

An Introduction to the Mathematics of  
Loop Quantum Gravity

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September 2022

Submitted in partial fulfilment of the requirements for the degree of  
Master of Science of Imperial College London.



# Abstract

Two revolutionary and experimentally validated theories of the early twentieth century gave rise to much of the research in theoretical physics until now. These theories are Quantum Mechanics and General Relativity. Quantum Mechanics (QM) is the theory which precisely describes the counter-intuitive behaviour of particles at a very small scale. General relativity (GR) is the theory which changed how we think about gravity at a very large scale. Both of these theories have respectively been experimentally validated time and time again. However, when attempting to unify QM and GR such that we may have a consistent understanding of the universe which we observe, they appear to be incompatible.

The search for a coherent theory which unifies QM and GR in a quantum theory of gravity is ongoing. Loop Quantum Gravity (LQG) is one contender for such a theory, alongside the popular String Theory and many others, and it is the aim of this paper to develop an understanding of the foundations of LQG.

Each of the three words Loop, Quantum, and Gravity are considered

somewhat independently. Chapter 1 covers much of the mathematical grounding of gravity and of loops. Chapter 2 explores the tools of quantisation via a generalised classical Hamiltonian formalism.

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# Acknowledgements

I would like to thank my supervisor, João Magueijo, for his guidance throughout this dissertation.

I would also like to thank Julia who has helped me in innumerable ways throughout my quest for knowledge and understanding in my last 4 years of study. Thank you for your unending love and support.

# Declaration

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# Notation and Definitions

GR = General Relativity

QM = Quantum Mechanics

LQG = Loop Quantum Gravity

EH = Einstein-Hilbert

EC = Einstein-Cartan

E-L = Euler-Lagrange

d.o.f. = degrees of freedom

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag} \left( -1 \quad +1 \quad +1 \quad +1 \right)$$

**bold** text indicates a definition

**bijection** = injection (one-to-one) and surjection (onto)

**equivalence relation** = a relation  $\sim$  which satisfies properties of reflexivity, symmetry and transitivity

**equivalence class** = all elements of a set which are equivalent  $\sim$  to each other

# Chapter 1

## The Mathematical Foundations of Gravity

The main idea of Loop Quantum Gravity is to use and assume what is already known from GR, in particular the assumption of diffeomorphism invariance, and to use this to formulate a Quantum Mechanical theory of gravity. [1, p.9]

To speak about geometry and curved spaces we must develop the necessary mathematical tools. This chapter builds up to this by providing useful and intuitive descriptions and definitions of the required mathematical structure. We begin with topological spaces and develop into manifolds, bundles, Riemannian geometry, and finally to loops and loop representations.

## 1.1 Topology

As humans we know from our intuition of everyday life that there exists continuity, be it in space or time. A topological space defines the simplest mathematical structure on which a notion of continuity can be made. [2, Lec.4] To develop a precise mathematical model of the universe it is therefore of fundamental importance to use this notion, defined as follows:

For some set  $X$ , a **topological space** is a coupling  $(X, T)$  such that  $T$  has the following properties:

1.  $X$  and  $\emptyset \in T$ ;
2. Any union of open subsets of  $X$  is in  $T$ ;
3. Any finite intersection of open subsets of  $X$  is in  $T$ .

In this case,  $T$  is called a **topology** on  $X$ .

For a full understanding of topological spaces it is important to investigate the notion of a set. This is not included here. As a brief outline, however, note that sets can be understood via axiomatic set theory, which in turn can be justified with an understanding of propositional logic. [2, Lec.1-3] Furthermore, an understanding of the foundational structure of mathematics as a discipline can be derived from the study of propositional logic.

Canonical examples of topological spaces include:

1. *Trivial topology* (or “chaotic topology” [2]): consists of only the empty set  $\emptyset$  and the entire set  $X$ ;

2. *Standard topology* (or “usual topology” [3, p.100]): consider a ball  $B_r(p)$  which in  $\mathbb{R}^1$  is some open interval of length (or diameter)  $2r$ , in  $\mathbb{R}^2$  is an open circle of radius  $r$ , in  $\mathbb{R}^3$  is an open ball of radius  $r$ , and so on into higher dimensions. Intuitively, any set  $U$  is in the standard topology if and only if there exists some ball  $B_r(p)$  about every point  $p \in U$  which is entirely contained in the set  $X$ . The radius  $r \in \mathbb{R}^+$  can be taken as small as needed to satisfy this requirement;

3. *Induced topology* (or “relative topology” [3, p.101]): Given some topological space  $(M, \theta)$  and subset  $N \subset M$ , the induced topology  $\theta|_N$  is defined as

$$\theta|_N = \{U \cap N : U \in \theta\}.$$

As stated above, we can define a notion of continuity on a topological space. A function  $f$  is **continuous** if the inverse of an open set (the codomain) is also an open set (the domain). Furthermore, if there exists some continuous function  $g$  such that two topological spaces are mapped onto one another by  $g$  and  $g^{-1}$ , then  $g$  is a **homeomorphism** and the two spaces are homeomorphic. More generally, however, two spaces are **homotopic** if they can be continuously deformed into each other with two a priori unrelated functions  $f$  and  $g$ . As such, a homeomorphism is a special type of homotopy which is bijective.

## 1.2 Manifolds

Manifolds are the mathematical spaces which physicists use to model space-time. A **manifold**  $M$  is a topological space which is locally homeomorphic to  $\mathbb{R}^n$ . As such, manifolds inherit the continuity conditions of topological spaces as described above, and they also inherit structure from our usual  $n$ -dimensional Euclidean geometry.

Although manifolds are locally flat, they can be globally curved. We must therefore establish tools to measure and generalise to curved spaces. Some of these tools are outlined below.

A **chart**  $(U, \phi)$  is an open set  $U \in M$  coupled with a mapping  $\phi$  from  $U$  to  $\mathbb{R}^n$ . Charts are intuitively always possible since manifolds are locally flat by definition. A **transition function**  $\psi_{ji}$  maps one chart  $(U_i, \phi_i)$  to another  $(U_j, \phi_j)$  via the manifold  $M$ , where  $U_i \cap U_j \neq \emptyset$ . Transition functions can be visualised as in figure 1.1, and can be written as follows:

$$\psi_{ji} = \phi_j \cdot \phi_i^{-1}. \quad (1.1)$$

A **differentiable manifold** is one for which the transition functions between charts are differentiable. A manifold is **smooth** if it is infinitely differentiable, denoted  $C^\infty$ , and we will henceforth assume the smoothness of differentiable manifolds unless stated otherwise.

A **tensor** is a mathematical object which is invariant (ie. it does not change) under coordinate transformations from one coordinate chart onto

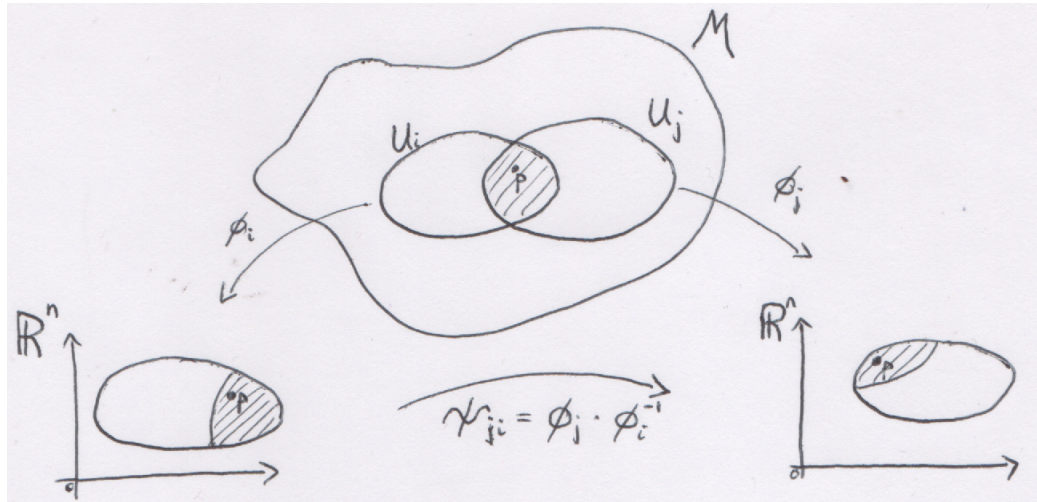


Figure 1.1: The transition function  $\psi_{ji}$  visualised from one chart  $(U_i, \phi_i)$  to another  $(U_j, \phi_j)$  via the manifold  $M$ , where  $U_i \cap U_j \neq \emptyset$ .

another. We can think of tensors as manifold invariants, since they can look very different in different charts but will always remain the same on the manifold itself. Tensors are often expressed in terms of components and basis vectors. Under a coordinate transformation the components transform proportionally to the basis vectors, acting to cancel any change to the tensor itself so that it remains invariant.

A simple example is that of the well known **vector**, which is a  $(1,0)$  tensor. A vector can be expressed familiarly as:

$$\mathbf{v} = v^i \mathbf{e}_i = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + \cdots + v^n \mathbf{e}_n \quad (1.2)$$

Under a coordinate transformation, if a basis vector  $\mathbf{e}_i$  transforms by a factor

of 2 then the components  $v^i$  will transform by a factor of  $\frac{1}{2}$  in order to leave the underlying vector invariant. This is precisely why tensors are used in GR: GR assumes symmetry under coordinate transformations, also known as diffeomorphism invariance or general covariance.

The **rank** of a tensor is the number of basis vectors needed in order to fully specify a tensor component. For example,

$$\mathbf{r} = r^{\mu\nu} \mathbf{e}_\mu \otimes \mathbf{e}_\nu \quad (1.3)$$

(where  $\mu, \nu \in \{0, 1, 2, \dots\}$ ) is a rank-2 tensor since it has 2 basis vectors, but the vector given in 1.2 above is a rank-1 tensor.

So far we have seen **contravariant tensor components** which are specified with superscript indices. There are also **covariant tensor components**, which are denoted by subscript indices. In particular, a **differential one-form** is a special type of tensor which is covariant and is totally anti-symmetric. We will first encounter one-forms in the Bundles section.

A **tensor density** is an object which transforms like a tensor but with an additional factor due to the Jacobian determinant of the transformation. The power of the determinant involved establishes the **weight** of the tensor density. We will see an important use of a tensor density in the EH action in the General Relativity section.

Since manifolds can have different curvature at different points (unlike a Euclidean manifold), it is necessary to have tools for moving and comparing

objects at different points or locations. For example, an **integral curve** is a curve on a manifold with a direction at every point equal to the direction of the tangent vector at that point. The **flow**  $\sigma(t, x_0)$  of a vector field on some manifold picks out an integral curve with starting point  $x_0$  and extends it for a distance  $t$ .

The **Lie derivative**  $\mathcal{L}_X Y$  calculates the change in a vector field  $Y$  as it moves along a flow of another vector field,  $X$ . The Lie derivative is analogous to the gradient of vector calculus.

### 1.2.1 Lie groups, Lie algebras and Representation

#### Theory

Similarly to topological spaces, a group  $(G, \circ)$  simply defines a structure or relationship between elements of a set  $G$ . In particular, in order to be a **group** a set of elements  $G$  must be coupled with a binary product or composition operation  $\circ$  between the elements, and must satisfy the following 4 axioms [4]:

1. **Closure** Composition of any 2 elements  $a$  and  $b$  of the set  $G$  must give another element also contained in  $G$ , ie.

$$a \circ b = c, \text{ where } a, b, c \in G; \tag{1.4}$$



2. **Associativity** For any elements  $a, b, c \in G$ ,

$$a \circ (b \circ c) = (a \circ b) \circ c; \quad (1.5)$$

3. **Identity** There exists an element  $e$  such that for any  $a \in G$ ,

$$a \circ e = a; \quad (1.6)$$

4. **Inverse** For any element  $a \in G$  there exists an element  $a^{-1} \in G$  such that

$$a \circ a^{-1} = e. \quad (1.7)$$

There is a fifth condition which sometimes holds. Group  $(G, \circ)$  elements may or may not commute with each other. If they do then it is called an **abelian group**.

5. **Abelian** For any elements  $a, b \in G$

$$a \circ b = b \circ a. \quad (1.8)$$

A **Lie Group**  $(G, \circ)$  is a group where the set  $G$  is also a differentiable manifold. [Clarification: there is no a priori relationship between Lie groups and Lie derivatives other than the name.] Lie groups encode many symme-

tries in physics, so understanding their structure is of vital importance to physicists. Given their differentiability, Lie groups are also continuous and we can therefore always find an element infinitesimally close to any other element. It follows that any group element  $\mathbf{A} \in G$  near the identity can be expressed as an exponential as follows:

$$\mathbf{A} = e^{\epsilon \mathbf{X}} = \mathbb{1} + \epsilon \mathbf{X} + \dots \quad (1.9)$$

where we have Taylor expanded the exponential up to first order, and  $0 < \epsilon \ll 1$  can be chosen to be arbitrarily close to zero. We can then use the  $\mathbf{X}$  from this construction to impart information about the local structure of the Lie group. In fact the set of  $\mathbf{X}$ , which generate all Lie group elements as above, form a Lie algebra  $\mathfrak{g}$  when coupled with a binary bracket operation  $[\cdot, \cdot]$ , known as a Lie bracket. A **Lie algebra** is simply a vector space coupled with a Lie bracket which by definition must satisfy conditions of antisymmetry, bilinearity and the Jacobi identity. Lie algebras are very useful for their relation to Lie groups as outlined above.

So far we have introduced the structure of Lie groups and Lie algebras abstractly. In order to apply this usefully to physics, however, we must represent these groups with sets of numbers which satisfy the structure of these Lie groups or Lie algebras. To do this we use matrices along with matrix multiplication. The structure preserving map which we use is the **homomorphism**, defined as  $H(a \circ b) = H(a)H(b) \forall a, b \in G$ , where  $(G, \circ)$

is a group and the product on the RHS is matrix multiplication. The study of matrices which fulfil the given abstract group structure (or any abstract algebraic structure) in this way is called **representation theory**. [4]

Two important examples of Lie groups for the study of gravity are  $SU(2)$  (special unitary group) and  $SO(n)$  (special orthogonal group). Both of these groups have elements with determinant 1 (ie. special).  $SU(2)$  can be represented by  $2 \times 2$  matrices  $U$  for which  $U^\dagger = U^{-1}$ , and  $SO(n)$  matrices can be represented by  $n \times n$  matrices  $O$  which satisfy  $O^T = O^{-1}$ . A common basis for  $SU(2)$  are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.10)$$

Following the procedure of exponentiating to find a Lie algebra as shown in eqn.1.9 above, we find that the elements  $\mathbf{X} \in \mathfrak{su}(2)$  must be traceless,  $\text{Tr } \mathbf{X} = 0$ , and hermitian,  $\mathbf{X}^\dagger = \mathbf{X}$ . These Lie algebra element properties come directly from the group element properties of being special and unitary, and are therefore the only restrictions on the elements of  $\mathfrak{su}(2)$ .

The dimension (or dimensionality)  $D$  of any Lie group is the number of linearly independent matrices in its Lie algebra (see [5]). Since  $\mathfrak{su}(2)$  elements are hermitian and traceless for example, the dimension of  $SU(2)$  is the number of degrees of freedom of these  $2 \times 2$  matrices, which is  $(2 \times 2 - 1) = 3$ . In fact, all of the information of a Lie algebra  $\mathfrak{g}$  can be deduced from its

structure constants  $f^{abc}$ , which are constants which satisfy

$$[\mathbf{X}^a, \mathbf{X}^b] = f^{abc} \mathbf{X}^c, \tag{1.11}$$

where  $\mathbf{X}^a, \mathbf{X}^b, \mathbf{X}^c \in \mathfrak{g}$ .

For any group we can choose from multiple representations (or reps), depending on the context. For example:

- the **trivial representation** maps all elements of a group to the matrix identity element,  $\mathbb{1}$ ;
- the **fundamental/defining representation** is used to define the group. The defining rep. of  $SU(2)$  are  $2 \times 2$  matrices which are special (ie. they have determinant 1) and unitary, such as the Pauli matrices above;
- the **adjoint representation** elements are  $D \times D$  matrices, where  $D$  is the dimension of the group as defined above. Elements of the adjoint representation are given by  $(T^a)_{bc} = -if^{abc}$ , and as such they are strongly related to the group's Lie algebra.

### 1.3 Bundles

In the previous section we constructed manifolds from our previous knowledge of topology. Bundles present the next level of structure. In fact, much of physics is formulated on different types of bundles.

A **bundle**  $(E, \pi, M)$  is a topological space defined by the two manifolds  $E$  and  $M$  and by the map  $\pi$  between them.  $E$  is called the **total space**,  $M$  is the **base space**, and  $\pi$  is a continuous and surjective map called the **projection**, where

$$\pi : E \longrightarrow M. \tag{1.12}$$

The **pre-image** of some point  $p \in M$  is the set  $\{q \in E : \pi(q) = p\}$ . Given some  $p \in M$ , the pre-image of the point  $p$  is called a **fibre**  $F$ . The fibre is therefore the set of all points in the total space which map to  $p$  on the base space (via the projection  $\pi$ ). A **fibre bundle** is a bundle for which all of the fibres are homeomorphic to each other. In other words, they all have the same structure. A fibre bundle is denoted  $(E, \pi, M, F)$ , and its fibres are called **typical fibres**.

A common example of a fibre bundle is the **tangent bundle**, which has fibres which are the tangent spaces  $T_pM$  to the base space  $M$  at every point  $p \in M$ . The disjoint union of all of these tangent spaces is denoted  $TM$ , which is the total space of the tangent bundle. Given the projection  $\pi$  the bundle can be written as the map from each tangent space  $T_pM$  to the points  $p \in M$  to which they are tangent:

$$\pi : TM \longrightarrow M \tag{1.13}$$

Since each of the tangent spaces of the tangent bundle are in fact vector



Figure 1.2: The cylinder  $E = S^1 \times \mathbb{R}$  extended to infinity in both directions [6] is an example of a trivial bundle. The vertical lines are the fibres  $\mathbb{R}$ , and the circle  $S^1$  is the base space. There is a projection  $\pi$  from the total space  $E$  to the base space  $S^1$ .

spaces, a tangent bundle is also a **vector bundle**.

A **trivial bundle** is a fibre bundle  $(E, \pi, M, F)$  with total space which is isomorphic to a product bundle,  $E = M \times F$ . A canonical example of a trivial bundle (which is therefore also a fibre bundle) is a cylinder, since the total space is the product between the circle  $S^1$  (base space) and the real line  $\mathbb{R}$  (fibres),  $E = S^1 \times \mathbb{R}$ , as shown in figure 1.2.

To see that this is indeed a fibre bundle note that the fibres are all straight, real lines, so they are clearly homeomorphic to each other and hence are typical fibres.

A **bundle morphism** (or bundle map) is defined so that two bundles  $(E, M, \pi)$  and  $(E', M', \pi')$  have diffeomorphisms  $u$  and  $f$  as per the diagram below, such that the diagram commutes.

$$\begin{array}{ccc}
 E & \xrightarrow{u} & E' \\
 \pi \downarrow & & \downarrow \pi' \\
 M & \xrightarrow{f} & B'
 \end{array}$$

In other words, for a bundle morphism we must have  $\pi' \circ u = f \circ \pi$ .

### 1.3.1 Principal Bundles

The principal bundle is important to us in particular since it is the structure which forms the basis of the Einstein-Cartan formulation of GR, as discussed in the GR subsection below. More generally, however, principal bundles are ubiquitous in physics, and Yang-Mills theories in general can be understood in terms of them. What follows is a practical introduction to principal bundles.

Continuing from the discussion of fibre bundles above, sometimes  $G$  is also included when denoting a fibre bundle  $(E, \pi, M, F, G)$ . Here  $G$  represents the **structure group**, which is a Lie group whose elements are those which transform coordinates from different (overlapping) charts into each other. For example, as per p.349 of [3], given two charts  $U_i$  and  $U_j$  of  $\mathbb{R}^n$  with coordinates  $x^i$  and  $y^j$  respectively, where  $U_i \cap U_j \neq \emptyset$ , we have a vector  $V \in U_i \cap U_j$  such that:

$$V = V^i \frac{\partial}{\partial x^i} = V^j \frac{\partial}{\partial y^j}. \quad (1.14)$$

It follows that

$$V^j = \frac{\partial y^j}{\partial x^i} V^i \quad (1.15)$$

where  $\frac{\partial y^j}{\partial x^i} \in GL(n, \mathbb{R})$  since it is an invertible matrix (GL stands for general linear group, which includes all invertible matrices). Eqn.1.15 clearly connects the charts  $U_i$  to  $U_j$  using the Lie group element which is the coordinate transformation. Barring any restriction on the change of coordinates in this case,  $G = GL(n, \mathbb{R})$  is the structure group.

Loosely speaking, a **principal bundle** is a fibre bundle for which the fibres  $F$  are the same as the structure group  $G$ . Given that we represent groups with matrices which are not always abelian, it is important to specify how the group elements act on the manifold points. Every principal bundle therefore comes equipped with a right group action  $\triangleleft$ , by definition. Although the above definition of a principal bundle is not false, we need some further structure in order to define it more precisely. This structure is given below.

The **orbit**  $\theta_p$  of a group element is the set of points  $q \in M$  in the base manifold which are mapped to each other by the action of *all* of the elements of the group  $G$  on  $p$ ,  $\theta_p = \{q \in M \mid \exists g \in G : p \triangleleft g = q\}$ .

Each of the points of the base manifold belong to exactly one orbit, and the set of all of these orbits, which partition the manifold, is called the **orbit space**. In fact, each orbit defines an equivalence class,  $\sim$ , and the orbit space is the quotient space of the manifold, often denoted  $M / \sim \equiv M/G$ .



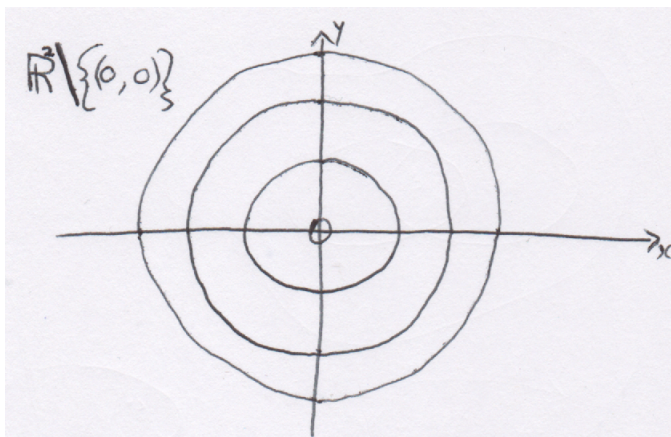


Figure 1.3: Example of an orbit space, with concentric  $SO(2)$  circles in the space  $\mathbb{R}^2 \setminus \{(0,0)\}$ . Since the origin is not included in the space, this example has a free action,  $\triangleright$ .

A very intuitive example from [2] considers a Euclidean manifold without the origin,  $\mathbb{R}^2 \setminus \{(0,0)\}$ . Acting on points  $p \in \mathbb{R}^2 \setminus \{(0,0)\}$  from the left (with  $\triangleright$ ) with elements  $g \in SO(2)$ , we find that the equivalence classes  $\sim$  are concentric circles around the origin, as per figure 1.3. In fact, each of these circles is isomorphic to  $SO(2)$ , which as we will see shortly is always the case for a free action.

The **stabilizer**  $S_p$  of a point  $p \in M$  is the set of group elements  $g \in G$  which leave the point unchanged. An action (right  $\triangleleft$  or left  $\triangleright$ ) is **free** if all of the points  $p \in M$  are left unchanged by only the identity. In other words, an action is free if all of the points of the manifold have stabilizer  $S_p = \{e\} \forall p \in M$ . We have already seen an example of this in the figure of concentric circles above.

Finally, a **principal bundle**  $(E, \pi, M, F, G)$  is defined as a fibre bundle with the following properties:

- The total space manifold  $E$  is equipped with a right  $G$ -action  $\triangleleft$ ,
- The action  $\triangleleft$  is free, and
- There is a bundle morphism (or bundle map)  $\cong$  such that

$$\begin{array}{ccc} E & & E \\ \pi \downarrow & \cong & \downarrow \rho \\ M & & E/G \end{array}$$

where  $\rho$  maps points of the total space  $E$  to their respective orbit (or equivalence class,  $\sim$ ).

Now, given some free action  $\triangleleft$ , all of the group elements  $g$  (except for the identity) map the point  $p$  to some other point/s. The group structure is therefore preserved by any free action, since for some  $g_1, g_2, \dots, g_n \in G$  we have

$$\theta_p = \{p, p \triangleleft g_1, p \triangleleft g_2, \dots, p \triangleleft g_n\}$$

As such, all of the orbits in the orbit space  $E/G$  are necessarily diffeomorphic to the group  $(G, \circ)$ . From our morphism diagram in the definition above we deduce that the base space  $M$  of a principal bundle is diffeomorphic to an orbit space  $E/G$ , with each orbit making up one fibre of the bundle, and we have just found that each orbit/fibre is diffeomorphic to the fundamental group,  $(G, \circ)$ . We have therefore now made precise the statement with which

we started above: a principal bundle is a fibre bundle equipped with an action for which the fibres are the same as the structure group  $(G, \circ)$ .

There is one more type of bundle which is important to mention in the context of GR, which is the associated bundle. Rather than using the most general definition, in our case an associated bundle is a fibre bundle associated to a principal bundle, which is defined as follows. Given some principal  $G$ -bundle  $P$  and some smooth manifold  $F$  with a left  $G$ -action, the **associated bundle** is  $\pi_F : P_F \rightarrow M$  where

1.  $P_F = (P \times F) / \sim_G$ , where  $(p, f) \sim_G (p', f') \iff \exists g \in G : p' = p \triangleleft g$  and  $f' = g^{-1} \triangleright f$ , and
2.  $\pi_F : [(p, f)] \mapsto \pi(p)$ .

We can form an example of an associated bundle from the frame bundle, which is a principal bundle. Similar to the tangent bundle, the **frame bundle** has a total space which is the disjoint set of bases (or frames)  $L_p M$  of the tangent spaces  $T_p M$  to  $M$  at every point  $p \in M$ . Since bases transform into each other via the  $GL(n, \mathbb{R})$  group, this is the structure group of the frame bundle. We can choose some  $F = \mathbb{R}^n$  and define a left action

$$\begin{aligned} \triangleright : GL(n, \mathbb{R}) \times F &\longrightarrow F \\ (g \triangleright f)^a &= (g^{-1})^a_b f^b. \end{aligned} \tag{1.16}$$

We then have the associated bundle  $\pi_{\mathbb{R}^d} : LM_{\mathbb{R}^d} \longrightarrow M$  [2, Lec.20].

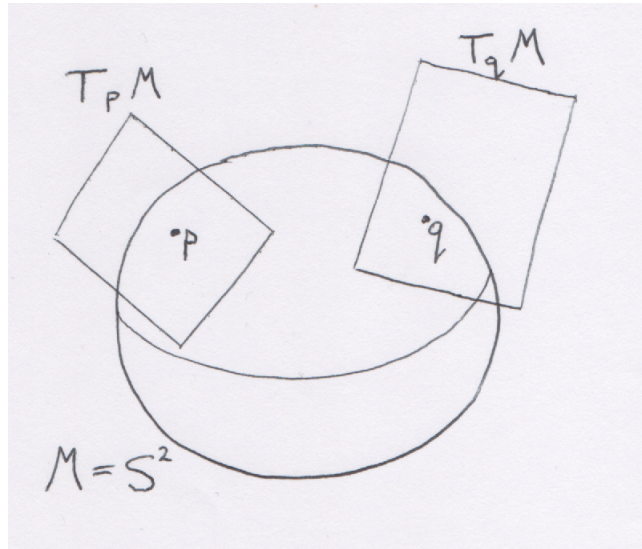


Figure 1.4: Example of the 2-sphere with tangent spaces  $T_p M$  at  $p$  and  $T_q M$  at  $q$ . Since these tangents are not coplanar, they must have different bases.

**Structure on Principal Bundles:** Vectors on different tangent spaces of a curved manifold often cannot be compared with each other as they are. Consider for example the 2-sphere. The basis of one tangent space will inevitably be different from that of a different tangent space, as shown in figure 1.4.

There is a similar situation for the fibres of a principal bundle. The solution is to find a tool to connect different fibres, which we call a **connection**. As it turns out, we can always choose a differential one-form with suitable limiting properties (which we do not specify here) as the connection [2, Lec.21], and it is far more convenient to do so in practice. Furthermore, this one-form connection on the principal bundle is Lie algebra valued, and

from it we can define a local, Lie algebra valued one-form connection on the base manifold itself. We call these **local connection coefficients**, and they are precisely the notion which we use in GR and Yang-Mills theories where they are usually denoted by  $\Gamma_\mu$  and  $A_a$  respectively.

## 1.4 Riemannian Geometry

A **metric tensor**  $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$  gives a measure of distance on a manifold. It does so by introducing an inner product,  $\cdot$ . In the familiar case of Pythagoras' Theorem, for example, where a flat Euclidean manifold is assumed, the distance between two points  $(0, 0)$  and  $(x, y)$  in 2D is  $\sqrt{x^2 + y^2}$ . The way this is calculated is as follows:

$$\begin{aligned} & \sqrt{\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} x & y \end{pmatrix}} \\ &= \sqrt{\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} \\ &= \sqrt{x^2 + y^2}, \end{aligned} \tag{1.17}$$

where  $g_{\mu\nu} = \delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is the Euclidean metric (notice that we often refer to a metric tensor simply by its components,  $g_{\mu\nu}$ , and this applies to any other tensor as well). The inclusion of a metric in this otherwise familiar example sheds light on an overlooked subtlety in high school mathematics.

As we know, manifolds are not Euclidean in general and the metric  $g_{\mu\nu}$  used in the inner product in eqn.1.17 will vary according to the manifold we are considering.

A **Riemannian manifold** is a real manifold with a positive definite metric, which means that the metric is symmetric and has only positive eigenvalues. A **pseudo-Riemannian manifold** has, instead, a pseudo-Riemannian metric, which means that some of the metric eigenvalues may be negative.

Common metrics include the Euclidean metric,  $\delta_{ij}$ , as seen above, the Minkowski metric,  $\eta_{\mu\nu}$ , as well as the Schwarzschild metric, Reissner-Nordstrom metric, and the Kerr metric which are used to investigate black holes. Each of these metrics describes a very different geometry.

Whereas a Lie derivative is analogous to the gradient of vector calculus, we need a notion of a directional derivative  $\nabla$  (see [7] for more details). Furthermore we use this directional derivative to connect different tangent spaces (or fibres) in a curved manifold, and we therefore call it a **connection**. This is indeed the same concept as the connection encountered in the Principal Bundles sub-section above, however in this case it is significantly restricted. From the general one-form connection defined previously we can establish a notion of parallel transport on a principal bundle, and then on an associated bundle, and finally on a vector bundle. We can then introduce a difference between points on this vector bundle, from which we may define the derivative which is the connection  $\nabla$  which we encounter here.

For our current use we only need to consider parallel transport on a

vector bundle. As such, note that a tangent vector is said to be **parallel transported** if it is moved in such a way that it remains parallel to itself at all times. As we already know, basis vectors  $e_\mu$  are not constant in general on a curved manifold, so their values may change when a vector is parallel transported. The derivative connection compensates for this change, and it is found as follows [8]:

$$\begin{aligned}
\nabla V &= \nabla(V^\mu e_\mu) \\
&= \frac{\partial}{\partial x^\mu}(V^\alpha e_\alpha) \\
&= \frac{\partial V^\alpha}{\partial x^\mu} e_\alpha + V^\alpha \frac{\partial e_\alpha}{\partial x^\mu} && \text{(Leibniz rule)} \\
&= \frac{\partial V^\alpha}{\partial x^\mu} e_\alpha + V^\alpha \Gamma_{\alpha\mu}^\lambda e_\lambda \\
&= \frac{\partial V^\alpha}{\partial x^\mu} e_\alpha + V^\lambda \Gamma_{\lambda\mu}^\alpha e_\alpha && \text{(re-labelling indices)} \\
&= \left( \frac{\partial V^\alpha}{\partial x^\mu} + V^\lambda \Gamma_{\lambda\mu}^\alpha \right) e_\alpha && (1.18)
\end{aligned}$$

The  $\Gamma_{\alpha\mu}^\lambda$  is called the (local) **connection coefficient**, and the connection/directional derivative  $\nabla$  is called a **covariant derivative**, since it transforms covariantly under a general coordinate transformation.

As defined above, the covariant derivative is a very commonly used type of connection called an affine connection. **Affine connections** use parallel transport to connect tensors. The properties of an affine connection are as

follows [3, p.268]:

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z \quad (1.19)$$

$$\nabla_{(X+Y)}Z = \nabla_X Z + \nabla_Y Z \quad (1.20)$$

$$\nabla_{(fX)}Y = f\nabla_X Y \quad (1.21)$$

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y \quad (1.22)$$

The following example is given to shed further light on the affine connection, and it is also relevant to the loop formulation which we develop in a subsequent chapter. A **closed curve** is simply a curve with the same start and end point. On a curved manifold, a parallel transported vector does not in general return to the same value when it is transported around a closed curve (or loop, see the Loops and Loop Representations section below). The connection coefficient  $\Gamma^\lambda_{\alpha\mu}$  of an affine connection removes this difference in a vector value due to parallel transport (ie due to the curvature of the manifold), thereby enabling comparison between vectors in different tangent spaces. In summary, the covariant derivative measures how different one object is from a different, parallel transported object.

Lastly, a **metric connection** is an affine connection which is metric compatible, which means that the metric  $\mathbf{g}$  is covariantly constant at all points on the manifold ([3, p.272]), or equivalently if “ $\mathbf{g}$  is parallel with respect to  $\nabla$ ” [9]. In particular, the condition for a metric connection is



$\nabla g = 0$ . The connection coefficient of the metric connection is as follows:

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \Gamma^\lambda_{(\mu\nu)} + \Gamma^\lambda_{[\mu\nu]} \\ &= \{\lambda_{\mu\nu}\} + K^\lambda_{\mu\nu}\end{aligned}\tag{1.23}$$

where

$$\{\lambda_{\mu\nu}\} = \frac{1}{2}g^{\lambda\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha})\tag{1.24}$$

is the **Christoffel symbol** and

$$K^\lambda_{\mu\nu} = \frac{1}{2}(T_\nu^\lambda{}_\mu + T_\mu^\lambda{}_\nu + T^\lambda_{\mu\nu})\tag{1.25}$$

is the **contorsion**, with **torsion tensor**

$$T^\lambda_{\mu\nu} = -T^\lambda_{\nu\mu}.\tag{1.26}$$

Notice that in eqn.1.23 we have split the connection coefficient into its symmetric (rounded brackets) and antisymmetric (square brackets) parts. Taking torsion  $T^\lambda_{\mu\nu} = 0$  the antisymmetric part vanishes, and we are left with the Christoffel symbol as the connection coefficient,  $\Gamma^\lambda_{\mu\nu} = \{\lambda_{\mu\nu}\}$ . In this case the connection  $\nabla$  is called the Levi-Civita connection, which is precisely what is used in the usual classical Einstein formalism of GR.

A **geodesic curve** is a curve which has been formed from the parallel

transport of a vector along itself. Geodesics generalise the notion of a straight line in Euclidean space to curved spaces, often giving the shortest distance between two points on a manifold.

It is clear from the example given above that connection coefficients  $\Gamma_{\mu\nu}^{\lambda}$  encode information about the curvature of a manifold. However, these coefficients are not tensors and so are not coordinate (or diffeomorphism) invariant. The coordinate invariant object that we use instead is the **Riemann tensor**:

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}{}_{\nu\rho,\sigma} - \Gamma^{\mu}{}_{\nu\sigma,\rho} + \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu} - \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\nu} \quad (1.27)$$

[Note once again that we have given only the components of the tensor here, and we would need to include the basis vectors to show the tensor in full.] Contracting the 1st and 3rd indices we get  $R_{\nu\rho} = R^{\mu}{}_{\nu\mu\rho}$ , called the **Ricci tensor**, and contracting once again we get the **Ricci scalar**  $R = R^{\nu}{}_{\nu} = g^{\mu\nu}R_{\mu\nu}$ .

An important identity involving the covariant derivative of the Riemann curvature is the following **Bianchi identity**:

$$R_{\mu\nu\alpha\beta;\sigma} + R_{\mu\nu\sigma\alpha;\beta} + R_{\mu\nu\beta\sigma;\alpha} = R_{\mu\nu[\alpha\beta;\sigma]} = 0. \quad (1.28)$$

where ‘ $;\eta$ ’ is a shorthand to mean the ‘covariant derivative with respect to  $\eta$ ’.

### 1.4.1 General Relativity

Einstein's classical theory of General Relativity is formulated in the language of Riemannian geometry, and (in a vacuum) can be stated as follows:

$$G_{\mu\nu} = 0. \tag{1.29}$$

where  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$  is called the **Einstein tensor**. This is the equation of motion which results from varying the Einstein-Hilbert action (again excluding matter),

$$S_{EH} = \int d^4x \sqrt{-g}R, \tag{1.30}$$

where here we have  $g = \det g_{\mu\nu}$ . The metric determinant in this action is a tensor density which is used to make the 4-integral coordinate invariant (or diffeomorphism invariant). [10] As we know, the Ricci scalar  $R$  comes from the Riemann tensor  $R_{\sigma\mu\rho\nu}$  via the Ricci tensor  $R_{\mu\nu} = g^{\sigma\rho}R_{\sigma\mu\rho\nu}$ , since  $R = g^{\mu\nu}R_{\mu\nu}$ , and each of these objects provides information about the curvature of a manifold (see [11] for an intuition of their relationship). This realisation sheds light on the interpretation of gravity as the curvature or geometry of spacetime.

So far our consideration of GR has implicitly been on the tangent bundle, where the tangents are the bundle fibres. There is however another way to formulate GR which is known as the **Einstein-Cartan** (EC) formalism, or

the tetrad or non-coordinate formalism. Although it largely imparts the same information as the EH (Einstein-Hilbert) action  $S_{EH}$  above, the structure of the EC formalism lends itself to being interpreted as a gauge theory, and it is what we use in quantum gravity and String theory, amongst others [12, p.42][13, p.5].

In a coordinate basis “every tangent vector [to a manifold] at a point  $p$  can be expressed as a linear combination of the coordinate derivatives  $\frac{\partial}{\partial x^\mu}$ ” [14]. As implied above, the EC formalism replaces the usual coordinate basis  $\frac{\partial}{\partial x^\mu}$  with a non-coordinate basis, in particular with an orthonormal basis  $e^I$ , where  $I \in \{0, 1, 2, 3\}$ . This new basis  $e^I$  is in fact a one-form which we call the **tetrad**. Although we use Einstein’s local equivalence principle whilst transitioning the metric  $\mathbf{g}$  to the tetrad in the beginning of [13], we find that all of the information of our usual metric  $\mathbf{g}$  (or “spacetime dynamics”) is in fact encoded in the tetrad globally. We have the following relationship:

$$\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{IJ} e^I \otimes e^J \quad (1.31)$$

Since we have not introduced any sort of restriction on our new basis  $e^I$ , which is a  $4 \times 4$  Lie algebra valued one-form matrix, it has 16 degrees of freedom. In our previous metric  $\mathbf{g}$  we had 10 d.o.f., however, and this discrepancy is famously resolved by assuming 6 gauge d.o.f. This introduces the Lorentz group (gauge group)  $SO(3, 1)$ .

To find the connection in EC theory we start by simply changing the basis

of our usual Levi-Civita connection  $\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu\alpha} dx^\alpha$  to the non-coordinate (tetrad) basis (which we denote with indices I,J,K,... as above) to find:

$$\Gamma^\mu{}_\nu \rightarrow \Gamma^I{}_J = e^I{}_\mu \Gamma^\mu{}_\nu e^\nu{}_J - de^I{}_\rho e^\rho{}_J \quad (1.32)$$

Writing the transformed expression on the RHS of 1.32 in the following form,

$${}^g A = gAg^{-1} - dg g^{-1} \quad (1.33)$$

for some group  $g \in G$ , it is clearly recognizable as the usual gauge connection transformation from Lie algebra theory. Our connection in the EC formalism is therefore nothing more than the analogue of the Lie algebra valued connection which we know so well from Yang-Mills gauge theories. Furthermore, it can be shown that when subjected to the metric compatibility condition of the Levi-Civita connection,  $\Gamma^I{}_J$  is in fact an element of the  $\mathfrak{so}(3,1)$  Lie algebra, legitimising our choice of the Lorentz gauge group  $SO(3,1)$  above. As an aside, note that this choice is not unique, since a Lie algebra does not in general uniquely specify a Lie group. We could have chosen the double cover  $Spin(3,1)$  of  $SO(3,1)$ , for example.

Unlike our usual Levi-Civita connection, in EC theory we do not necessarily restrict to vanishing torsion  $\mathbf{T} = 0$ . Instead we have contorsion

$$C_{IJK} = T_{KIJ} - T_{JKI} - T_{IJK}, \quad (1.34)$$

and our complete connection in EC theory is

$$\omega^I{}_J = \Gamma^I{}_J + C^I{}_J, \quad (1.35)$$

which we call the **spin connection**. For simplicity, however, by assuming vanishing torsion we can transform our Riemann curvature to the orthonormal (non-coordinate) basis as follows:

$$R^\mu{}_{\nu\alpha\beta} \rightarrow R^I{}_J = e^I{}_\mu e^\nu{}_J R^\mu{}_{\nu\alpha\beta} \frac{1}{2} dx^\alpha \wedge dx^\beta, \quad (1.36)$$

where this curvature is the analogue of our field strength tensor in Yang-Mills.

We now have the fundamental tools of the EC formulation, namely the tetrad  $e^I$  and the connection  $\omega^I{}_J$ . To shed light on the construction from a mathematical perspective (see [13]), consider a principal  $G$ -bundle with  $G = SO(3, 1)$  and an associated vector bundle with trivial fibre the  $SO(3, 1)$  representation space  $\mathbb{V}$ . Following the steps encountered in the ‘Principal bundles’ chapter to find local connection coefficients, the connection which we find is the spin connection  $\omega^I{}_J$ . Furthermore, the tetrad  $e^I$  comes from an invertible and differentiable map  $e : TM \rightarrow \mathbb{V}$ . We define an inner product  $\langle \cdot, \cdot \rangle$  from an object called the Killing-Cartan form, from which we find  $\mathbf{g} = \eta_{IJ} e^I \otimes e^J$  as given in eqn.1.31. This is the mathematical structure underlying the EC formalism. [15]

Lastly, if instead of  $SO(3, 1)$  in the principal bundle construction above we choose its double cover,  $G = Spin(3, 1)$ , then it is possible to couple

spinors to gravity. This is of obvious necessity for a complete theory of gravity, though we do not provide any further details here.

## 1.5 Loops and Loop Representations

In addition to the Riemannian geometry outlined above, a **holonomy** is a measure of the change to a tensor or spinor when it is parallel transported around a closed curve (or loop) in a smooth manifold. We consider closed curves  $l$  and  $m$  to be **equivalent** if they have the same holonomy. We then define a **loop** to be any equivalence class  $[l]$  of closed curves. In other words, a loop is defined as the set of closed curves with the same holonomy.

We denote the set of closed curves which start and end at  $o$  by  $L_o$ . Similarly, we denote the set of loops (ie. equivalence classes) basepointed at  $o$  by  $\mathcal{L}_o$ .

Along with loop composition  $\circ$ , loops satisfy the group axioms of closure, associativity, identity and inverse. The identity is the loop with zero holonomy, and the inverse of a loop is the same set (or equivalence class) of closed curves but traversed in the opposite direction. We know that the set is closed under loop composition since  $l \circ m$  gives another loop at  $o$ ,  $\forall l, m \in \mathcal{L}_o$ . Lastly, loop composition is associative. The resulting group is called the **group of loops**, which is non abelian [16] since loops do not commute.

As for the relationship between loops and closed curves just described, we define a **path** to be any equivalence class of **open curves** (ie. curves

with different start and end points). The notation for an open curve  $p$  from the origin  $o$  to some point  $x$  is  $p_o^x$ .

A **tree** is a closed curve which does not enclose any area. In other words, a tree is an open curve composed with its inverse. A synonym for a path which is a tree is a **thin** path. Finally, two (open) paths  $p_o^x$  and  $q_o^x$  are equivalent if and only if  $p_o^x \circ q_o^{-1x}$  is a tree. Unlike loops, paths do not form a group under path composition.

In the context of bundles, it is the connections  $A_a$  (which we know from the previous chapter to be Lie-algebra-valued one-forms on the base manifold,  $M$ ) which are parallel transported around closed curves to find holonomies. For a trivial bundle, given a closed curve  $l$  from the fibre at  $o$  back to itself via connections  $A_a$ , we can therefore write the holonomy as

$$H_A(l) = P \exp \left( \int_l A_a(y) dy^a \right). \quad (1.37)$$

Here  $P$  is a path ordering operator, which ensures that the parallel transport occurs in the right order around the closed curve  $l$ .

Furthermore, on a principal  $G$ -bundle the holonomies are group elements,  $H_A(l) \in G$  (in fact they are representations of the group of loops). Given a parametrisation of the curve path which starts at  $l(0)$  and ends at  $l(1)$  we have the following product

$$l(1) = l(0)H_A(l), \quad (1.38)$$



which defines the right action of the principal bundle.

As we did in previous chapter sections, it is useful to define tools to compare objects as they move through the continuous space of loops (see pg.7 of [16] for more detail about loop space continuity). For some function  $\Psi$ , given an infinitesimal variation  $\delta\gamma$  to a loop  $\gamma$ , we define the **loop derivative** as the operator  $\Delta_{\mathbf{ab}}$  which acts on the original loop function  $\Psi(\gamma)$  to give the difference between  $\Psi(\gamma)$  and  $\Psi(\gamma + \delta\gamma)$ , as per the following equation [16]:

$$\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = \left( 1 + \frac{1}{2} \sigma^{ab}(x) \Delta_{\mathbf{ab}}(\pi_o^x) \right) \Psi(\gamma). \quad (1.39)$$

In this expression,  $\sigma^{ab} = 2\epsilon_1\epsilon_2(u^{[a}v^{b]})$  is the area of  $\delta\gamma$ , and  $\Delta_{\mathbf{ab}}$  is anti-symmetric. It is not obvious what an infinitesimal variation  $\delta\gamma$  of a loop looks like. As per figure 1.5, it includes a small loop at  $x$  and an open path connecting the points  $o$  and  $x$ . A very similar notion of a derivative can be defined for open paths, though this is not our focus here.

The loop derivative is in fact a tensor and, given suitable conditions, it satisfies the Bianchi identity. Furthermore, it is a generator of the group of loops.

In the EC theory given previously, we found that gravity is usefully formulated as a gauge theory. We know that gauge theories have observables which must be gauge invariant. Taking the trace of our holonomy provides

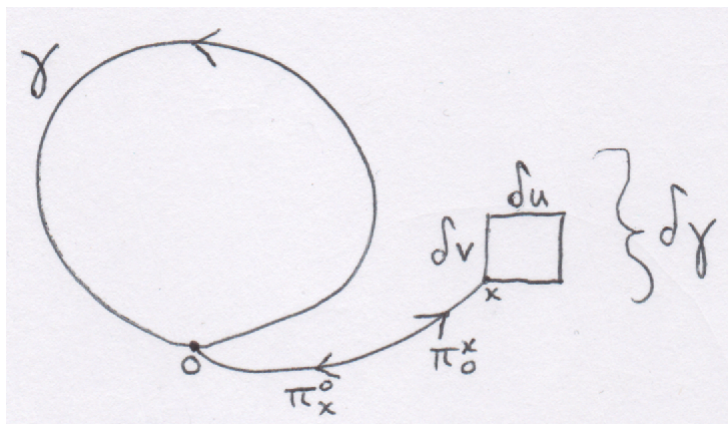


Figure 1.5: Infinitesimal loop  $\delta\gamma$  extending from the loop  $\gamma$ . [16]

such a quantity,

$$W_{\mathbf{A}}(\gamma) = \text{Tr} \left[ P \exp \left( \oint_{\gamma} \mathbf{A}_a(y) dy^a \right) \right], \quad (1.40)$$

and it is called the **Wilson loop**, [17]. In fact, “all the information present in a holonomy can be reconstructed from the Wilson loops.” [16]

Using the tools given so far in this section and previously writing gravity as a gauge theory in the EC formulation, it is possible to rewrite our best theory of gravity to date, GR, using the group of loops. Before we can quantise gravity from this theory, however, we must learn how to rewrite it in the language of quantum theory, namely the Hamilton formulation. This is the subject of the next chapter.

# Chapter 2

## Generalized Hamiltonian Formulation

### 2.1 Canonical formulation

In order to quantise gravity we need first to derive a canonical formulation of general relativity, namely a Hamiltonian formalism. As outlined by Dirac in his 1964 New York Lectures [18], in general this proceeds as follows:

1. Find an action for the theory from which the equations of motion can be derived,

$$S = \int L dt \tag{2.1}$$

where the integrand  $L$  is the Lagrangian of the theory, then

2. Rewrite the action in terms of a Hamiltonian  $H$  rather than a La-

grangian.

We denote the  $n = 1, \dots, N$  general degrees of freedom in the Lagrangian formulation by  $q_n$  and  $\dot{q}_n = \frac{dq_n}{dt}$ , so that  $L = L(q_n, \dot{q}_n)$ , and we treat  $q_n$  as independent of  $\dot{q}_n$  and vice versa. We then find the usual Euler-Lagrange equation,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) = \frac{\partial L}{\partial q_n} \quad (2.2)$$

and introduce the general momentum as,

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \quad (2.3)$$

### 2.1.1 Primary and Secondary Constraints

However, in our case we need a generalized Hamiltonian formalism, which necessarily introduces the need for constraints. An excellent and simple explanation of the need for constraints is given in section 4.4 of [12], which uses an example of making measurements in a classical Newtonian system but with a consistently and non-linearly late (or early) clock. It turns out that it is in fact possible to make precise measurements in this system, if we initially generalise the time coordinate and then apply appropriate constraints. This is similar to our case. In order to increase generality we do not assume that the momentum  $p_n$  is independent of velocity  $\dot{q}_n$ . As a result of this extra

freedom we introduce constraints which take the following form:

$$\phi_m(q_n, p_n) = 0. \quad (2.4)$$

Since these are independent of the Euler-Lagrange (E-L) equations of motion they are called **primary constraints**.

We know that the Hamiltonian is given by  $H = p_n \dot{q}_n - L$ , so

$$\delta H = \dot{q}_n \delta p_n + p_n \delta \dot{q}_n - \frac{\partial L}{\partial q_n} \delta q_n - \frac{\partial L}{\partial \dot{q}_n} \delta \dot{q}_n, \quad (2.5)$$

using the product rule and the multivariable chain rule. We have already defined  $p_n = \frac{\partial L}{\partial \dot{q}_n}$  in eqn.2.3 above, so the 2<sup>nd</sup> and 4<sup>th</sup> terms in this eqn.2.5 cancel, which gives

$$\delta H = \dot{q}_n \delta p_n + \frac{\partial L}{\partial q} \delta q_n. \quad (2.6)$$

Furthermore, using the Euler-Lagrange equations we know that  $\frac{\partial L}{\partial q_n} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) = \frac{d}{dt} (p_n) = \dot{p}_n$ , so we get

$$\delta H = \dot{q}_n \delta p_n + \dot{p}_n \delta q_n. \quad (2.7)$$

Rearranging this result we find the Hamiltonian equations,

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + \dot{p}_n \frac{\partial q_n}{\partial p_n} \quad (2.8)$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} + \dot{q}_n \frac{\partial p_n}{\partial q_n}. \quad (2.9)$$

However, Dirac [18] writes these as

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n} \quad (2.10)$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} + u_m \frac{\partial \phi_m}{\partial q_n}, \quad (2.11)$$

where by comparison it is clear that each of the  $\phi_m$  terms is  $q_n$  and  $p_n$  dependent only, whereas  $u_m$  can also have other dependence such as velocity  $\dot{q}$ , for example. These dependencies will be important to remember in what follows.

Now, using the multivariable chain rule for some function  $g$  which is dependent on  $q$  and  $p$ , we have

$$\begin{aligned} \dot{g} &= \frac{dg}{dt} = \frac{\partial g}{\partial q_n} \frac{dq_n}{dt} + \frac{\partial g}{\partial p_n} \frac{dp_n}{dt} \\ &= \frac{\partial g}{\partial q_n} \dot{q}_n + \frac{\partial g}{\partial p_n} \dot{p}_n \end{aligned} \quad (2.12)$$

Substituting the results 2.10 and 2.11 into 2.12 we get

$$\begin{aligned} \dot{g} &= \frac{\partial g}{\partial q_n} \left( \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n} \right) + \frac{\partial g}{\partial p_n} \left( -\frac{\partial H}{\partial q_n} + u_m \frac{\partial \phi_m}{\partial q_n} \right) \\ &= [g, H] + u_m [g, \phi_m], \end{aligned} \quad (2.13)$$

where we have used Poisson brackets similarly to our usual Hamiltonian

formalism, satisfying the usual anti-symmetry and linear properties and the Jacobi identity. To summarise, we now have both of Hamilton's equations 2.10 and 2.11 incorporated into one single equation 2.13, which therefore encodes all of the same dynamics as the E-L equations of motion.

Note that we can rewrite eqn.2.13 in a more compact form

$$\dot{g} = [g, H + u_m \phi_m], \quad (2.14)$$

which includes the function  $u_m$  for which we know that the Poisson bracket is not defined. The reason this works is that, by the bracket linearity and product rule we have

$$\begin{aligned} [g, H + u_m \phi_m] &= [g, H] + [g, u_m \phi_m] && \text{(linearity)} \\ &= [g, H] + [g, u_m] \phi_m + u_m [g, \phi_m], && \text{(product rule)} \end{aligned} \quad (2.15)$$

and the second term on the second line is clearly zero by the primary constraint,  $\phi_m \approx 0$ , regardless of the value of the (undefined) Poisson bracket. We are therefore once again left with eqn.2.13, as promised.

Another important point to note at this point is that the  $\phi_m$  from eqn.2.4 should only be taken to be zero *after* all Poisson brackets have been worked out. Dirac [18] therefore refers to these as “weak equations”, with notation  $\phi_m \approx 0$ . Furthermore, since Poisson brackets can only be applied to functions with  $q$  and  $p$  dependencies they are not defined for  $u_m$ .

We stated above that primary constraints are those which hold independently of the E-L equations (or equivalently in our Hamiltonian case, of eqn.2.13). By contrast, **secondary constraints** are those which must hold *after* the E-L equations have been satisfied, but not necessarily before.

For example, if we let  $g = \phi_m$  in eqn.2.13 then we get

$$0 \approx [\phi_m, H] + u'_m [\phi_m, \phi'_m], \quad (2.16)$$

since  $\phi_m$  remains zero throughout time, so  $\dot{\phi}_m = 0$ . Once the Poisson brackets have been worked out on the RHS and the primary constraints have been implemented, we either find that eqn.2.16 is automatically satisfied, or we find that we must put another constraint  $\chi_M \approx 0$  in place such that it is satisfied, where  $\chi_M$  is independent of  $u_m$  so  $\chi_M = \chi_M(p, q)$ . If the latter, then we call  $\chi_M \approx 0$  a secondary constraint. We may find more secondary constraints by inserting  $\chi_M$  into eqn.2.13 once again, and this continues until we have found all of the constraints.

There is a useful notation whereby all  $S$  constraints, including both primary and secondary, can be written as  $\phi_S = \phi_1, \dots, \phi_m, \phi_{m+1}, \dots, \phi_{m+M}$ . Here there are  $m$  primary constraints and  $M$  secondary constraints, though they are both included in the  $\phi_S$  notation.



### 2.1.2 First and Second Class constraints

We argued above that eqn.2.16 is either automatically satisfied or we must impose a secondary constraint to make it so, independent of  $u_m$ . There is however a third option in which neither of these holds true, which is when the RHS remains dependent on  $u_m$ . After some further analysis [18], this leads to a notion of first and second class constraints.

A function  $R = R(p, q)$  is **first class** if it satisfies  $[R, \phi_S] \approx 0$  for all of the primary and secondary constraints  $\phi_S$ . An equivalent way to express this is  $[R, \phi_S] = r_{ij}\phi_j$  since  $\phi_j \approx 0$ . The function is **second class** if this condition does not hold for all constraints  $\phi_S$ .

The notions of first and second class constraints are in fact especially relevant to the quantisation of gravity. [16, Chp.7][19][20]

## 2.2 ADM Hamiltonian

In order to present Einstein's theory of GR as a quantum (field) theory, canonically quantised, we must first write it in the Hamiltonian formalism. Using our usual methods as described in the Canonical Formulation section above, we first write our action in the Lagrangian formalism, which we have seen previously as:

$$S_{EH} = \int d^4x \sqrt{-g} R. \quad (2.17)$$

We then transform the Lagrangian  $\sqrt{-g}R$  according to  $H = p\dot{q} - L$ .

This poses a problem. In general, field theories in the Lagrangian or Hamiltonian formalism treat space and time differently, since we need some notion of time in order to define the velocity  $\dot{q}$ . On the other hand, we know that GR treats space and time on the same footing via the requirement for diffeomorphism invariance. The solution most often used to sort out this discrepancy is to split space and time and consider them separately, making a 3+1 space which we will call  $M$ . The reason this works is that the time component which we choose is left arbitrary, thereby avoiding setting a preference for any particular direction of time. [12, p.40]

A **hypersurface** is an  $(n - 1)$ -dimensional manifold embedded in an  $n$ -dimensional manifold. For example we usually picture the 2-dimensional surface of a smooth kitchen table as embedded in the 3-dimensional space of the kitchen, or kitchen space. The topology which we get as a result of the splitting outlined above is  $M = {}^3\Sigma \times \mathbb{R}$ , where  ${}^3\Sigma$  is some 3-dimensional hypersurface embedded in  $M$ , and  $\mathbb{R}$  is some parameter  $t$  on the real line. In our analogy,  $M$  is analogous to the kitchen space and  ${}^3\Sigma$  is the surface of the table.

Now, each different value of  $t$  specifies a different foliation of the  ${}^3\Sigma$  space, which can be denoted as  ${}^3\Sigma_t$ , and since  ${}^3\Sigma$  is left completely arbitrary diffeomorphism invariance is in fact preserved. To help with picturing the foliations, this is similar to a bunch of numbered papers being stacked on top of each other forming a pile of papers, and then using the number parameter

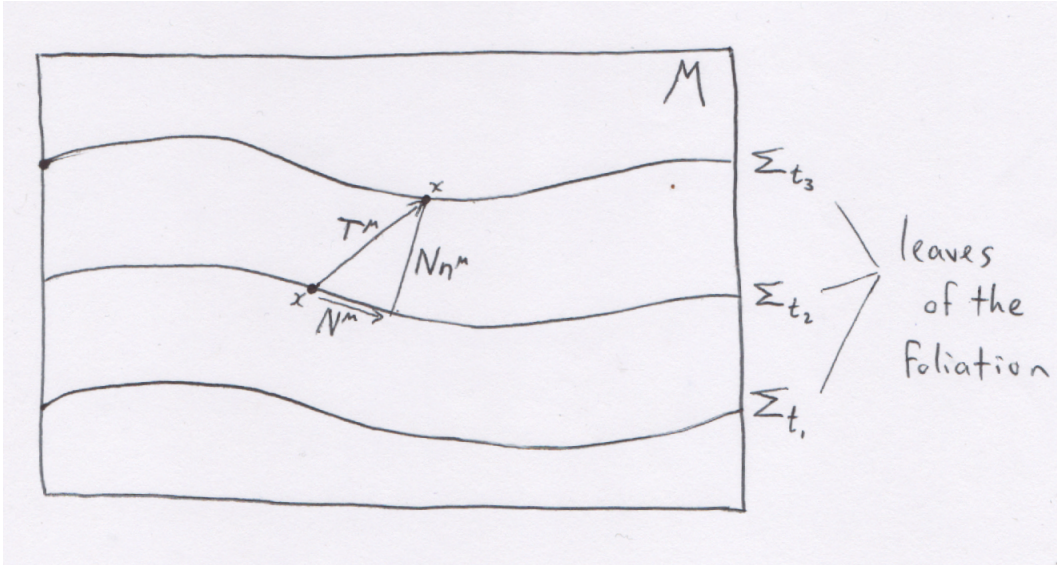


Figure 2.1: Shift and lapse functions shown explicitly in the context of a foliation of hypersurfaces  ${}^3\Sigma$ . [1]

to distinguish one paper from another.

Although it is of arbitrary topology by construction, it is of course useful and necessary to parametrise the  ${}^3\Sigma$  space. We do this by introducing some unit vector  $n^\mu$  normal to  ${}^3\Sigma$  at every point. We then define a parameter  $T^\mu$  such that

$$T^\mu = N n^\mu + N^\mu, \quad (2.18)$$

where  $N^\mu$  is tangent to the  ${}^3\Sigma$  surface, and it is called the **shift vector**.  $N$  is called the **lapse function**. Furthermore, it is an obvious choice to make  $n^\mu$  timelike, which makes  ${}^3\Sigma$  spacelike. See figure 2.1 for further clarity.

We introduce a metric  $q_{ab}$  on the 3-space  ${}^3\Sigma$ ,

$$q_{ab} = g_{ab} + n_a n_b \quad (2.19)$$

and the extrinsic curvature  $K_{ab}$  [1, p.42],

$$K_{ab} = q_a^\rho q_b^\sigma \nabla_\rho n_\sigma \quad (2.20)$$

$$= \frac{1}{2} \mathcal{L}_n q_{ab} \quad (2.21)$$

Sometimes the metric  $q$  and curvature  $K$  are called the **first and second fundamental form** of  ${}^3\Sigma$ , respectively. Notice that the letter which we have used to denote the  ${}^3\Sigma$  space metric  $q$  is the same as that used for the generalized position  $q$  in eqn.2.2. This similarity is intentional, since the metric plays the role of the generalized position in our (generalized) Hamiltonian formulation.

From the structure which we have developed up to now we can construct the velocity,

$$\dot{q}_{ab} = \mathcal{L}_t q_{ab} = 2NK_{ab} + \mathcal{L}_{N^\mu} q_{ab} \quad (2.22)$$

and the conjugate momentum,

$$\pi_{ab} = \frac{\delta L}{\dot{q}_{ab}} = \sqrt{q}(K^{ab} - Kq^{ab}) \quad (2.23)$$

as per the outline in Chapter 7 of [16]. Rewriting the EH Lagrangian in terms of these variables, and then transforming it into the Hamiltonian formalism we get

$$H = \int d^3x \left( N(-\sqrt{\det q}R + (\sqrt{\det q})^{-1}(\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi^{ab}q_{ab})^2)) - 2N^b D_a \pi_b^a \right) \quad (2.24)$$

where  $R$  is the intrinsic curvature of  ${}^3\Sigma$ , and  $D_a$  is the covariant derivative compatible with  $q_{ab}$ . Just like any other Hamiltonian formulation, here we have conjugate variables  $q$  and  $\pi$ , which are related as follows:

$$\{q_{ab}, \pi^{cd}\} = \delta_a^c \delta_b^d \delta(x-y). \quad (2.25)$$

This should be recognizable from QM (although we have not yet quantised our system), where as usual in the natural system of units we set  $\hbar = 1$ . For completeness we include the full action of the ADM Hamiltonian formulation:

$$S = \int dt \int d^3x \left( \pi_{ab} \dot{q}^{ab} + (\sqrt{\det q}^{-1} N)(-qR + (\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi^{ab}q_{ab})^2)) - 2N^b D_a \pi_b^a \right) \quad (2.26)$$

This method of splitting space and time in such a way to get the Hamiltonian above was developed by Arnowitt, Deser and Misner. We therefore call the resulting Hamiltonian the **ADM Hamiltonian**.

Now that we have found a generalized Hamiltonian the next step is to

find the constraints of our theory. Having found all of the constraints, we proceed to quantisation.

## 2.3 Loop Quantum Gravity

We now have many of the tools needed to quantise gravity, which brings us to the end of the paper. Given more time, the next step is naturally to explore the application of all of these tools to the quantisation of GR.

Areas left to be explored include the use of Ashtekar variables in the quantisation of GR, especially using the loop formalism. From here we would go on to study the research of Rovelli, Smolin, Gambini and Pullin, into the uses and relevance of spin networks and spin foams, and including applications to black holes. An exploration of the Wheeler-DeWitt equation is also in order, including a look into the so called problem of time.

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