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Moduli Space of supersymmetric field theories in 3d, 4d and 5d

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A dissertation submitted for the degree of

MSc in Quantum Fields and Fundamental Forces

September 22, 2022

Acknowledgements

Firstly, I want to thank Professor Amihay Hanany for opening the door to me and showing me the richness of supersymmetric field theory and moduli space, and for the countless time he spent in answering my questions in a inspiring way and discussing the calculation result with me.

I want to thank Julius Grimminger for his help in both my early stage learning in SUSY and later study in quiver gauge theory. I want to thank Lorenzo Mansi for those valuable discussion we had in both the common room and on the bus to campus. I want to thank Rudolph Kalveks for his Mathematica code to make most calculation possible, and his patient explanation of different concepts.

I want to thank my "elder brother" Professor Yang-Hui He, and Jia-Kang Bao for their hospitality in LIMS (and the Hagoromo chalk making writing on broad even more enjoyable), and for helping me use *Sage* to understand group structures and simplify the calculation. Special thanks for JK's examination of my code and helpful discussion of my calculation result.

I want to thank Yu-Sheng Jiao for his explanation and discussion in superstring. I want to thank Chun-Hao Li and Guhesh Kumaren for useful discussion in quiver gauge theory. I want to thanks all the other QFFF friends to make this year more meaningful.

I want to thank Xiu-Hong, Yi-Wen, Sai-Jie for their support of my decision to continue my study, and I also want to thank all my family for the continuous support and understanding.

Lastly, I want to thank my cat to be adopted, Hilbert, who gives me a lot mental support in my imagination.

Abstract

In this dissertation, we will review the moduli space of gauge theories in 3d, 4d and 5d with 8 supercharges and the engineering with brane system. We will also discuss the higher form symmetry which generalise the symmetry to a topological operator related with a arbitrary dimensional objective and not necessarily a group. With non-invertible 1-form symmetry we can construct a collection of $3d$ $\mathcal{N} = 6$ theories with 12 supercharges whose moduli space are $\mathbb{C}^{4n}/\Gamma$, Γ is complex reflection group. When Γ belongs to one of the infinite families of complex reflection group under classification of Shephard-Todd, it can be identified with ABJ(M) theories; When Γ is exceptional complex reflection group, it's a new theory.

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Introduction

In the traditional way, we approach a QFT with Lagrangian and correlation functions, both are local function of fields. This formulation is invalid for theories have no Lagrangian, such as $6d$ $\mathcal{N} = (1, 0)$ theories and many-body system, or multiple Lagrangian, such as duality, or the Lagrangian is too complicated to analysis. The non-Lagrangian construction of supersymmetric theories from string/M/F theories has highly enriched our understanding of QFTs in the past few decades. Moreover, the local operators (or more precisely, mutual local) are not necessarily to be defined as local functions of fields, and play a crucial role in duality. As we will analysis in this article, the GNO monopole in $3d$ $\mathcal{N} = 4$ theories is defined by boundary condition and the t'Hooft-Wilson line in $4d$ $\mathcal{N} = 4$ is theories defined on a path integral.

A new language to describe symmetries has be purposed in [1], which generalises the description of Noether and Landau. This new types of symmetries emerge both in string theory and condensed matter theory, for example, the categorical description of topological phase beyond Landau-Ginzburg theory. In this formulation, symmetries are allowed to be 1. higher form symmetries depend on a manifold topologically, 2. higher group symmetries when higher form symmetries in different degrees affect each other, 3. non-invertible symmetries of non-invertible topological defects and 4. subsystem symmetries depends on a manifold geometrically. All those generalised symmetries can be spontaneously broken. The structure of symmetries are no longer limited to groups, now it fits into the language of category theory more naturally. Global symmetries give constrains in QFT dynamics. For example, the gauging of global symmetries are obstructed by the well-known t'Hooft anomaly in RG flow, and also in duality, the global symmetries (or spectrum of operators) are required to be matched. Hence, the extended understanding of symmetries provides us with a new tool to find new theories and duality.

Moduli space, on which holomorphic functions are expected to be identified with chiral ring of gauge invariant operators, plays an important role in our analysis of supersymmetric field theories and their duality. Many tools we have developed to analysis the moduli space in the past few years, such as Hilbert series[2, 3] counting the invariants of moduli space, highest weight generating function[4] encoding the Hilbert series in a more compact way and etc. It's recommend to check A for readers unfamiliar with those tools.

In $3d$ $\mathcal{N} = 4$ theories we have used brane construction in Type IIB string theory[5] to describe the moduli spaces successfully. We can further classify the *hyper-Kähler* quotient with corresponding magnetic quiver of brane system. It's a huge convenience to represent the Higgs branch of $3d$, $4d$, $5d$ and $6d$ theories with 8 supercharges into magnetic quiver, which is better studied.

In chapter 2, we will provide a brief introduction of $4d$ $\mathcal{N} = 2$, and also $3d$ $\mathcal{N} = 4$ theories, which can be obtained through dimensional reduction preserving all SUSY[6]. The structure of moduli space will be emphasised. We will also discuss the emerging of higher form symmetry in the last section.

In chapter 3, we will focus on the Type IIB string theory approach to $3d$ $\mathcal{N} = 4$ quiver gauge theories, and also the Hasse diagram technique to encode the symplectic structure of moduli space (or equivalently the Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism).

In chapter 4, we will dive into the brane web and toric construction of $5d$ $\mathcal{N} = 1$ theories[7][8] and analysis the strong coupling non-perturbative properties through webs. The strong coupling non- trivial fixed point of $5d$ $\mathcal{N} = 1$ is studied in [9, 10, 11]. Different gauge theories can be reached by different mass deformation of the same SCFT, we call it "UV duality", which means a SCFT may have different low energy description. There is a topological $U(1)_I$ symmetry with current $J = \star trF \wedge F$, instantons charged under this symmetry. The global symmetry can be enhanced by instanton in UV fixed point, $G_{enhanced} \supset G \times U(1)_I$. For example, the $SU(2)$ theories with N_f fundamental matters can has enhanced symmetries from $SO(2N_f) \to E_{N_f+1}$, $E_1 = SU(2), E_2 = SU(2) \times U(1), E_3 = SU(3) \times SU(2), E_2 = SU(5), E_6, E_7, E_8$. We will focus on $SU(N_c)$ gauge theory with N_f flavours and Chern-Simons level k, and obeys the condition $N_c - \frac{1}{2}N_f + 2 - |k| \geq 0$[12].

In chapter 5, we study the novel $3d$ $\mathcal{N} = 6$ theories beyond ABJ(M) model [13, 14]. $4d$ $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{g}$ has moduli space $\mathbb{C}^{3n}/\mathcal{W}(\mathfrak{g})$. Compactifying $4d$ $\mathcal{N} = 4$ SYM with a twist along S^1 to $3d$, we can obtain theories with $\mathcal{N} = 2, 4, 6$[15, 16, 17]. For certain value of τ_{YM}, it's stabliser in $SL(2, \mathbb{Z})$ may form a symmetry. However, the $SL(2, \mathbb{Z})$ symmetry genericly change the spectrum of line operator, therefore, the symmetry that leave τ_{YM} invariant may no longer be a symmetry. By gauging a 1-form symmetry, we can secure the spectrum, and result in a non-invertible symmetry. Moduli spaces of those $3d$ $\mathcal{N} = 6$ theories have the form of $\mathbb{C}^{4n}/\Gamma$, where Γ is a finite subgroup of $\mathcal{W}(\mathfrak{g})$ and also a *complex reflection group*[18, 19, 20]. When Γ belongs to one of the infinite families of complex reflection group under classification of Shephard-Todd, it can be identified with ABJ(M) theories; When Γ is exceptional complex reflection group, we have a new theory beyond ABJ(M).

2

$4d\ \mathcal{N}=2$ and $3d\ \mathcal{N}=4$

$4d\ \mathcal{N}=2$ theories has the superalgebra

$$\{\mathcal{Q}^I_\alpha, \mathcal{Q}^{\dagger J}_{\dot\beta}\} = \delta^{IJ} P_\mu \sigma^\mu_{\alpha\dot\beta},\ \{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} = \epsilon^{IJ}\epsilon_{\alpha\beta} Z, \tag{2.1}$$

where $I, J = 1, 2$ labels the two collection of supersymmetry generators, Z is central charge. We can obtain $3d\ \mathcal{N}=4$ theories through dimensional reduction preserving all SUSY[6]. The aim of this chapter is to provide a brief introduction of $4d\ \mathcal{N}=2$ and $3d\ \mathcal{N}=4$ theories, especially the structure of moduli space. We will also discuss the emerging of higher form symmetry in the last section. This chapter will follow the logic of [21], with some notation taken from [22, 23].

2.1 Electromagnetic duality

Let's start with Maxwell theory. The free Maxwell equations in differential form are

$$dF = d\star F = 0. \tag{2.2}$$

Where $(\star F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. It's easy to see the equations are invariant under Hodge dual:

$$F \leftrightarrow \star F. \tag{2.3}$$

This operation is also called S-transformation. Define the electromagnetic charge (n, m), where n is the electric charge under $U(1)$, m is magnetic charge. Wick-rotating to Euclidean space-time, we have the following relation:

$$\begin{aligned} &\int_{S^2} \frac{4\pi}{e^2} \star F = 2\pi n\ (\textit{enclose an electric particle}),\\ &\int_{S^2} F = 2\pi m\ (\textit{enclose a Dirac monopole}). \end{aligned} \tag{2.4}$$

The action is given by

$$\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\vartheta}{32\pi^2} F_{\mu\nu} \star F^{\mu\nu}. \tag{2.5}$$

The second term is a total derivative:

$$\frac{1}{2} F_{\mu\nu} \star F^{\mu\nu} = \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma). \tag{2.6}$$

This term has a topological contribution in quantum theory, which we will discuss later. Define the *complexcoupling* as:

$$\tau = \frac{4\pi i}{e^2} + \frac{\vartheta}{2\pi}. \tag{2.7}$$

The theory is invariant, quantum mechanically, under $\vartheta \to \vartheta + 2\pi$ (In Yang-Mills theory, it can be explained as instanton contribution is periodic in ϑ). This operation is also called T-transformation.

S and T transformation generate a $SL(2, \mathbb{Z})$ group on the space spanned by (n, m):

$$\begin{aligned} S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; S \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} -m \\ n \end{pmatrix}, \; S\tau = -\frac{1}{\tau}, \\ T &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; T \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} n+m \\ m \end{pmatrix}, \; T\tau = \tau + 1. \end{aligned} \tag{2.8}$$

Generally, the complex coupling transform under $SL(2, \mathbb{Z})$ as $\tau \to \frac{a\tau+b}{c\tau+d}$.

2.2 't Hooft-Polyakov monopole

't Hooft-Polyakov monopole can be treated as a solitonic analogue to Dirac monopole in non-Abelian theory. Let's start with the classical setup of 't Hooft-Polyakov monopole. Consider a SU(N) theory with a scalar transforms in adjoint representation, the action is:

$$tr \int d^4x \; \frac{1}{g^2} (\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi D^\mu \Phi). \tag{2.9}$$

Trace taken over $SU(N)$, Φ is $N \times N$ traceless Hermitean matrix. To find the solution, we can apply the same trick as Bogomoln'yi [24]. First we set $\partial_0 = A_0 = 0$, s.t, the solution is time-independent and pure magnetic. Then we written down the monopole energy:

$$\mathcal{E} = tr \int d^3x \; \frac{1}{g^2}(B_iB^i + D_i\Phi D^i\Phi) \tag{2.10}$$

$$= tr \int d^3x \; \frac{1}{g^2}[(B_i \mp D_i\Phi)^2 \pm 2B_iD^i\Phi] \tag{2.11}$$

$$\geq \frac{2}{g^2} \int d^3x \; \partial_i tr(B^i\Phi). \tag{2.12}$$

The Bogomoln'yi bound satisfied when $B_i = \mp D_i\Phi$ (it's also called BPS (Bogomoln'yi–Prasad–Sommerfield) bound[24, 25], more generally). (A general inequity of mass of a particle with electric n, magnetic m and flavor charge f required by BPS bound is $M \geq |Z| = |n \cdot a + m \cdot 2\tau a + f \cdot \mu|$, where $|2a|$ is the mass of W-boson, μ is the mass term constant.) There's no constraint for Φ, so we can pick the vev we want. Here we set

$$\langle\Phi\rangle = \begin{pmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_N \end{pmatrix} = \vec{\Phi} \cdot \vec{H}, \tag{2.13}$$

$\vec{\Phi}$ is root vector with $\Phi_a > 0, \forall a \neq N$ and $\sum_{a=0}^{N} \Phi_a = 0$. $\vec{H}$ is basis for $SU(N)$ Cartan subalgebra (simple root for $SU(N)$ is $\vec{\alpha_i} = (0, \cdots, 1_{i-th}, -1_{i+1-th}, \cdots, 0)$). If $\Phi_i \neq \Phi_j, \forall i \neq j$, the gauge group broken into the maximal torus $SU(N) \to U(1)^{N-1}$. The moduli space for vev given by $SU(N)/U(1)^{N-1}$. The monopole is particle-like object with co-dimension 1, and in our construction, it localised in space, so we can consider the vev on the spatial boundary S^2_∞, which defines a map $S^2_\infty \to SU(N)/U(1)^{N-1}$. This map can be classified by second fundamental group:

$$\Pi_2(SU(N)/U(1)^{N-1}) \cong \Pi_1(U(1)^{N-1}) \cong \mathbb{Z}^{N-1}. \tag{2.14}$$

From the vacuum condition $D_\mu\Phi_a = 0$ when $\Phi = \langle\Phi\rangle$, we can derive the expression for B_i, then we will find the relation between magnetic charge and the winding number (we omit the proof here):

$$\vec{m} = \int_{S^2_\infty} d^2xB_i = \frac{4\pi\vec{\nu}}{g}, \; \vec{\nu} \in \mathbb{Z}^{N-1}. \tag{2.15}$$

The energy bound of monopole in different topological sectors labelled by winding number is $\mathcal{E} \geq \frac{2\pi}{g^2} \sum_{a=1}^{N-1} \nu_a\Phi_a$. More generally, $\vec{m}$ take value in dual of root lattice of $\mathfrak{g}$ modulo the Weyl group, or weight lattice of dual of $\mathfrak{g}$ modulo the Weyl group.

2.3 Vector multiplets and hypermultiplets

We use the $\mathcal{N}=1$ formulation to construct the multiplets for $4d$ $\mathcal{N}=2$ theory. An $\mathcal{N}=2$ vector multiplet consist of a $\mathcal{N}=1$ vector multiplet and a $\mathcal{N}=1$ chiral multiplet in adjoint representation of G.

$$\begin{array}{ccccccl} & \swarrow\!\!\nearrow & \lambda_\alpha & \leftrightarrow & A_\mu & & \mathcal{N}=1 \text{ vector multiplet} \\ \Phi & \leftrightarrow & \tilde{\lambda}_\alpha & \swarrow\!\!\nearrow & & & \mathcal{N}=1 \text{ chiral multiplet} \end{array} \tag{2.16}$$

An $\mathcal{N}=2$ hypermultiplet consist of a $\mathcal{N}=1$ chiral multiplet and a $\mathcal{N}=1$ anti-chiral multiplet in bi-fundamental representation of G.

$$\begin{array}{ccccccl} & \swarrow\!\!\nearrow & Q & \leftrightarrow & \psi & & \mathcal{N}=1 \text{ chiral multiplet} \\ \tilde{\psi}^\dagger & \leftrightarrow & \tilde{Q}^\dagger & \swarrow\!\!\nearrow & & & \mathcal{N}=1 \; anti-chiral \; multiplet \end{array} \tag{2.17}$$

We demand a symmetry does not commute with the supersymmetry generators, the R-symmetry. Here we have $SU(2)_R$ symmetry among gaguinos λ_α, $\tilde{\lambda}_\alpha$ and also among Weyl fermions Q and $\tilde{Q}^\dagger$.

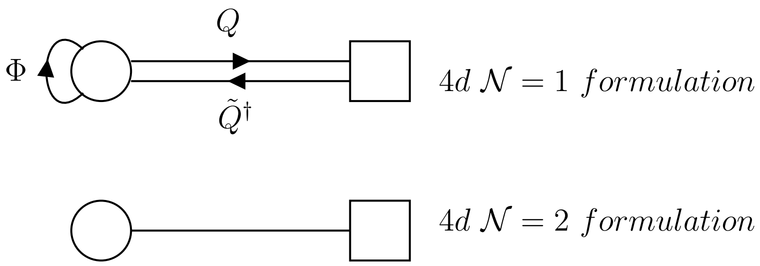
In $3d$ $\mathcal{N}=4$, we can get the ingredients of multiplets through dimensional reduction. According to the branching rule:

$$\begin{aligned} &[1,0]_{so(4)} \to [1]_{su(2)}, \\ &[0,1]_{so(4)} \to [1]_{su(2)}, \\ &[1,1]_{so(4)} \to [2]_{su(2)} \oplus [0]_{su(2)}. \end{aligned} \tag{2.18}$$

The hypermultiplets remain intact, and in vector multiplets, one component of vector boson A_μ is fixed to be scalar.

2.4 Quiver gauge theory

Now we consider quiver representation of theories[26]. Start from unitary quiver with $4d$ $\mathcal{N}=1$ formulation, circle node with label N represents a $U(N)$ gauge group, square node with label k represents a $SU(k)$ flavour group. Line with arrow from node A to B means chiralmultiplet transform in anti-fundamental representation of group of A and fundamental representation of group of B. When $A=B$, it's a vector multiplet (or hypermultiplet) transform in adjoint representation of group A. The superpotential can be written as $\mathcal{W} \sim tr\tilde{Q}^\dagger \Phi Q$. In $4d$ $\mathcal{N}=2$ formulation, a pair of lines with different arrow direction is replaced by a line without arrow means hypermultiplet transform in bi-fundamental representation, and we assign a vector multiplet to every circle node automatically.



2.5 Coulomb branch and Higgs branch

The terms of Coulomb branch and Higgs branch firstly appears in [6]. The classical moduli space of supersymmetric vacua is the zero locus of the scalar potential. In our case, the moduli space can be split into Coulomb branch, Higgs branch and Mixed branch depending on how the symmetry are broken.

1. The Coulomb branch $\mathcal{M}_C$ is where the scalars in vector multiplets take generic vev in Cartan subalgebra $\mathfrak{h}$ of gauge algebra $\mathfrak{g}$ quotient by Weyl group $\mathcal{W}$, while the scalars in hypermultiplets vanish. The flavour symmetry and $SU(2)_R$ symmetry is preserved. The complex dimension of Coulomb branch is

$$dim_{\mathbb{C}}(\mathcal{M}_C) = rank(G). \tag{2.19}$$

   In general, Coulomb branch will receive quantum corrections, it's required to be a *special Kähler* manifold after q.c.

2. The Higgs branch $\mathcal{M}_H$ is where the scalars in vector multiplets vanish, and the scalars in hypermultiplets take vev with respect to D-term and F-term equation, quotient by gauge group G. The flavour symmetry and $SU(2)_R$ symmetry is broken. The complex dimension of Higgs branch is

$$dim_{\mathbb{C}}(\mathcal{M}_H) = 2(n_H - n_V). \tag{2.20}$$

   Where n_H is the number of hypermultiplets and n_V is the number of vector multiplets. Higgs branch is a *hyper-Kähler* manifold by definition. It doesn't receive any perturbative quantum correction due to the supersymmetric non-renormalisation theorem[27].

3. The mixed branch $\mathcal{M}_{Mixed}$ is where scalars from vector multiplets and hypermultiplets both take generic non-vanishing vev. Gauge symmetry, flavour symmetry and $SU(2)_R$ are broken. The Mixed branch is locally a metric product of a *special Kähler* base and a *hyper-Kähler* fiber. The $4d$ $\mathcal{N} = 4$ SYM theories, which can be seen as $4d$ $\mathcal{N} = 2$ SYM coupled with a adjoint hypermultiplet, have moduli space $\mathbb{C}^{3r}/\Gamma$ parameterised by adjoint scalars from

$N = 2$ vector multiplets and $N = 2$ adjoint hypermultiplets, this is an example of mixed branch.

With our definition of $\mathcal{M}_C, \mathcal{M}_H, \mathcal{M}_{Mixed}$, the whole moduli space can be written as a union

$$\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H \cup \mathcal{M}_{Mixed}. \tag{2.21}$$

In $3d$ $\mathcal{N} = 4$, the Higgs branch will be the same as in $4d$ $\mathcal{N} = 2$, in fact, this will also apply to $5d$ and $6d$ theories with 8 supercharges. The Coulomb branch in $3d$ is *hyper-Kähler*, there is a hidden ingredient after we perform the dimensional reduction to $3d$. Recall the electromagnetic duality we have mentioned before, we say the vector A_μ in hypermultiplet is a 1-form gauge field. Then the field strength $F = dA$ is a 2-form. Now we perform the Hodge dual, and resulting dual field strength $F_D \sim \star F$ is a 1-from (in d-dim manifold, the Hodge dual of a p-form is a (d-p)-form), which can be described by a 0-from (scalar) dual field. Now we have 2 scalars (from $4d$ $N = 2$ vector multiplet) + 1 scalar (from dimensional reduction of vector) + 1 scalar (from the dual description of vector) to parameterise the Coulomb branch, which result in a *hyper-Kähler* manifold[6, 28]. From [29, 27], when we promote constants to background superfields:

1. Scalars in gauge coupling transform in $([0]+[2])\times[0]$ representation of $SU(2)_L \times SU(2)_R$, which indicates that it can only appears in Coulomb branch. Hence, Higgs branch is non-renormalised;

2. Scalars in mass term transform in $[2] \times [0]$ representation of $SU(2)_L \times SU(2)_R$, hence, only affected the metric on Coulomb branch;

3. Scalars in Fayet-Iliopoulos term of $U(1)$ factors of gauge group transform $[2] \times [0]$ of $SU(2)_L \times SU(2)_R$, hence, only affected the metric on Higgs branch.

FI term associated with a hidden global symmetry which doesn't manifest in Lagrangian. For every $U(1)$ factors in gauge group, there is a topological conserved current $\star trF$, which give rise to FI term. The conservation law of the non-Abelian part of gauge group is violated by instantons [30]. All the $U(1)$s forms the Cartan $U(1)^r$ of a non-Abelian hidden global symmetry.

2.6 Monopole operator

We can generalised the idea of 't Hooft-Polyakov monopole to operator. Different from point operators, which are defined as local function of fields, monopole operator cannot be supported on an local point. The 't Hooft monopole operator in $3d$ may be created by dimensional reduction from $4d$ 't Hooft line operator. The monopole operator $V_m(x)$ can be defined by insert a Dirac monopole-like singularity at point x with magnetic charge m, i.e, we can set the boundary condition as $r \to 0$ to be[31, 32]:

$$\begin{aligned} A_\pm \sim \frac{m}{2}(\pm 1 - cos\theta)d\phi, \\ \sigma \sim \frac{m}{2r}. \end{aligned} \tag{2.22}$$

The $V_m(x)$ we just defined is a half BPS bare monopole operator (preserving SUSY of $3d$ $\mathcal{N}=2$), satisfying the BPS equation $(d - iA)\sigma = - \star F$. $A_\pm$ is connection 1-from in northern/southern patch of S^2 enclosing x, m take value in weight lattice of GNO (or Langlands) dual group of gauge group G[33], quotient by Weyl group, i.e, $\Lambda_w(\hat{\mathfrak{g}})/\mathcal{W}(\hat{\mathfrak{g}})$, and satisfies the quantization condition in section 1.2. The monopole operator may or may not charged under the topological symmetry group, the center of the dual group $Z(\hat{\mathfrak{g}}) = \Lambda_w(\hat{\mathfrak{g}})/\Lambda_r(\hat{\mathfrak{g}})$, which is the weight lattice quotient by root lattice of dual group. $V_m(x)$ can be dressed by constant background adjoint complex scalar ϕ commute with m, with same SUSY preserved.

2.7 Monopole formula

[34] Bare and dressed monopole operators take vev on Coulomb branch of $3d$ $\mathcal{N}=4$ theory and contribute to chiral ring. The R-charge of bare BPS monopole V_m is

$$\Delta(m) = -\sum_{\alpha \in \Delta_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)|. \tag{2.23}$$

Where the first term summing over positive roots Δ_+ of gauge group is contributed by vector multiplets, and the second term summing over weights of matter fields representation $\mathcal{R}_i$ under gauge group is contributed by hypermultiplets. In good ($\Delta > \frac{1}{2}$) or ugly ($\Delta = \frac{1}{2}$) theory, the R-charge coincides with the conformal dimension. [35] The Hilbert series of Coulomb branch of $3d$ $\mathcal{N}=4$ good or ugly theory given by monopole formula:

$$HS = \sum_{m \in \Lambda_w(\hat{\mathfrak{g}})/\mathcal{W}(\hat{\mathfrak{g}})} z^{J(m)} t^{\Delta(m)} P_G(t, m), \tag{2.24}$$

where t is the fugacity for R-symmetry, z is the fugacity for topological symmetry, $J(m)$ is topological charge, P_G is the dressing factor counting the Casimir invariants of residual gauge group compatible with m,

$$P_G(t, m) = \prod_{i=1}^{r} \frac{1}{1 - t^{d_i(m)}}, \tag{2.25}$$

where d_m is the degree of Casimir invariants.

2.8 Higher form symmetry

Higher form symmetry, also known as generalised global symmetry[1], generalise the concept of symmetry to a topological operator. In duality, the global symmetries of dual theories have to be matched, which including all higher form global symmetry. Analysis for 't Hooft-Wilson line operator is studied in [36]. Start with ordinary symmetry, when the symmetry group is continuous, the conserved charge Q is integral of Noether current $d-1$-form j,

$$Q(M^{(d-1)}) = \oint_{M^{(d-1)}} j, \tag{2.26}$$

where $M^{(d-1)}$ is a co-dimension 1 manifold. Consider symmetry transformation as an topological operator $U_g(M^{(d-1)})$ where g is element of global symmetry. $U_g(M^{(d-1)})$ is defined for both continuous and discrete symmetry, $U_g(M^{(d-1)}) \sim e^Q$ when continuous. The transformations satisfy associativity,

$$U_{g_1}(M^{(d-1)})U_{g_2}(M^{(d-1)}) = U_{g_1 g_2}(M^{(d-1)}). \tag{2.27}$$

The transformation we defined is topological, it changes only when the deformation of $M^{(d-1)}$ crosses an operator $V(P)$, charged under the symmetry, on a space time point P. When $M^{(d-1)}$ a closed $d-1$ sphere surrounding P,

$$U_g(S^{d-1})V_i(P) = R_i^j(g)V_j(P). \tag{2.28}$$

When $M^{(d-1)}$ is the entire space (which can be seen as an analog for ETCR),

$$U_g(M^{(d-1)})V_i(P) = R_i^j(g)V_j(P)U_g(M^{(d-1)}), \tag{2.29}$$

where $R_i^j(g)$ is the representation of g acts on V. The ordinary symmetry is a 0-form symmetry, means the charged operator V supported on a 0-dimensional point P. For higher p-form symmetry, associated with $d-p-1$-form Noether current when continuous, we define the symmetry operator associated to a co-dimension $p+1$ manifold

$$U_{g_1}(M^{(d-p-1)})U_{g_2}(M^{(d-p-1)}) = U_{g_1 g_2}(M^{(d-p-1)}). \tag{2.30}$$

The charged operator V under p-form symmetry supported on a p-dimensional manifold $C^{(p)}$. Then we have the following relations, when $M^{(d-1)}$ a closed $d-p-1$

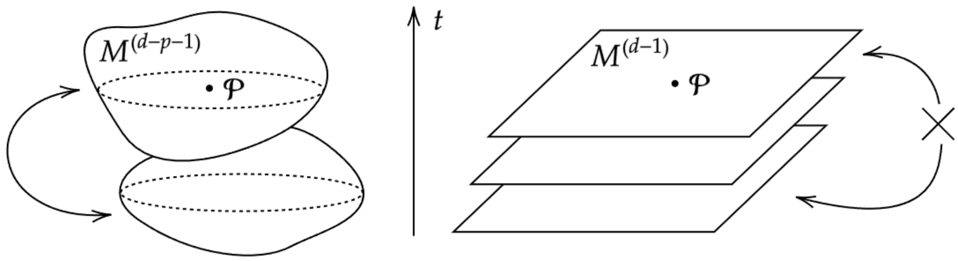


Figure 2.1: Abelian and non-Abelian symmetry demonstration

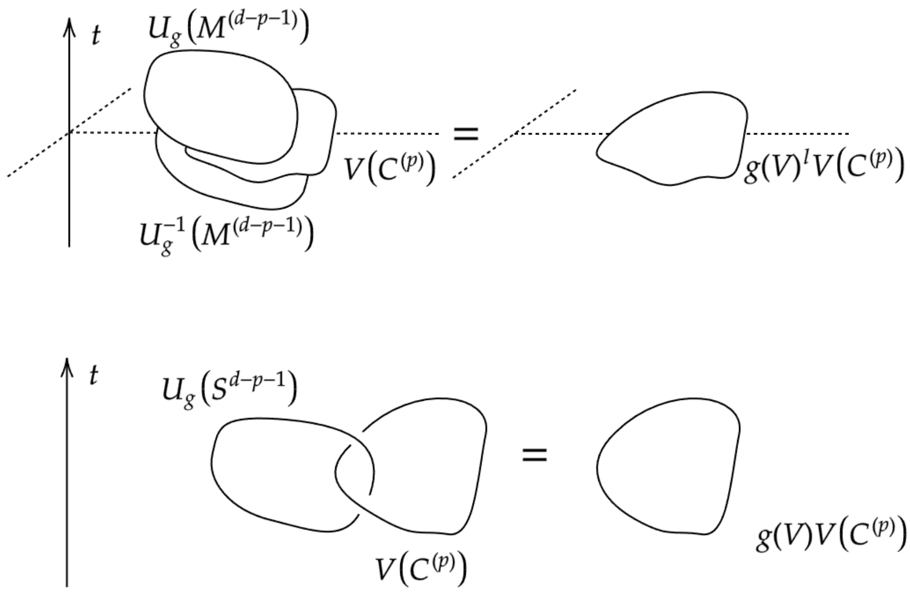


Figure 2.2: Ward identity / ETCR demonstration

sphere linking C,

$$U_g(S^{d-p-1})V(C^{(p)}) = g(V)V(C^{(p)}), \tag{2.31}$$

when $M^{(d-p-1)}$ is the entire space,

$$U_g(M^{(d-p-1)})V(C^{(p)} = g(V)^{l(C^{(p)},M^{(d-p-1)})}V(C^{(p)})U_g(M^{(d-p-1)}), \tag{2.32}$$

where $g(V)$ is the representation of g acts on V, $l(C^{(p)}, M^{(d-p-1)})$ is the linking number between $C^{(p)}$ and $M^{(d-p-1)}$.

p-form symmetry also has the following properties:

1. When $p \geq 2$, the symmetry is Abelian;

2. When $p \geq d-2$ and continuous, the symmetry can never be broken;

3. When $p \geq d - 1$ and discrete, the symmetry can never be broken;

In practice, when we have a theory with a simply connected gauge group G and no matters transforming under Γ, subgroup of center of G, the theory has a 1-form global symmetry Γ. (In standard model, there is an ambiguity to choose the true gauge group since all the matters are blind to the center of $U(1) \times SU(2) \times SU(3)$. Here is the reason.) In a $4d$ non-Abelian theory with gauge algebra $\mathfrak{g}$, a 't Hooft-Wilson line operator (defect) is labelled by charge $(n, m) \in \Lambda_w(\Lambda_w(\mathfrak{g}) \times \hat{\mathfrak{g}})$. The mutual locality of two lines imposed by $m \cdot n' - n \cdot m' = 0$ $(mod\ k)$ to guarantee the correlation function of line operators are local. Consider a $SU(2)$ gauge theory, the center $Z(SU(2)) = \mathbb{Z}_2$. We are able to further classify the theory by specifying the the representation of line operators under 1-form symmetry $\mathbb{Z}_2$[36]:

1. $\mathsf{SU(2)}$ Spectrum includes Wilson line in fundamental representation of gauge group $SU(2)$ $(n, m) = (1, 0)$. Due to mutual locality condition, the allowed spectrum (n, m) satisfies $n \in \mathbb{Z},\ m \in 2\mathbb{Z}$.

2. $\mathsf{SO(3)}_+$ Spectrum includes t' Hooft line in fundamental representation of dual gauge group $SU(2)$ $(n, m) = (1, 0)$, the allowed spectrum (n, m) satisfies $n \in 2\mathbb{Z},\ m \in \mathbb{Z}$.

3. $\mathsf{SO(3)}\_$ Spectrum includes dyonic line $(1, 1)$, the allowed spectrum (n, m) satisfies $n + m \in 2\mathbb{Z}$.

For a general $SU(N)$ theory, it can be further classified into $(SU(N)/\Gamma)_i$, Γ is subgroup of center $Z(SU(N))$, i labels the spectrum of line operators. For example, allowed spectrum of $SU(4)$ with discrete subgroup $\mathbb{Z}_4$ and $\mathbb{Z}_2$ are

1. $\mathsf{SU(4)}$ $n \in \mathbb{Z},\ m \in 4\mathbb{Z}$,

2. $(\mathsf{SU(4)}/\mathbb{Z}_2)_0$ $2n + m \in 4\mathbb{Z}$,

3. $(\mathsf{SU(4)}/\mathbb{Z}_2)_1$ $n \in 2\mathbb{Z},\ m \in 2\mathbb{Z}$,

4. $(\mathsf{SU(4)}/\mathbb{Z}_4)_0$ $n \in 4\mathbb{Z},\ m \in \mathbb{Z}$,

5. $(\mathsf{SU(4)}/\mathbb{Z}_4)_1$ $n - m \in 4\mathbb{Z}$,

6. $(\mathsf{SU(4)}/\mathbb{Z}_4)_2$ $n + 2m \in 4\mathbb{Z}$,

7. $(\mathsf{SU(4)}/\mathbb{Z}_4)_3$ $n + m \in 4\mathbb{Z}$.

Under S and T transformation, spectrum changes follows the rule we introduced in Section 2.1, hence, the theories change between each other. The duality orbits of $SU(2)$ and $SU(4)$ are shown in Figure 2.3.

$$
\begin{tikzcd}
SU(2) \arrow[loop left, "T"] \arrow[r, leftrightarrow, "S"] & SO(3)_+ \arrow[r, leftrightarrow, "T"] & SO(3)_- \arrow[loop right, "S"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & (SU(4)/\mathbb{Z}_4)_1 \arrow[dr, "T"] \arrow[dd, leftrightarrow, "S"'] & \\
SU(4) \arrow[loop left, "T"] \arrow[r, leftrightarrow, "S"] & (SU(4)/\mathbb{Z}_4)_0 \arrow[ur, "T"] \arrow[dr, "T"'] & & (SU(4)/\mathbb{Z}_4)_2 \arrow[d, leftrightarrow, "S"'] \\
(SU(4)/\mathbb{Z}_2)_+ \arrow[loop right, "{S, T}"] & & (SU(4)/\mathbb{Z}_4)_3 \arrow[ur, "T"'] & (SU(4)/\mathbb{Z}_2)_- \arrow[loop right, "T"]
\end{tikzcd}
$$

Figure 2.3: Duality orbit of $4d$ $\mathcal{N} = 4$ theories with $\mathfrak{g} = \mathfrak{su}(2), \mathfrak{su}(4)$[36]

.

3

Brane construction and quiver

Supersymmetric field theory can be constructed from string/M theory with gravity decoupled, which are genericly non-Lagrangian. This chapter we will focus on the Type IIB string theory approach to $3d\,\mathcal{N}=4$ quiver gauge theories.

3.1 Type IIB configuration

We use the same construction from[5]. The objects we need to engineer the $3d\,\mathcal{N}=4$ quiver gauge theory are D3 branes, D5 branes and NS5 branes. We put the D3 branes along x_0, x_1, x_2, D5 branes along $x_0, x_1, x_2, x_7, x_8, x_9$, and NS5 branes along $x_0, x_1, x_2, x_3, x_4, x_5$.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	•	•	•				•			
D5	•	•	•					•	•	•
NS5	•	•	•	•	•	•				

Table 3.1: Spacetime direction of Type IIB configuration.

All the brane solutions in Type IIB are half-BPS states, which means preserve half of the supersymmetry. For a D5 brane located at fixed x_3, x_4, x_5, x_6, to preserve a half SUSY, we require the Killing spinor satisfies

$$\epsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9\epsilon_R. \tag{3.1}$$

For a NS5 brane located at fixed x_6, x_7, x_8, x_9, the Killing spinor satisfies

$$\epsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon_L,\ \ \epsilon_R = -\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\epsilon_R. \tag{3.2}$$

With both D5 branes and NS5 branes as our setup, one quarter SUSY are preserved, i.e, we have $32/4 = 8$ supercharges. We can further derive from the above that if the Killing spinor satisfy the conditions for D5 branes and NS5 branes above, it will automatically satisfy

$$\epsilon_L = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_6 \epsilon_R. \tag{3.3}$$

This means we can import D3 branes, as describe at the beginning, without breaking SUSY any further. The Lorentz group $SO(1,9)$ breaks into $SO(1,2) \times SO(3)_V \times SO(3)_H$, with $SO(3)_V$ acts on $\vec{m}$ and $SO(3)_H$ acts on $\vec{w}$. The double cover of $SO(3)_V$ and $SO(3)_H$, which are $SU(2)_V$ and $SU(2)_V$, will act as symmetry on Coulomb branch and Higgs branch respectively. We denote $(\vec{m}, z_i) = (x_3, x_4, x_5, x_6)$ and $(t_j, \vec{w}) = (x_6, x_7, x_8, x_9)$, as position of *i-th* D5 brane and *j-th* NS5 branes respectively, and $(\vec{x}_k, \vec{y}_k)$ as $(\vec{m}, \vec{w})$ position of *k-th* D3 brane. Instead of infinite D3 branes, here we will consider D3 branes ends on fivebranes, which has three infinite directions (x_0, x_1, x_2), and on finite direction x_6. The worldvolume theories on finite D3 branes are $2{+}1d$ theories, with R-symmetry exactly $SU(2)_V \times SU(2)_V$. There are three cases for D3 branes:

1. When ends on NS5-NS5, each D3 brane gives a $U(1)$ vector multiplet, which will be enhanced from $U(1)^n$ to $U(n)$ via Chan-Paton factors when n D3 branes coincide. The vector multiplets can be identified with open strings ends on same or different D3 branes. The position condition to allow this happen is $\vec{y} = \vec{w}_1 = \vec{w}_2$, D3 is free to move along $\vec{m}$;

2. When ends on D5-D5, each D3 brane gives a massless hypermultiplet. The position condition is $\vec{x} = \vec{m}_1 = \vec{m}_2$, D3 is free to move along $\vec{w}$. Under mirror symmetry the D5 branes convert into NS5 branes and massless hypermultiplets convert to vector multiplets, this gives $U(n)$ theory;

3. When ends on D5-NS5, no massless modes, no moduli, only one D3 brane is allowed between each pair of D5 and NS5 branes. The position condition is $\vec{x} = \vec{m}, \vec{y} = \vec{w}$.

We start with electric theory, i.e, D3 branes ends on D5 branes. The coupling constant of D3 worldvolume theory stretch between *i-th* and $i+1$-*th* NS5 branes is $\frac{1}{g^2} \sim |t_i - t_{i+1}|$. We call the theory after duality magnetic theory, with coupling constant $\frac{1}{g^2} \sim |z_j - z_{j+1}|$. In electric theory, the massless hypermultiplets arise when the D3 branes on different side of NS5 brane meet, which can be identified as string stretch between D3 branes on different sides. For example, k_1 D3 branes on the left and k_2 D3 branes on the right give rise to hypermultiplets transform in $[k_1, \bar{k_2}]$ representation of gauge group $U(k_1) \times U(k_2)$. Same logic applies for magnetic theory. Also, when a single D3 brane "cross" fivebrane, massless hypermultiplets arise due to string stretch between D3 brane and fivebrane.

Now let's look at the brane creation or annihilation process, the Hanany-Witten transition. As proposed in [5], the linking number of fivebranes invariant under transition. The linking number of A brane is $L_A = \frac{1}{2}(r_B - l_B) + (L - R)$, where L and R is the number of D3 branes end on the left and right side of A brane respectively, r_B and l_B is the number of B branes to the right and left side of A brane respectively, (A,B) = (D5, NS5) or (NS5, D5). When a D5 brane go cross a NS5 brane, 1. if a D3 connected in between, it annihilates; 2. If no D3 connected in between, a D3 brane is created between them after crossing. To find the corresponding quiver of a brane system, first we perform the HW transition until no D3 branes attached to D5 branes, then for each interval of two successive NS5 branes having n coinciding D3 branes, there is a $U(n)$ gauge node, and nodes of adjacent interval are connected with link 1. The quiver theory we acquire will have $\prod_i U(n_i)$ gauge symmetry. For each interval with k D5 branes, there is a square node with $U(k)$ flavour symmetry. We also need to decouple a $U(1)$ factor either from gauge or flavour symmetry, due to the translation symmetry (we can think of it as center of mass). Usually, like in K_G theory[37], we will decouple it from gauge symmetry when the rank of gauge group ≥ 2.

3.2 Mirror symmetry

In our configuration of Type IIB, the mirror symmetry can be explained as symmetry between D5 branes and NS5 branes, generated by $S \in SL(2, \mathbb{Z})$ exchange the type of D5 and NS5 branes and a rotation R change x_i to $x_i + 4$ and $x_i + 4$ to $-x_i$ for $i = 3, 4, 5$. Let take the configuration in Figure 3.1 as example. Take the vertical axis as x_3 and horizontal axis as x_6, in this setup, NS5 brane is a infinite vertical line, the D5 brane is point-like, and D3 is finite horizontal line. There is a D3 brane end on two NS5 branes, and N D5 branes between the two NS5 branes. The worldvolume theory corresponding to quiver (1)-[N]. Now we perform the RS transformation, in the mirror image, the vertical axis change to x_7 and horizontal axis remains x_6.

We summarise the properties of mirror symmetry [28, 37] in $3d$ $\mathcal{N} = 4$ theories as follow:

	Mirror	
Complex Structure		Complexified *Kähler* form
Coulomb branch $\mathcal{M}_C$		Higgs branch $\mathcal{M}_H$
Mass term $\vec{m}$	$\leftrightarrows$	FI term $\vec{w}$
R-symmetry $SU(2)_L$		R-symmetry $SU(2)_R$
Gauge symmetry G		Global symmetry G_F

Table 3.2: Mirror symmetry in $3d$ $\mathcal{N} = 4$

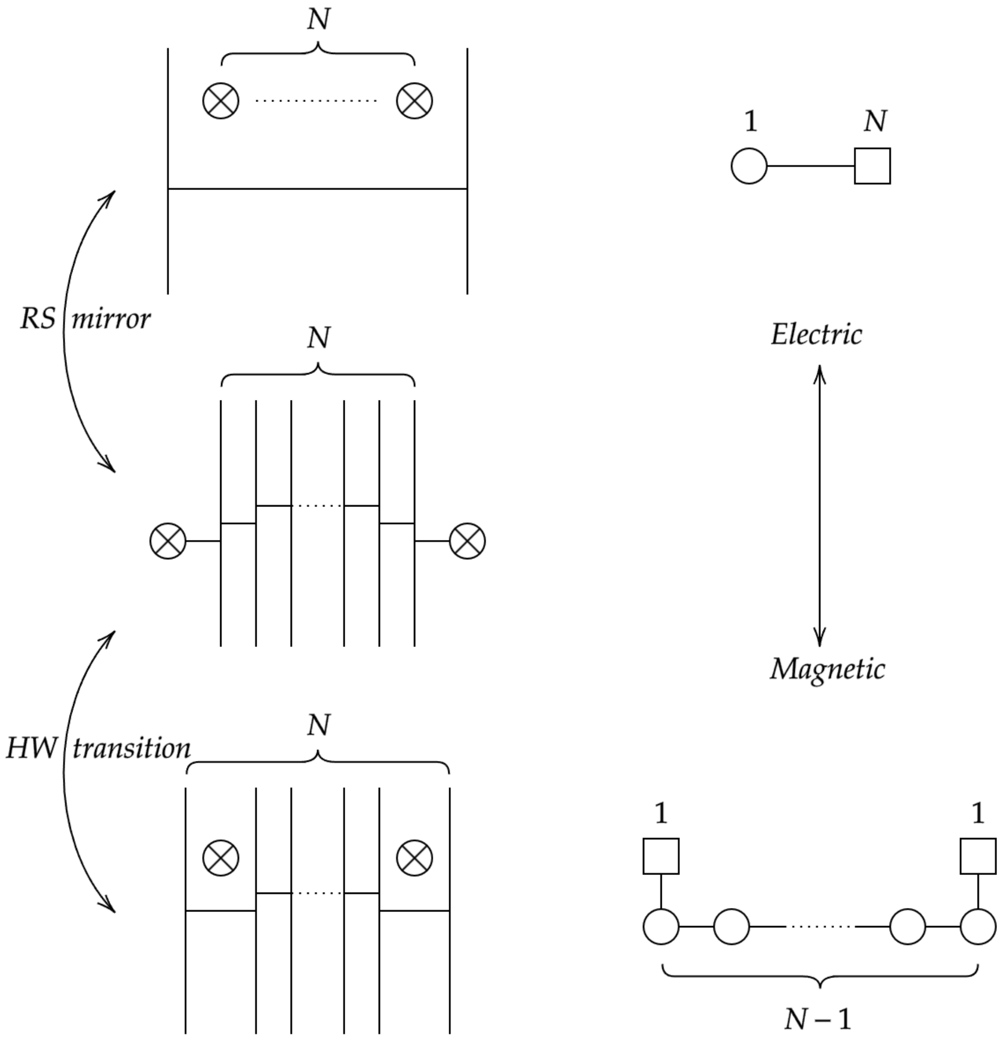


Figure 3.1: Mirror symmetry in practice: brane realisation of SQED A_N quiver and its dual a_N quiver

3.3 Hasse diagram

Every symplectic singularity (*hyper-Kähler* cone) admits a finite stratification $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$, s.t, the singular part of X_i+1 is X_i and the normalisation of any reducible components of X_i is a symplectic singularity[38]. The stratification is not unique generally.

With the technique of Kraft-Procesi transition[39, 40], later generalised by quiver subtraction[41], we can embed the symplectic structure of the moduli space of a quiver gauge theory into a finite partially ordered set, also called Hasse diagram. We call the node in Hasse diagram leaves and line transverse slices (also a symplectic singularity). It's conjectured in [42] that the slices for generic symplectic singularities are the same as for nilpotent orbit closures. A more recent job to proceed quiver subtraction as affine Grassmannian is discussed in [43]. To construct Hasse diagram, we start from a magnetic Q,

1. We find all subquiver D which are elementary slices (The elementary slice we commonly use can be found in Appendix B.), align Q and D;
2. Subtract the ranks of D from D, and add $U(1)$ nodes to add Q', which matches the balance of remaining nodes with Q.
3. Identify D as line of transverse slice between nodes of Q' and Q'.

This agrees with the result from partial Higgsing[44, 45], or more precisely, Kibbling[46, 47].

Let's check for the following cases:

Higgs branch of $SU(3)$ with N fundamental hypermultiplets

The gauge symmetry $SU(3)$ can be potentially broken into $SU(2)\times U(1)$, $SU(2)$, $U(1)\times U(1)$, $U(1)$, $\{1\}$. Firstly consider $SU(3)\to SU(2)\times U(1)$, the branching rule is

1. $[1,0]_{SU(3)} \to q[1]_{SU(2)} + q^{-2}[0]_{SU(2)}$;
2. $[0,1]_{SU(3)} \to q^{-1}[1]_{SU(2)} + q^{2}[0]_{SU(2)}$;
3. $[1,1]_{SU(3)} \to [2]_{SU(2)} + (q[1]_{SU(2)} + q^{-1}[1]_{SU(2)} + [0]_{SU(2)})_{\ acquire\ mass}$.

The W-bosons acquire mass by absorbing components of hypermultiplets in the same representation, which cannot happen due to the $U(1)$ charge. We can make the observation that $SU(3)\to SU(2)$ is allowed. There will be

1. $1[0]_{SU(2)}$ absorbed and $(2(N-1)+1)[0]_{SU(2)}$ becomes transverse (trivial under $SU(2)$), forming a $U(1)$ theory with $N-1$ fundamental hypermultiplets;
2. $[2]_{SU(2)}+2(N-2)[1]_{SU(2)}$ remaining form $SU(2)$ theory with $N-2$ fundamental hypermultiplets.

(We should be careful about the bi-fundamental when counting numbers in $4d$ $N=1$ formulation.)

Higgs branch of $SU(4)$ with N fundamental hypermultiplets

The same argument can be conducted for $SU(4)$, first we consider $SU(4)\to SU(3)$, the branching rule is

1. $[1,0,0]_{SU(4)} \to [1,0]SU(3) + [0,0]SU(3)$;
2. $[0,0,1]_{SU(4)} \to [0,1]SU(3) + [0,0]SU(3)$;

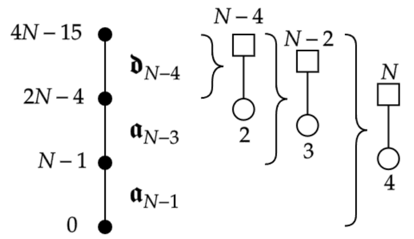


Figure 3.2: Hasse diagram of $SU(4)$ with N fundamental hypermultiplets

3. $[1,0,1]_{SU(4)} \to [1,1]_{SU(3)} + ([1,0]_{SU(3)} + [0,1]_{SU(3)} + [0,0]_{SU(3)})_{\ acquire\ mass}$.

The transverse and remaining theory are

1. $1[0]_{SU(3)}$ absorbed and $(2(N-1)+1)[0]_{SU(2)}$ becomes transverse (trivial under $SU(3)$), forming a $U(1)$ theory with $N-1$ fundamental hypermultiplets;

2. $[1,1]_{SU(3)} + (N-2)[1,0]_{SU(3)} + (N-2)[0,1]_{SU(3)}$ remaining form $SU(3)$ theory with $N-2$ fundamental hypermultiplets.

Compared with the Hasse diagram from quiver subtraction, we find agreement.

The above examples show the Hasse diagram of Higgs branch (or Coulomb branch of magnetic theory), now we try to construct the Hasse diagram for full moduli space. Invertible Hasse diagram [48]. First we define the inversion of Hasse diagram with only ADE or nilpotent orbit closure type slices as inverting the partial ordering and exchange the singularity type of ADE with nilpotent orbit closure with each other. For example, the inverse of partial ordering $\mathfrak{d}_5 \to \mathfrak{a}_7$ is $A_7 \to D_5$. Once we have the Hasse diagram of Higgs branch (or Coulomb branch), we can use inversion to generate the diagram for the other branch, then for every node we add the inversion of sub-diagram with higher partial order attach to the node, then identify all the nodes can reach to a same node by subtracting same slices up to permutation. Examples will be given in Section 4.6.

4

$5d$ brane webs and torics

The strong coupling non- trivial fixed point of $5d$ $\mathcal{N}=1$ is studied in [9, 10, 11]. Different gauge theories can be reached by different mass deformation of the same SCFT, we call it "UV duality", which means a SCFT may have different low energy description.

The prepotential description of $5d$ $\mathcal{N}=1$ theory is:

$$\mathcal{F}=\frac{1}{2}m_0 h\delta_{ij}\phi^i\phi^j+\frac{1}{6}kd_{ijk}\phi^i\phi^j\phi^k+\frac{1}{12}(\sum_{e\in\Lambda_r(\mathfrak{g})}|e\cdot\phi|^3-\sum_i\sum_{w\in\mathcal{R}_i}|w\cdot\phi+m_i|^3),\quad(4.1)$$

where $m_0=\frac{1}{g_0^2}$ is the ground instanton mass, h is the Dynkin index of gauge group G, k is the Chern-Simons (CS) level, d_{ijk} is the anomaly index of G, $\Lambda_r(\mathfrak{g})$ is the root lattice of gauge algebra $\mathfrak{g}$, w is the weight of hypermultiplet in representation $\mathcal{R}_i$. The magnetic monopole string tension is $T=\partial_i\mathcal{F}$, the effective gauge coupling is $\frac{1}{g^2}=\partial_i\partial_j\mathcal{F}$, the cubic Chern-Simons coupling $C=\partial_i\partial_j\partial_k\mathcal{F}\in\mathbb{Z}$. There is a topological $U(1)_I$ symmetry with current $J=\star trF\wedge F$, instantons charged under this symmetry. The global symmetry can be enhanced by instanton in UV fixed point, $G_{enhanced}\supset G\times U(1)_I$. For example, the $SU(2)$ theories with N_f fundamental matters can has enhanced symmetries from $SO(2N_f)\to E_{N_f+1}$, $E_1=SU(2), E_2=SU(2)\times U(1), E_3=SU(3)\times SU(2), E_2=SU(5), E_6, E_7, E_8$.

Here we will focus on theories which can be represented by brane webs, which are $SU(N_c)$ gauge theory with N_f flavours and Chern-Simons level k, and the condition $N_c-\frac{1}{2}N_f+2-|k|\geq 0$[12].

4.1 Type IIB again

Now let's see the brane web configuration[49]. As in the last chapter, we construct brane web in Type IIB. We introduce (p,q) fivebrane as bound state of D5 brane and NS5 brane, with tension $T_{p,q}=|p+\tau q|T_{D5}$, where τ is the axiodilaton. The

$(1,0)$ fivebrane is D5 brane, $(0,1)$ fivebrane is NS5 brane.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
(p,q) *fivebrane*	•	•	•	•	•	p	q			
(p,q) *sevenbrane*	•	•	•	•	•			•	•	•

Table 4.1: Spacetime direction of 5D brane web configuration

By setting $\tau = i$, we can represent fivebrane with charge (p,q) by lines with slope $\Delta y/\Delta x = q/p$. Vertices are allowed if charge neutral, i.e, $\sum p_i = \sum q_i = 0$. The Lorentz group $SO(1,9)$ is broken into $SO(1,4) \times SO(3)$, the double cover of $SO(3)$ forms the R-symmetry of $5d$ $\mathcal{N} = 1$ theory, $SU(2)_R \cong Sp(1)_R$. Brane web also has $SL(2,\mathbb{Z})$ symmetry on (x,y) plane imported by τ.

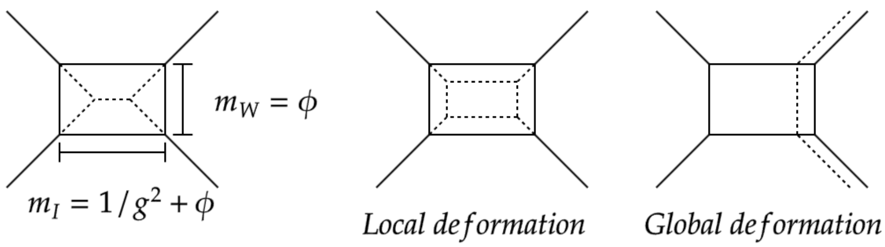


Figure 4.1: Spectrum and deformation of 5d brane web

Take brane web for pure SU(2) theory with $\theta = 0$ as example, we can identify the number # of local deformation with the dimension of Coulomb branch, the number # of global deformation minus 3 (due to the positions of three external legs are redundant to determine the shape) with the rank of global symmetry group, which is the topological symmetry group $U(1)_I$ here, instanton operators charged under this symmetry. We can also the read the spectrum by (as shown in figure 4.1, we fixed $\tau = i$)

1. identifying the fundamental string stretch between two D5 brane with W-boson, whose mass is the length times string tension, $m_W = T_s\Delta y = \phi$;

2. Identifying the D string stretch between two NS5 brane with instanton, whose mass is the length times string tension, $m_I = T_s\Delta x = \phi + \frac{1}{g^2}$, where g is the coupling constant;

3. Identifying the 3-brane wrapping on a face in web with monopole[49], the monopole tension is given by the area of the face in unit of T_s^2, which is $T_m = T_s^2\Delta x\Delta y = \phi(\phi + \frac{1}{g^2})$.

They are all BPS saturated states. (Spectrum with mixed charge can be found in [50].) On Higgs branch $\phi = 0$, we have massless boson and monopole. We set

$g \to \infty$ to reach the infinite coupling fixed point, where instanton also becomes massless. The massless instanton turns on the Higgs branch by modifying the chiral ring relation from $S^2 = 0$ to $S^2 = I^+I^-$. With $\phi = \frac{1}{g^2} = 0$, we have a conformal field theory with enhanced $E_1 = SU(2)$ global symmetry.

On top of pure gauge theory, we add N_f external (p,q) fivebrane to introduce matter fields, the charge of all external fivebrane determines the CS level, the quark mass identified with the distance between external fivebrane and D5 brane. We can also equivalently let each external semi-infinite (p,q) fivebrane ends on a (p,q) sevenbranes with an $SL(2,\mathbb{Z})$ monodromy $M_{p,q}$ for τ,

$$M_{p,q} = \begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix}, \quad \tau \to \frac{(1+pq)\tau - p^2}{q^2\tau + 1 - pq}. \tag{4.2}$$

A (p',q') brane crossing the monodromy becomes $M_{p,q}(p',q')^T$. An element $g \in SL(2,\mathbb{Z})$ act on monodromy as $gM_{p,q}g^{-1}$. Now we can move the branes with respect to Hanany-Witten transition:

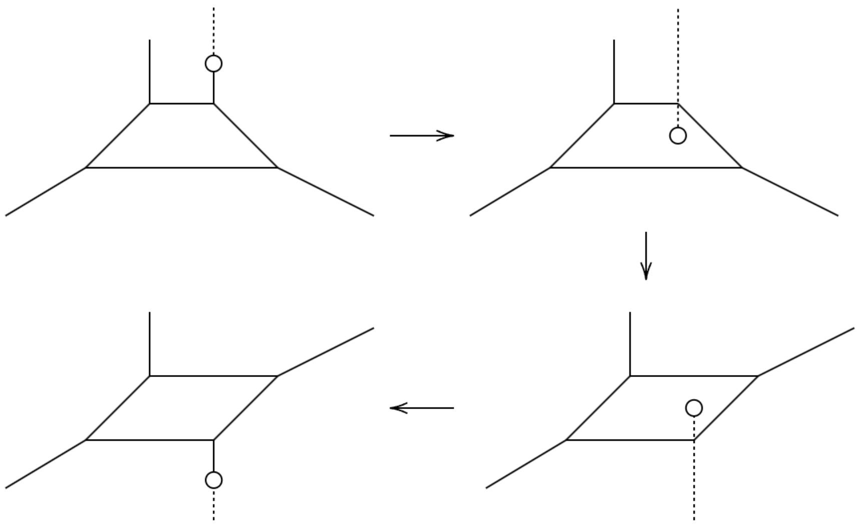


Figure 4.2: HW transition from $\mathbb{F}_2$ to $\mathbb{F}_0$
Figure 6. in [51]

We can identify the string stretch between color brane and flavour brane, and the quark mass with length of the string times tension, $m_Q = |m \pm \frac{1}{2}|$. We can perform the integrating out by move external fivebranes to infinite, start from a theory with N_f and k, we can result with $N_f' = N_f - n$ and $k - \frac{n}{2} \leq k' \leq k + \frac{n}{2}$ [52][49][53].

A technique to reach the theories violate the bound $N_c - \frac{1}{2}N_f + 2 - |k| \geq 0$ by introducing and integrating out "pseudo" hypermultiplets was proposed in [54].

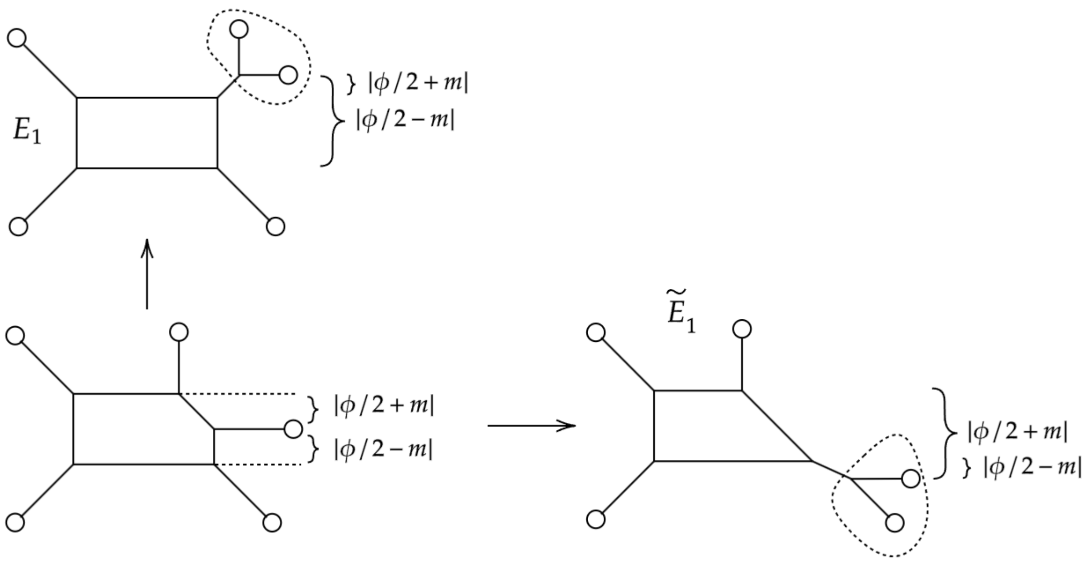


Figure 4.3: Integrating out from $SU(2) + 1\mathbf{F}$

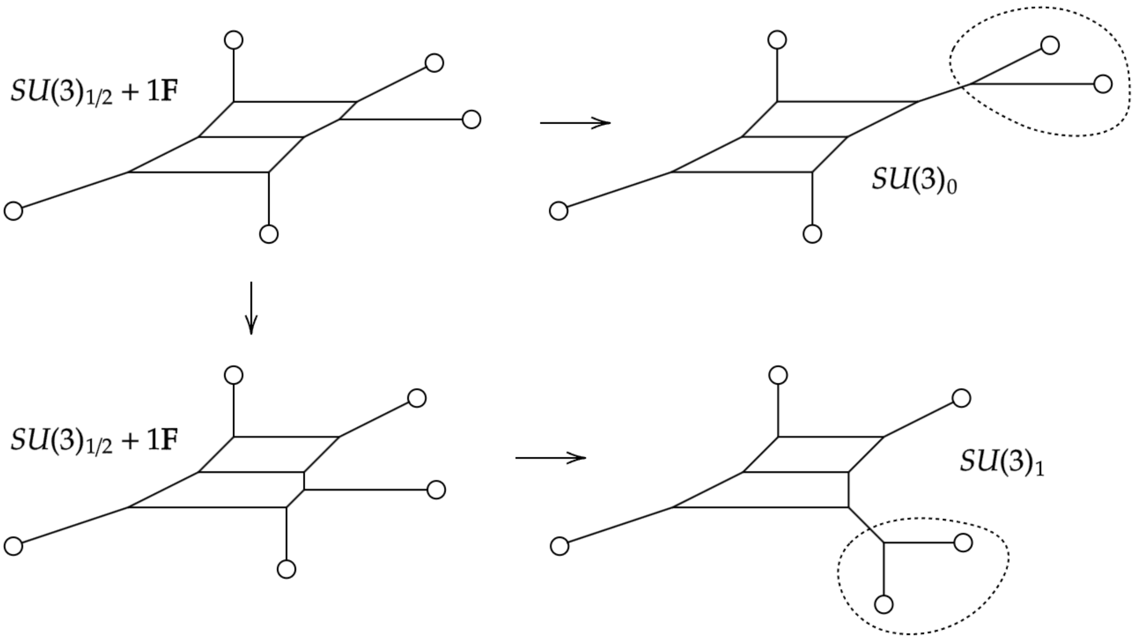


Figure 4.4: Integrating out from $SU(3)_{\frac{1}{2}+1\mathbf{F}}$

$SU(2)$ with 8 flavours is in fact a $6d$ theory (small E_8 instanton)[55] compactified on a circle S^1, we can calculate the monodromy and lead to identity. All $SU(N)_0 + (2N+4)\mathbf{F}$ theories are also KK theories compactified from $6d$ theory with/without twist.

We can represent brane web with its dual toric diagram[50], exchanging the vertices with polygons, faces with points, and (p,q) edge with $\pm(-q,p)$ line orthogonal to it. The similar idea as we associated web with toric singularity. It's can be proofed that the toric diagram has to be convex. We use two type of points in toric diagram[56]: the solid node $\bullet$ and empty node $\circ$:

1. A line as $\bullet - \bullet$ means a fivebrane ends on a sevenbrane;

2. A line as $\bullet - n \circ - \bullet$ means $n+1$ fivebranes ends on the same sevenbrane.

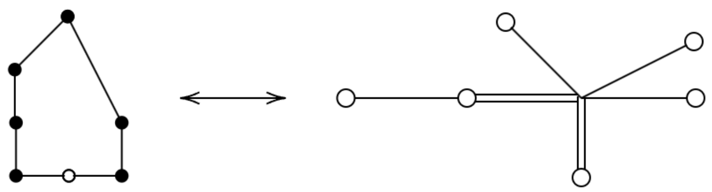


Figure 4.5: Web and its dual toric

The toric diagram was generalised in [8], with s-rule and r-rule required.

4.2 Generalised toric polygon

A generalised toric polygon on a $2d$ lattice consist of

1. a set of black vertices $\bullet$, $V_b = v_i \in \mathbb{Z}^2$,

2. a set of edges, $E = E_\alpha \in V(\mathbb{Z}^2)$, connect a subset of vertices,

3. a set of white vertices $\circ$, $V_w = (\partial E \cap \mathbb{Z}^2) \setminus V_b$,

specified by the following data:

1. $\lambda_\alpha = gcd(E_\alpha)$, the greatest common divisor of two components of E_α as a $\mathbb{Z}^2$ vector;

2. $L_\alpha = \frac{E_\alpha}{\lambda_\alpha}$, the reduced vector of edges;

3. $\mu_\alpha = (\mu_{\alpha,1}, \cdots, \mu_{\alpha,b_\alpha+1}) \in \mathcal{P}(\lambda_\alpha)$, the partition of λ_α as the distribution of $\bullet$ and $\circ$ on E_α, where b_α is the number of internal $\bullet$ on E_α.

For example, the $SU(3)_1 + 6\mathbf{F}$ theory at infinite coupling can be represented by

$$\begin{aligned} &L = ((0,-1),(1,0),(0,1),(-1,1),(-1,0)), \\ &\lambda = (4,2,3,1,1), \ \mu = ((1^4),(2),(1^3),(1),(1)) \end{aligned} \tag{4.3}$$

4.3 S-rule, r-rule

We require all GTPs of $5d$ theories to satisfy the s-rule, i.e. the rule to preserve the SUSY. Here we follow the same rule from [8] which is a modification of [56]. We can define a "Tile" as a convex polygon consist of E_1, E_2, E_3, E_4, with requirements

1. $L_2 = -L_4$ with $\lambda_2 \geq \lambda_4$, λ_4 can be zero, which means the tile is either trapezoid or triangle;

2. $\mu_\alpha = (\lambda_\alpha)$, no $\bullet$ as internal point of edges;

3. define $\tilde{L} = (L_1, L_2, L_3)$, $\tilde{\lambda} = (\lambda_1, \lambda_2 - \lambda_4, \lambda_3)$, then, $\tilde{\lambda}_\alpha \tilde{\lambda}_\beta |det(\tilde{\lambda}_\alpha \tilde{\lambda}_\beta)| \geq \tilde{\lambda}_\gamma^2$, $\forall \alpha \neq \beta \neq \gamma$.

The s-rule requires the GTPs can be tessellated with tiles. Also, an r-rule is required, which means r, the number local deformation, the rank of the gauge theory it represent, is no smaller than zero. We define the rank of a GTP as

$$r = Area + 1 - \frac{1}{2} \sum_\alpha (\mu_\alpha)^2. \tag{4.4}$$

Web with $E = \{(-N, 1), (N, 1), (0, 1), (0, 1)\}$ obeys s-rule so long as N $\geq$ 2.

4.4 Magnetic quiver, from decomposing webs

Now we can move to the next step, [7] proposed to use maximal decomposition and tropical intersection[57] to determine the magnetic quiver of the 5D theory

$$\mathcal{H}_{3\mathcal{D}}(Brane\ Web) = \bigcup_{Max.Decops} \mathcal{C}_{5\mathcal{D}}(Magnetic\ Quiver).$$

First, we decompose the web into subwebs maximally, each n coincided subweb represent a unitary n node of magnetic quiver. The intersection between two intersected fivebranes with charge (p_1, q_1) and (p_2, q_2) is

$$I = det \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}, \tag{4.5}$$

the stable intersection between subwebs are the total intersection stable under a shift. It will also receive correction ± 1 if fivebranes from different subwebs end on the same/different side of a same sevenbrane. Stable intersection represent the number of links between corresponding nodes. Since the decomposition is not unique genericly, we will get multiple magnetic quivers. Then, the overall moduli we have is the union of cones we get from magnetic quivers, the intersection can also be calculated easily with web. However, the discrete contribution to the moduli space of $5d$ $\mathcal{N}=1$ theories by glueball operators cannot be reflected by brane web, we will see in the later examples.

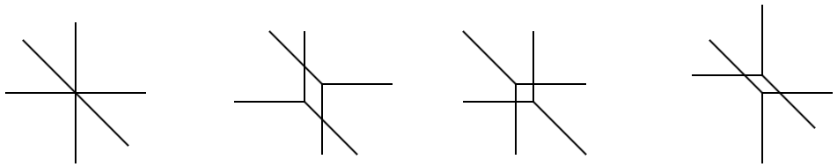


Figure 4.6: Webs with same stable intersection 2

4.5 Tropical quiver, from coloring torics

[8] proposed a algorithm to determined magnetic quiver from GTPs by coloring the edges, which is the dual algorithm as web decomposition. We define a color partition $\sum_{i}^{n_c} \lambda_\alpha^i \in \mathcal{P}(\lambda_\alpha)$, where n_c is the number of colors. We required colored GTPs to satisfy the following conditions:

1. $\sum_\alpha \lambda_\alpha^i L_\alpha = 0$, each color can form a sub-polygon S_i by shifting;
2. Each S_i is a $m_i = gcd(\lambda^i)$ times refined Minkowski sum (sum of vertices with respect to sum of the partitions) of unique IMP (irreducible and minimal polygon, minimal means cannot form a new GTP by turn a $\bullet$ into $\circ$, irreducible means cannot be decompose into Minkowski sum of two GTPs) T_i, which satisfies r-rule;
3. $\mu_\alpha \leq \sum_{i}^{n_c} \mu_\alpha^i$, where μ_α^i is the partition of colored edge induced by λ_α^i, all sub-polygons fit into GTP simultaneously.

By add colored lines internally, a colored GTP can be further divided into polygons with same color edges and parallelograms with same color parallel edge. The mixed volume of S_i, S_j is the total area of parallelograms with color i, j. We can identify each color with a node of m_i, link between two nodes i and j receive positive contribution from mixed volume of S_i, S_j, and negative contribution from both i, j colored edge, $l_{i,j} = \frac{1}{m_i m_j}(MV(S_i, S_j) - \sum_\alpha \mu_\alpha^i \cdot \mu_\alpha^j)$. Furthermore, the self-intersection is $l_{i,i} = \frac{1}{m_i^2}(2Area(S_i) - \sum_\alpha (\mu_\alpha^i)^2) = 2(r(T_i) - 1)$, the number of loops (adjoint hypermultiplets) is $1 + \frac{l_{i,i}}{2}$. Also, each E_α contribute to a b_α sequence of nodes with rank

$(m_{\alpha,1}, \cdots, m_{\alpha,b_\alpha})$, where $m_{\alpha,k} = \sum_x^k (-\mu_{\alpha,x} + \sum_i^{n_c} \mu^i_{\alpha,x})$. The link between neighbor nodes in the sequence is one, no loops can be attached to the nodes. The link between k-th sequence node of E_α and i color node is $l_{\alpha,i,k} = \frac{1}{m_i}(\mu^i_{\alpha,k} - \mu^i_{\alpha,k+1})$.

4.6 $SU(3)$ with single cone

Let's take $SU(3)_{\frac{3}{2}} + 5\mathbf{F}$ and $SU(3)_1 + 6\mathbf{F}$ at infinite coupling (UV theories) as example, both cases has a single cone.

The colored GTP and web decomposition of $SU(3)_{\frac{3}{2}}+5\mathbf{F}$ are shown in Figure.4.7. We can write down the following partition and color partitions for colored GTP, and calculate all the non-zero links:

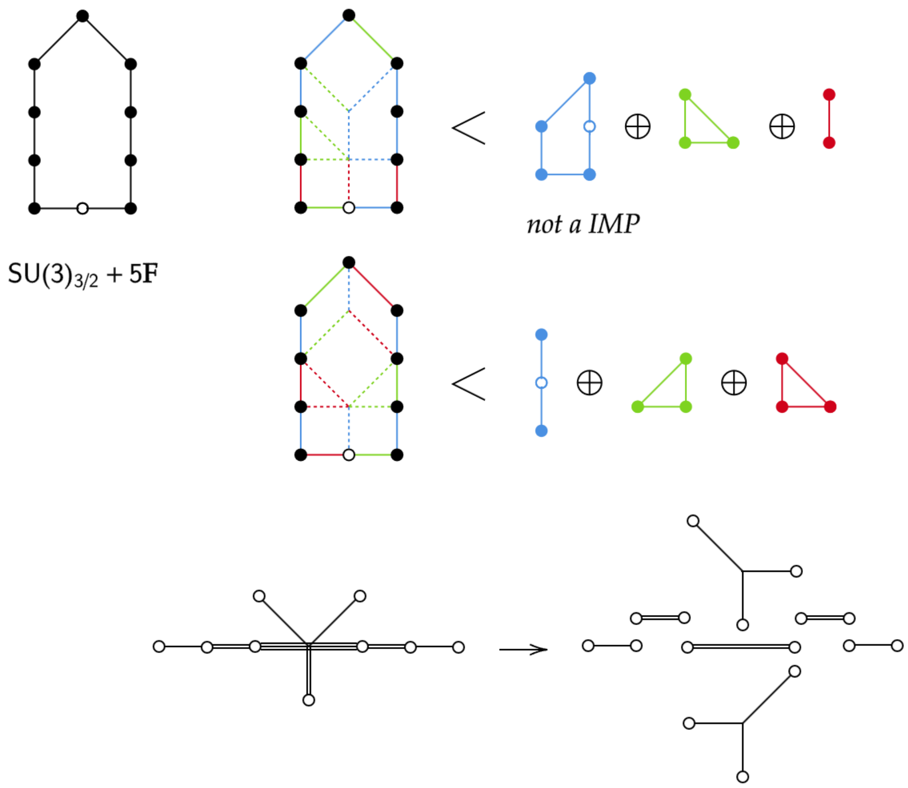


Figure 4.7: Web and GTP of $SU(3)_{\frac{3}{2}} + 5\mathbf{F}$: Wrong coloring, right coloring and web decomposition

$$
\begin{aligned}
&\mu = \left(\left(1^3\right), (2), \left(1^3\right), (1), (1)\right), \\
&\mu_b = ((2), -, (2), -, -), \ \ \mu_g = (-, (1), (1), -, (1)), \ \ \mu_r = ((1), (1), -, (1), -) \\
&m_b = 2, \ \ m_g = 1, \ \ m_r = 1 \\
&m_1 = (2, 1), \ \ m_3 = (2, 1) \\
&l_{b11} = l_{r11} = 1, \ \ l_{b31} = l_{g31} = 1, \ \ l_{gr} = 1
\end{aligned}
\tag{4.6}
$$

The tropical quiver we get is in 4.8. It's easy to check the magnetic quiver deduced from web decomposition is the same as tropical, which agrees our statement. Now we can construct Hasse diagram both from quiver subtraction, or web manipulation or from Minkowski sum and tessellation of GTP (more details in [8]). The Hasse diagram is $\mathfrak{a}_4, \mathfrak{a}_6, \{0\}$.

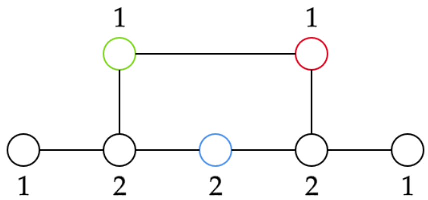


Figure 4.8: Tropical quiver of $SU(3)_{\frac{3}{2}} + 5\mathbf{F}$

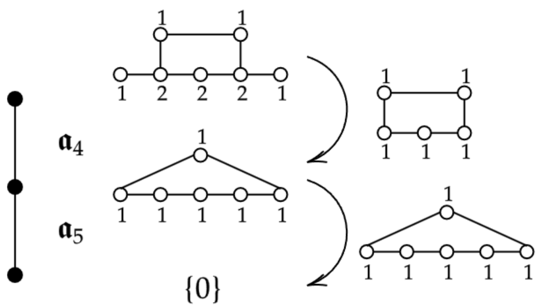


Figure 4.9: Hasse diagram of Higgs branch of $SU(3)_{\frac{3}{2}} + 5\mathbf{F}$

Same logic applies to $SU(3)_1 + 6\mathbf{F}$ are shown in Figure.4.10. The Hasse diagram is $\mathfrak{d}_5, \mathfrak{a}_7, \{0\}$.

$$
\begin{aligned}
&\mu = \left(\left(1^4\right), (2), \left(1^3\right), (1), (1)\right), \\
&\mu_b = ((3), -, (3), -, -), \;\; \mu_g = ((1), (1), -, (1), -), \;\; \mu_r = (-, (1), -, -, (1)) \\
&m_b = 3, \;\; m_g = 1, \;\; m_r = 1 \\
&m_1 = (3, 2, 1), \;\; m_3 = (2, 1) \\
&l_{b11} = l_{g11} = 1, \;\; l_{b32} = 1, \;\; l_{br} = 1
\end{aligned}
\tag{4.7}
$$

The theories at finite coupling can also be represented by GTP with certain tessellation, we will skip the discussion here. It's not hard to see that the sub-web and sub-GTP is indeed dual to each other. We can also, which will be much more convenient when calculate the stable intersection and move between different phase on web, label the subweb with colors as in [42].

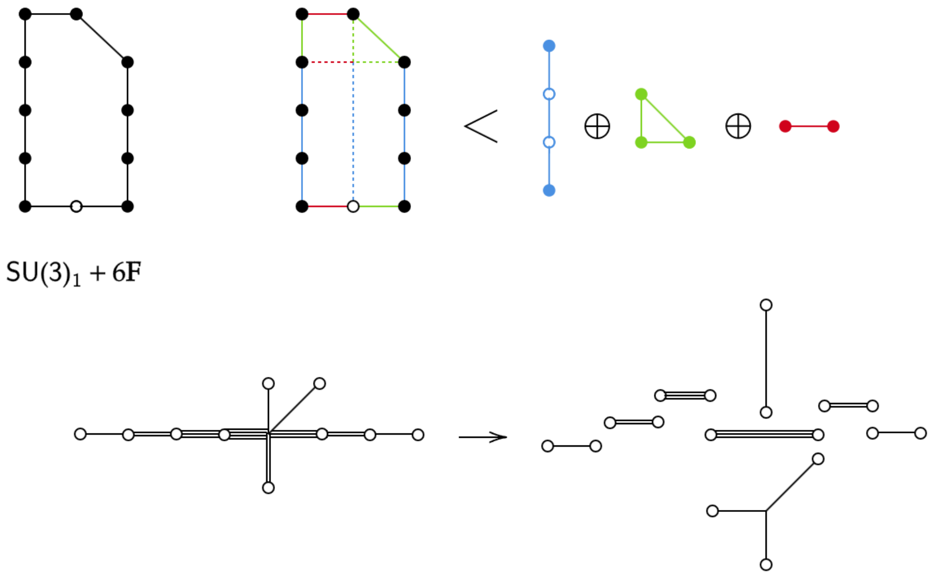


Figure 4.10: GTP coloring and web decomposition of $SU(3)_1 + 6\mathbf{F}$

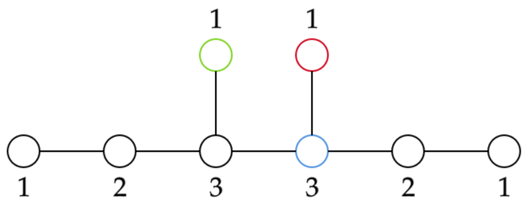


Figure 4.11: Tropical quiver of $SU(3)_1 + 6\mathbf{F}$

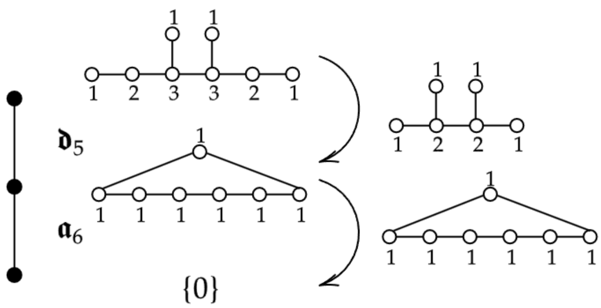


Figure 4.12: Hasse diagram of Higgs branch of $SU(3)_1 + 6\mathbf{F}$

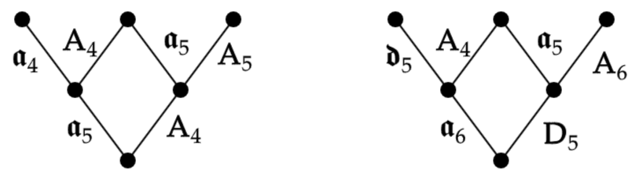


Figure 4.13: Hasse diagram of full moduli space of $SU(3)_{\frac{3}{2}} + 5\mathbf{F}$ and $SU(3)_1 + 6\mathbf{F}$

4.7 $SU(3)$ with two cones

We take $SU(3)_0 + 4\mathbf{F}$ as example, there are two cones with empty intersection, and we have a "Y"-shape Higgs branch Hasse diagram.

$$\begin{aligned}
&1.\\
&\mu = \left(\left(1^3\right), (2), \left(1^2\right), (1), (1)\right),\\
&\mu_b{=}((1),(2),-,(1),(1)),\ \ \mu_g{=}((2),-,(2),-,-)\\
&m_b{=}\ 1,\ \ m_g{=}\ 2\\
&m_1 = (2,1),\ \ m_3 = (1)\\
&l_{b11} = l_{g11} = 1,\ \ l_{g31} = 1,\ \ l_{bg} = 1
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&2.\\
&\mu = \left(\left(1^3\right), (2), \left(1^2\right), (1), (1)\right),\\
&\mu_b = \left(\left(1^2\right), (1), -, (1), -\right),\ \ \mu_g{=}(-,(1),(1),-,(1)),\ \ \mu_r{=}((1),-,(1),-,-)\\
&m_b{=}\ 1,\ \ m_g{=}\ 1,\ \ m_r{=}\ 1\\
&m_1 = (1,1),\ \ m_3 = (1)\\
&l_{b12} = l_{r11} = 1,\ \ l_{g31} = l_{r31} = 1,\ \ l_{bg} = 2
\end{aligned} \tag{4.9}$$

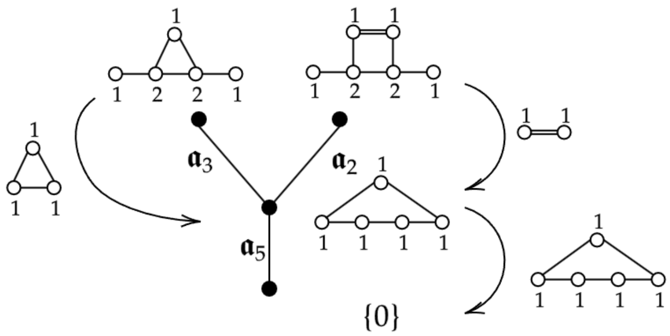


Figure 4.14: Hasse diagram of Higgs branch of $SU(3)_0 + 4\mathbf{F}$

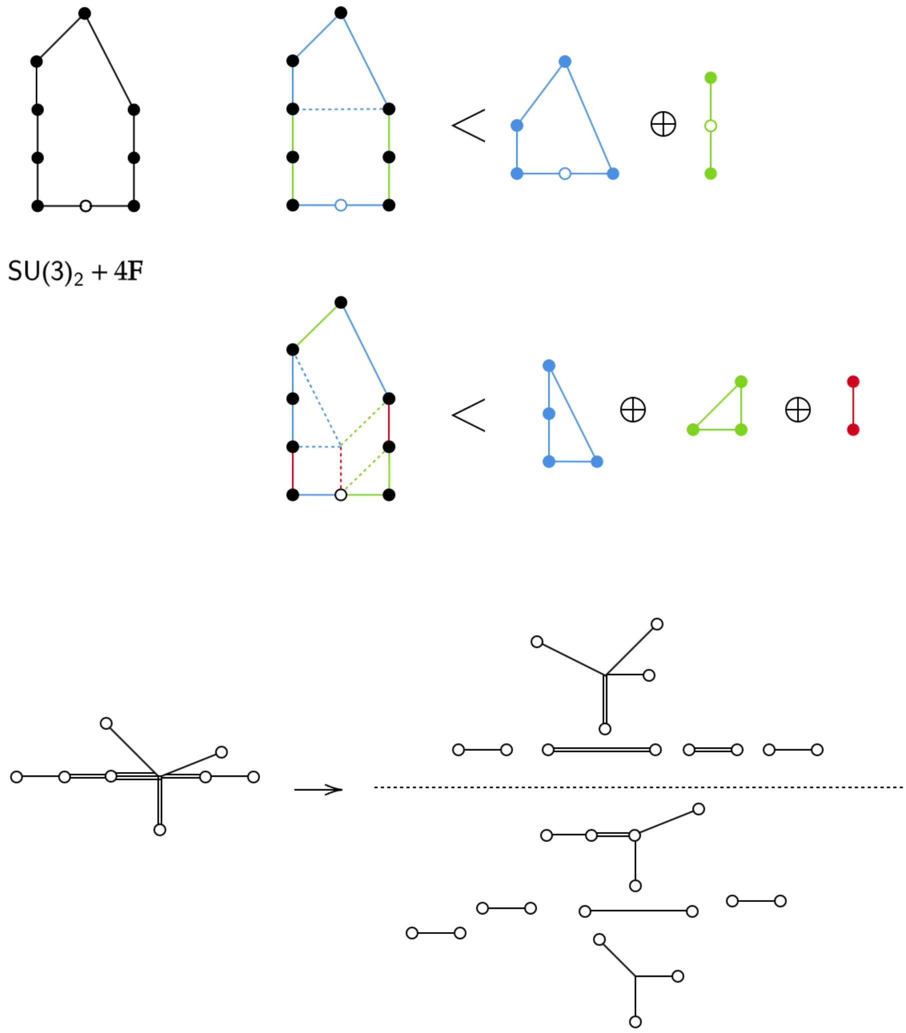


Figure 4.15: GTP coloring and web decomposition of $SU(3)_0 + 4\mathbf{F}$

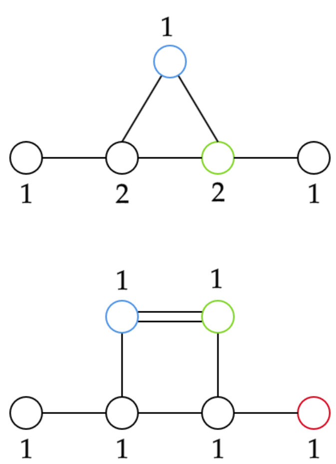


Figure 4.16: Tropical quiver of $SU(3)_0 + 4\mathbf{F}$

4.8 Global symmetry conjecture

The *i-th* node i is balance if the excess [35]

$$e_i = \sum_{i,j\ connected} r_j - 2r_i, \tag{4.10}$$

where r_i is the rank of *i-th* node. Affine Dynkin diagram with dual Coxeter label is balanced. BGS algorithm to compute the global symmetry was proposed in [58]:

1. When the quiver is framed, and has s balanced sub-Dynkin diagram each gives a Lie group G_i and k unbalanced node, the global symmetry is

$$G = U(1)^k \times \prod_i^s G_i; \tag{4.11}$$

2. When the quiver is unframed:

   (a) if there is a unbalanced node of rank one ungauge it and use the formula for framed quiver;

   (b) if there is no unbalance node of rank one, ungauge one from an unbalanced node of any rank, then the global symmetry is

$$G = U(1)^{k-1} \times \prod_i^s G_i. \tag{4.12}$$

4.9 Equations From SQCD

Here we use the F-term equations from $4d$ $\mathcal{N} = 2$ to help us describe the classical Higgs branch of $5d$ $(N) = 1$. For $SU(N_c) + N_f\mathbf{F}$, the equations can be written as following[59]:

$$\begin{aligned}
& MM' = 0; \\
When\ N_f \geq N_c: & \\
& * B\tilde{B} = *M^{N_c}; \\
& M'B = \tilde{B}M' = 0; \\
& B^{[i_1 i_2 \ldots i_{N_c}} B^{j_1] j_2 \ldots j_{N_c}} = 0; \\
When\ N_f \geq N_c + 1: & \\
& M * B = *\tilde{B}M = 0,
\end{aligned} \tag{4.13}$$

where quark Q^i_a is the scalar component of hypermultiplets in the form of $N_c \times N_f$ matrix, meson $M^i_j = \tilde{Q}^a_j Q^i_a$, $M' = M - \frac{1}{N_c}Tr(M)$, baryon $B^{i_1 \cdots i_{N_c}} = \epsilon^{a_1 \cdots a_{N_c}} Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}}$.

The corresponding algebraic varieties of brane web follow the above equations correctly. In the cases we will discuss later, some of the equations may be converted to nilpotent equations. For $N_f < N_c$, there is only mesonic branch, no baryon exists. The chiral ring of Higgs branch is

$$\mathcal{C} = \frac{\mathbb{C}[M]}{\langle MM' \rangle}. \tag{4.14}$$

$MM' \to M^2$ as we change N_c to ∞, however, the ring structure is invariant. Note that $M^2 = 0$ implies $(trM)^{N_f+1} = 0$. The glueball operator [60] and the instanton operator will give correction to the equations, whose analysis is generally non-trivial. With the techniques we learnt so far, we can try to written down the vacuum equation when infinite coupling. It's easy to be done when the Higgs branch is a reduced instanton moduli space, but generally, it's challenging.

4.10 Examples

4.10.1 $SU(2)$

When the gauge group is SU(2) there is no CS level, but the theory still have two different phases, denoted by E_1 ($\theta = 0$) and $\widehat{E}_1$ ($\theta = \pi$).

$\widehat{E}_1$

At both finite and infinite coupling, the Higgs branch of $\widehat{E}_1$ is the reduced moduli space one-$U(1)$ instanton. In ADHM construction, we have two possible reduced moduli space for one-$U(1)$ instanton, trivial or $\mathbb{Z}_2$, the right answer here is the latter. We can write down the chiral ring relation as $S^2 = 0$, where S is the glueball operator bilinear in gaugino. There is no enhancement for global symmetry $U(1)_I$.

E_1

At finite coupling, the Higgs branch of E_1 is also $\mathbb{Z}_2$. At infinite coupling, the global symmetry of E_1 is enhanced from $U(1)_I$ to $E_1 = SU(2)$, the Higgs branch is the same as reduced one-E_1 instanton moduli space $\mathbb{C}^2/\mathbb{Z}_2$, the magnetic quiver is affine A_1. We can write down the chiral ring relation as $S^2 = I^+I^-$, and the Hilbert series

$$HS[E_1]_\infty = \frac{1 - t^4}{(1 - t^2)(1 - qt^2)(1 - q^{-1}t^2)}, \tag{4.15}$$

where t is fugacity for $SU(2)_R$ symmetry, q is the fugacity for $U(1)_I$ which is identified with the Cartan of global symmetry $SU(2)$. The $\frac{1}{1-t^2}$ factor is generated by S. I^+, I^- and S form a triplet of $SU(2)$.

4.10.2 $SU(2) + 1\mathbf{F}$

From the brane web or toric, when finite coupling, the web is non-decomposible and toric is non-colorable, we have discrete moduli space defined by $M^2 = S^2 = SM = 0$, which is $\mathbb{Z}_2 \cup \mathbb{Z}_2$ with empty intersection. At infinite coupling, we can read the cone of moduli space from magnetic quiver as a_1, and also a $\mathbb{Z}_2$ with empty intersection, determined by equation $S^2 = I^+I^-$, $(S - M)^2 = (S - M)S = 0$. The enhanced global symmetry is $E_2 = SU(2) \times U(1)$. The Hilbert series is

$$HS[E_2]_\infty = \frac{1 - t^4}{(1 - t^2)(1 - qt^2)(1 - q^{-1}t^2)} + t^2. \tag{4.16}$$

4.10.3 $SU(2) + 2\mathbf{F}$

Calculate from web or toric, the classical Higgs branch of $SU(2)+2\mathbf{F}$ is composed of two cones: a mesonic cone and a baryonic cone. At finite coupling, we have equation $M^2 = B\tilde{B}$, $M^2 = \frac{1}{2}MtrM$, $S^2 = B(M - \frac{1}{2}trM) = SB = SM = 0$. When B take non-vanishing vev, the baryonic symmetry $SU(2)_R$ manifest, we are on baryonic cone. When B vanishing, it's mesonic cone. The two cones has empty intersection. the Higgs branch is $a_1 \cup a_1 \cup \mathbb{Z}_2$ with $a_1 \cap a_1 = a_1 \cap \mathbb{Z}_2 = 0$. At infinite coupling, mesonic branch receives further correction from instanton operator, changes from a_1 to a_2. Now the Higgs branch is $a_1 \cup a_2 \cup \mathbb{Z}_2$ with $a_1 \cap a_2 = a_1 \cap \mathbb{Z}_2 = a_2 \cap \mathbb{Z}_2 = 0$. The enhanced global symmetry is $E_3 = SU(3) \times SU(2)$. The Hilbert series is

$$HS[E_3]_\infty = \frac{1 - t^4}{(1 - t^2)(1 - \mu^2 t^2)(1 - q\mu t^2)(1 - q^{-1}\mu t^2)} + \frac{1}{1 - \nu^2 t^2} - 1, \tag{4.17}$$

where μ is the fugacity of mesonic symmetry $SU(2)_M$ as a subgroup of $SU(3)$, ν is fugacity of baryonic symmetry $SU(2)_B$.

More cases of $SU(2)$ are studied in [61].

4.10.4 $SU(N)_0$

At finite coupling, we have moduli space $\mathbb{Z}_N$ defined by $S^N = 0$. At infinite coupling, with correction from instanton operators, the moduli space is $\mathbb{C}^2/\mathbb{Z}_N$ defined by $S^N = I^+I^-$.

4.10.5 $SU(N)_{\frac{1}{2}} + 1\mathbf{F}$

At finite coupling, we have moduli space $\mathbb{Z}_2 \cup \mathbb{Z}_N$ with empty intersection, defined by $M^2 = S^N = 0$. At infinite coupling, with correction from instanton operators, the moduli space is $\mathbb{C}^2/\mathbb{Z}_N \cup \mathbb{Z}_2$ with empty intersection, defined by $S^N = I^+I^-$, $M^2 = SM = 0$.

4.10.6 $SU(N)_1 + 2\mathbf{F}$

At finite coupling, the Higgs branch is $\mathbb{C}/\mathbb{Z}_N \cup \mathbb{C}/\mathbb{Z}_2 \cup \mathbb{Z}_2$ with empty intersection, defined by equations $M^2 = B\tilde{B}$, $M^2 = \frac{1}{2}MtrM$, $S^N = B(M - \frac{1}{2}trM) = SB = SM = 0$. At infinite coupling, both the mesonic cone and baryonic cone receive correction from instanton operators.

4.10.7 $SU(N)_{\frac{N}{2}} + N\mathbf{F}$

For $N_c > 2$, the baryonic branch receives correction from instanton too, as shown in Figure 4.17. The center node of mensonic branch magnetic quiver connected with a new U(1) node with bond 2. The two nodes of baryonic branch magnetic quiver connected with a new U(1) node with bond 1, the bond between the two nodes 1 less.

$SU(N)$ theories which can be categorised into "Exceptional Sequences" are studied in [62].

4.11 Geometric classification

From geometric engineering perspective [63, 51], we can classify families of $5D$ theories start form a father theory which is a $5d$ KK theory related to $6d$ theory on S^1. Consider M-theory compactified on a smooth non-compact Calabi-Yau threefold CY_3, in the singular limit when all Kähler parameters vanishing, we have a $5d$ SCFT. The Calabi-Yau threefold associated to the $5d$ SCFT can be described as a local neighborhood of a union of Kähler surfaces $\bigcup_i^r S_i \in CY_3$ glued together, where r is the rank of gauge group in $5d$. With certain gluing rule we can construct the family of $5d$ theories from $5d$ SCFT.

For $SU(3)$ theories with only fundamental matters, we have $SU(3)_0 + 10\mathbf{F}$, $SU(3)_{-\frac{3}{2}} + 9\mathbf{F}$, $SU(3)_4 + 6\mathbf{F}$, $SU(3)_9$ as father theories.

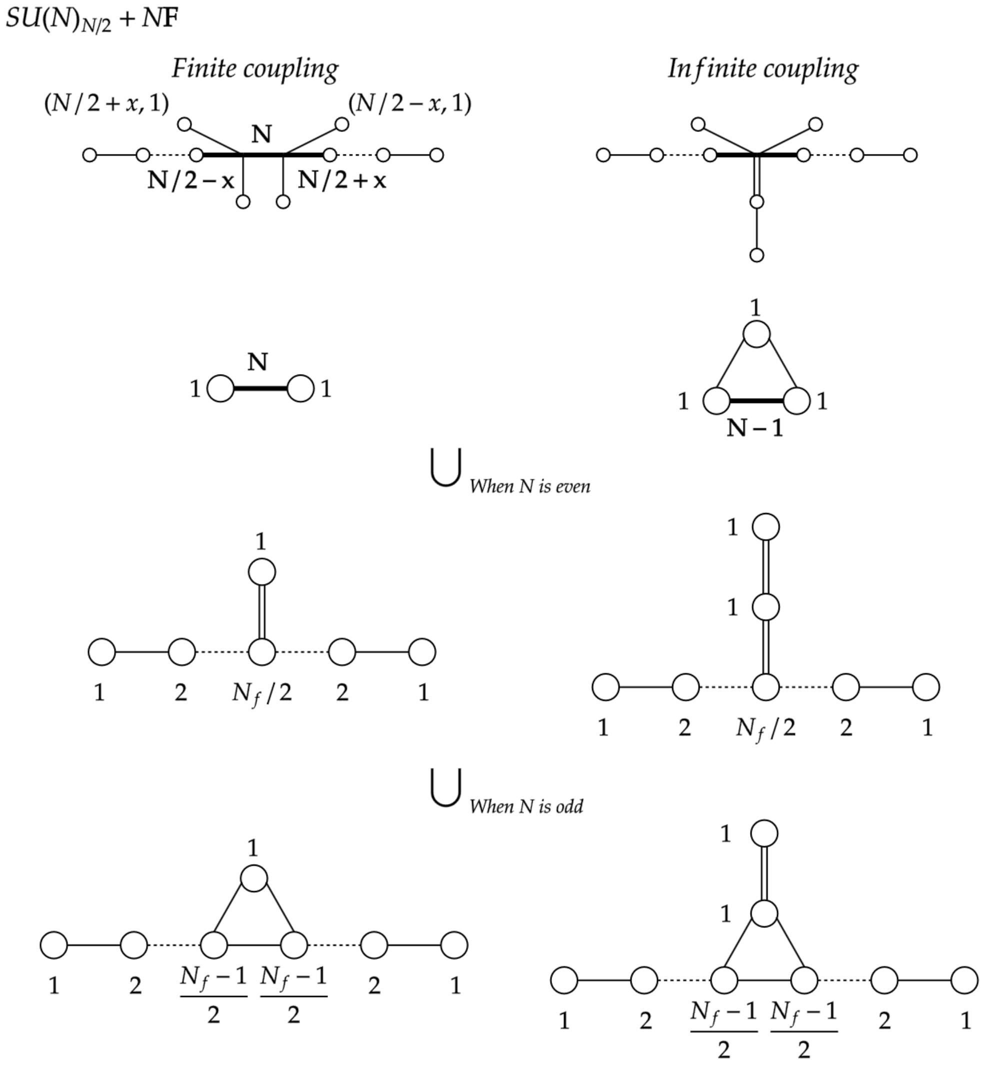


Figure 4.17: Web and magnetic quiver of $SU(N)_{\frac{N}{2}} + N\mathbf{F}$

5

Moduli space of $3D$ $\mathcal{N} = 6$ theories

$4d$ $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{g}$ has moduli space $\mathbb{C}^{3n}/\mathcal{W}(\mathfrak{g})$. Compactifying $4d$ $\mathcal{N} = 4$ SYM with a twist along S^1 to $3d$, we can obtain theories with $\mathcal{N} = 2, 4, 6$[15, 16, 17]. For certain value of τ_{YM}, it's stabliser in $SL(2, \mathbb{Z})$ may form a symmetry. However, the $SL(2, \mathbb{Z})$ symmetry genericly change the spectrum of line operator, as we have stated in Section 2.8, therefore, the symmetry that leave τ_{YM} invariant may no longer be a symmetry. By gauging a 1-form symmetry, we can secure the spectrum, and result in a non-invertible symmetry. Moduli spaces of those $3d$ $\mathcal{N} = 6$ theories have the form of $\mathbb{C}^{4n}/\Gamma$, where Γ is a finite subgroup of $\mathcal{W}(\mathfrak{g})$ and also a *complex reflection group*[18, 19, 20]. When Γ belongs to one of the infinite families of complex reflection group under classification of Shephard-Todd, it can be identified with ABJ(M) theories; When Γ is exceptional complex reflection group, we have a new theory.

5.1 Non-invertible symmetry

The topological operation we import to secure the symmetry is σ: gauging the one form and ι: stacking with an invertible phase, for more discussion of non-invertible defect please check[64, 65, 66, 67, 68, 69, 70]. The origin duality orbit only hold up to a local counterterm imported by topological operation, we can further label the theories according to the counterterm. In $\mathfrak{su}(2)$ theories, the duality orbit is modified to Figure 5.1.

5.2 Twisted compactification

Denote supercharges of $4d$ $\mathcal{N} = 4$ SYM by $Q_{a\alpha}, \bar{Q}^a_{\dot{\alpha}}$, with $\alpha = 1, 2$ and $a = 1, \cdots, 4$, transform in $(\mathbf{2}, \mathbf{4})$ and $(\bar{\mathbf{2}}, \bar{\mathbf{4}})$ of $SO(1, 3) \times SO(6)_R$ respectively. The R-symmetry $SO(6)_R \cong SU(4)_R$.

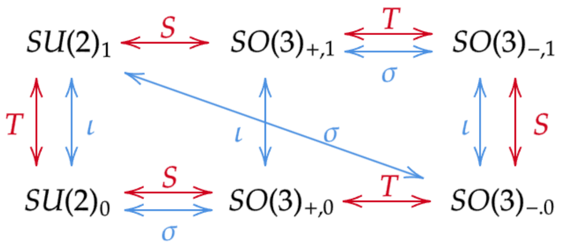


Figure 5.1: Topological operation is blue, S,T transformation is red. σ, ι generate a $SL(2, \mathbb{Z}_2)$ group.

5.2.1 S-duality twist

The S duality group $SL(2, \mathbb{Z})$, under which the Yang-Mills coupling $\tau \to \frac{a\tau+b}{c\tau+d}$, can be associated with a $U(1)$ bundle with transition function e^{iv} in which $v = arg(c\tau + d)$[71]. $Q_{a\alpha}, \bar{Q}^a_{\dot{\alpha}}$ have charge $\mp\frac{1}{2}$ under $U(1)$ modular transformation:

$$\gamma: \; Q_{a\alpha} \to e^{-i\frac{v}{2}} Q_{a\alpha}, \; \bar{Q}^a_{\dot{\alpha}} \to e^{i\frac{v}{2}} \bar{Q}^a_{\dot{\alpha}}. \tag{5.1}$$

The S-duality twist acts on supercharges non-trivially, therefore, the SUSY is broken. To recover the SUSY we want, we need R-symmetry twist to compensate the additional phase[72, 73, 74].

5.2.2 R-symmetry twist

R-symmetry twist introduce additional phase for charged operators. Here we will use the basis of $SU(4)$, s.t, the R-twist takes the form

$$r = \begin{pmatrix} e^i\phi_1 & & & \\ & e^i\phi_2 & & \\ & & e^i\phi_3 & \\ & & & e^i\phi_4 \end{pmatrix} \in SU(4). \tag{5.2}$$

It acts on supercharges as

$$r: \; Q_{a\alpha} \to e^{i\phi_a} Q_{a\alpha}, \; \bar{Q}^a_{\dot{\alpha}} \to e^{-i\phi_a} \bar{Q}^a_{\dot{\alpha}}, \; \sum_{i=1}^{4} \phi_i = 0. \tag{5.3}$$

5.3 S-R twist with dressing

By choosing $\phi_1 = \phi_2 = \phi_3 = \frac{v}{2}, \phi_4 = -\frac{3v}{2}$, we preserve $\frac{3}{4}$ of 16 supercharges, so result in $3d$ $\mathcal{N} = 6$ theory. With different choice of γ, The overall S-R twist can be elements from $\mathbb{Z}_k,\ k = 3, 4, 6$.

As mentioned above, we need to dress γ with topological operation σ, ι to make it a non-invertible symmetry we want. A good news is that the action on supercharges is unchanged after dressing, which makes the analysis easy to proceed.

Now we needs to consider the how the invariant polynomials of $\mathcal{W}(\mathfrak{g})$ transform under the twist, for simply-laced $\mathfrak{g}$, $f_n \to e^{2n\pi/k} f_n$, f_n is homogeneous invariant polynomial of $\mathcal{W}(\mathfrak{g})$ with degree n. By counting the remaining invariants, we can easily identify the complex reflection group Γ. For example, consider $4d$ SYM with $SU(N)_0$, the invariant of $\mathcal{W}(\mathfrak{su}_N)$ have degree $2, 3, \cdots, N$. for $k = 3$, the remaining degrees are $3, 6, \cdots$, which coincide with invariant degrees of $G(k, 1, n)$, where $n = [N/k]$ denote the rank. It's proposed in [75] that the theory is exactly $U(n+r)_k \times U(n)_{-k}$ ABJ(M) theory, where $N = nk + r$ and $r < k$[13, 14]. We are interested in novel theories cannot given by ABJ(M), the simplest example is obtained by twist compactifying $\mathfrak{so}_8$ with $k = 3$, the moduli space is $\mathbb{C}^8/G_4$, where G_4 is the smallest exceptional Shephard-Todd group. $\mathfrak{e}_6$, $\mathbb{C}^8/G_{25}$ with $k = 3$, $\mathbb{C}^8/G_8$ with $k = 4$, $\mathbb{C}^8/G_5$ with $k = 6$. $\mathfrak{e}_7$, $\mathbb{C}^8/G_{26}$ with $k = 3, 6$, $\mathbb{C}^8/G_8$ with $k = 4$. More about complex reflection group can be found in Appendix F.

5.4 Orbifold $\mathbb{C}^{4n}/\Gamma$

5.4.1 ADE revisit

The ADE singularity, as known as local K-3 singularity, Kleinian singularity and Du Val singularity, lies on the origin of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$. Subgroups of $SU(2)$ can be classified into ADE type.

1. A_n: cyclic group $\mathbb{Z}_{n+1}$;
2. D_n: binary dihedral group $D_{2n-4} \ltimes \mathbb{Z}_2$;
3. E_6: binary tetrahedral group $T \ltimes \mathbb{Z}_2$;
4. E_7: binary octahedral group $O \ltimes \mathbb{Z}_2$;
5. E_8: binary icosahedral group $I \ltimes \mathbb{Z}_2$.

The followings are the action of Γ on $\mathbb{C}^2$, defining equation of $\mathbb{C}^2/\Gamma$ and unrefined Hilbert series[2]:

Type	Generator	Defining equation	HS (unrefined)
A_n	$\left\langle \begin{pmatrix} w_n & 0 \\ 0 & w_n^{-1} \end{pmatrix} \right\rangle$	$uv = w^{n+1}$	$\frac{1+t^{n+1}}{(1-t^2)(1-t^{n+1})}$
D_n	$\left\langle \begin{pmatrix} w_{2n} & 0 \\ 0 & w_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$	$u^2 + v^2 w = w^{n-1}$	$\frac{1+t^{2n-2}}{(1-t^4)(1-t^{2n-4})}$
E_6	$\langle S, T \rangle$	$u^2 + v^3 + w^4 = 0$	$\frac{1-t^4+t^8}{1-t^4-t^6+t^{10}}$
E_7	$\langle S, U \rangle$	$u^2 + v^3 + vw^3 = 0$	$\frac{1-t^6+t^{12}}{1-t^6-t^8+t^{14}}$
E_8	$\langle S, T, V \rangle$	$u^2 + v^3 + w^5 = 0$	$\frac{1+t^2-t^6-t^8-t^{10}+t^{14}+t^{16}}{1+t^2-t^6-t^8-t^{10}-t^{12}+t^{16}+t^{18}}$

$$
\begin{aligned}
S &= \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, \; T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\
U &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \; V = \begin{pmatrix} \frac{i}{2} & \frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} \\ -\frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} & -\frac{i}{2} \end{pmatrix}.
\end{aligned}
\tag{5.4}
$$

5.4.2 Orbifolding on $\mathbb{C}^4$

For our construction of $\mathbb{C}^4$, we don't expected it to be *hyper-Kähler*, since the action doesn't satisfy the Calabi-Yau relation: $det = 1$ in general. We preserve the $SU(4)$ symmetry of $\mathbb{C}^4$ when orbifolding, which can be identified as the R-symmetry of $3d$ $\mathcal{N} = 6$ theories above. (For the classification of orbifolding $\mathbb{C}^4$ with action as a finite subgroup of $SU(4)$, please see [76])

The Molien formula is

$$
HS = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\prod_i^4 \mathbb{1}_n - gt_i}, \tag{5.5}
$$

n is the rank of G, the product times over all fugacities of SU(4). Then we the fugacity map $t_1 \to xt, t_2 \to \frac{y}{x}t, t_3 \to \frac{z}{y}t, t_4 \to \frac{1}{z}t$.

$\mathbb{C}^4/\mathbb{Z}_n$			
Orbifold	HS(Unrefined)	PL	HWG
$\mathbb{C}^4/\mathbb{Z}_2$	$\frac{1+6t^2+t^4}{(1-t^2)^4}$	$[2,0,0]t^2 - [0,2,0]t^4 + [1,1,1]t^6 + O(t^8)$	$PE[\mu_1^2 t^2]$
$\mathbb{C}^4/\mathbb{Z}_3$	$\frac{1+16t^3+10t^6}{(1-t^3)^4}$	$[3,0,0]t^3 - [2,2,0]t^6 + ([1,1,2] + [1,4,0]+[2,2,1]+[4,1,1])t^9+O(t^{12})$	$PE[\mu_1^3 t^2]$
$\mathbb{C}^4/\mathbb{Z}_4$	$\frac{1+31t^4+31t^8+t^{12}}{(1-t^4)^4}$	$[4,0,0]t^4 - ([0,4,0] + [4,2,0])t^8 + ([1,1,3]+[1,4,1]+[2,2,2]+[2,5,0]+[3,3,1]+[4,1,2]+[4,4,0]+[5,2,1]+[7,1,1])t^{12} + O(t^{16})$	$PE[\mu_1^4 t^2]$
$\mathbb{C}^4/\mathbb{Z}_5$	$\frac{1+52t^5+68t^{10}+4t^{15}}{(1-t^5)^4}$	$[5,0,0]t^5 - ([2,4,0] + [6,2,0])t^{10} + O(t^{15})$	$PE[\mu_1^5 t^2]$

5.4.3 $\mathbb{C}^4/\mathbb{Z}_n$ $(\mathbb{C}^4/G(n,1,1),\ \mathbb{C}^4/G(np,p,1))$

The orbifolding action we required of $\mathbb{Z}_n$ on $\mathbb{C}^4$ is generated by $w\mathbb{1}_4$, $w = e^{i2\pi/n}$. This action, which preserves $SU(4)$, has different form compared with *A-type* singularity on $\mathbb{C}^2$, $\left(\begin{smallmatrix} w & 0 \\ 0 & w^{-1}\end{smallmatrix}\right)$, which breaks the $SU(2)$ symmetry into $U(1)$.

We can calculate the Hilbert series for $\mathbb{C}^4/\mathbb{Z}_n$ using Molien formula:

$$\begin{aligned}
HS &= \sum_{i=0}^{n-1} \frac{1}{(1-w^i t_1)(1-w^i t_2)(1-w^i t_3)(1-w^i t_4)} \\
&= \sum_{i=0}^{n-1} \sum_{j_1,j_2,j_3,j_4=0}^{\infty} w^{i(j_1+j_2+j_3+j_4)} t_1^{j_1} t_2^{j_2} t_3^{j_3} t_4^{j_4} \\
&= \sum\nolimits_{j_1+j_2+j_3+j_4=nm} \frac{1-w^{n(j_1+j_2+j_3+j_4)}}{1-w^{j_1+j_2+j_3+j_4}} t_1^{j_1} t_2^{j_2} t_3^{j_3} t_4^{j_4} \\
&= \sum_{m=0}^{\infty} Sym^{nm}[t_1+t_2+t_3+t_4]
\end{aligned} \tag{5.6}$$

using the the fugacity map $t_1 \to xt, t_2 \to \frac{y}{x}t, t_3 \to \frac{z}{y}t, t_4 \to \frac{1}{z}t$ to identifying the fundamental representation,

$$\to \sum_{m=0}^{\infty} Sym^{nm}[1,0,0]_{su(4)} t^{nm} = \sum_{m=0}^{\infty} [nm,0,0]_{su(4)} t^{nm}. \tag{5.7}$$

The HWG is:

$$HWG = \sum_{m=0}^{\infty} \mu_1^{nm} t^{nm} = PE[\mu_1^n t^n]. \tag{5.8}$$

Now we use the PL to read the generators and relations:

$$PL = [n,0,0]t^n + \sum_{i=1}^{[\frac{n}{2}]} [2n-4i, 2i, 0] t^{2n} + O(t^{3n}). \tag{5.9}$$

It remains the same form for $\mathbb{C}^m/\mathbb{Z}_n$

$$PL = [n,0,\cdots]_{SU(m)} t^n + \sum_{i=1}^{[\frac{n}{2}]} [2n-4i, 2i, 0, \cdots]_{SU(m)} t^{2n} + O(t^{3n}). \tag{5.10}$$

The generators transform in the *n-th* symmetric product of fundamental representation, whose elements can be identified with monomials of degree n, we denote it as $M_{i_1,\cdots,i_n} = x_{i_1}\cdots x_{i_n}$, $M_{i_1,\cdots,i_n} = M_{(i_1,\cdots,i_n)}$. The relations can be identified with the collection of Young Tableaux with partition $(2^{n-2i}, 1^{2i})$. The generating function of relations of $\mathbb{C}^m/\mathbb{Z}_n$, for all m, can be written as

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{i=1}^{[n/2]}\mu_1^{2n-4i}\mu_2^{2i}t^{2n} &= \sum_{n'=0}^{\infty}\sum_{i=1}^{n'}\mu_1^{2(2n')-4i}\mu_2^{2i}t^{2n'} + \sum_{n'=0}^{\infty}\sum_{i=1}^{n'}\mu_1^{2(2n'+1)-4i}\mu_2^{2i}t^{2n'+1} \\ &= PE[\mu_1^2 t + \mu_2^2 t^2]\mu_2^2 t^2.\end{aligned} \tag{5.11}$$

$\mathbb{C}^m/Z_n$ is *hyper-Kähler* only when n is divisor of m, where the HS is palindromic. $\mathbb{C}^4/Z_3$ can be realised by $4d$ $\mathfrak{su}_3$ theory and $\mathfrak{so}_6$ theory with $k=3$, $\mathbb{C}^4/Z_4$ can be realised by $4d$ $\mathfrak{su}_4$ theory and $\mathfrak{so}_6$ theory with $k=4$, $\mathbb{C}^4/Z_6$ can be realised by $4d$ $\mathfrak{su}_6$ theory, $\mathfrak{so}_8$ theory and $\mathfrak{so}_{10}$ theory with $k=6$.

5.4.4 $\mathbb{C}^8/G(3,1,2)$

$G(3,1,2)$ is a novel complex reflection group with order 18 and rank 2. The braid relation of group generators are $r^2=s^3=1, rsrs=srsr$.

The Hilbert series of $\mathbb{C}^8/G(3,1,2)$ is:

$$\begin{aligned}HS\ (unrefined) =&(1+16t^3+216t^6+776t^9+1636t^{12}+1676t^{15}+1116t^{18} \\ &+340t^{21}+55t^{24})/((1-t^3)^4(1+t^6)^4).\end{aligned} \tag{5.12}$$

The PL of HS is:

$$\begin{aligned}PL =&[3,0,0]t^3+[6,0,0]t^6-([0,3,1]+[3,0,2]+[3,3,0]+[5,2,0])t^9 \\ &+([0,1,2]+[0,3,2]+[1,1,3]+[1,2,1]+[1,4,1]+[2,1,0] \\ &+[2,2,2]+[2,3,0]+[2,5,0]+[3,0,3]+[3,1,1]+[3,3,1] \\ &+[4,1,2]+[5,0,1]+[5,2,1]-[0,6,0]-[4,4,0]-[8,2,0])t^{12}+O(t^{15}).\end{aligned} \tag{5.13}$$

The HWG is:

$$HWG = PE[\mu_1^3t^3+\mu_2^6t^12+\mu_1^6t^6)](1+\mu_2^2\mu_1^2t^6+\mu_2^3\mu_1^3t^9+\mu_2\mu_1^7t^9+\mu_2^4\mu_1^4t^{12}+\mu_2^5\mu_1^5t^{15}). \tag{5.14}$$

5.4.5 $\mathbb{C}^8/G(3,3,2)$ ($\mathbb{C}^4/D_3,\ \mathbb{C}^4/S_3$)

$G(3,3,2)$ has order 6 and rank 2. It is isomorphic to *Symmetric group* S_3 and *Dihedral group* D_3. S_3 can be represented by permutation of coordinates on $\mathbb{C}^3$, which is 3-*dim* reducible with invariant subspace spanned by $x+y+z$, we can easily obtain the 2-*dim* reducible representation by projection, generated by $r=\left(\begin{smallmatrix}-1 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $s=\left(\begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix}\right)$, with respect to the relation $\{r^2=s^2=(sr)^3=\mathbb{1}\}$ the set of generators is not unique. by choosing sr as one of the fundamental generators, we get the generators and relation of D_3, which is $\{r^2 = (sr)^3 = (sr^2)^2 = \mathbb{1}\}$, again, the generators are not unique, we can choose the other set of generators so long as the relation holds. More generally, the relation of generators of S_n is $\{r_1^2 = \cdots = r_{n-1}^2 = (r_1 r_2)^n = \cdots = (r_{n-2} r_{n-1})^n = \mathbb{1}\}$, the relation of generators of D_n is $\{r^2 = s^n = (rs)^2 = \mathbb{1}\}$. The Hilbert series is independent of the choice of generator basis.

The Hilbert series of $\mathbb{C}^8/G(3,3,2)$ is:

$$\begin{aligned} HS\ (unrefined) =& (1 + 6t^2 + 16t^3 + 21t^4 + 36t^5 + 56t^6 + 36t^7 + 21t^8 \\ & + 16t^9 + 6t^{10} + t^{12})/((1 - t^2)^4(1 - t^3)^4). \end{aligned} \tag{5.15}$$

The PL of HS is:

$$PL = [2,0,0]t^2 + [3,0,0]t^3 - [1,2,0]t^5 - ([0,0,2] + [2,2,0])t^6 + O(t^7). \tag{5.16}$$

The HWG is:

$$HWG = PE[\mu_1^2 t^2 + \mu_1^3 t^3 + \mu_2^2 t^4 + \mu_2 \mu_1^3 t^5 - \mu_2^2 \mu_1^6 t^{10}]. \tag{5.17}$$

From the palindromic Hilbert series we can see $\mathbb{C}^8/G(3,3,2)$ is *hyper-Kähler*.

5.4.6 $\mathbb{C}^8/G_4$

The calculation of $\mathbb{C}^8/G_4$ as follow:

$$\begin{aligned} HS\ (unrefined) =& (1 + 31t^4 + 130t^6 + 391t^8 + 970t^{10} + 1766t^{12} + 2310t^{14} \\ & + 2595t^{16} + 2410t^{18} + 1695t^{20} + 920t^{22} + 445t^{24} \\ & + 140t^{26} + 20t^{28})/((1 - t^6)^4(1 - t^4)^4). \end{aligned} \tag{5.18}$$

$$\begin{aligned} PL =& [4,0,0]t^4 + ([0,3,0] + [6,0,0])t^6 - [0,4,0]t^8 - ([1,0,3] \\ & + [2,1,2] + [2,4,0] + [3,2,1] + [4,3,0] + [6,2,0])t^{10} + O(t^{12}). \end{aligned} \tag{5.19}$$

$$HWG = PE[\mu_1^4 t^4 + (\mu_1^6 + \mu_2^3)t^6](1 + \mu_1^4 \mu_2^2 t^8 + \mu_1^8 \mu_2 t^{10}). \tag{5.20}$$

However, in PL the $[0,3,0]t^6$ is not what we wanted. When a group act on a vector space V, the invariants of degree n is in the ring $Sym^n[V^*]$, so it should be stay in $Sym^n[1,0,0] = [n,0,0]$ representation of $SU(4)$ intuitively. The emergency of $[0,3,0]$ remains a problem to us. Either we should modify the Molien sum and fugacity map at the beginning or we will have a more exotic explanation of it.

6

Conclusion and Discussion

Die Grenzen meiner Sprache bedeuten die Grenzen meiner Welt.

– Ludwig Wittgenstein

In this article, we reviewed the brane construction of $3d/\mathcal{N} = 4$ theories and the corresponding magnetic quiver. We analysis the Higgs branch, especially the non-perturbative correction when infinite coupling, in $5d$ $\mathcal{N} = 1$ theories with two dual algorithm: web decomposition and toric coloring, and we represent the cones in the language of magnetic quivers. We use SR twist and topological operation related with non-invertible defect in $4d$ $\mathcal{N} = 4$ theories to construct a collection of new $3d$ $\mathcal{N} = 6$ theories which are out of the scope of ABJ(M) model.

Moduli spaces as a orbifold of complex reflection groups also appear when S-folding $6d$ $\mathcal{N} = 2$ theories[77]. The complex reflection group provide us a new collection of theories and duality, many new results are waiting to be explored.

Review and outlook of the generalised global symmetry can be found in [78]. With the technique of manipulating generalised symmetries, we can construct more theories which are prohibited by the observation with only ordinary symmetry, which brings us one step closer to the core of non-perturbative and non-Lagrangian features of quantum world.

Appendix A

Useful math tool

A.1 Invariants and moduli space

More details of the definitions in this section can be found in [79, 80].

A.1.1 With Gröbner elimination

Gröbner elimination provides a systemic way to analysis the invariants and relations. In practice, with a set of fundamental generators, $A = \{f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\}$, we extend A to $B = \{g_1 = f_1(x_1, \cdots, x_n), \cdots, g_n = f_n(x_1, \cdots, x_n)\}$. Then eliminate $(x_1, \cdots, x_n)$ from B, we can get the set of relations in terms of generators. For example, for A-type singularity on $\mathbb{C}^2$, the generators are $u = x^n$, $v = y^n$ of degree n, $w = xy$ of degree 2, eliminate x, y from generators we get the relation $uv = w^n$ of degree $2n$. The complication to get the relations increases rapidly with more generators.

Hilbert Basis Theorem. Every ideal of polynomial ring $K[X]$ is finitely generated.

Lexicographic Order. Given two polynomials x^a and x^b, we say $x^a >_{lex} x^b$ if the left most entry of $a - b \in \mathbb{Z}^n$ is positive. For example, in $K[x, y, z]$, $x^5y^2z^2 >_{lex} x^2y^6z^7$, $yz^3 >_{lex} z^4$.

Division Algorithm. Let $F = (f_1, f_2, ..., f_m)$ be an ordered set $(m - tuple)$ of polynomials in $K[X]$, then every polynomial f in $K[X]$ can be written as $f = a_1f_1 + a_2f_2 + ... + a_mf_m + r$, where $a_i, r \in K[X]$. We denote r as $\bar{f}^F$.

Gröbner Basis. Fix a monomial order $<$, $G = g_1, g_2, ..., g_m \subset I$ is a *Gröbner Basis* of ideal I if for $\forall f \in I$, $\exists\ i$, s.t, the leading term of f under order $>$ is divisible by leading term of g_i. A *Gröbner Basis* of ideal I is indeed a basis of I, i.e, $I =< G >$. G is *reduced*, if $\forall g \in G$, the leading monomial of g only appear in g. G is monic if G is reduced and the coefficient of leading monomial is 1. *Monic reduced Gröbner Basis* is unique for each I.

Elimination Ideal. The *i-th* elimination ideal of ideal I in $K[x_1, ..., x_n]$ is $I_i = I \cap K[x_{i+1}, ..., x_n]$.

Elimination Theorem. Fix order $>$, if G is a *Gröbner Basis* for I, then $G_i = G \cap K[x_{i+1}, ..., x_n]$ is a *Gröbner Basis* for I_i.

A.1.2 With Hilbert series

A more compact way to analysis the invariants and relations is to count them first.

Hilbert Series. Let $R = K[X]$, $f \in R$ can be decompose into sum of homogeneous polynomials $f = f_0 + ... + f_m$, in which f_d has degree d. $S = \bigoplus_{d \geq 0} S_d$, where S_d is the subspace (not subring) of homogeneous polynomials with degree d. Under group action G, the ring of invariants is $S/G = \bigoplus_{d \geq 0} S/G \cap S_d$ We can call S and S/G graded ring. The *Hilbert series* of graded ring S/G is:

$$HS = \sum_{d \geq 0} dim[S/G \cap S_d]t^d \ \in \mathbb{Z}[t]. \tag{A.1}$$

If S/G is generated by homogeneous polynomials with degree $d_1, ..., d_m$, it can be written as:

$$HS = \sum_{d \geq 0} \frac{F(t)}{(1 - t^{d_1}) \cdots (1 - t^{d_n})}, \tag{A.2}$$

$F(t) \in \mathbb{Z}[t]$, and when S/G is complete intersection with relations at degree $r_1, ..., r_m$, $F(t) = (1 - t^{r_1}) \cdots (1 - t^{r_m})$, when S/G is *hyper-Kähler*, $F(t)$ is palindromic.

Molien's Theorem. When $G \subset GL(n)$ is a finite group, *Hilbert series* of S/G is given by:

$$HS = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\mathbb{1}_n - gt_i}. \tag{A.3}$$

A.2 Plethystic Exponential

The definition follows [2][3].

$$PE[f(t)] = exp(\sum_{n=1}^{\infty} \frac{f(t^n) - f(0)}{n}). \tag{A.4}$$

$$PEF[f(t)] = PE[f(-t)] = exp(\sum_{n=1}^{\infty} \frac{f((-t)^n) - f(0)}{n}). \tag{A.5}$$

$$PL[g(t)] = PE^{-1}[g(t)] = \sum_{k=1}^{\infty} \frac{\mu k}{k} log(g(t^k)). \tag{A.6}$$

$$\mu(k)=\begin{cases}0 & \text{if } k\ has\ repeated\ prime\ factors\\ 1 & \text{if } k=1\\ (-1)^n & \text{if } k\ is\ product\ of\ n\ distinct\ prime\ factors\end{cases} \tag{A.7}$$

They have the following useful properties:

$$PE[\sum_{n=1}^{\infty} a_n t^n]=\prod_{n=1}^{\infty}\frac{1}{(1-t^n)^{a_n}}. \tag{A.8}$$

$$PL[\frac{\prod_{m=1}^{\infty}(1-t^m)^{b_m}}{\prod_{n=1}^{\infty}(1-t^n)^{a_n}}]=\sum_{n=1}^{\infty} a_n t^n-\sum_{m=1}^{\infty} b_m t^m. \tag{A.9}$$

$$PE[f(x)t]=\sum_{n=1}^{\infty} Sym^k[f(x)]t^k. \tag{A.10}$$

$$PEF[f(x)t]=\sum_{n=1}^{\infty}\Lambda^k[f(x)]t^k. \tag{A.11}$$

We perform PE on all the variables by default.

A.3 Highest weight generating function

Highest generating function is introduced in [4] to encode the information of refined Hilbert series in a more compact form. With the orthogonal relation of irreducible representation,

$$\mu_1{}^{n_1}...\mu_r{}^{n_r}=\sum_{m_1,...,m_r=0}^{\infty}\int_G d\mu_G[m_1,...,m_r]^*[n_1,...,n_r]\mu_1{}^{m_1}...\mu_r{}^{m_r}, \tag{A.12}$$

where the integral is over global symmetry G with rank r, $d\mu_G$ is Haar measure of G. Hence, we can derive the HWG formula:

$$HWG=\sum_{m_1,...,m_r=0}^{\infty}\int_G d\mu_G[m_1,...,m_r]^*\mu_1{}^{m_1}...\mu_r{}^{m_r}HS. \tag{A.13}$$

For example,

$$HS=PE[[2,0,\cdots,0]_{su(n)}t]\rightarrow HWG=PE[\sum_{i=1}^{n-1}\mu_i^2t^i+t^n]. \tag{A.14}$$

A.4 Fugacity map and branching rule

Here we discuss the branching rule and fugacity map from Dynkin diagram. We start from *A-type* diagram. $A_3 \to A_2 \times A_0$ We label the fugacity of $U(1)$ (A_0) charge with q. For fundamental representation, with the fact that the net $U(1)$ charge is 0 for $SU(4)$, we can fix the branching as: $[1,0,0] \to [1,0]q + [0,0]q^{-3}$. Now we write down the characters in coordinates: $x + \frac{y}{x} + \frac{z}{y} + \frac{1}{z} \to (x + \frac{y}{x} + \frac{1}{y})q + q^{-3}$. Here we are free to fix one variable, let's take $z \to q^3$. Then we can derive the full fugacity map as $x \to xq, y \to yq^2, z \to q^3$.

We always start with affine Dynkin diagram for the other types. Take D_4 as an example. We label the node with root fugacities z_0, z_1, z_2, z_3, z_4, the constraint can be read from dual Coxeter label (which is the same as conformal dimension when we balance the quiver): $z_0 z_1 z_2^2 z_3 z_4 = 1$, the roots can be mapped to coordinates with fugacity map $z_i \to x_j^{A_{ij}}$ where A_{ij} is the Cartan matrix. $z_1 = \frac{x^2}{w}, z_3 = \frac{y^2}{w}, z_4 = \frac{z^2}{w}, z_2 = \frac{w}{xyz}, z_0 = \frac{1}{w}$. Now let's consider the branching $D_4 \to A_3(z_1, z_2, z_3)$. We can set $z_4/z_0 \to 1$. Then we identify the coinciding nodes. We can get the fugacity map $x \to x, y \to y, z \to 1, w \to w$.

This can also be done with given projection matrix P. For example, the projection matrix for $Sp(10) \to SU(6)$ is,

$$P = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 3 & 2 \end{pmatrix}. \tag{A.15}$$

Then we can write the fugacity map as $x_i \to P^T_{ij} x_j$, which maps $[1,0,0,0,0,0,0,0,0,0]_{Sp(10)} \to [0,0,1,0,0]_{SU(6)}$.

A.5 Nilpotent orbit

We take the definition from book[81]. $X \in \mathfrak{g}$ is nilpotent if the adjoint operator $ad_x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, i.e, $(ad_x)^m = 0, \exists\ m \in \mathbb{Z}^+$. Take $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ as an example. We denote the set of all partitions λ of n as $\mathcal{P}(n)$, for n=3,

$$\mathcal{P}(3) = \{(3), (2,1), (1^3)\}. \tag{A.16}$$

We can use Young tableaux to represent partition, and the transposed partition identify with Young tableaux reflected with respect to diagonal. For example, $\lambda = (4, 2^3, 1) \to (5, 4, 1^2) = \lambda^t$. Define the Jordan block J_i as $i \times i$ matrix with the form:

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}. \tag{A.17}
$$

Every nilpotent elements $X \in \mathfrak{sl}(n, \mathbb{C})$ can be taken into Jordan normal form by action of $PSL(n) = SL(n)/Z$, Z is the center of $SL(n)$. Jordan normal Form determined uniquely by partition $\lambda = (\lambda_1, \cdots, \lambda_k)$:

$$
X_\lambda = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}. \tag{A.18}
$$

Then the nilpotent orbit is defined as:

$$
\mathcal{O}_\lambda := PSL(n) \cdot X_\lambda. \tag{A.19}
$$

Every partition λ defined a nilpotent orbit uniquely, which has empty intersection between each other, due to the uniqueness of Jordan normal form. We can also define the closure of nilpotent orbit $\mathcal{O}_\lambda$ as

$$
\bar{\mathcal{O}}_\lambda = \bigcup_{\lambda' \leq \lambda} \mathcal{O}_{\lambda'}. \tag{A.20}
$$

where the partition ordering defines as

$$
for\ \lambda \neq \lambda',\ we\ say\ \lambda > \lambda',\ iff\ \lambda_i \geq \lambda'_i,\ \forall i. \tag{A.21}
$$

For example, $\bar{\mathcal{O}}_{(3,1)} = \mathcal{O}_{(3,1)} \cup \mathcal{O}_{(2^2)} \cup \mathcal{O}_{(2,1^2)} \cup \mathcal{O}_{(1^4)}$. We can compute that the Higgs branch of (1)-[N] quiver are closure of minimal nilpotent orbit of the $\mathfrak{sl}_N$. The Higgs branch is defined as

$$
\{M_{N\times N} | M^2 = 0, trM = 0, rank(M) \leq 1\}, \tag{A.22}
$$

M is a nilpotent element of $\mathfrak{sl}_N$, the corresponding Jordan normal forms satisfy $rank \leq 1$ are ones with trivial partition (1^N) and minimal partition $(2, 1^{N-2})$. Easy to check from the definition, $\bar{\mathcal{O}}_{(2,1^{N-1})} = \mathcal{O}_{(1^N)} \cup \mathcal{O}_{(2,1^{N-1})}$ is exactly the Higgs branch. Furthermore, the Coulomb branch of ADE type quiver is the closure of minimal nilpotent orbit of the corresponding algebra of Dynkin diagram.

Appendix B

Simply-laced magnetic quiver

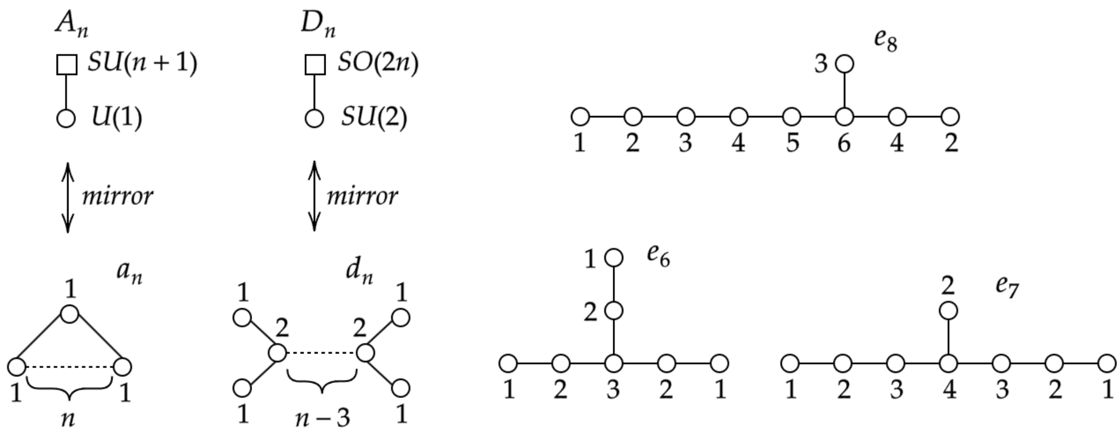


Figure B.1: ADE magnetic quivers and their Coulomb branches. A_n stands for orbifold with A_n type singularity, a_n stands for closure of minimal nilpotent orbit of a_n.

Appendix C

Table of 5D global symmetry

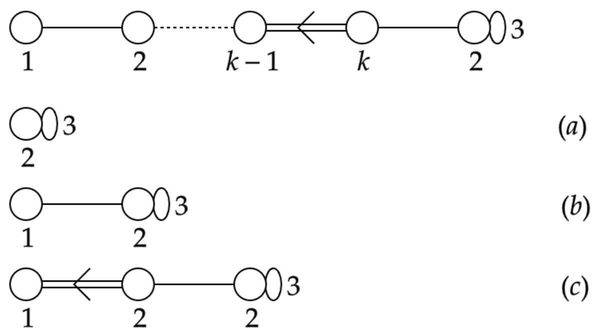
Parameter region	Global symmetry at UV	Global symmetries on components
$N_c - \frac{1}{2}N_f > \lvert k\rvert = 0, \frac{1}{2}$	$SU(N_f) \times U(1) \times U(1)$	$SU(N_f) \times U(1)$
$N_c - \frac{1}{2}N_f > \lvert k\rvert > \frac{1}{2}$	$SU(N_f) \times U(1) \times U(1)$	$SU(N_f) \times U(1), \quad SU(N_f)$
$N_c - \frac{1}{2}N_f = \lvert k\rvert = 0$	$SU(N_f) \times SU(2) \times SU(2)$	$SU(N_f) \times SU(2) \times SU(2)$
$N_c - \frac{1}{2}N_f = \lvert k\rvert = \frac{1}{2}$	$SU(N_f) \times SU(2) \times U(1)$	$SU(N_f) \times SU(2) \times U(1)$
$N_c - \frac{1}{2}N_f = \lvert k\rvert = 1$	$SU(N_f) \times SU(2) \times U(1)$	$SU(N_f) \times SU(2) \times U(1), \quad SU(N_f)$
$N_c - \frac{1}{2}N_f = \lvert k\rvert > 1$	$SU(N_f) \times SU(2) \times U(1)$	$SU(N_f) \times SU(2) \times U(1), \quad SU(N_f) \times SU(2)$
$N_c - \frac{1}{2}N_f + 1 = \lvert k\rvert = 0$	$SU(N_f + 2)$	$SU(N_f + 2)$
$N_c - \frac{1}{2}N_f + 1 = \lvert k\rvert = \frac{1}{2}$	$SU(N_f + 1) \times SU(2)$	$SU(N_f + 1) \times SU(2)$
$N_c - \frac{1}{2}N_f + 1 = \lvert k\rvert = 1, \frac{3}{2}$	$SU(N_f + 1) \times U(1)$	$SU(N_f + 1) \times U(1)$
$N_c - \frac{1}{2}N_f + 1 = \lvert k\rvert > \frac{3}{2}$	$SU(N_f + 1) \times U(1)$	$SU(N_f + 1) \times U(1), \quad SU(N_f + 1)$
$N_c - \frac{1}{2}N_f + 2 = \lvert k\rvert = 0$	(6d)	(6d)
$N_c - \frac{1}{2}N_f + 2 = \lvert k\rvert = \frac{1}{2}$	$SO(2N_f + 2)$	$SO(2N_f + 2)$
$N_c - \frac{1}{2}N_f + 2 = \lvert k\rvert = 1$	$SO(2N_f) \times SU(2)$	$SO(2N_f) \times SU(2)$
$N_c - \frac{1}{2}N_f + 2 = \lvert k\rvert = \frac{3}{2}, 2$	$SO(2N_f) \times U(1)$	$SO(2N_f) \times U(1)$
$N_c - \frac{1}{2}N_f + 2 = \lvert k\rvert > 2$	$SO(2N_f) \times U(1)$	$SO(2N_f) \times U(1), \quad SO(2N_f)$

Figure C.1: Global symmetry of Higgs branch of $5d$ $\mathcal{N} = 1$ theories, table 1 in [7]

Appendix D

Monopole formula in practice

Here we consider the first two case of family of quivers in form of (1)-(2)$\cdots$(k-1)=(k)-(2)3adj.



a

Start from the simplest case (a). We decouple a $U(1)$ factor from $U(2)$ node with 3 adjoint hypermultiplets (the same for quiver$(b),(c)$), then the monopole charge is $(m,0)$, the R-charge is $\Delta(m) = (3-1)(m-0) = 2m$, The monopole formula for quiver (a) is

$$HS = \sum_{m\geq 0}^{\infty} P_{SU(2)}(m) z^m (t^2)^{2m},\ P_{SU(2)}(t^2;m) = \begin{cases} \frac{1}{1-t^4} & \text{if } m=0 \\ \frac{1}{1-t^2} & \text{if } m>0 \end{cases}. \tag{D.1}$$

Note that we use t^2 instead of t to match the power of t with the degree of polynomial invariants. It's easy to calculate that

$$HS\ (unrefined) = \frac{1-t^{12}}{(1-t^4)^2(1-t^6)}. \tag{D.2}$$

This is the Hilbert series of D_4 singularity.

b

The monopole charge is $(m_1, 0), (m_2)$, the R-charge is $\Delta(m_1, m_2) = 2m_1 + \frac{1}{2}|m_1 - m_2|$, The monopole formula for quiver (b) is

$$\begin{aligned} HS &= \sum_{m_1 \geq 0, m_2 \geq 0}^{\infty} P_{U(1)}(t) P_{SU(2)}(t; m_1) z_1^{m_1} z_2^{m_2} (t^2)^{2m_1 + \frac{1}{2}|m_1 - m_2|}, \\ P_{U(1)}(t) &= \frac{1}{1 - t^2} \end{aligned} \tag{D.3}$$

The unrefined HS is

$$HS\ (unrefined) = \frac{1 - t^{12}}{(1 - t^2)^3 (1 - t^5)^2} \tag{D.4}$$

The global symmetry is A_1, with the fugacity map $z_1 \to \frac{1}{x^2}, z_2 \to x^2$. The highest weight generating function is

$$HWG = PE[\mu_1^2 t^2 + t^4 + \mu_1^2 t^5 + \mu_1^2 t^7 - \mu_1^4 t^{14}]. \tag{D.5}$$

c

The monopole charge is $(m_1, 0), (m_{21}, m_{22}), (m_3)$, the R-charge is $\Delta(m_1, m_2, m_3) = 2m_2 + \frac{1}{2}(|m_1 - m_{21}| + |m_1 - m_{22}| + m_{21} + m_{22}) - |m_{21} - m_{22}| + \frac{1}{2}(|2m_{21} - m_3| + |2m_{23} - m_3|)$, The monopole formula for quiver (c) is

$$\begin{aligned} HS &= \sum_{m_1 \geq 0, m_{21} \geq m_{22}, m_3 \geq 0}^{\infty} P_{U(1)}(t) P_{SU(2)}(t; m_1) P_{U(2)}(t; m_2) z_1^{m_1} z_2^{m_{21} + m_{22}} z_3^{m_3} (t^2)^{\Delta(m_1, m_2, m_3)}, \\ P_{U(2)}(m2; t) &= \begin{cases} \frac{1}{(1-t^2)(1-t^4)} & \text{if } m_{21} = m_{22} \\ \frac{1}{(1-t^2)^2} & \text{if } m_{21} > m_{22} \end{cases}. \end{aligned} \tag{D.6}$$

The unrefined HS is

$$\begin{aligned} &HS\ (unrefined) = \\ &\frac{1 + 6t^2 + 16t^4 + 32t^6 + 47t^8 + 57t^{10} + 62t^{12} + 57t^{14} + 47t^{16} + 32t^{18} + 16t^{20} + 6t^{22} + t^{24}}{(1 - t^2)^4 (1 - t^6)^4}. \end{aligned} \tag{D.7}$$

The global symmetry is C_2, with the fugacity map $z_1 \to \frac{x^2}{y}, z_2 \to y^2 x^2$ and constrain $z_1 z_2 z_3 = 1$, we can calculate the HWG

$$HWG = PE[t^4 + \mu_1^2 t^2 + \mu_2^2 t^4 + \mu_2 t^6 + \mu_2 t^8 - \mu_2^2 t^{16}]. \tag{D.8}$$

Appendix E

Brane system of D_4

Brane system of D_4 consists of four D-branes, a orientifold plane, images of D-branes under orientifold plane and strings stretch in between, with boundary condition that string cannot end on both D-brane and its image. We identify the Cartan with string with both ends on the same D-brane. All positive roots shown in Figure E.1,

1. $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_3 + e_4$ (simply roots);
2. $\alpha_1 + \alpha_2 = e_1 - e_3,\ \alpha_2 + \alpha_3 = e_2 - e_4,\ \alpha_2 + \alpha_4 = e_2 + e_4$;
3. $\alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4,\ \alpha_1 + \alpha_2 + \alpha_4 = e_1 + e_4,\ \alpha_2 + \alpha_3 + \alpha_4 = e_2 + e_3$;
4. $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = e_1 + e_3$;
5. $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = e_1 + e_2$.

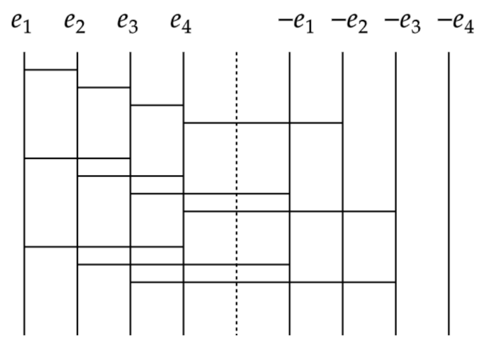


Figure E.1: Brane system of D_4

The dimension of representation $[n_1, n_2, n_3, n_4]$ can be derived as:

$$\begin{aligned} dim = (n_1+1)(n_2+1)(n_3+1)(n_4+1)(n_1+n_2+2)(n_2+n_3+2)(n_2+n_4+2) \\ \times(n_1+n_2+n_3+3)(n_1+n_2+n_4+3)(n_2+n_3+n_4+3) \\ \times(n_1+n_2+n_3+n_4+4)(n_1+2n_2+n_3+n_4+5) \\ /(1^4\cdot 2^3\cdot 3^3\cdot 4\cdot 5). \end{aligned} \tag{E.1}$$

It's not easy to see the triality from brane system, but from roots, we can see the triality between $\alpha_1, \alpha_2, \alpha_3$, which result in the equivalence of vector representation $[1, 0, 0, 0]$, spinor representation $[0, 1, 0, 0]$ and conjugate representation $[0, 0, 1, 0]$. We can construct the brane system of A-type, B-type and C-type group with different boundary condition. It's a challenging task to construct brane system of exceptional groups. For E_8, we may introduce a NS5 brane on the orientifold consist of a half brane and its image, to contribute half linking number.

Appendix F

Complex Reflection Group

A complex reflection group is a group generated by (pseudo)-reflection on vector space V over $\mathbb{C}$, whose action preserves a complex hyperplane. Complex reflection group is classified by Shepherd and Todd[20]. Irreducible complex reflection groups can be classified into infinity family $G(m,p,n)$ and 34 exceptional cases. $G(m,p,n)$ can be realised as a semiproduct of Abelian group of order m^n/p and $Sym(n)$. $G(m,p,n)$ has rank n ($n-1$ when $m=p=1$ or $m=n=p=2$), order $m^n \times n!/p$, degrees of primitive polynomial ring invariants $m, 2m, \cdots, (n-1)m, mn/p$. (For a mathematical reference please see [19], an for physicist we recommend [18])

Special cases: $G(1,1,n) = \mathcal{W}(A_{n-1}),\ G(2,1,n) = \mathcal{W}(B_n),\ G(1,1,n) = Sym(n)$, $G(2,2,n) = \mathcal{W}(D_n),\ G(m,p,1) = G(m/p,1,1) = \mathbb{Z}_{m/p},\ G(p,p,2) = D_p$, $G(2,2,2)$ is Klein four group, $G(m,p,n)$ is reducible if and only if $m=p=n=2$.

The invariants of degree 2 of group G can always be given by invariant forms defined by:

$$f = \{f \in GL(\mathbb{C}^n) | \forall g \in G, f = g \cdot f \cdot g^T\}. \tag{F.1}$$

The invariants given by $x \cdot f \cdot x^T$. The calculation of higher degree invariants will get messier. The good news is that we can always use *Sage* (or other algebraic geometry computing software) to study the group structure, including the elements, the braid relation of generators, invariant forms and invariants. Given the invariants of complex reflection group on $\mathbb{C}^n[x,y]$, we can generalise it to $\mathbb{C}^{4n}[x_i,y_i],\ i = 1,\cdots,4$. The the primitive invariants of $\mathbb{C}/\mathbb{Z}_2$ are x^2, of $\mathbb{C}^4/\mathbb{Z}_2$ are x_ix_j, where $i,j=1,\cdots,4$. The the primitive invariants of $\mathbb{C}^2/D_3$ are $x^2-xy+y^2,\ x^2y-xy^2$, of $\mathbb{C}^6/D_3$ are $x_ix_j - x_iy_j + y_iy_j,\ x_ix_jy_k - x_iy_jy_k$, up to a permutation of i,j,k, where $i,j,k = 1,\cdots,4$.

F.1 Data of exceptional complex reflection group

Data of the exceptional complex reflection group we meet above:

ST name	rank	order	braid relation
G_4	2	24	$r^3 = s^3 = 1, rsr = srs$
G_5	2	72	$r^3 = s^3 = 1, rsrs = srsr$
G_8	2	96	$r^4 = s^4 = 1, rsr = srs$
G_{25}	3	648	$r^3 = s^3 = t^3, rsr = srs, sts = tst, rt = tr$
G_{26}	3	1296	$r^2 = s^3 = t^3, rsrs = srsr, sts = tst, rt = tr$

Table F.1: Rank, order and braid relation of G_4, G_5, G_8, G_{25}, G_{26}

Invariants:

1. $G_4 : x^4 - 8xy^3,\ x^6 + 20x^3y^3 - 8y^6;$

2. $G_5 : x^6 + 20x^3y^3 - 8y^6,\ x^9y^3 + 3x^6y^6 + 3x^3y^9 + y^{12};$

3. $G_8 : x^8 + 14x^4y^4 + y^8,\ x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}\ ;$

4. G_{25} :
   $x^6 - 10x^3y^3 + y^6 - 10x^3z^3 - 10y^3z^3 + z^6,\ x^6y^3 - x^3y^6 - x^6z^3 + y^6z^3 + x^3z^6 - y^3z^6,$
   $x^9y^3 - 4x^6y^6 + x^3y^9 + x^9z^3 + 2x^6y^3z^3 + 2x^3y^6z^3 + y^9z^3 - 4x^6z^6 + 2x^3y^3z^6 - 4y^6z^6 + x^3z^9 + y^3z^9;$

5. G_{26} :
   $x^6 - 10x^3y^3 + y^6 - 10x^3z^3 - 10y^3z^3 + z^6,$
   $x^9y^3 - 4x^6y^6 + x^3y^9 + x^9z^3 + 2x^6y^3z^3 + 2x^3y^6z^3 + y^9z^3 - 4x^6z^6 + 2x^3y^3z^6 - 4y^6z^6 + x^3z^9 + y^3z^9,$
   $x^{12}y^6 - 2x^9y^9 + x^6y^{12} - 2x^{12}y^3z^3 + 2x^9y^6z^3 + 2x^6y^9z^3 - 2x^3y^{12}z^3 + x^12z^6 + 2x^9y^3z^6 - 6x^6y^6z^6 + 2x^3y^9z^6 + y^{12}z^6 - 2x^9z^9 + 2x^6y^3z^9 + 2x^3y^6z^9 - 2y^9z^9 + x^6z^12 - 2x^3y^3z^{12} + y^6z^{12}.$

F.2 Character table of G_4 and ADE classification

From the character table, we can derive the product representation of G_4 (binary tetrahedral group) by multiplying the characters.

Name/Size	1	1	4	4	6	4	4
1	1	1	1	1	1	1	1
1′	1	1	ω	ω^2	1	ω	ω^2
1″	1	1	ω^2	ω	1	ω^2	ω
2	2	-2	-1	-1	0	1	1
2′	2	-2	-ω	-ω^2	0	ω	ω^2
2″	2	-2	-ω^2	-ω	0	ω^2	ω
3	3	3	0	0	-1	0	0

Table F.2: Character table of G_4, ω is a 2nd root of unity.

We can encode the representation tensor product into Dynkin diagram.

1. $\mathbf{2} \times \mathbf{1} = \mathbf{2}$

2. $\mathbf{2} \times \mathbf{1}' = \mathbf{2}'$

3. $\mathbf{2} \times \mathbf{1}'' = \mathbf{2}''$

4. $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$

5. $\mathbf{2} \times \mathbf{2}' = \mathbf{1}' + \mathbf{3}$

6. $\mathbf{2} \times \mathbf{2}'' = \mathbf{1}'' + \mathbf{3}$

7. $\mathbf{2} \times \mathbf{3} = \mathbf{2} + \mathbf{2}' + \mathbf{2}''$

We draw the representations as nodes, if two nodes are related by times representation **2**, connect them. Here we have an affine E_6 Dynkin diagram with dual Coxeter label.

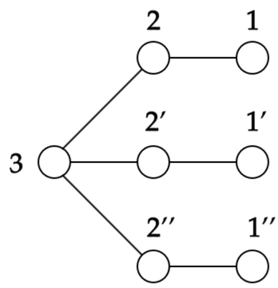


Figure F.1: Dynkin diagram of $\hat{E}_6$

Following the same logic, we can find the corresponding Dynkin diagram of G_{12} (binary octahedral group) is affine E_7. The finite group with Dynkin diagram affine E_8 is binary icosahedral group, which is not a complex reflection group.

Name/Size	1	1	12	8	6	8	6	6
$\mathbf{1}$	1	1	1	1	1	1	1	1
$\mathbf{1}'$	1	1	-1	1	1	1	-1	-1
$\mathbf{2}$	2	-2	0	-1	0	1	ω^2	-ω^2
$\mathbf{2}'$	2	-2	0	-1	0	1	-ω^2	ω^2
$\mathbf{2}''$	2	2	0	-1	2	-1	0	0
$\mathbf{3}$	3	3	1	0	-1	0	1	1
$\mathbf{3}'$	3	3	-1	0	-1	0	-1	-1
$\mathbf{4}$	4	-4	0	1	0	-1	0	0

Table F.3: Character table of G_{12}, ω is a 8th root of unity.

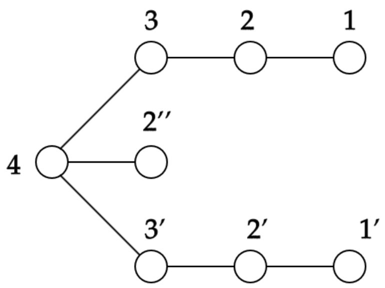


Figure F.2: Dynkin diagram of $\hat{E}_7$

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