifically of the closed form:

$$[\delta_{\xi+\alpha}, \delta_{\xi'+\alpha'}]\varphi = \delta_{\xi''+\alpha''}\varphi \tag{14}$$

A comparison with the Lie algebroid above confirm that the brackets both have the same structure.

We are however interested in a supergravity which shall require more general structures than Lie algebroids to enable us to accommodate two form tensors and the dilaton. We shall see the machinery needed in the next section on generalised geometry but we shall close this section by making a step towards this by introducing Courant algebroids.

3.2 Courant algebroids

Courant algebroids derive from the study of Lie Bialgebroids where in addition to a regular bilinear bracket; an inner product has also been introduced [8][9] [10].

Definition 3.2 (Courant Algebroid) *Let $E \xrightarrow{\pi} M$ be a vector bundle; let $[\cdot,\cdot]$ be a bilinear bracket [3] defined on $\Gamma(E)\times\Gamma(E)\to\Gamma)E)$; let $\rho : E \to TM$ be the anchor, and let $\langle\cdot,\cdot\rangle : \Gamma(E)\times\Gamma(E)\to C^\infty(M)$ be a non-degenerate bilinear symmetric form. Together $(E, [\cdot,\cdot], \rho, \langle\cdot,\cdot\rangle)$ form a Courant Algebroid if the following are true: For $u, v, w \in \Gamma(E)$:*

1. $[u,[v,w]] = [[u,v],w] + [v[u,w]]$

[3] Not necessarily a Lie bracket or Skew-Symmetric

2. $[u, fv] = \rho(u)fv + f[u, v]$
3. $[u, v] = \hat{d}\langle u, v\rangle$
4. $\rho(u)\langle v, w\rangle = \langle [u, v], w\rangle + \langle v, [u, w]\rangle$

where f is a smooth function on M and $\hat{d}f \in \Gamma(E)$ such that $\langle u, \hat{d}f\rangle = \rho(u)f$.

If the sequence:

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0, \tag{15}$$

is exact then we say that the Courant algebroid is exact.

A relation between a Lie algebroid and a Courant algebroid is present if we were to consider restrict the base space M to a sngle point. In both cases the resulting object would be a Lie algebra however in Courant Algebroid case we would have the added bracket $\langle \cdot, \cdot\rangle$.

4 Generalised Geometry

We shall define the generalised structure bundle using a conformal basis which is best understood by first introducing another generalised structure.

Let the following sequence

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0, \tag{16}$$

for a spin manifold M of dimension d be exact. Then it can be shown, using the splitting lemma, that the E which we shall call the generalised tangent space is isomorphic to $T^*M \rtimes TM$. It can be further shown [6] (though no canonical construction exists) that the semi-direct product can be promoted to a direct product.

We can define an additional bundle $\tilde{E} = \det T^*M \otimes E$. This slight extension is required in order for the dilation to appear in this generalised construction.

We can now write a conformal basis for $\tilde{E}$ which shall enable us to write down the definition of a generalised structure bundle. Specifically we say $\{\hat{E}_A\}$ where $A = 1, 2, \ldots, 2d$ is a conformal basis of $\tilde{E}$ such that:

$$\langle \hat{E}_A, \hat{E}_B\rangle = \Phi^2 \eta_{AB} \tag{17}$$

where,

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi \in \Gamma(\det T^*M) \tag{18}$$

Now we can define a generalised structure bundle:

Definition 4.1 (Generalised Structure Bundle) $\tilde{F} = \{(x, \{\hat{E}_A\}) : x \in M, \{\hat{E}_A\}$ *is the conformal basis of* $\tilde{E}$ *as defined above*$\}$

On the generalised tangent space shared with this bundle we can introduce a generalisation of a Lie derivative $\mathcal{L}_v$ called the Dorfman derivatives given by:

$$L_v W = \mathcal{L}_v W + \mathcal{L}_v \zeta - i_W d\lambda \tag{19}$$

where $W = w + \zeta$.

We can in fact use these derivatives to build a courant bracket:

$$\begin{aligned} \|V, W\| &= \frac{1}{2}(L_v W - L_W V) \\ &= [V, W] + \mathcal{L}_V \zeta - \mathcal{L}_w \lambda - \frac{1}{2} d(i_v \zeta - i_w \lambda) \end{aligned} \tag{20}$$

We wish to show that the metric g along with the patched B-field and dilaton ϕ structure, as one would expect to find in the NSNS sector of type II supergravity, can be recovered from the generalised structure bundle $\tilde{F}$.

We start by considering a sub-bundle of $\tilde{F}$ of the form $O(p,q) \times O(q,p)$. It is possible to split the bundle E into the sub-bundles $C_+ \oplus C_-$ [6]. The metric:

$$\langle V, V \rangle = i_\nu \lambda \tag{21}$$

is restricted by the signature (p,q) and (q,p) on C_+ and C_- such that they are both homeomorphic to TM.

$$\begin{aligned} \langle \hat{E}^+_a, \hat{E}^+_b \rangle &= \Phi^2 \eta_{ab} \\ \langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \rangle &= -\Phi^2 \eta_{\bar{a}\bar{b}} \\ \langle \hat{E}^+_a, \hat{E}^-_{\bar{a}} \rangle &= 0 \end{aligned} \tag{22}$$

where Φ is defined over a non-vanishing section which has now been fixed. The metric η_{ab} is flat and has the signature (p,q).

We introduce a new basis:

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & \text{for } A = a \\ \hat{E}^-_{\bar{a}} & \text{for } A = \bar{a} + d \end{cases} \tag{23}$$

which can be given by the bracket:

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB} \tag{24}$$

where,

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix} \tag{25}$$

We can calculate the form of $\hat{E}^+_a$ and $\hat{E}^-_{\bar{a}}$ explicitly:

$$\begin{aligned} \hat{E}^+_a &= e^{-2\phi} \sqrt{-g} \left(\hat{e}^+_a + e^+_a + i_{\hat{e}^+_a} B \right) \\ \hat{E}^-_{\bar{a}} &= e^{-2\phi} \sqrt{-g} \left(\hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B \right) \end{aligned} \tag{26}$$

We can express the metric in this frame as follows:

$$g\left(\hat{e}_a^+, \hat{e}_b^+\right) = \eta_a b \quad \text{and} \quad g\left(\hat{e}_{\bar{a}}^-, \hat{e}_{\bar{b}}^-\right) = \eta_{\bar{a}\bar{b}} \tag{27}$$

It is possible to construct an invariant generalised metric [6]:

$$G = \Phi^2 \left(\eta^{ab}\hat{E}_a^+ \otimes \hat{E}_b^+ + \eta^{\bar{a}\bar{b}}\hat{E}_{\bar{a}}^- \otimes \hat{E}_{\bar{b}}^-\right) \tag{28}$$

where we have an invariant density:

$$\Phi = e^{-2\phi}\sqrt{-g} \tag{29}$$

Thinking back to the non-generalised Reimannian geometry, we know that manifolds always admit what is known as the Levi-Civita connection. This is a unique torsion free connection. We shall show that it is both possible to construct such a connection however we shall discuss the finding that this is not unique in the generalised case.

We can also construct a generalised connection D. To ensure that it is consistent we require:

$$DG = 0 \quad \text{or} \quad D\Phi = 0 \tag{30}$$

For $w \in \Gamma(\tilde{E})$ this is equivalent as requiring D to have the form:

$$D_M W^A = \begin{cases} \partial_M w_+^a + \Omega_M{}^a{}_b w_+^b & A = a \\ \partial_M w_-^{\bar{a}} + \Omega_M{}^{\bar{a}}{}_{\bar{b}} w_-^{\bar{b}} & A = \bar{a} \end{cases} \tag{31}$$

where:

$$\Omega_{M\,ab} = -\Omega_{M\,ba} \quad \text{and} \quad \Omega_{M\,\bar{a}\bar{b}} = -\Omega_{M\,\bar{b}\bar{a}} \tag{32}$$

5 Consistent Truncations

Consistent truncations are known to exist for the spheres S^3, S^5 and S^7. We shall look at S^3 more thoroughly since evidence points to the consistent truncation on this sphere as capturing the bosonic fields and dilaton of type IIA supergravity.

The existence of a consistent truncation is based on the parallelisability of the space, this is why we take interest in spheres since which spheres are parallelisable is a classic problem in the field of mathematics. S^2 famous for being non-parallelisable.

5.1 Consistent Truncations on S^3

There exists consistent truncations for type II supergravity for both S^5 and S^3 spheres. Specifically the S^5 sphere relates to IIB supergravity and considers the contributions of fermions which is not within the scope of this work. Instead we shall be looking at the S^3 case which describes the near-horizon NS-fivebrane background which we understand as relating to the bosonic fields of type II supergravity.

The solutions of type II exist on:

$$\underbrace{\mathbb{R}^{5,1} \times \mathbb{R}_t}_{\text{Dilaton}} \times S^3 \tag{33}$$

We can write the volume element as:

$$ds^2 = ds^2(\mathbb{R}^{5,1}) + dt^2 + R^2 ds^2(S^3) \tag{34}$$

where R is the radius of the 3-sphere.

We can write two additional constants of note:

$$H = 2R^{-1}\text{vol}(g) \quad \text{and} \quad \phi = \frac{-t}{R} \tag{35}$$

The generalised frame for $SO(4)$ is given by:

$$\hat{E}_{ij} = \nu_{ij} + \sigma_{ij} - i_{v_{ij}} B \tag{36}$$

Ultimately we can recover the structure $\mathfrak{so}(2) \times \mathfrak{so}(2)$ when we calculate algebra give by the generalised Lie derivatives [11]:

$$\begin{aligned} L_{\hat{E}^L_{\bar{a}}} \hat{E}^L_{\bar{b}} &= \|\hat{E}^L_{\bar{a}}, \hat{E}^L_{\bar{b}}\| = R^{-1}\epsilon_{\bar{a}\bar{b}\bar{c}}\hat{E}^L_{\bar{c}} \\ L_{\hat{E}^R_a} \hat{E}^R_b &= \|\hat{E}^R_a, \hat{E}^R_b\| = R^{-1}\epsilon_{abc}\hat{E}^R_c \\ L_{\hat{E}^L_{\bar{a}}} \hat{E}^R_a &= \|\hat{E}^L_{\bar{a}}, \hat{E}^R_a\| = 0 \end{aligned} \tag{37}$$

where R is the radius of the sphere.

5.2 Categorising consistent truncations

We have shown that it is possible to find a consistent truncation by identifying a viable frames that are parallelisable. This is not an easy problem, however we are fortunate that there exists a means of simplifying this to an algebraic problem. There exists a general structure known as a G-Algebroid which can be used to express various structures including: Lie algebroids, Courant algebroids and Elgebroids (constructed using exceptional groups). We shall proceed by introducing this new object and consider how pullbacks on these algebroids provide a means of simplifying our characterisation of Leibniz Parallelisable structures.

5.2.1 G-Algebroids

Definition 5.1 (G-Algebroid) *Let $P(M, G)$ be a principal bundle and let $E \to M$ and $N \to M$ be associated vector bundles endowed with the following structures:*

- $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$
- $\rho : E \to TM$

- $\mathcal{D} : \Gamma(N) \to \Gamma(E)$

Together they form a G-Algebroid under the following conditions:

1. $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$
2. $[u, fv] = (\rho(u)f)v + f[u, v]$
3. $[u, v] + [v, u] = \mathcal{D}(u \otimes v)_N$
4. $\mathcal{D}(fn) = f\mathcal{D}_n + (\hat{d}f \otimes n)_E$

where $\hat{d} = \rho^t \circ d$ *;* $u, v, w \in \Gamma(E)$ *;* $n \in \Gamma(N)$ *, and* $f \in C^\infty(M)$*. Furthermore we require that* $[u, \cdot]$ *forms an action that preserves the G-structure.*

The G-algebroid can be restricted to recover several familiar objects. If we set the associated bundle N to vanish then conditions 3. and 4. are trivial. Furthermore we have by virtue of condition 2. a Lie algebroid. In other words if the triple $(G, E, N) = (GL(n, \mathbb{R}), \mathbb{R}^n, 0)$ then our restricted G-algebroid coincides with a Lie algebroid. If we restrict our algebroid, so that M is a point, the resulting *G-algebra* coincides with a standard Lie algebra.

We can also recover a Courant algebroid if we set the triple $(G, E, N) = (O(p, q), \mathbb{R}^{p+q}, \mathbb{R})$ and set the operator $\mathcal{D} = \hat{d}$. The choice of operator is made clearer by noting that $O(p, q)$ is its own dual, or in other words $E \cong E^*$. Thus our new operator is of the form $\hat{d} : C^\infty(M) \to \Gamma(E)$. It is also important to note that the structure-preserving action $[u, \cdot]$ is essential for recovering axiom 4. in definition 3.2.

We saw in section 3.1 that we can use pullbacks to build new fibre bundles, we can likewise use them to build G-algebroids.

Proposition 5.2 *Let* $E \to M$ *be a G-algebroids and let* $f : M' \to M$ *be a surjective submersion* [4]*. Then* $f^*E \to M'$ *is a G-algebroid endowed with the G-structure* $([\cdot, \cdot]_{E'}, \rho', \mathcal{D}')$ *if the following conditions are met:*

- $[f^*u, f^*v]_{E'} = f^*[u, v]$
- $f_*\rho'(f^*u) = \rho(u)$
- $\mathcal{D}' f^* n = f^* \mathcal{D} n$

where $u, v \in \Gamma(E)$ $n \in \Gamma(N)$ *and* $E' := f^*E$*.*

Note that in the second condition we have used f_* to denote a *pushforward* [5] another naturally induced map of f.

Theorem 5.3 *A G-algebroid is Liebniz paralellisable if it can be expressed as the pullback of a a G-algebra*

[4] By this we mean that f is a differentiable map where both f and its differential are surjective everywhere

[5] also known as a differenetial map

Now that we have identified pullbacks as a key means of obtaining Liebniz paralellisable algebroids we may consider the question of what conditions are need to guarantee a pullback exists.

In [12] it is demonstrated for elgebroids (exceptional algebroids) that an action is promoted to a pullback of an exact elgebroid if and only if the stabilisers of action are co-lagrangian. To understand this we shall consider the simpler case of Courant Algebroids.

5.2.2 Courant Algebroids

Let us first note that we can define a pullback Courant Algebroid by adapting the conditions of proposition 5.2. We can identify E as a Courant algebroid by imposing the same restrictions as we did in the previous section. Furthermore, the last condition of proposition 5.2 on the operator is dropped and a new condition is applied to the brackets $\langle\cdot,\cdot\rangle$:

$$\langle f^*u, f^*v\rangle_{E'} = f^*\langle u,v\rangle \qquad \text{for } u,v\in\Gamma(E) \tag{38}$$

Together this is sufficient to define Courant Algebroid structure over the pullback [10]. The following proposition shall allow us to go a step further by refining these conditions.

Proposition 5.4 *Any Courant algebroid structure on the pullback f^*E is characterised uniquely by the anchor map $\rho' : f^*E \to TM'$. Furthermore, the structure exists if and only if all the following condition hold for $u, v \in \Gamma(E)$*

- $f_*(\rho'(f^*(u)) = \rho(u)$
- $[\rho'(f^*u), \rho'(f^*v)] = \rho(f^*[u,v])$
- *for any $p \in M'$ the kernel of ρ' at p is a subspace of E at the point $f(p)$ such that $\ker(\rho'_p) \subset E_{f(p)}$ is coisotropic with respect to the bracket $\langle\cdot,\cdot\rangle$.*

We can better realise this by considering the case of when the Courant Algebroid E has a base space M that is a single point. E is therefore a Lie algebra $\mathfrak{g}$ with the additional bracket $\langle\cdot,\cdot\rangle$. The first two conditions of proposition 5.4 tells us that if we let ρ' be an action of $\mathfrak{g}$ then it will preserve the brackets the sections $\Gamma(E)$ which are constants. The pullback built using $\mathfrak{g}$ and manifold M' is $E' = \mathfrak{g}\times M'$ [13] [10]. The last condition of proposition 5.4 tells us for every point $p \in M'$ the stabilisers of action ρ' must be coisotropic in order for E' to be a Courant algebroid.

We also have the further result from [13] that if M' is quotient in form, that is $M' = G/H$, and $\mathfrak{h}^\perp = \mathfrak{h}$ for the Lie group of H then E is furthermore exact. The condition on $\mathfrak{h}$ is also known as Lagrangian or maximally isotropic. [6] Had we chosen $O(d,d)$ as our prospective Courant algebroid we would have also required $\dim\mathfrak{h} = d$ to achieve this.

[6] in the case where $\mathfrak{h}^\perp \subset \mathfrak{h}$, in other words coisotropic but not necessarily Lagrangian, then the Courant algebroid is transative rather than exact.

In summary, we have linked the problem of classifying whether our Courant Algebroid is Liebniz parallelisable to a far more straight forward problem of assessing the isotropy of algebras. This is still not a trivial problem and may not be approachable canonically but is still a distinct improvement [12].

5.2.3 Elgebroids

While our interest has been limited to the scope of Courant Algebroids and type IIA supergravity, for completeness it is worth taking a quick look at how these results are applicable more generally to G-algebroids and elgabroids associated with other supergravity theories.

As before we can define an action to characterise the G-algebroid.

Definition 5.5 (Action of a G-algebra) *Let E be a G-algebra. The action of the G-algebra on the manifold M' is a map $\chi : E \to \Gamma(TM')$ that preserves the bracket $[\cdot, \cdot]$.*

For each point $p \in M'$ we can define a stabiliser for χ_p as the kernal of the map.

We can now state for elgebroids a similar result to proposition 5.4.

Theorem 5.6 *The transitive action of an elgebra E on an n-dimensional manifold M' defines an M-exact pullback elgebroid on $E' = E \times M'$ if and only if the stabilisers of the action are co-Lagrangian of co-dimension n.*

M-exact is understood to mean E' is exact and $\dim M$ is equal to n, the dimension of the corresponding exceptional algebra.

It can be shown [12] from this theorem that for the pair (E, V), where E is an elgebra and $V \subset E$ a subalgebra, that there exists a natural action of E on M' which generates a Leibniz parallelisable elgebroid over M' in complete analogy to the courant algebroid. Here the condition is that the subalgebra is co-Lagrangian with codimension n.

This result means all Leibniz parallelisable elgebroids which are M-exact are characterised by (E, V). This leads to an analogous simplification of the problem of classifying Leibniz parallelisable elgebroids by transfering the problem to an algebraic one.

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