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Supergravity and Branes on Curved Manifolds

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Abstract

This dissertation discusses supergravity theories and their brane solutions on various backgrounds. We started by introducing the 11D and type II supergravities and solve the M2, M5, and D3 brane solutions on the flat background. We then considered the special holonomy of the curved background manifolds that preserve supersymmetry. We calculated the M2 and D3 branes at the apex of the Calabi-Yau manifolds. We then discussed branes wrapped on calibrated cycles and how they preserve supersymmetry. We derived the 5D and 4D minimal gauged supergravity via dimension reduction on Sasaki-Einstein manifolds. In the 5D and 4D minimal gauged supergravity, we calculated the D3 branes and M2 branes wrapped on H^2, S^2, and spindle. In the end, we introduced the GK geometry and used it to obtain the D3 and M2 branes wrapped on H^2, S^2, and spindle solutions.

Contents

Chapter 1

Introduction

Supergravity theories play an important role in the study of string theory and M-theory. Although M-theory is not yet well understood, in low energy, it is believed to give rise to the 11D supergravity theory which has the maximum amount of supersymmetry. Under various compactifications, the M-theory leads to five types of 10-dimensional superstring theory. The types of superstring theory that are of the most interest here are type II string theories, IIA and IIB with different chiralities. They give rise to 10-dimensional type II supergravity theories in the low energy limit.

String theory and M-theory have dynamic higher-dimensional objects known as branes. In string theory, D-branes are where open strings are attached to and receive perturbative effects from open string dynamics. These branes are also non-perturbatively solutions of supergravity. A brane in spacetime will have a back reaction on the spacetime which is described by supergravity. The brane solutions of the supergravity normally break half the supersymmetry of the supergravity and have asymptotic anti-de-Sitter (AdS) spacetime structures.

The AdS and Conformal Field Theory correspondence (AdS-CFT) is a conjecture which states that a quantum gravity theory with an asymptotic AdS structure is dual to some conformal field theory living on the boundary of the asymptotic AdS space. With the conjecture, the supergravity is related to the conformal field theory on the AdS boundary due to the asymptotic AdS structure of these brane solutions. This provides a useful tool to study supergravity, string and M-theory.

Branes can also be wrapped on compact cycles in curved manifolds. The supersymmetry of these wrapped branes is preserved via the notion of calibration or topological twist. To solve the wrapped brane solutions, one can perform the dimension reduction from the 11D or 10D type II supergravities to lower-dimension minimal gauged supergravities and then uplift the solution back to 11D or 10D. In the length scale much smaller than the radius of the cycle, the brane world volume is approximately the same as unwrapped branes. In the low energy limit where the length scale is much larger than the radius of the cycle, the wrapped dimensions become undynamic, and then the brane world volume looks like a lower dimension theory. Some of these solutions also have AdS factors, and hence are dual to some superconformal field theory. Then the dual superconformal field theory could flow across dimensions, interpolated by supergravity.

Recently, a new class of solutions of the lower-dimensional minimal gauged supergravity was discovered, which describes branes wrapped on 2-dimensional spindles.[28][29] The spindles are orbifolds with two conical singularities. Different from the calibrated cycles, the way branes wrapping on spindles preserve the supersymmetry is not via the topological twist. When the wrapped brane solution is uplifted to 10/11D, it was found that the solution can be made to be completely regular without singularities.

Through the calculations, one can find that the AdS2 solutions of the M2 branes wrapping 2-cycles/spindles and the AdS3 solutions of D3 branes 2-cycles/wrapping spindles share very similar geometry structures. In fact, they can be classified into the same class of geometry known as the Gauntlett-Kim (GK) geometry.[34]

We will start the discussion in chapter 2 by briefly introducing the 11D and type II supergravity theories. Following this, we will use the supergravity theories to find some brane solutions on a flat background spacetime. Then we will discuss the world volume theory of branes. In chapter 3, we will talk about AdS-CFT correspondence with relates supergravity, string and M-theory with the brane solutions obtained in section 2 to some conformal field theory. We will start by introducing the correspondence. We will then discuss how is the correspondence applied in the context of supergravity, string theory, and large N gauge theory.

From chapter 4, we will generalise the discussion to branes on curved background manifolds. We will first discuss the condition of the background geometry in order to preserve supersymmetry. These background manifolds are classified by the special holonomy of the manifold. Through supergravity, we then solve the D3 and M2 branes transverse to the Calabi-Yau manifolds. In chapter 5, we will discuss the other way of putting branes on curved manifolds which is to wrap them on cycles in the curved manifolds. We will start the discussion by wrapping probe branes around calibrated cycles to preserve supersymmetry via topological twist. Then through the dimension reduction, we derive the 5D and 4D minimal gauged supergravity. In these minimal gauged supergravities, we will discuss the M2 and D3 brane wrapping H^2, S^2, and spindles. These solutions share similar geometrical structures. They both belong to the GK geometry which is discussed in chapter 6.

Chapter 2

Supergravity and Branes

There are five types of 10-dimensional superstring theories, type IIA, type IIB, type I string theory, and heterotic string theory with $E_8 \times E_8$ and $SO(32)$ gauge symmetry. These string theories are related to each other via various dualities. It is now believed that there exists an 11-dimensional M-theory which with different compactifications gives rise to each superstring theory. The actual formulation of the M-theory is still unknown. Nevertheless, the low energy limit of the M-theory gives rise to the 11-dimensional supergravity. With various compactifications, the 11D supergravity is related to other 10-dimensional supergravities which are the low energy limit of the corresponding string theory. Here we are mainly interested in the 11D and type II supergravity. Although the supergravity theories are non-renormalizable as a quantum theory, it is fine that they are the effective theory of the string theory and M-theory. Moreover, it is interesting to study their non-perturbative solutions.

These supergravity theories admit non-perturbative solutions known as branes. Branes are extended objects in spacetime, which is a generalized notion of point particles and strings. These branes are dynamic in the string theory and M-theory. In string theory, the ends of the open strings are attached to the branes. Due to the open string dynamics, branes receive perturbative effects giving rise to the brane world volume theory. Branes in M-theory do not receive the same perturbative effect as in string theory since there is no string in the theory.

In this chapter, we will first introduce the 11D and type II supergravities. Then we will find certain brane solutions to these supergravities and discuss the brane world volume theorem in string theory and M-theory.

2.1 D=11 Supergravity

The 11-dimension supergravity [1] is the low energy effective field theory of the M-theory. The 11D supergravity has the maximum supersymmetry with 32 supercharges. It is the maximum amount of supercharges a theory can have without having fields with spin higher than 2 in 4D. Also, in dimensions higher than 11, the fermionic degrees of freedom became much more than the bosonic degrees of freedom, and hence the corresponding supersymmetric theory can not be constructed.

The corresponding graviton multiplet in the 11D supergravity contains elfbeins (metric) e_μ^A where $\eta_{AB}e_\mu^A e_\nu^B = g_{\mu\nu}$, vector fermion gravitinos Ψ_μ, and three-form gauge field $A_{\mu\nu\rho}$ with gauge transformation $\delta A_{[3]} = d\Lambda_{[2]}$ and field strength $F_{[4]} = dA_{[3]}$. The Action of the theory is given by

$$S_{11} = \int \mathrm{d}x^{11}\sqrt{-g}\Big(R - \frac{1}{48}F^2\Big) + F_{[4]} \wedge F_{[4]} \wedge A_{[3]}, \tag{2.1}$$

plus terms involving gravitinos Ψ_μ. Here we have set the gravitinos Ψ_μ to zero to obtain the bosonic sector of the action. The equations of motion are

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{12}\Big(F_{\mu\nu}^2 - \frac{1}{12}g_{\mu\nu}F^2\Big), \\ d * F &= -\frac{1}{2}F \wedge F, \\ dF &= 0, \end{aligned} \tag{2.2}$$

where $F_{\mu\nu}^2$ denotes $F_{\mu\alpha\beta\rho}F_\nu{}^{\alpha\beta\rho}$ and F^2 denotes $F_{\alpha\beta\rho\sigma}F^{\alpha\beta\rho\sigma}$.

The theory is supersymmetric with the action invariant under infinitesimal supersymmetric transformations generated by a 32 components spinor parameter ϵ

$$\begin{aligned} \delta e_\mu^A &= \bar{\epsilon}\Gamma^A\Psi_\mu, \\ \delta A_{\mu\nu\rho} &= \bar{\epsilon}\Gamma_{[\mu\nu}\Psi_{\rho]}, \\ \delta\Psi_\mu &= D_\mu\epsilon = \nabla_\mu\epsilon + \frac{1}{288}(\Gamma_\mu{}^{\alpha\beta\rho\sigma} - 8\delta_\mu{}^\alpha\Gamma^{\beta\rho\sigma})F_{\alpha\beta\rho\sigma}\epsilon, \end{aligned} \tag{2.3}$$

where gamma matrices $\Gamma^A = e_\mu^A\Gamma^\mu$ are in the elfbein basis and $\Gamma_{\alpha\beta} = \Gamma_{[\alpha}\Gamma_{\beta]}$, and $\nabla_\mu\epsilon$ is the spin connection

$$\nabla_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu{}^A{}_B\Gamma_A^B. \tag{2.4}$$

To truncate the theory down to the bosonic sector, we have set gravitino Ψ_μ to zero. Hence the supersymmetric variation of the elfbein e_μ^A and three-form gauge field $A_{\mu\nu\rho}$ vanishes. The supersymmetric transformation of the gravitino, however, is non-zero unless

$D_\mu\epsilon = 0$. Therefore, in order for the bosonic action Eq.(2.1) and equations of motion Eq.(2.2) to preserve supersymmetry, solutions to the equation of motion need to admit a spinor ϵ such that

$$D_\mu\epsilon = \Big[\nabla_\mu + \frac{1}{288}({\Gamma_\mu}^{\alpha\beta\rho\sigma} - 8{\delta_\mu}^\alpha\Gamma^{\beta\rho\sigma})F_{\alpha\beta\rho\sigma}\Big]\epsilon = 0. \tag{2.5}$$

The spinor satisfying the condition is called the Killing spinor. A supersymmetric solution preserving some portions of the supersymmetry needs to admit the same amount of the Killing spinor components.

From the Killing spinors, one can construct Killing vectors,

$$K_\mu^{ij} = \bar{\epsilon}^i\Gamma_\mu\epsilon^j, \tag{2.6}$$

where K^{ii} can be either time-like or null. The equations of motion and Killing spinor condition implies $\mathcal{L}_{Kij}(g) = \mathcal{L}_{Kij}(F) = 0$. [2] It can be shown from the integrability condition of the Killing spinor $[D_\mu, D_\nu]\epsilon = 0$ that if $d * F = -\frac{1}{2}F \wedge F$ and $dF = 0$ are satisfied, then one is left with the condition

$$E_{\mu\nu}\Gamma^\mu\epsilon = 0, \tag{2.7}$$

where

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{12}\Big(F_{\mu\nu}^2 - \frac{1}{12}g_{\mu\nu}F^2\Big). \tag{2.8}$$

By imposing $\bar{\epsilon}$ and $E_{\mu\alpha}\Gamma^\alpha$ from the right, the expression becomes

$$\bar{\epsilon}E_{\mu\nu}\Gamma^\mu\epsilon = E_{\mu\nu}K^\nu = 0, \qquad E_{\mu\alpha}E_{\mu\beta}\{\Gamma^\alpha, \Gamma^\beta\}\epsilon = E_{\mu\alpha}{E_\mu}^\alpha = 0. \tag{2.9}$$

For a time like Killing vectors, this implies $E_{\mu 0} = 0$ in the appropriate basis. Then Eq.(2.9) sum only the spacial indices which imply $E_{\mu\nu} = 0$ and all of the equations of motion can be satisfied by requiring the Killing spinor and $d * F = -\frac{1}{2}F \wedge F$ and $dF = 0$. [2]

2.2 Type II Supergravity

Type II supergravity is a 10 dimensional $\mathcal{N} = 2$ supergravity theory which is the low energy limit of type II string theory. String theory has closed string modes, followed by an infinite tower of massive modes. In the low energy limit where the string length is small, the massive modes decoupled, and the spectrum is left with only the massless modes of the string theory, giving rise to the type II supergravity spectrum. The spectrum includes NS-NS fields metric $g_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$, and a dilaton ϕ, R-R n-form fields $C_{[n]}$, gravitino Ψ_μ, and dilatino λ. Depending on chirality, two types of type II string theory can arise, the type IIA and the type IIB string theory, related to each other by T-duality. The type IIA string theory has fermion degrees of freedom in opposite chiralities while the type IIB string theory has the same chiralities. These two type II string theories give rise to type IIA and type IIB supergravity respectively.

2.2a Type IIA

Type IIA supergravity is the low energy effective theory of type IIA string theory. The spectrum of the theory contains metric $g_{\mu\nu}$, a dilaton ϕ, anti-symmetric two-form field $B_{\mu\nu}$, one-form R-R vector A_μ, and three-form R-R field $C_{\mu\nu\rho}$, and fermion fields gravitino Ψ_μ and dilatino λ. The type IIA supergravity can be obtained from the consistent truncation of the Kazula-Klein dimension reduction from D=11 supergravity compactified on a circle to 10 dimensions. This followed from the fact that M-theory compactified on a circle gives the type IIA string theory.[3]

In the Kazula-Klein reduction, the 11D metric G_{MN} in the 11D supergravity can be decomposed into a 10D metric $g_{\mu\nu}$, a Kazula-Klein vector A_μ, and a dilaton ϕ,

$$G_{MN} = e^{-2\phi/3} \begin{pmatrix} g_{\mu\nu} + e^{2\phi} A_\mu A_\nu & e^{2\phi} A_\mu \\ e^{2\phi} A_\nu & e^{2\phi} \end{pmatrix}, \tag{2.10}$$

which can also be written as

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dx^{(11)} + A_\mu dx^\mu)^2. \tag{2.11}$$

Elfbeins E of the metric are given by

$$E^A = \{e^{-\phi/3} e^a_\mu dx^\mu, e^{2\phi/3}(dx^{(11)} + A_\mu dx^\mu)\}, \tag{2.12}$$

where e^a_μ are zehnbeins and $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$, and

$$E^M_A = \begin{pmatrix} e^{\phi/3} e^\mu_a & 0 \\ -e^{\phi/3} A_a & e^{-2\phi/3} \end{pmatrix}. \tag{2.13}$$

The spin connection of the metric is given by

$$\omega^a{}_b = \tilde{\omega}^a{}_b - \frac{1}{3} e^{\phi/3} \partial_b \phi e^a - \frac{1}{2} e^{4\phi/3} F_{ab} e^{11}, \qquad \omega^{11}{}_b = -\frac{2}{3} e^{\phi/3} \partial_b \phi e^{11} - \frac{1}{2} e^{4\phi/3} F_{ab} e^a. \tag{2.14}$$

Then for the metric $G_{\mu\nu}$, one obtains $\sqrt{-G} = \sqrt{-g}e^{-8\phi/3}$ and

$$\sqrt{-G}R(G) = \sqrt{-g}\Big[e^{-2\phi}\Big(R(g) + 4\partial_\mu\phi\partial^\mu\phi\Big) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big]. \tag{2.15}$$

The three-form gauge field $A^{(11)}_{LMN}$ can be decomposed into a three-form gauge field $C_{\mu\nu\rho} = A^{(11)}_{\mu\nu\rho}$ and a two-form gauge field $B_{\mu\nu} = A^{(11)}_{\mu\nu 11}$ in the dimension reduction. And the four-form field strength of the three-form gauge field $F^{(11)}_{KLMN} = 4\partial_{[K}\ A^{(11)}_{LMN]}$ is decomposed into

$$\begin{aligned} F^{(11)}_{\mu\nu\rho 11} &= 3\partial_{[\mu}A^{(11)}_{\nu\rho]11} = 3\partial_{[\mu}B_{\mu\rho]} = H_{\mu\nu\rho}, \\ F^{(11)}_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}A^{(11)}_{\nu\rho\sigma]} = 4\partial_{[\mu}C_{\nu\rho\sigma]} = F_{\mu\nu\rho\sigma}. \end{aligned} \tag{2.16}$$

Writing the field strength in the vielbein basis F_{ABCD}, we get

$$\begin{aligned} F^{(11)}_{abcd} &= e^{4\phi/3}\tilde{F}_{abcd} = e^{4\phi/3}(F_{abcd} + 4A_{[a}H_{bcd]}), \\ F^{(11)}_{abc11} &= e^{\phi/3}H_{abc}. \end{aligned} \tag{2.17}$$

Then we get

$$\frac{1}{48}F^{(11)}_{ABCD}F^{ABCD}_{(11)} = \frac{1}{48}e^{8\phi/3}\tilde{F}_{abcd}\tilde{F}^{abcd} + \frac{1}{12}e^{2\phi/3}H_{abc}H^{abc}. \tag{2.18}$$

And the Chern-Simons term becomes

$$F^{(11)}_{[4]} \wedge F^{(11)}_{[4]} \wedge A^{(11)}_{[3]} = F_{[4]} \wedge F_{[4]} \wedge B_{[2]}. \tag{2.19}$$

Substituting this and Eq.(2.15) into the bosonic 11D supergravity action, we get bosonic action for type IIA supergravity

$$S_{IIA} = \int \mathrm{d}x^{10}\sqrt{-g}\Big[e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H^2_{[3]}\Big) - \frac{1}{4}F^2_{[2]} - \frac{1}{48}F^2_{[4]}\Big], \tag{2.20}$$

plus the Chern-Simons term

$$S_{CS} = \int F_{[4]} \wedge F_{[4]} \wedge B_{[2]}. \tag{2.21}$$

The action is exactly the effective action of the type IIA string theory massless modes requiring the cancellation of Weyl anomaly. The above action is written in the string frame. With the rescaling $g = e^{\phi/2}g'$, we can switch the action to the Einstein frame given by

$$S_{IIA} = \int \mathrm{d}x^{10}\sqrt{-g'}\Big(R' + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-\phi}H^2_{[3]} - \frac{1}{4}e^{\frac{3}{2}\phi}F^2_{[2]} - \frac{1}{48}e^{\frac{1}{2}\phi}F^2_{[4]}\Big) + S_{CS}. \tag{2.22}$$

With the dimension reduced from 11 to 10, the vector spinor gravitino is decomposed to Ψ_μ and Ψ_{11}. And 11D spinors can also be decomposed into two 10 Majornan-Weyl spinors with different chirality. Therefore, the 11D gravitino is decomposed into two 10D gravitinos Ψ_μ and two 10D dilatinos $\lambda = \Psi_{11}$, each having opposite chiralities. Again to preserve the supersymmetry in a theory with the fermionic degrees of freedom truncated, the Killing spinor needs to be considered. Substitute the spin connection and RR gauge fields into the 11D Killing spinor condition, with a field redefinition

$$\tilde{\lambda} = e^{-\phi/6}\lambda, \qquad \tilde{\Psi_\mu} = e^{-\phi/6}\Big(\Psi_\mu + \frac{1}{2}\Gamma_\mu\Gamma_{11}\lambda\Big), \qquad \tilde{\epsilon} = e^{\phi/6}\epsilon, \tag{2.23}$$

one gets the supersymmetric condition for type IIA supersymmetry,

$$\begin{aligned}\delta\Psi_\mu &= \Big(\nabla_\mu - \frac{1}{4}H_{\mu\nu\rho}\Gamma^{\nu\rho 11} - \frac{1}{8}e^{\phi}F_{\alpha\beta}\Gamma_\mu^{\ \alpha\beta 11} + \frac{1}{8}e^{\phi}F_{\alpha\beta\rho\sigma}\Gamma^{\alpha\beta\rho\sigma}\Gamma_\mu\Big)\epsilon = 0,\\ \delta\lambda &= \Big(-\frac{1}{3}\partial_\mu\phi\Gamma^{\mu 11} + \frac{1}{6}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - \frac{1}{4}e^{\phi}F_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{12}e^{\phi}F_{\alpha\beta\rho\sigma}\Gamma^{\alpha\beta\rho\sigma 11}\Big)\epsilon = 0.\end{aligned} \tag{2.24}$$

2.2b Type IIB

Type IIB supergravity is the low energy effective theory of type IIB string theory. The spectrum of the theory contains metric $g_{\mu\nu}$, a dilaton ϕ, anti-symmetric two-form field $B_{\mu\nu}$, 0-form R-R gauge field C_0 (an axion), 2-form R-R gauge field $C_{\mu\nu}$, and a self-dual 4-form R-R gauge field $C_{\mu\nu\rho\sigma}$, and two left-handed Majorana-Weyl fermion fields gravitino Ψ_μ and two left-handed Majorana-Weyl dilatino λ. Type IIB supergravity cannot be obtained from the dimension reduction from 11D supergravity. Nevertheless, we can construct a similar gauge invariant but problematic action [3]

$$S_{IIB} = \int \mathrm{d}x^{10}\sqrt{-g}\Big[e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H^2_{[3]}\Big) - \frac{1}{2}F^2_{[1]} - \frac{1}{12}F^2_{[3]} - \frac{1}{2\times 48}F^2_{[5]}\Big], \tag{2.25}$$

plus the Chern-Simons term

$$S_{CS} = \int C_{[4]} \wedge H_{[3]} \wedge F_{[3]}. \tag{2.26}$$

Where $F_{[1]}$, $F_{[3]}$, and $F_{[5]}$ are the field strength for the R-R 0-form, 2-from, and 4-form. In addition to the action, we need a self-dual constraint on the four-form R-R gauge field $C_{\mu\nu\rho\sigma}$ which can not be built within the Lagrangian

$$F_{[5]} = *F_{[5]}. \tag{2.27}$$

However, the action description of the type IIB supergravity is problematic. The first thing is that the action is not supersymmetric. Since the four-form self-dual constraint cannot be built in the action, the four-form in the action has more components than demanded before applying the constraint. Hence the Lagrangian has more bosonic degrees of freedom than fermionic degrees of freedom, breaking the supersymmetry. Also, with the self-dual constraint, the kinetic term of the five-form vanishes. One thing one can do is to describe the type IIB supergravity via equations of motion. Fortunately, the equations of motion including the self-dual constraint, are supersymmetric.

To have a bosonic solution preserving the supersymmetry, we again require a vanishing supersymmetric transformation. The supersymmetric variations of the bosonic fields are again dependent on the fermionic fields, hence vanishing by setting fermions to zero. The supersymmetric variations of the fermionic fields are given by

$$\begin{aligned}\delta\Psi_\mu &= \Big(\nabla_\mu + \frac{i}{8}e^\phi + \frac{i}{16\cdot 5!}e^\phi F_{\alpha\beta\rho\sigma\nu}\Gamma^{\alpha\beta\rho\sigma\nu}\Gamma_\mu\Big)\epsilon - \frac{1}{8}(2H_{\mu\alpha\beta}\Gamma^{\alpha\beta} + ie^\phi F_{\alpha\beta\rho}\Gamma^{\alpha\beta\rho}\Gamma_\mu)\epsilon^*,\\ \delta\lambda &= \frac{1}{2}(\partial_\mu\phi - ie^\phi\partial_\mu C_0)\Gamma^\mu\epsilon + \frac{1}{4}(ie^\phi F_{\alpha\beta\rho}\Gamma^{\alpha\beta\rho} - H_{\mu\alpha\beta}\Gamma^{\mu\alpha\beta})\epsilon^*,\end{aligned} \tag{2.28}$$

Where ϵ is 10d Majorana Weyl spinor satisfying the chiral projection $\Gamma^{11}\epsilon = \epsilon$. Requiring these transformation to vanish leads to the type IIB Killing spinor condition.

For simplicity, the type IIB spectrum can be truncated to contain only a metric, a scalar dilaton field, and the self-dual 4-form gauge field with the rest of the spectrum set to zero. Such truncated type IIB supergravity is of the most interest in the following discussion. The action of the truncated theory in the string frame is given by

$$S = \int dx^{10}\sqrt{-g}\Big(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4\cdot 5!}F_{[5]}^2\Big), \tag{2.29}$$

with the self-dual constraint on the 4-form. And one gets the equations of motion given by

$$\begin{aligned}R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4\cdot 4!}F_{\mu\alpha\beta\rho\sigma}F_\nu{}^{\alpha\beta\rho\sigma},\\ \nabla_\mu(F_{\mu\nu\alpha\beta}) &= 0,\\ \partial_\mu\partial^\mu\phi &= 0.\end{aligned} \tag{2.30}$$

And the Killing spinor condition becomes requiring a constant scalar and

$$\delta\Psi_\mu = D_\mu\epsilon = \Big(\nabla_\mu + \frac{i}{16\cdot 5!}F_{\alpha\beta\rho\sigma\nu}\Gamma^{\alpha\beta\rho\sigma\nu}\Gamma_\mu\Big)\epsilon = 0. \tag{2.31}$$

2.3 Brane Solutions of Supergravity

The D=11 and type II supergravity have non-perturbative extended objects sourcing the n-form gauge fields. Like the zero-dimensional particle sourcing the one form vector potential of the electric and magnetic field, we can have a d dimension objects either electrically couple to (d)-form gauge field or magnetically couple to a $(D-d-4)$-form gauge field in the supergravity. We call these extended objects p-branes, where $p=(d-1)$ corresponds to the spatial dimension of the brane world volume. These p-branes have n-form charges, therefore, are the source of the n-form gauge fields. Having a brane on a flat spacetime will have a back reaction to the spacetime. These back reactions are described by the Einstein equation. Solutions to the 11D supergravity are called M-branes, which satisfy the equations of motion Eq.(2.2). M-branes are the extended objects in the M-theory. There are two M-branes, the M2-branes that are coupled electrically to the 3-form field and the M5-branes that are coupled magnetically to the 3-form field. The M-branes are also supersymmetric, meaning they preserve a portion of the supersymmetry by admitting some Killing spinors. D branes are solutions to the type II supergravities, which satisfy the type II supergravity equations of motions and supersymmetric conditions. D-branes are dynamic objects in string theory with the end of the open string attached to them and receive perturbative effects.[4]

We now study the back reaction of various branes on flat backgrounds to obtain the brane solutions of supergravity theories. Start with p-brane on a D-dimensional flat spacetime with Poincare symmetry $ISO(1,D-1)$. The p+1 dimensional brane world volume naturally breaks $ISO(1,D-1)$ into $ISO(1,p)\times SO(D-p-1)$, where $ISO(1,p)$ is the symmetry of the $(p+1)$-dimensional world volume and $SO(D-p-1)$ is the symmetry of the transverse space. The most general metric with such symmetry can be written as

$$ds^2 = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}(dr^2 + r^2 d\Omega^2_{D-p-2}), \tag{2.32}$$

where $\{x^\mu\}$ is the coordinate of the brane world volume and $d\Omega^2_{d-p-2}$ gives the standard metric on a D-p-2 sphere parametrized by the coordinate $\{y^m\}$. The brane with such metric is located at r=0 and is transverse to the y^m coordinate. The vielbeins $\{e^A, e^R, e^M\}$ of the metric is given by

$$e^A = e^{A(r)}\delta^A_\mu dx^\mu \qquad e^R = e^{B(r)}e^r \qquad e^M = e^{B(r)}\tilde{e}^m, \tag{2.33}$$

where $\{\tilde{e}^M\}$ are the vielbeins on the D-p-2 sphere. The convention used here is to have Greek indices labelling the brane world volume directions and Latin indices labelling the spherical directions. While in the vielbein coordinate, the world volume directions are labelled by $\{ABCDE\}$ and the spherical directions are labelled by $\{MNIJK\}$. With $de^a = -\omega^a{}_b \wedge e^b$, one can calculate the spin connections

$$
\begin{aligned}
\omega^{A}_{\ R} &= e^{-B(r)}A'(r)e^{A} = e^{A(r)-B(r)}A'(r)\delta^{A}_{a}dx^{a},\\
\omega^{M}_{\ R} &= e^{-B(r)}\Big(B'(r)+\frac{1}{r}\Big)e^{M} = \Big(B'(r)+\frac{1}{r}\Big)\tilde{e}^{M},\\
\omega^{M}_{\ N} &= \tilde{\omega}^{M}_{\ N},
\end{aligned}
\tag{2.34}
$$

where $\tilde{\omega}^{M}_{\ N}$ is the spin connection on the D-p-2 sphere, and its Ricci tensor is given by $R_{mn} = (D-p-2)g_{mn}$, where g_{mn} is the metric on the sphere. One finds the non-zero components of the Ricci tensor

$$
\begin{aligned}
R_{\mu\nu} &= -\eta_{\mu\nu}e^{2(A-B)}\Big(A'' + d(A')^2 + \tilde{d}A'B' + \frac{\tilde{d}+1}{r}A'\Big),\\
R_{mn} &= -\tilde{g}_{mn}r^2\Big(B'' + dA'B' + \tilde{d}(b')^2 + \frac{2\tilde{d}+1}{r}B' + \frac{d}{r}A'\Big),\\
R_{rr} &= -\Big(dA'' + (\tilde{d}+1)B'' + d(A')^2 - dB'A' + \frac{\tilde{d}+1}{r}B'\Big),
\end{aligned}
\tag{2.35}
$$

where $\tilde{g}_{mn}$ is the metric on the D-p-2 sphere, and $d = p+1$, $\tilde{d} = D-p-2$.

We now derive the M2 brane and M5 brane solution in the 11D supergravity, and D3 brane solution in the type IIB supergravity.

2.3a M2-Branes

M2 brane is a solution of 11D supergravity that is electrically coupled to a 3-form gauge field with the world volume dimension three. With the symmetry ansatz, the 3-form gauge field coupled to the M2-brane world volume can be written as

$$
A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}e^{C(r)},
\tag{2.36}
$$

and the non-vanishing component of the field strength is

$$
F_{r\mu\nu\rho} = \epsilon_{\mu\nu\rho}\partial_r e^{C(r)}.
\tag{2.37}
$$

Einstein equation become

$$
\begin{aligned}
A'' + 3(A')^2 + 6A'B' + \frac{7}{r}A' &= \frac{1}{3}(c')^2e^{2C-6A},\\
B'' + 3A'B' + 6(b')^2 + \frac{13}{r}B' + \frac{3}{r}A' &= -\frac{1}{6}(c')^2e^{2C-6A},\\
3A'' + 7B'' + 3(A')^2 - 3B'A' + \frac{7}{r}B' &= \frac{1}{3}(c')^2e^{2C-6A}.
\end{aligned}
\tag{2.38}
$$

And the equation of the field strength becomes

$$\nabla^2 C + C'(C' + 6B' - 3A') = 0, \tag{2.39}$$

where ∇^2 is the Laplacian of $\tilde{d}+1$ flat space.

To have a supersymmetric solution, we still need to consider the supersymmetry condition given by the Killing spinor equation Eq.(2.5). To solve the Killing spinor condition, we first split the Killing spinor ϵ according to the symmetric ansatz

$$\epsilon = \epsilon_0 \otimes \eta(r), \tag{2.40}$$

where ϵ_0 is a constant spinor in 3 dimensions and $\eta(r)$ is a spinor in 8 dimensions. The corresponding gamma matrices in the vielbein basis are $\Gamma_A = \gamma_A \otimes \sigma_1 \otimes \mathbb{I}$, $\Gamma_R = \mathbb{I} \otimes \sigma_2 \otimes \mathbb{I}$, and $\Gamma_M = \mathbb{I} \otimes i\sigma_3 \otimes \Sigma_M$ where γ_A are the gamma matrices in 3d Minkowski spacetime, Σ_M are the gamma matrices in 7d Euclidean space, and σ_i are Pauli matrices. Substituting the gamma matrices, spin connection and field strength, the Killing spinor condition can be rewritten as

$$\begin{aligned}
D_\mu\epsilon =& \Big(\partial_\mu - \frac{1}{2}e^{-A-B}\partial e^A \gamma_\mu \otimes \sigma_2 \cdot \sigma_1 \otimes \mathbb{I} - \frac{1}{6}e^{-3A-B}\partial_r e^C \gamma_\mu \otimes \sigma_2 \otimes \mathbb{I}\Big)\epsilon = 0,\\
D_r\epsilon =& \Big(\partial_r - \frac{1}{6}e^{-3A-B}\partial_r e^C \mathbb{I} \otimes \sigma_1 \otimes \mathbb{I}\Big)\epsilon = 0,\\
D_m\epsilon =& \Big(\tilde{\nabla}_m - \frac{1}{2}\mathbb{I} \otimes \sigma_1 \otimes \tilde{\Sigma}_m\\
&- \frac{1}{2}e^{-2B}\partial_r e^B \mathbb{I} \otimes \sigma_1 \otimes \Sigma_m + \frac{1}{12}e^{-3A-B}\partial_r e^C \mathbb{I} \otimes \mathbb{I} \otimes \Sigma_m\Big)\epsilon = 0,
\end{aligned} \tag{2.41}$$

where we have used the condition for all gamma matrices [5]

$$\Gamma_{a1..ak} = \alpha \epsilon_{a1...ad}\Gamma^{a(k+1)...ad}\Gamma_{d+1}, \tag{2.42}$$

where

$$\alpha = \frac{1}{(d-k)!}(-1)^{k(k-1)/2 + d(d-1)/2}. \tag{2.43}$$

It can be shown that it is possible to find a spinor on an n-sphere with odd n satisfying $\left(\tilde{\nabla}_m - \frac{1}{2}\tilde{\Sigma}_m\right)\epsilon = 0$ [6], where $\tilde{\nabla}_m$ and $\tilde{\Sigma}_m$ are the covariant derivative and gamma matrices on the n-sphere. Therefore, the Killing spinor condition is solved by

$$3A = -6B = C, \tag{2.44}$$

$$\eta(r) = \eta_0 e^{-\frac{1}{6}C}, \tag{2.45}$$

satisfying the chiral projection $\sigma_1 \otimes \mathbb{I} \cdot \eta = -\eta$, or equivalently $\Gamma_{012}\epsilon = -\epsilon$, and a constant spinor ϵ_0 on the brane world volume. Substitute this condition into the equation of motion Eq.(2.39), and we get

$$\nabla^2 e^{-C} = 0. \tag{2.46}$$

Demanding an asymptotically flat solution, solutions to the equation are the 6d harmonic functions

$$e^{-C} = H_8(r) = 1 + \frac{\kappa}{r^6}. \tag{2.47}$$

From this, we get the supersymmetric solution of the 11D supergravity

$$\begin{aligned} dS^2 &= \Big(1 + \frac{\kappa}{r^6}\Big)^{-\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + \Big(1 + \frac{\kappa}{r^6}\Big)^{\frac{1}{3}} (dr^2 + r^2 d\Omega_7^2), \\ A_{\mu\nu\rho} &= \epsilon_{\mu\nu\rho}\Big(1 + \frac{\kappa}{r^6}\Big)^{-1}. \end{aligned} \tag{2.48}$$

The chiral condition on the Killing spinors reduces the components of Killing spinors by half. This means the M2-brane solution preserves half of the supersymmetry of the 11D supergravity corresponding to 16 supercharges.

The metric of the solution has a coordinate singularity at $r = 0$, which corresponds to a horizon. In the near horizon limit $r = 0$, we can approximate $e^{-C} = \frac{\kappa}{r^6}$, then with a coordinate transformation, the solution becomes

$$dS^2 = \frac{1}{4}\kappa^{\frac{1}{3}}\Big[\frac{1}{\rho^2}\Big(dx^\mu dx^\nu \eta_{\mu\nu} + d\rho^2\Big) + 4d\Omega_7^2\Big], \tag{2.49}$$

where $\rho = \frac{\sqrt{\kappa}}{2r^2}$. The metric in the parentheses is just an anti-de-Sitter metric. Hence the near horizon limit of the M2 brane is AdS4 $\times$ S^7.

Note the M2 brane solution is solved from the Laplacian equation Eq.(2.46). Any linear superposition of the harmonic functions is also a solution [7]

$$e^{-C} = 1 + \frac{\kappa}{r^6} \to 1 + \sum_i \frac{\kappa}{(r - r_i)^6}. \tag{2.50}$$

Hence we can generalize the solution of a single M2 brane to multiple parallel M2 branes given by

$$dS^2 = \Big(1 + \sum_i \frac{\kappa}{(r - r_i)^6}\Big)^{-\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + \Big(1 + \sum_i \frac{\kappa}{(r - r_i)^6}\Big)^{\frac{1}{3}} dy^m dy^n \delta_{mn}. \tag{2.51}$$

The reason for the possibility of stacking branes is due to the no-force condition. One can calculate the ADM mass to be equal to the electric charge of the brane solution, which saturates the BPS bound. Hence gravitational attraction is cancelled by the electric force. This is the same situation as the extremal Reissner–Nordström black hole. In fact, the M2 brane is also in the extremal limit of the general black branes, where two horizons of the black brane coincide.

2.3b M5-Branes

M5 brane is a solution of D=11 supergravity that is magnetically coupled to a 3-form gauge field with the world volume dimension six. With the symmetry ansatz, the field strength of the magnetic 3-form gauge field can be written as

$$F_{mnij} = -\epsilon_{mnijr}\partial_r e^{C(r)}. \tag{2.52}$$

Then the condition $d * F = -\frac{1}{2}F \wedge F$ is automatically satisfied. And the condition $dF = 0$ requires $\nabla^2 e^C = 0$, which is again related to harmonic functions. The solution to the 5d laplacian is

$$e^C = H_5(r) = 1 + \frac{\kappa}{r^3}. \tag{2.53}$$

And the Einstein equation become

$$\begin{aligned} A'' + 6(A')^2 + 3A'B' + \frac{4}{r}A' &= \frac{1}{6}(c')^2 e^{2C-6B} \\ B'' + 6A'B' + 3(b')^2 + \frac{7}{r}B' + \frac{6}{r}A' &= -\frac{1}{3}(c')^2 e^{2C-6B} \\ 6A'' + 4B'' + 6(A')^2 - 6B'A' + \frac{4}{r}B' &= \frac{1}{6}(c')^2 e^{2C-6B}. \end{aligned} \tag{2.54}$$

We then check the constraint of the Killing spinor conditions on the solution. To solve the Killing spinor condition, the spinor is again split into a constant 6d spinor and a 5d spinor

$$\epsilon = \epsilon_0 \otimes \eta(r). \tag{2.55}$$

The corresponding gamma matrices are given by

$$\Gamma_P = (\gamma_A \otimes \mathbb{I}, \gamma^7 \otimes \Sigma_R, \gamma^7 \otimes \Sigma_M), \tag{2.56}$$

where $\gamma^7 = \gamma_0 ... \gamma_5$. Then the Killing spinor condition with the presence of the magnetic

4-form field strength becomes

$$\begin{aligned}
D_\mu\epsilon =&\Big(\partial_\mu - \frac{1}{2}e^{-A}\partial e^A\gamma_\mu\cdot\gamma^7\otimes\Sigma^r - \frac{1}{12}e^{-3B}\partial_r e^C\gamma_\mu\otimes\Sigma^r\Big)\epsilon = 0\\
D_r\epsilon =&\Big(\partial_r + \frac{1}{12}e^{-3B}\partial_r e^C\gamma^7\otimes\mathbb{I}\Big)\epsilon = 0\\
D_m\epsilon =&\Big(\tilde\nabla_m - \frac{1}{2}\Sigma^R\tilde\Sigma_m\\
&+\frac{1}{2}e^{-B}\partial_r e^B{\Sigma_m}^r - \frac{1}{6}e^{-3B}\partial_r e^C\gamma^7\otimes{\Sigma_m}^r\Big)\epsilon = 0.
\end{aligned} \tag{2.57}$$

Again it can be shown that for an even-dimensional sphere, $\tilde\nabla_m\epsilon = \frac{1}{2}\Sigma^R\tilde\Sigma_m\epsilon$ with $\Sigma^R = \Sigma_1 ... \Sigma_n$ can be satisfied.[6] With the projection $\gamma^7\epsilon_0 = \epsilon_0$, or $\Gamma_0 ... \Gamma_5\epsilon = \epsilon$, the Killing Spinor condition is solved by

$$\epsilon = e^{-C/12}\epsilon_0\otimes\eta_0, \tag{2.58}$$

and

$$A = \frac{1}{6}C, \qquad\qquad B = -\frac{1}{3}C. \tag{2.59}$$

Therefore, the supersymmetric solution of the M5-brane is given by

$$dS_{11}^2 = \Big(1+\frac{\kappa}{r^3}\Big)^{-\frac{1}{3}}dx^\mu dx^\nu\eta_{\mu\nu} + \Big(1+\frac{\kappa}{r^3}\Big)^{\frac{2}{3}}(dr^2 + r^2 d\Omega_4^2), \tag{2.60}$$

with the four-form field strength

$$F_{mnij} = 3\kappa\epsilon_{mnijr}\frac{1}{r^4}. \tag{2.61}$$

With the projection condition on the Killing spinor, the solution breaks half of the supersymmetry. The solution again has the asymptotic AdS structure. In the near horizon limit, the metric approximate

$$dS^2 = 4\kappa^{\frac{2}{3}}\Big[\frac{1}{\rho^2}(dx^\mu dx^\nu\eta_{\mu\nu} + d\rho^2) + \frac{1}{4}d\Omega_4^2\Big], \tag{2.62}$$

with $\rho^2 = 4\kappa\frac{1}{r}$. Therefore, the near horizon limit of the solution is AdS7$\times S^4$.

2.3c D3-Branes

D3 brane is a solution of type IIB supergravity that is coupled to a 4-form self-dual gauge field with the world volume dimension Four. For the D3-brane solution, the type

IIB supergravity spectrum is truncated to the metric, a scalar dilation field, and the self-dual 4-form. The self-dual 4-form can be written as $F = (1 + *)G$, and the non-vanishing components of F are given by

$$\begin{aligned} F_{\mu\nu\alpha\beta r} &= G_{\mu\nu\alpha\beta r} = \epsilon_{\mu\nu\alpha\beta}\partial_r e^C \\ F_{mnijk} &= (*G)_{mnijk} = \epsilon_{mnijkr}e^{4(B-A)}\partial_r e^C. \end{aligned} \tag{2.63}$$

In the presence of only the scalar field and the self-dual five-form, the system is described by the equation of motion Eq.(2.30) and Killing spinor condition Eq.(2.31). Recall the Killing spinor condition requires a constant dilaton field. Then from the Einstein equation, we get

$$\begin{aligned} A'' + 4(A')^2 + 4A'B' + \frac{5}{r}A' &= \frac{1}{4}(c')^2 e^{2C-8A} \\ B'' + 4A'B' + 4(b')^2 + \frac{9}{r}B' + \frac{4}{r}A' &= -\frac{1}{4}(c')^2 e^{2C-8A} \\ 4A'' + 5B'' + 4(A')^2 - 4B'A' + \frac{5}{r}B' &= \frac{1}{4}(c')^2 e^{2C-8A}. \end{aligned} \tag{2.64}$$

And from the equation of motion of the field strength, we get

$$\nabla^2 C + C'(C' + 6B' - 3A') = 0. \tag{2.65}$$

Again we need to consider the Killing spinor condition to have a supersymmetric solution. To solve the Killing spinor condition, the 10d Majorana-Weyl spinor satisfying $\Gamma^{11}\epsilon = \epsilon$ is split into a 4d constant spinor ϵ_0 and a 6d spinor $\eta(r)$

$$\epsilon = \epsilon_0 \otimes \eta(r). \tag{2.66}$$

And the corresponding gamma matrices in the vielbein coordinates are given by

$$\Gamma_P = (\gamma_A \otimes \mathbb{I}, -i\gamma^5 \otimes \Sigma_R, -i\gamma^5 \otimes \Sigma_M), \tag{2.67}$$

where $\gamma^5 = \gamma_0...\gamma_3$ and $\Sigma^7 = \sigma_R\Sigma_1...\sigma_5$. With the projection

$$\gamma^5\epsilon_0 = i\epsilon_0, \qquad\qquad \Sigma^7\eta = -i\eta, \tag{2.68}$$

or, in other words, $\Gamma_{0123}\epsilon = \epsilon$, such that

$$\Gamma^{11}\epsilon = \gamma^5 \otimes \Sigma^7\epsilon = \epsilon, \tag{2.69}$$

the Killing spinor condition becomes

$$\begin{aligned} &\Big(\partial_\mu + \Big(\frac{1}{2}e^{-A}\partial_r e^A - \frac{1}{8}e^{-4A}\partial_r e^C\Big)\gamma^\mu\Sigma^r\Big)\epsilon = 0, \\ &\Big(\partial_r + \frac{1}{8}e^{-4A}\partial_r e^C\Big)\epsilon = 0, \\ &\Big(\tilde{\nabla}_m - \frac{1}{2}\tilde{\Sigma}_m\Big)\epsilon + \Big(\frac{1}{2}e^{-B}\partial_r e^B + \frac{1}{8}e^{-4A}\partial_r e^C\Big)\Sigma_m{}^r\epsilon = 0. \end{aligned} \tag{2.70}$$

The condition is solved by a constant spinor ϵ_0,

$$\eta(r) = e^{-C/8}\eta_0, \tag{2.71}$$

and

$$A = \frac{1}{4}C, \qquad\qquad B = -\frac{1}{4}C. \tag{2.72}$$

Substitute the supersymmetric condition into the equation of motion Eq.(2.65), we once again get the Laplacian equation

$$\nabla^2 e^- C = 0, \tag{2.73}$$

which is solved by

$$e^{-C} = H = 1 + \frac{\kappa}{r^4}. \tag{2.74}$$

Hence the D3 brane solution is given by

$$dS_{10}^2 = \Big(1 + \frac{\kappa}{r^4}\Big)^{-\frac{1}{2}} dx^\mu dx^\nu \eta_{\mu\nu} + \Big(1 + \frac{\kappa}{r^4}\Big)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2). \tag{2.75}$$

The solution again breaks half of the supersymmetry due to the projection on the spinor. In the limit of $r \to 0$, the metric of the solution is asymptotically

$$dS_{10}^2 = \kappa^{\frac{1}{2}} \Big[\frac{1}{\rho^2}(dx^\mu dx^\nu \eta_{\mu\nu} + d\rho^2) + d\Omega_5^2\Big], \tag{2.76}$$

with $\rho = \kappa^{\frac{1}{2}}/r$. Therefore, the near horizon limit of the solution is given by AdS5$\times S^5$.

2.4 Brane World-volume Action

The D-branes and M-branes are dynamic objects in string theory and M-theory. Therefore, they are described by some brane world-volume effective action. The classical action of a p-brane is given by the Nambu-Goto action,

$$S = \int d\sigma^{p+1} \sqrt{\det(G_{ab})} = \int d\sigma^{p+1} \sqrt{\det(\partial_a X^\mu \partial_b X_\mu)}, \tag{2.77}$$

which is given by the world volume of the brane. The Polyakov version of the action is given by

$$S = \frac{1}{2}\int d\sigma^{p+1} \sqrt{-G}(G^{ab}\partial_a X^\mu \partial_b X_\nu - p + 1), \tag{2.78}$$

where G^{ab} is the induced metric on the brane world volume.

In bosonic string theory, the end of the open strings can attach to a stack of D branes, i.e. open strings are subjected to the Newmann boundary condition in the D-brane world volume directions and the Dirichlet boundary condition in the D-brane transverse direction. The open strings attached to branes have excitations corresponding to a massless vector field, leading to an electromagnetic theory described by the Dirac-Born-Infeld action [8] of D-branes

$$S = \int d\sigma^{p+1}\sqrt{\det(h_{ab} + 2\pi\alpha' F_{ab})}, \tag{2.79}$$

where F_{ab} is the field strength of the massless vector field.

In order to have a supersymmetric theory, a brane action with target space supersymmetry is needed. The construction of the supersymmetry action is in analogy to the Green-Schwarz formalism.[3] To have $\mathcal{N} = 2$ spacetime supersymmetry in the action, the coordinates X^μ are accompanied by two Majorana-Weyl fermionic coordinates Θ^1 and Θ^2 satisfying the chiral projection of type II supergravities. The supersymmetric transformation is given by

$$\delta X^\mu = \bar{\epsilon}_A \Gamma^\mu \Theta^A, \qquad\qquad \delta\Theta^A = \epsilon^A, \tag{2.80}$$

where A=1,2. The supersymmetric version of the Born-Infeld action is

$$S = \int d\sigma^{p+1}\sqrt{\det(G_{ab} + 2\pi\alpha' \mathcal{F}_{ab})}. \tag{2.81}$$

Where

$$G_{ab} = \eta_{\mu\nu}\Pi_a{}^\mu \Pi_{b\mu}, \tag{2.82}$$

with

$$\Pi_a^\mu = \partial_a X^\mu - \bar{\Theta}_A \Gamma^\mu \partial_a \Theta^A, \tag{2.83}$$

and $\mathcal{F}_{ab}$ is the supersymmetric combination of the field strength in the Born-Infeld action, which is given by

$$\mathcal{F}_{ab} = F_{ab} + b_{ab}, \tag{2.84}$$

where b_{ab} is a 2-from depend on Θ.

In Green-Schwarz string theory, it was found that there are twice the fermionic degrees of freedom with half of them being gauge components. This fact indicates that there is a hidden symmetry in the theory known as kappa symmetry [9] in addition to the supersymmetry. To have the kappa symmetry in the action, an additional supersymmetric Wess-Zumino term needs to include. A similar story happens to D-branes. In order to have the correct fermionic degree of freedom, the theory now has the kappa symmetry generated by Majorana-Weyl spinor κ, [3]

$$\delta X^{\mu} = \bar{\Theta}_A \Gamma^{\mu} \delta\Theta^A, \qquad \delta\Theta^1 = P_- \kappa^1, \qquad \delta\Theta^2 = P_+ \kappa^2, \tag{2.85}$$

where the projection operator $P_{\pm}$ is given by

$$P_{\pm} = \frac{1}{2}(1 \mp \frac{1}{(p+1)!\sqrt{-G}} \epsilon^{a...b} \Pi^{\mu}_a ... \Pi^{\nu}_b \Gamma_{\mu...\nu}). \tag{2.86}$$

To have the kappa symmetry generated by Eq.(2.85) in the theory, the supersymmetric Wess-Zumino term is needed in addition to the Born-Infeld action. The bosonic part of the term is given by

$$S_{CS} = \mu_p \int \sum_n C_{[n]} \wedge e^{B+2\pi\alpha' F} = \mu_p \int C_{[p+1]} + C_{[p-1]} \wedge (B + 2\pi\alpha' F) + ... \tag{2.87}$$

with the integral forms pulled back to the brane volume. $C_{[n]}$ in the Wess-Zumino term correspond to RR fields which equal the RR fields that are electrically coupled to the branes and equal the dual fields of RR fields that are magnetically coupled to the branes. And μ_p is the charge of the branes. Note the first term in the expansion describe how the (p+1)-form is coupled to the p-branes as mentioned in the last section. And the term gives the (p+1)-form current coupled to the (p+1)-form.

A similar construction of the supersymmetric D-branes world volume action can be done to M-branes.[10] For M2-brane, the action is known as the super-membrane action, given by

$$S = \int d\sigma^3 \sqrt{-G} \Big(\frac{1}{2} G^{ab} \partial_a X^{\mu} \partial_b X^{\nu} g_{\mu\nu} - \frac{1}{2} + \bar{\Theta} \Gamma^a \nabla_a \Theta + \frac{1}{3!} \epsilon^{abc} A_{\mu\nu\alpha} \partial_a X^{\mu} \partial_b X^{\nu} \partial_c X^{\alpha} + ... \Big), \tag{2.88}$$

which is the Polyakov action plus the Wess-Zumino term describing M2-brane coupling to the 3-form. Θ now is an 11D spinor coordinate. The same as the D-brane action, the M2-brane action has supersymmetry with variations generated by 11D spinor ϵ

$$\delta_{\epsilon} X^{\mu} = \bar{\epsilon} \Gamma^{\mu} \Theta, \qquad \delta_{\epsilon} \Theta = \epsilon, \tag{2.89}$$

as well as the kappa symmetry generated by κ [11]

$$\delta_{\kappa} X^{\mu} = 2\bar{\Theta} \Gamma^{\mu} P_+ \kappa, \qquad \delta_{\kappa} \Theta = 2 P_+ \kappa, \tag{2.90}$$

with $P_{\pm}$ given by

$$P_{\pm} = \frac{1}{2}(1 \pm \frac{1}{3!\sqrt{-G}} \epsilon^{abc} \partial_a X^{\mu} \partial_b X^{\nu} \partial_c X^{\alpha} \Gamma_{\mu\nu\alpha}). \tag{2.91}$$

satisfying $P_+P_- = 0$. A bosonic configuration of M2 brane described by X^μ breaks the supersymmetry since the supersymmetry transformation $\delta_\epsilon\Theta$ does not vanish by requiring $\Theta = 0$. However, the supersymmetry can be preserved if the non-vanishing $\delta_\epsilon\Theta$ can be gauged away by the kappa symmetry, meaning

$$\delta_\epsilon\Theta + \delta_\kappa\Theta = \epsilon + 2P_+\kappa = 0. \tag{2.92}$$

Using $P_+P_- = 0$ we get the condition on ϵ

$$P_-\epsilon = 0. \tag{2.93}$$

Branes also have backreaction on the geometry of supergravity described by

$$S = S_{\text{SUGRA}} + S_{\text{pBrane}}. \tag{2.94}$$

Chapter 3

AdS-CFT Correspondence

In the previous chapter, we obtained various solutions for the supergravities with the asymptotic anti-de-Sitter spacetime structure. It was found that through Maldacena's Anti-de-Sitter and Conformal Field Theory (AdS-CFT) Correspondence conjecture, [18] Supergravity or String and M-theory on asymptotic AdS spacetime is related to some conformal field theory on the boundary of the AdS spacetime. This correspondence is a powerful tool which provided a new approach to studying string theory, M theory, and quantum field theory. In this chapter, we will first briefly introduce the AdS-CFT correspondence and then discuss it in the context of type IIB string theory and supergravity.

3.1 Conformal Symmetry and AdS Geometry

Conformal Field Theory: The conformal symmetry[12] is an extension of the Poincare symmetry. In addition to the translations, rotations, and boosts in the Poincare group, the conformal group includes two extra transformations, the dilations and the special conformal transformations(SCT). Under conformal transformations, the metric transforms as

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{3.1}$$

for some function $\Lambda(x)$. The conformal transformations are generated by vector fields satisfying the conformal Killing equation

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu} \nabla_\alpha \epsilon^\alpha. \tag{3.2}$$

And the conformal transformations on the coordinates are given by

$$\begin{aligned} \text{Translation :} \quad & x'^{\mu} = x^{\mu} + a^{\mu} \\ \text{Dilation :} \quad & x'^{\mu} = \alpha x^{\mu} \\ \text{Rotation :} \quad & x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \\ \text{SCT :} \quad & x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}. \end{aligned} \tag{3.3}$$

The corresponding generators of these transformations on functions are given by

$$\begin{aligned} \text{Translation :} \quad & P_{\mu} = i\partial_{\mu} \\ \text{Dilation :} \quad & D = i x^{\mu} \partial_{\mu} \\ \text{Rotation :} \quad & M_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \\ \text{SCT :} \quad & K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}). \end{aligned} \tag{3.4}$$

It can be shown that the conformal generators obey the lie algebra

$$\begin{aligned} & [M_{\mu\nu}, D] = 0 & & [D, P_{\mu}] = -iP_{\mu} \\ & [M_{\mu\nu}, P_{\rho}] = -2i\eta_{\rho[\mu}P_{\nu]} & & [D, K_{\mu}] = iK_{\mu} \\ & [M_{\mu\nu}, K_{\rho}] = -2i\eta_{\rho[\mu}K_{\nu]} & & [P_{\mu}, K_{\nu}] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D \\ & [M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}). & & \end{aligned} \tag{3.5}$$

We can rearrange these generators into

$$J_{\mu\nu} = M_{\mu\nu}, \qquad J_{\mu d} = \frac{1}{2}(K_{\mu} - P_{\mu}), \qquad J_{\mu(d+1)} = \frac{1}{2}(K_{\mu} + P_{\mu}), \qquad J_{d(d+1)} = D, \tag{3.6}$$

labelled by $(0, 1, ..., d, d+1)$. These generators satisfy the $\mathcal{SO}$(d,2) algebra with signature $(-, +, ..., +, -)$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho}). \tag{3.7}$$

Therefore, the lie algebra of the conformal group is isomorphic to the lie algebra of SO(d,2).

An operator $\hat{O}(x)$ with scaling dimension Δ transform under dilation $x^{\mu} \to \alpha x^{\mu}$ as $\hat{O}(x) \to \alpha^{\Delta}\hat{O}(\alpha x)$. The dilation operator act on the operator as

$$[D, \hat{O}(x)] = i(x^{\mu}\partial_{\mu} - \Delta)\hat{O}(x). \tag{3.8}$$

For the operator at origin, this expression is diagonalized to $[D, \hat{O}(0)] = -i\Delta\hat{O}(0)$. With the commutation relation between D and P_{μ}, K_{μ}, it is easy to verify that P_{μ} increases the scaling dimension by one whereas K_{μ} lowers the scaling dimension by one,

$$[D, P_{\mu}\hat{O}(0)] = -i(\Delta + 1)P_{\mu}\hat{O}(0), \qquad [D, K_{\mu}\hat{O}(0)] = -i(\Delta - 1)K_{\mu}\hat{O}(0). \tag{3.9}$$

For a unitary theory, there is a lower bound on the operator [14], therefore, there exists an operator satisfying $[K_\mu, \hat{O}(0)] = 0$. We call the operators satisfying the condition the primary operators, while operators constructed by acting P_μ are called the descendant. A conformal field theory does not have asymptotic states, so it is natural to consider operators.[15] The representations of conformal groups are labelled by the "lowest weights". The representations correspond to operators with the primary operator as the lowest weight. [13]

The form of correlation functions of operators is restricted by conformal symmetry. For example, the form of the two-point function of scalar operators is completely fixed by the conformal symmetry, which is given by

$$\left\langle \hat{O}_1(x_1)\hat{O}_2(x_2) \right\rangle = \frac{C}{(x_1 - x_2)^{2\Delta}}. \tag{3.10}$$

Anti deSitter space: Anti-de-Sitter space is a Lorentzian maximum symmetric space. It is a Lorentzian analogy of hyperbolic space. An AdS(d+1) space with radius R can be embedded in a (d,2) Minkowski space via

$$X_{-1}^2 + X_0^2 - X_1^2 - ... - X_d^2 = R^2. \tag{3.11}$$

AdS space is maximally symmetric, and the isometry of AdS(d+1) is SO(d,2) which is isomorphic to the d-dimensional conformal group. Parameterizing the coordinates with Poincare coordinate chart, [14]

$$X_{-1} = \frac{R}{2z}(1 + |x_i|^2 - t^2 + z^2), \qquad X_0 = \frac{R}{z}t, \qquad X_i = \frac{R}{z}x_i, \qquad X_d = \frac{R}{2z}(1 - |x_i|^2 + t^2 - z^2), \tag{3.12}$$

and pulling back the Minkowski metric, one gets the AdS(d+1) metric in the Poincare chart

$$ds^2 = R^2 \frac{1}{z^2}(-dt^2 + dx_i dx^i + dz^2). \tag{3.13}$$

which is identical to the brane solutions in the near horizon limit discussed in the last chapter. We now consider the geometry near the boundary of AdS(d+1) space at $z \to 0$. As $z \to 0$, $dz \to 0$, the metric become

$$ds^2 = R^2 \frac{1}{z^2}(-dt^2 + dx_i dx^i). \tag{3.14}$$

which is just the (d-1,1) Minkowski space up to scaling. Therefore, the boundary of AdS(d+1) space is equipped with the SO(d-1,1) isometry. Note the metric is also invariant under dilation $\{x^\mu, z\} \to \{\alpha x^\mu, \alpha z\}$. Hence the boundary is actually conformal. A spacetime is asymptotically AdS if its boundary approaches the AdS boundary, and the boundary of the asymptotic AdS spacetime is also conformal.[14]

The correspondence: CFT and AdS geometry are connected via the field-operator correspondence. A field living in the bulk with an asymptotic AdS geometry is dual to an operator with the same spin of the conformal field theory living on the AdS boundary.[15] For example, it can be shown from correlation functions that the scalar field of the bulk theory and a corresponding operator in CFT are matched via

$$\mathcal{Z}_{\text{Bulk}}\Big[\Phi(x,z)|_{z=0} = \Phi_0(x)\Big] = \mathcal{Z}_{\text{CFT}}\Big[\Phi_0(x)\Big] = \Big\langle e^{\int dx^d \Phi_0(x)\hat{O}(x)}\Big\rangle, \tag{3.15}$$

where $\mathcal{Z}_{\text{Bulk}}[\Phi(x,z)|_{z=0} = \Phi_0(x)]$ is the partition function of the bulk theory with scalar $\Phi(x,z)$ restricted to the boundary condition. The action in the partition function is on shell and regularized. And $\mathcal{Z}_{\text{CFT}}[\Phi_0(x)]$ is the generating functional of the operator $\hat{O}(x)$ with the boundary condition of the scalar field in the bulk $\Phi_0(x)$ as its source. The same relation can also apply to other fields and operators. The stress-energy tensor is sourced by the metric, therefore the stress-energy tensor of the CFT is in correspondence with the metric tensor in the bulk. Since the bulk theory can have a dynamical metric, it is a quantum gravity theory. This leads to the famous AdS-CFT conjecture of Maldacena, which states a quantum gravity theory in (d+1)-dimensional spacetime with an asymptotic AdS geometry is dual to a d-dimensional conformal field theory without gravity on the boundary of AdS spacetime.

3.2 4D $\mathcal{N}$=4 SYM and D3-Brane World Volume

The best-studied AdS/CFT example is the correspondence between the D3 brane world volume theory of type IIB string theory and the 4D $\mathcal{N} = 4$ super Yang-Mills theory. We previously obtained the asymptotically AdS5$\times S^5$ solution for the D3 brane with 16 supercharges preserved. Due to the AdS/CFT correspondence, the AdS boundary of the theory is described by a 4D $\mathcal{N} = 4$ superconformal field theory.

The 4d Yang-Mills theory is classically invariant under scaling. This conformal invariance, however, is broken by quantum effects, for example, momentum cut-off. The 4D $\mathcal{N}$=4 super Yang-Mills, however, scales invariant. [14] This can be read from the fact that the Beta function of super Yang-Mills theory at one loop order,

$$\frac{\partial g}{\partial \ln\mu} = -\frac{g^3}{48\pi^2}(11 - 4\times 4 - 6) = 0, \tag{3.16}$$

and higher-order perturbations. The theory is superconformal with the superconformal group $PSU(2,2|4)$. The spectrum of the vector multiplet has a gauge field A_μ, six scalar fields

ϕ_I in the fundamental representation of SO(6), and 4 Weyl spinors in the fundamental representation of SU(4). The super Yang-Mills action of the vector multiplet is given by [14]

$$\begin{aligned} S = & \int dx^4 \frac{1}{g_{\mathrm{YM}^2}} \mathrm{Tr}\Big[-F_{\mu\nu}F^{\mu\nu} - D_\mu \phi_I D^\mu \phi^I + \bar{\lambda}_i \not{D} \lambda^i - \sum_{i,j} [\phi_I, \phi_J]^2 + \bar{\lambda}_i \Gamma^I \phi_I \lambda^i \Big] \\ & + \theta \int F \wedge F. \end{aligned} \tag{3.17}$$

Written in the superpotential form, the action has a complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \tag{3.18}$$

This coupling has an SL(2,$\mathbb{Z}$) symmetry.

We now consider the string theory side of the story. Open strings ended on D3 brane give rise to the D3 brane dynamics described by the brane world volume theory. The theory has a 4D gauge field, 6 scalars describing the brane transverse degrees of freedom, and the same amount of the fermionic degrees of freedom, the string massive modes again decoupled in the low energy limit. The bosonic part of the theory is described by the Born-Infeld action, which for D3-brane in the Einstein frame is given by

$$S_{\mathrm{BI}} = \int d\sigma^4 e^{(p-3)\phi/4} \sqrt{-\det(G + e^{-\phi/2}F)} = \int d\sigma^4 \sqrt{-\det G}(1 + \frac{1}{4}\mathrm{e}^{-\phi} F_{\mu\nu}F^{\mu\nu} + \ldots) \tag{3.19}$$

The higher order terms can be ignored in the low energy limit where $\alpha' = 0$. And the Wess-Zumino action Eq.(2.87) can be expended into

$$S_{\mathrm{WZ}} = \int (C_4 + F \wedge C_2 + \frac{1}{2} F \wedge F \wedge C_0 + \ldots). \tag{3.20}$$

where the first term in the action is the 4-form gauge field coupling to the D3 brane. Note the third term in this expansion, Eq.(3.20), and the second term in Born-Infeld action Eq.(3.19) gives

$$S = -\int \mathrm{d}x^4 (\mathrm{e}^{-\phi} F_{\mu\nu}F^{\mu\nu} + C_0 \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}). \tag{3.21}$$

One can immediately recognise that the above action recovers the kinetic and Chern-Simons term of the field strength in the super Yang-Mills action Eq.(3.17) with complexified gauge coupling

$$\tau = \frac{C_0}{2\pi} + i\frac{4\pi}{g_s \mathrm{e}^{\phi}}. \tag{3.22}$$

Notice that there is a discrete symmetry $\tau \to \tau + 1$ due to the axionic properties of $C_0 \to C_0 + 2\pi$. In fact, this coupling also has a $SL(2,\mathbb{Z})$ invariance. We can read from the coupling that

$$g_s = g_{\mathrm{YM}}^2, \tag{3.23}$$

and the expectation value of the dilaton shifts the sting coupling constant.

One can consider a change to the super Yang-Mills coupling. Such change is given by the deformation of a marginal operator in the super Yang-Mills theory. The marginal operator in the generating functional is sourced by a scalar. According to the field-operator correspondence Eq.(3.15), this deformation by the marginal operator will change the scalar dilaton field boundary condition. The change in the expectation value of the dilaton field will shift the string coupling. [15] This exactly relates the string coupling with the super Yang-Mills coupling.

3.3 Large N Gauge Theory and Supergravity

In the discussion from the previous section, we ignored the massive higher spin modes and the gravitational coupling to the brane world volume field. The former assumption makes sure that supergravity gives a good approximation to the bulk theory and Yang-Mills gives a good description of the theory on the brane. And the latter leaves the brane world volume decoupled from the gravitons and other closed string modes such that one is left separately with a supergravity theory in the bulk and a Super Yang-Mills on the conformal boundary. These require a small string coupling constant g_s and a small string length $l_s^2 \sim \alpha'$ compared with the radius of the AdS space L. [16] From the t'Hooft planar diagram, one is able to read that Feynman diagrams of a U(N) gauge theory look like world sheets with t'Hooft coupling given by $\lambda = g_{YM}^2 N = g_s N$.[17] It is also found that $2g_{YM}^2 N = L^4/l_s^4$. Hence, to have a weakly coupled gravity and small string length, one needs a large N gauge theory and the gauge theory becomes strongly coupled. Also, as one goes to the near horizon limit, the energy is redshifted, and one is left in the low energy regime. Then we are able to correspond the D3 brane world volume on AdS5 to large N 4D super Yang-Mills.

A similar story can also be applied to M-theory and 11D supergravity. M5-branes on AdS7 × S^4 is dual to 6D $\mathcal{N} = (0, 2)$ superconformal field theory. M2-branes on AdS4 × S^7 is dual to 3D $\mathcal{N} = 6$ ABJM theory. [18]

One way to check the correspondence is via global symmetries. For example, the superconformal group of the 4D $\mathcal{N} = 4$ super Yang-Mills theory is PSU(2,2|4) which has a subgroup of SO(4,2) × SU(4) where the SO(4,2) is the 4D conformal group and SU(4) is the R symmetry of the theory. The group is isomorphic to the SO(4,2) × SO(6) isometry of

AdS5 $\times$ S^5. There are other ways of checking the correspondence via correlation functions, operators, moduli space, matching anomalies, and central charges. [15]

Chapter 4

Branes on Curved Background

In chapter2, we considered branes on a flat background spacetime as the supergravity solutions. In this chapter, instead of flat backgrounds, we consider the background to be the product of flat spacetime and some curved manifolds. A curved manifold has a non-zero spin connection, which affects the Killing spinor condition. Therefore, the supersymmetry condition restricts the possible geometry of the background manifold. Branes on curved manifolds can either transverse to the curved manifolds or wrap around a cycle in the curved manifold. In this chapter, we will first discuss the geometry of the background manifold which is classified by special holonomy and then talk about the supergravity solutions of branes transverse to Calabi-Yau manifolds. Branes wrapping on curved cycles will be discussed in the next chapter.

4.1 Special Holonomy Manifolds

We first consider the geometry of the D=11 supergravity on a more general background spacetime with a vanishing 3-form gauge field. Then the supergravity equation of motion requires the spacetime to be Ricci flat. And the Killing spinor condition requires spacetime to admit a covariant constant spinor $\nabla_\mu \epsilon = 0$, which implies

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{4} R_{\mu\nu\alpha\beta}\Gamma^{\alpha\gamma}\epsilon = 0. \tag{4.1}$$

$R_{\mu\nu\alpha\beta}\Gamma^{\alpha\gamma}$ generates the infinitesimal holonomy transformation of the manifold. Hence a necessary supersymmetric condition is for the Killing spinor to be invariant under the holonomy group of the manifold. This means the 32-component Killing spinor needs to decompose into the trivial representation of the special holonomy group.[19]

Here we consider the spacetime to be the product of flat spacetime with some Ricci flat manifold $\mathbb{R}^{1,D-d} \times \mathcal{M}^d$. The holonomy group of spacetime is the holonomy group of the Ricci flat manifold. The Ricci flat manifolds that can be considered here are Calabi-Yau N-folds with SU(N) holonomy, hyper Kähler manifolds with Sp(N) holonomy, G2-holonomy manifolds and Spin(7)-holonomy manifolds. The manifold we will be talking about the most here is the Calabi-Yau manifold. The Calabi-Yau manifold admits two nowhere vanishing closed forms, the Kähler form and a holomorphic top form Ω.

To count preserved supersymmetries, one needs to count the number of spinor components in the spinor 32 representation of Spin(10,1) decomposes into the trivial representations of the special holonomy group of the manifold. For example, consider the 8-dimensional manifolds, then the 32 representation of Spin(10,1) decomposes into Spin(2,1)$\times$Spin(8). Spin representations of Spin(8) can be written as $8_s \oplus 8_c$. First, consider SU(4) holonomy. The representations of Spin(8) 8_s decompose into $6 \oplus 1 \oplus 1$ and 8_c decompose into $4 \oplus 4$ representation of SU(4). There are two trivial representations, each is a 3D spinor with 2 components. Therefore the SU(4) holonomy background preserves 4 supersymmetry. For the Spin(7)-holonomy manifold, 8_s decompose into $7 \oplus 1$ and 8_c decompose into 8 representation of Spin(7), preserving 2 supersymmetry.[19]

dimension	Holonomy	SUSY
10	SU(5)	2
10	SU(3)×SU(2)	4
8	Spin(7)	2
8	SU(4)	4
8	Sp(2)	6
8	SU(2)×SU(2)	8
7	G2	4
6	SU(3)	8
4	SU(2)	16

Table 4.1: The supersymmetry preserved by each background manifold with special holonomy. [19]

We have discussed the supersymmetry by the special holonomy manifolds, or by probe branes on the special holonomy manifolds. We can consider the back reactions of the branes on these special holonomy manifolds. The branes back reactions are described by the supergravity and in most cases break half supersymmetry of the special holonomy manifold. We will first consider branes on the product manifold of flat spacetime and a Calabi-Yau manifold, with the brane transverse to the Calabi-Yau manifold. We will then in the next chapter consider brane wrap a compact cycle within the Calabi-Yau manifold.

4.2 Branes Transverse to Calabi-Yau Manifolds

We now consider M2 and D3 branes on the background manifold with special holonomy by switching on the n-form gauge fields. We first look at the M2 and D3 brane transverse to and on the apex of the Calabi-Yau cone. With the background as the tensor product of a flat spacetime and a Calabi-Yau cone $\mathbb{R}^{1,p} \times \mathrm{CY}^{(D-p-1)/2}$.

4.2a Calabi-Yau Cone and Sasaki Geometry

A Calabi-Yau n-cone is a cone over a Sasaki-Einstein $2n-1$ manifold. The cone metric of the Calabi-Yau cone can be written as

$$dS^2 = dr^2 + r^2 dS^2(\mathrm{SE}2n-1), \tag{4.2}$$

where $ds^2(\mathrm{SE})$ is the metric on the Sasaki-Einstein manifold.

An odd-dimensional manifold is Sasakian if and only if the cone over the manifold is Kähler.[20] The Kähler cone over the Sasakian manifold has a complex structure $\mathcal{I}$ which acting on the radial direction of the Kähler cone gives a normed Killing vector $\xi = \mathcal{I}(r\partial_r)$ on the Sasakian manifold. The Killing vector is called the Reeb vector and corresponds to the R-symmetry of the dual field theory discussed in the next subsection. The metric of the Sasakian manifold can be written as

$$dS^2(\mathrm{SE}2n-1) = (dz+\sigma)^2 + dS^2(\mathrm{KE}n-1) \tag{4.3}$$

where $\eta = dz + \sigma$ is the one form dual to the Reeb vector such that $\eta(\xi) = 1$ and the $dS^2(\mathrm{KE}n-1)$ is a transverse Kähler metric on a transverse Kähler $n-1$ manifold. And η satisfy

$$d\eta = 2J \tag{4.4}$$

where J is the Kähler form of the transverse Kähler manifold. Therefore, the Sasakian manifold is in the middle of two Kähler manifolds.

If the Kähler cone is also Ricci-flat or in other words Calabi-Yau, then the Sasakian manifold is also Einstein with the Ricci tensor

$$R_{\mu\nu} = (2n-2)g_{\mu\nu} \tag{4.5}$$

where n is the complex dimension of the Calabi-Yau cone and $g_{\mu\nu}$ is the metric on the Sasaki-Einstein manifold. And for the Sasaki-Einstein manifold, the transverse Kähler manifold is also Einstein with the Ricci tensor

$$R_{\mu\nu}(T) = 2ng_{\mu\nu}(T) \tag{4.6}$$

where $g_{\mu\nu}(T)$ is the transverse Kähler metric on the transverse Kähler Einstein manifold.

4.2b M2 and D3 Branes at Apex of the CY Manifolds

M2 and D3 branes have transverse space even-dimensional, hence we can replace the transverse flat space with a Calabi-Yau cone and put branes at the apex of the cone which corresponds to $r = 0$. For M2-Brane the background spacetime is $\mathbb{R}^{1,2} \times \mathrm{CY}^4$, and for D3 is $\mathbb{R}^{1,3} \times \mathrm{CY}^3$. The same n-form gauge fields are switched on as in chapter(2). The metric can be written as

$$ds^2 = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}(dr^2 + r^2 ds^2(SE)), \tag{4.7}$$

where the second term is the Calabi-Yau cone metric with $ds^2(SE)$ the Sasaki-Einstein metric. Note the only difference between the flat space and the CY cone is the odd-dimensional sphere is now replaced with the Sasaki-Einstein manifold. Sasaki-Einstein manifolds have the same Ricci tensor as the spheres. Therefore, solutions to the equations of motion are the same as the flat background in chapter(2), which together with the asymptotic structures are given by

$$\begin{aligned}
&\mathrm{M2} \qquad dS^2 = \left(1+\frac{\kappa}{r^6}\right)^{-\frac{2}{3}} dx^\mu dx_\nu + \left(1+\frac{\kappa}{r^6}\right)^{\frac{1}{3}} (dr^2 + r^2 ds^2(\mathrm{SE7})) \rightarrow \mathrm{AdS4}\times\mathrm{SE7} \\
&\mathrm{D3} \qquad dS^2 = \left(1+\frac{\kappa}{r^4}\right)^{-\frac{1}{2}} dx^\mu dx_\nu + \left(1+\frac{\kappa}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 ds^2(\mathrm{SE5})) \rightarrow \mathrm{AdS5}\times\mathrm{SE5}
\end{aligned} \tag{4.8}$$

To have a supersymmetric solution, we also have to check the Killing spinor condition. The condition of requiring spinors to satisfy $(\tilde{\nabla}_m - \alpha\tilde{\Sigma}_m)\epsilon = 0$ on a sphere now has to satisfy on the Sasaki-Einstein manifold. For a general Sasaki-Einstein manifold, at least one spinor solution can be found for an odd complex dimension of the Calabi-Yau cone which corresponds to the D3 brane case, and at least two solutions can be found for an even dimension of the Calabi-Yau cone which corresponds to the M2 brane case.[20] Hence a portion of supersymmetry is broken. If more spinor solutions are found, then the Sasaki-Einstein manifold is a sphere, which recovers the flat background manifold which preserves all of the spinor solutions.

The D3 brane case has four spinor components on the 5-dimensional Sasaki-Einstein manifold with only one component left. Hence the D3 brane on the CY3 cone preserves a quarter of the supersymmetry and is left with 4 supercharges. This breaks half of the supersymmetry of the CY3 background. The AdS/CFT dual of the D3 brane solutions on CY3 is now a 4D $\mathcal{N} = 1$ superconformal field theory. The R symmetry of the dual field theory corresponds to the Reeb Killing vector mentioned earlier. And for the M2 brane on the CY4 case, eight spinor components on the 7-dimensional Sasaki-Einstein manifold are reduced to two. This also preserves a quarter of the supersymmetry and left with 4 supercharges. Hence the AdS/CFT dual of M2 brane on CY4 is a 3D $\mathcal{N} = 2$ superconformal field theory. Note M2 brane on CY4 does not break half supersymmetry of the CY4 manifold. This is a special case where the projections on spinors overlap.

Chapter 5

Branes Wrapped on Cycles

In the last chapter, we discussed curved background manifolds for brane solutions and branes transverse to the Calabi-Yau manifolds. In this chapter, we discuss another way to put branes on curved background manifolds, which is to wrap them around a supersymmetric cycle in the curve manifold. As mentioned earlier, for branes wrapped around curved cycles, the non-trivial spin connection of the cycle will appear in the Killing spinor condition. Hence the supersymmetry condition restricts the possible geometry structure a brane solution can have. There are several ways for wrapped brane solutions to preserve supersymmetry. One way for branes to be wrapped around curved cycles is via calibration. The other way is to turn on the gauge field connection on the normal bundle of the cycle such that the spin connection on the cycle is cancelled by the gauge field leaving a constant spinor. This is known as the topological twist, which is closely related to the calibration. For a brane wrapped around a compact cycle, in the IR limit where the scale is much larger than the cycle, the fluctuations on the cycle become undynamic, hence the dual field theory can have fewer dimensions than in the UV where the scale is much smaller. These solutions also have AdS structures in different dimensions, therefore they are also dual to superconformal field theories in various dimensions, and the renormalization group flow of the superconformal field theories could go across different dimensions. [21]

In this chapter, we will first introduce the idea of calibration and discuss how probe-branes wrapped around calibrated cycle preserve supersymmetry and how the calibration is related to the topological twist. Then we will talk about the D3 and M2 brane wrapped around H^2 and S^2 in the minimal gauged supergravity and discuss how the UV solution flows to the IR solution. We will also discuss D3 and M2 branes wrapped on a type of orbifold with conical singularities known as the spindle. For the spindle solution, it was found that there is no topological twist to preserve the supersymmetry, and although the spindle has singularities, the uplifted solution is free from the singularities. The M2 and D3 branes wrapped around H^2 and spindles share similar geometrical structures known as the GK geometry which will be introduced in the next chapter.

5.1 Calibration

In chapter 2 we talked about the world volume effective actions of branes and introduced the kappa symmetry to gauge away the extra degrees of freedom. In order for a brane with a bosonic configuration to preserve supersymmetry, the condition on the Killing spinor

$$P_{-}\epsilon = 0 \tag{5.1}$$

need to be satisfied, where

$$P_{-} = \frac{1}{2}(1 - \Gamma) \tag{5.2}$$

For M2-branes,

$$\Gamma = \frac{1}{3!\sqrt{-G}}\epsilon^{abc}\partial_a X^{\mu}\partial_b X^{\nu}\partial_c X^{\alpha}\Gamma_{\mu\nu\alpha}, \tag{5.3}$$

where

$$G = \det\Big(\partial_a X^{\mu}\partial_b X^{\nu} g_{\mu\nu}\Big), \tag{5.4}$$

for $a, b, c = 0, 1, 2$. Consider M2 brane wrapped around a 2-dimensional cycle. In the gauge the world volume parameter $\sigma_0 = X_0$ with the direction of σ_1 and σ_2 undetermined and wrapped around a cycle Σ, we get

$$\Gamma = \frac{1}{2!\sqrt{-G}}\epsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\Gamma_0\Gamma_{\mu\nu}, \tag{5.5}$$

with

$$G = \det\Big(\partial_a X^{i}\partial_b X^{j} g_{ij}\Big), \tag{5.6}$$

for $a, b = 1, 2$, where G is just the determinant of the metric pulled back to the cycle. $P_{\pm}$ are projection operators satisfying $P_{\pm}^2 = P_{\pm} = P_{\pm}^{\dagger} = P_{\pm}^{\dagger}P_{\pm}$, therefore

$$\epsilon^{\dagger}P_{-}\epsilon = \epsilon^{\dagger}P_{-}^{\dagger}P_{-}\epsilon = |P_{-}\epsilon|^2 \geqslant 0, \tag{5.7}$$

with equality only if the supersymmetry is preserved. This is equivalent to saying

$$\epsilon^{\dagger}\epsilon \geqslant \epsilon^{\dagger}\Gamma\epsilon \tag{5.8}$$

with Killing spinor normalized to $\epsilon^\dagger\epsilon = 1$, we get

$$\sqrt{-G} \geqslant \frac{1}{2!}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu \epsilon^\dagger \Gamma_0 \Gamma_{\mu\nu}\epsilon = \frac{1}{2!}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu \bar{\epsilon}\Gamma_{\mu\nu}\epsilon \tag{5.9}$$

which is equivalent to saying that the form $\psi = \bar{\epsilon}\Gamma_{\mu\nu}\epsilon$ pulled back to the cycle Σ is less or equal to the volume form of the cycle, and the form equals the volume form if the cycle is supersymmetric,

$$Vol_\Sigma \geqslant \psi|_\Sigma = \frac{1}{2!}\bar{\epsilon}\Gamma_{\mu\nu}\epsilon\partial_a X^\mu \partial_b X^\nu d\sigma^a d\sigma^b. \tag{5.10}$$

Moreover, as given by minimizing the Nambu-Goto action, the brane needs to be wrapped around cycles with minimum volume. Such conditions can be realized via the notion of calibration. [19][22]

The calibration form is a closed n-form ψ that is less than the volume form of any cycle when pulled back to the cycle, [19]

$$d\psi = 0, \qquad\qquad vol_\Sigma \geqslant \psi|_\Sigma, \quad \forall\Sigma. \tag{5.11}$$

Then an n-cycle Σ is called being calibrated if the calibration n-form pulled back to the cycle is equal to the volume form of the cycle,

$$vol_\Sigma = \psi|_\Sigma. \tag{5.12}$$

Also, since the form is closed, it can be proven that the calibrated cycle has the minimum volume. Therefore, a calibrated cycle is supersymmetric.

The Calabi-Yau manifolds admit two types of form that can be the calibrations forms, the wedge products of Kähler forms J^n, and the real part of the holomorphic volume form $\Omega_{(n,0)}$, which can be constructed by

$$J_{mn} = i\rho^\dagger\gamma_{mn}\rho, \tag{5.13}$$

$$\Omega_{m1..m2n} = \rho^T\gamma_{m1..m2n}\rho, \tag{5.14}$$

where ρ is the covariantly constant spinor in the Calabi-Yau manifolds.[19] The cycle calibrated by J^n has 2n dimensions and inherits the closed Kähler form, therefore is a Kähler 2n-cycle. The cycle calibrated by the holomorphic volume form $\Omega_{(n,0)}$ are called the special lagrangian cycles, they have the same dimensions as the complex dimensions of the Calabi-Yau manifold.

There are also Spin(7)-holonomy manifolds, which admits a nowhere vanishing self-dual closed Cayley 4-form

$$\psi_{mnij} = -\bar{\rho}\gamma_{mnij}\rho. \tag{5.15}$$

The Cayley 4-form calibrates the Cayley 4-cycles in the Spin(7)-holonomy manifolds. The G2-holonomy manifold admits a nowhere vanishing associative 3-form that is closed and co-closed, given by

$$\psi_{mni} = -i\bar{\rho}\gamma_{mni}\rho, \tag{5.16}$$

which calibrates the associative 3-cycles. Since the form is also co-closed, the Hodge dual of the form is also a calibration form, calibrating the co-associative 4-cycles. The hyper Kähler manifolds are a type of Calabi-Yau manifolds hence admit the Kähler and special Lagrangian calibration. In addition, the 8-dimensional HK2 manifolds also calibrate the Cayley cycles and quaternion Kähler 4-cycles. In later discussions, we will mainly focus on cycles in Calani-Yau manifolds. Just as the brane solutions in the flat space, a brane wrapping cycle in a special holonomy manifold in general breaks half of the supersymmetry of the special holonomy manifold.

5.2 Topological Twist

Since the spin connection ω on the curved cycle affects the supersymmetric condition, one can turn on the gauge connection A on the normal bundle of the cycle and enforce $\omega = -A$ to cancel the spin connection such that

$$D_\mu\epsilon \sim (\partial_\mu + \omega_\mu \cdot \Gamma + A_\mu)\epsilon = 0 \tag{5.17}$$

leaves a constant Killing spinor on the cycle. Since the normal bundle is in the direction transverse to the brane, the G structure of the normal bundle is a subgroup of the R-symmetry of the dual field theory. This is exactly the topological twist in the topological field theory, where the fields in the theory are coupled to a gauge field of the global symmetry to cancel the spin connection of the theory. [23]

For example, consider a probe D3-brane wrapping a 2-sphere in the Calabi-Yau 2-fold. [21] This corresponds to the case of Kähler or special Lagrangian 2-cycle calibration, which in Calabi-Yau 2-folds are equivalent. The symmetry of the tangent and normal bundle of a D3-brane on the flat space is given by SO(3,1)$\times$SO(6). And the symmetry of the canonical bundle of the Calabi-Yau two folds is SU(2). As the D3 brane is wrapped around a two-cycle in the CY2, the bundle SU(2) is split into two U(1)$\sim$SO(2) bundles with one tangent to the two-sphere and one normal to the two-sphere. The connection on the 2-sphere is in SO(2)$_{S^2}$. To perform the topological twist, we have to turn on the gauge field connection on the normal bundle of the sphere which is also in SO(2) to cancel the $\mathrm{SO}(2)_{S2}$ spin connection. And the two remaining world volume directions have the symmetry of SO(1,1) and the 4 transverse direction SO(4). This is equivalent to breaking the flat space SO(6)→SO(4)×SO(2). Then the supersymmetry of the tangent and normal bundle becomes $\mathrm{SO}(1,1)\times[\mathrm{SO}(2)_{S2}\times\mathrm{SO}(2)]\times\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$, with the SO(2) being twisted with $\mathrm{SO}(2)_{S2}$. The spinors then transform in the representation of the symmetry $(\pm', \pm, \mp, 2, 1)$ and $(\pm', \pm, \mp, 1, 2)$ corresponding to $\mathcal{N} = (4, 4)$ supersymmetry in 2-dimension.

It turns out the way that calibrated cycles preserve supersymmetry is exactly via topological twist.[19] The tangent bundle of the special holonomy manifold on the cycle can be decomposed into a tangent bundle and normal bundle of the cycle $T(M)|_\Sigma = T(\Sigma) \oplus N(\Sigma)$. The special Lagrangian cycles are calibrated by the holomorphic volume form. Except for the special Lagrangian two-cycle, the Kähler form restricted to the special Lagrangian cycles vanishes. Therefore, the complex structure maps the tangent space (tangent bundle) of the special Lagrangian cycle to the tangent space normal to the cycle (normal bundle). Hence the tangent bundle and the normal bundle of the special Lagrangian cycles are isomorphic to each other. This agrees with the M2 brane wrapping Kähler/Lgaranian 2-cycles in the CY2 example above where the tangent bundle is split into two equivalent bundles. The way a brane wrapped on a special Lagrangian cycle preserve supersymmetry is exactly to turn on the gauge connection on the normal bundle to cancel the spin connections.

5.3 D3 Branes Wrapped on Kähler 2-Cycles

We have discussed probe-branes wrapped on calibration cycles and how they preserve supersymmetry. For two cycles in the Calabi-Yau manifolds, the calibrated cycles are the Kähler 2-cycles. The back reaction of the D-brane on spacetime is described by type II supergravity. We now consider the detailed calculations of D3 and M2 branes wrapped on Kähler 2-cycles in a truncated supergravity theory in 5/4D which is the minimal gauged supergravity.

5.3a 5D Minimal Gauged Supergravity

The dynamics of the 10/11 dimensions in general can be very hard to solve. One can perform dimension reduction to a certain dimension to solve the equation of motion and then uplift back to the 10/11 dimension, such that a solution of the reduced theory is an exact solution to the original theory. For type IIB supergravity on AdS5×S^5, one can perform a dimension reduction on a 5-sphere to obtain the $\mathcal{N} = 8$ 5D SO(6) gauged supergravity. The SO(6) gauged supergravity can have further truncation into the Cartan subalgebra of SO(6), which is a $\mathcal{N} = 2$ 5D supergravity with three U(1) gauge fields known as the STU model. Earlier we obtained the D3 brane solution on AdS5×SE5. We can perform a dimension reduction on the Sasaki-Einstein 5 manifold to obtain the 5D $\mathcal{N} = 2$ minimal gauged supergravity. [24][25] This is equivalent to the supergravity obtained from setting the three U(1) gauge fields of the STU model to equal.

We now perform the dimension reduction of the type IIB supergravity on SE5 to obtain the 5D minimal gauged supergravity. Different from dimension reduction from 11D to type IIA supergravity discussed in chapter 2, type IIB supergravity is described by the equation of motions. Hence the dimension reduction is performed on the equation of motions. The type IIB supergravity in our consideration has the equation of motion Eq.(2.30) with vanishing dilaton field,

$$R_{AB} = \frac{1}{4 \cdot 4!} F^2_{[4]AB}, \tag{5.18}$$

with the self-dual 5-form field strength satisfying $dF = 0$ as in the D3 brane case. To perform the dimension reduction, the SE5 manifold with the metric written as Eq.(4.3) is fibered over $\mathbb{R}^{1,4}$ with metric dS_5^2 via

$$dS_{10}^2 = dS_5^2 + \Big(\frac{1}{3}d\psi + \sigma + \frac{2}{3}A\Big)^2 + dS^2(KE4) \tag{5.19}$$

where $d\sigma = 2J$ with J and $dS^2(KE4)$ the Kähler form and the Kähler metric on the transverse Kähler Einstein manifold, and A is an 1-form gauge field on $\mathbb{R}^{1,4}$. The self-dual 5-form field strength is given by [24]

$$\begin{aligned} F_{[5]} &= (1+*)\Big(4vol_5 - \frac{2}{3}(*_5F_{[2]}) \wedge J\Big) \\ &= 4vol_5 - \frac{2}{3}(*_5F_{[2]}) \wedge J + \Big(2J \wedge J - \frac{2}{3}F_{[2]} \wedge J) \wedge (\frac{1}{3}d\Psi + \sigma + \frac{2}{3}A\Big) \end{aligned} \tag{5.20}$$

where vol_5 is the volume form on $\mathbb{R}^{1,4}$ and F is the field strength of the 1-form A.

The metric can be written in the vielbein form,

$$e^A = e_5^A, \qquad e^\Psi = \frac{1}{3}d\Psi + \sigma + \frac{2}{3}A, \qquad e^M = e^M_{\text{KE4}}. \tag{5.21}$$

Then the spin connection of the metric is given by

$$\begin{aligned} \omega^A{}_B &= \tilde{\omega}_5{}^A{}_B - \frac{1}{3}F^A{}_B \cdot e^\Psi & \qquad \omega^\Psi{}_B &= \frac{1}{3}F_{BC} \cdot e^C \\ \omega^M{}_N &= \tilde{\omega}_{\text{KE4}}{}^M{}_N - J^M{}_N \cdot e^\Psi & \qquad \omega^\Psi{}_N &= J_{NK} \cdot e^K \end{aligned} \tag{5.22}$$

Where $\tilde{\omega}_5{}^A{}_B$ is the spin connection on $\mathbb{R}^{1,4}$ and $\tilde{\omega}_{\text{KE4}}{}^M{}_N$ is the spin connection on the transverse Kähler-Einstein space. And the components of the 10D Ricci tensor in the vielbein basis are given by

$$
\begin{aligned}
R_{AB} &= \tilde{R}_{AB} - \frac{2}{9}F^2_{AB} & R_{\Psi\Psi} &= \frac{1}{9}F^2 + J^2 = \frac{1}{9}F^2 + 4 \\
R_{A\Psi} &= -\frac{1}{3}\nabla_N F^N{}_A & R_{M\Psi} &= -\nabla_N J^N_{\ M} = 0 \\
R_{MN} &= \tilde{R}_{MN} + 2J_{MI}J^I{}_N = 4g_{MN} & R_{MA} &= 0
\end{aligned}
\tag{5.23}
$$

We have used the fact that the Ricci tensor of KE4 is $\tilde{R}_{mn} = 4g_{mn}$. And from the field strength side, we get

$$
\begin{aligned}
F^2_{[4]AB} &= 4\cdot 4!\Big(\frac{4}{9}F^2_{AB} - \frac{1}{9}F^2\tilde{g}_{AB} - 4\tilde{g}_{AB}\Big) \\
F^2_{[4]A\Psi} &= 4\cdot 4!\cdot\frac{2}{9} *_5 (F\wedge F)_A & F^2_{[4]MA} &= F^2_{[4]M\Psi} = 0 \\
F^2_{[4]MN} &= 4\cdot 4!\cdot 4\tilde{g}_{MN} & F^2_{[4]\Psi\Psi} &= 4\cdot 4!\Big(\frac{1}{9}F^2 + 4\Big)
\end{aligned}
\tag{5.24}
$$

Therefore, the equations of motion of the 5D minimal gauged supergravity are given by

$$
\begin{aligned}
R_{\mu\nu} &= -4g_{\mu\nu} + \frac{2}{3}F^2_{\mu\nu} - \frac{1}{9}F^2 g_{\mu\nu} \\
d*F &= -\frac{2}{3}F\wedge F
\end{aligned}
\tag{5.25}
$$

From the equations of motions, we can construct an action containing the cosmological constant

$$
S_5 = \int dx^5\sqrt{-g}\Big(R + 12 - \frac{1}{3}F^2\Big) + \int F\wedge F\wedge A. \tag{5.26}
$$

To have a supersymmetric theory, the Killing spinor condition must also be considered. We now find the condition on the solutions of 5D minimal gauged supergravity in order to preserve supersymmetry in the type IIB supergravity. The type IIB Killing spinor in the vielbein basis with only the self-dual 5-form field strength switched on is given by

$$
\begin{aligned}
\delta\Psi_P = D_P\epsilon &= \Big(\nabla_P + \frac{i}{16\cdot 5!}F_{Q_1\ldots Q_5}\Gamma^{Q_1\ldots Q_5}\Gamma_P\Big)\epsilon = 0 \\
&= \Big(e^\mu_P\partial_\mu + \frac{1}{4}\omega^{Q_1}_{PQ_2}\Gamma_{Q_1}{}^{Q_2} + \frac{i}{16\cdot 5!}F_{Q_1\ldots Q_5}\Gamma^{Q_1\ldots Q_5}\Gamma_P\Big)\epsilon,
\end{aligned}
\tag{5.27}
$$

with 10D Weyl spinor satisfying the chiral projection $\Gamma_{11}\epsilon = \epsilon$. e^A_μ are inverse vielbeins

$$
e^\mu_P = \begin{pmatrix} \tilde{e}^\mu_A & 0 & 0 \\ -A_A & 3 & -\sigma_M \\ 0 & 0 & \tilde{e}^m_M \end{pmatrix} \tag{5.28}
$$

We first make a 5-5 split on the spinor

$$\epsilon = \lambda \otimes \eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5.29}$$

with the corresponding gamma matrix given by $\Gamma_P = \{\gamma_A \otimes \mathbb{I} \otimes \sigma_1, \mathbb{I} \otimes \Sigma_M \otimes \sigma_2\}$ with $\gamma_0 ... \gamma_4 = -i$ and $\Sigma_R \Sigma_1 ... \Sigma_4 = 1$. Hence the chiral projection operator is given by $\Gamma_{11} = \Gamma_0 ... \Gamma_{10} = \mathbb{I} \otimes \mathbb{I} \otimes \sigma_3$. It is easy to verify that the Killing spinor is Weyl. Substituting the field strength we get

$$F_{Q_1...Q_5} \Gamma^{Q_1...Q_5} = 40\Big(12 \cdot (-i\sigma_1 + \sigma_2) - F_{BC}\gamma^{BC} J_{IJ} \Sigma^{IJ} \frac{1}{2}(i\sigma_1 - \sigma_2)\Big) \tag{5.30}$$

Then

$$\begin{aligned} F_{Q_1...Q_5} \Gamma^{Q_1...Q_5} \Gamma_A &= 40\Big(-i12 \cdot \gamma_A(\mathbb{I} + \sigma_3) - iF_{BC}\gamma^{BC} \gamma_A J_{IJ} \Sigma^{IJ} \frac{1}{2}(\mathbb{I} + \sigma_3)\Big) \\ F_{Q_1...Q_5} \Gamma^{Q_1...Q_5} \Gamma_M &= 40\Big(12 \cdot \Sigma_M(\mathbb{I} + \sigma_3) + F_{BC}\gamma^{BC} J_{IJ} \Sigma^{IJ} \Sigma_M \frac{1}{2}(\mathbb{I} + \sigma_3)\Big) \end{aligned} \tag{5.31}$$

Therefore,

$$\begin{aligned} D_A &= \tilde{\nabla}_A - 2A_A \partial_\psi + \frac{i}{6} F_{AB} \gamma^B \Sigma^R + \frac{1}{24} F_{BC} \gamma^{BC} \gamma_A J_{IJ} \Sigma^{IJ} + \frac{1}{2}\gamma_A \\ D_\Psi &= 3\partial_\psi - \frac{1}{12} F_{BC} \Gamma^{BC} - \frac{i}{24} F_{BC} \gamma^{BC} J_{IJ} \Sigma^{IJ} + \frac{i}{2}\Sigma_\Psi - \frac{1}{2} J_{IJ} \Sigma^{IJ} \\ D_M &= \partial_M - 3\sigma_M \partial_\psi + \frac{1}{4} \tilde{\omega}^{IJ}_{\text{KE4}} \Gamma_{IJ} + \frac{i}{24} F_{BC} \gamma^{BC} J_{IJ} \Sigma^{IJ} \Sigma_M + \frac{i}{2}\Sigma_M - \frac{1}{2} J_{NM} \Sigma^{N\Psi} \end{aligned} \tag{5.32}$$

The Kähler form can be written as $J = e^{12} + e^{23}$, hence

$$\frac{1}{2} J_{MN} \Sigma^{MN} = \Sigma^{12} + \Sigma^{34} \tag{5.33}$$

by requiring

$$\begin{aligned} &\Sigma^{12}\eta = \Sigma^{34}\eta = \frac{1}{2} J_{MN} \Sigma^{MN} = i\eta, \\ &\Sigma^{\Psi} = \Sigma^{12}\Sigma^{34} = -\eta, \end{aligned} \tag{5.34}$$

as the Killing spinor on SE5. With more detailed analysis,[24] the Killing spinor condition can be solved by

$$\eta = e^{\frac{i}{2}\psi} \eta_0 \tag{5.35}$$

and a 5D spinor satisfying

$$(\tilde{\nabla}_\mu - \frac{i}{12}({\gamma_\mu}^{\alpha\beta} - 4\delta^\alpha_\mu \gamma^\beta) F_{\alpha\beta} - \frac{1}{2}\gamma_\mu - iA_\mu)\epsilon = 0 \tag{5.36}$$

where we have used the fact that $\gamma_\mu \gamma^{\alpha\beta} = \gamma_\mu{}^{\alpha\beta} + 2\delta_\mu{}^{[\alpha}\gamma^{\beta]}$.Due to the projection Eq.(5.34), the solution preserves a quarter of the supersymmetry which corresponds to $\mathcal{N} = 2$ in 5D.

Hence the $\mathcal{N} = 2$ **5D minimal gauged supergravity** is described by the equations of motion given by

$$\begin{aligned} R_{\mu\nu} &= -4g_{\mu\nu} + \frac{2}{3}F^2_{\mu\nu} - \frac{1}{9}F^2 g_{\mu\nu} \\ d *_5 F &= -\frac{2}{3}F \wedge F \end{aligned} \tag{5.37}$$

and the 5D Killing spinor condition given by

$$(\tilde{\nabla}_\mu - \frac{i}{12}(\gamma_\mu{}^{\alpha\beta} - 4\delta^\alpha_\mu \gamma^\beta)F_{\alpha\beta} - \frac{1}{2}\gamma_\mu - iA_\mu)\epsilon = 0. \tag{5.38}$$

With the 5D solution, one can recover the 10D solution by uplifting the solution via

$$dS^2_{10} = dS^2_5 + \left(\frac{1}{3}d\psi + \sigma + \frac{2}{3}A\right)^2 + dS^2(KE4) \tag{5.39}$$

The usual D3 brane solution can not be found in the 5D minimal gauged supergravity. The AdS5 background can be recovered by setting the gauge field to zero.

5.3b D3 Brane wrapped on $\mathbf{H}^2$

We have mentioned branes wrapping Kähler 2-cycles via calibration in order to preserve supersymmetry. A calibrated cycle turns on the gauge field on the normal bundle to preserve the supersymmetry through the topological twist. We now use the 5D minimal gauged supergravity introduced earlier to see how branes are wrapped on the Kähler 2-cycles and how the topological twist is applied to preserve supersymmetry, and how the solution flow crosses different dimensions. [26]

We consider two types of Kähler-Einstein 2-cycles, H^2 and S^2, where H^2 is negatively curved and S^2 is positively curved. In order to be made compact, the cycles can be quotient by discrete subgroups. Here we first use H^2 as an example, which has the metric

$$dS^2(H^2) = \frac{1}{y^2}(dy^2 + dz^2). \tag{5.40}$$

We first consider the simpler IR solution of the form AdS3$\times H^2$ with the metric

$$dS^2_5 = pdS^2(\text{AdS3}) + qdS^2(H^2). \tag{5.41}$$

with p and q some undetermined constant. The spin connection of the metric is given by

$$\omega^A{}_B = \tilde{\omega}_{\rm AdS}{}^A{}_B, \qquad \omega^Z{}_Y = \tilde{\omega}_{\rm H2}{}^Z{}_Y = -\frac{1}{y}dz. \tag{5.42}$$

where $\{A, B\}$ labels vielbein basis on AdS3 and $\{Y, Z\}$ labels vielbein basis on H^2. To ensure the topological twist, a gauge field needs to be turned on

$$A = -\frac{1}{2}\omega = \frac{1}{2y}dz, \qquad F_{[2]} = -\frac{1}{2y^2}dy \wedge dz. \tag{5.43}$$

Now with the equation of motion Eq.(5.25), we can determine the factors p and q. The Ricci tensor of the AdS3 space is $R_{ab} = -2g_{ab}/p$ and for H^2 is $R_{mn} = -g_{mn}/q$. Then it is easy to get the solution to the 5D minimal gauged supergravity

$$dS_5^2 = \frac{4}{9}dS^2(\text{AdS3}) + \frac{1}{3}\frac{1}{y^2}(dy^2 + dz^2). \tag{5.44}$$

We now consider the Killing spinor condition, one gets

$$\begin{aligned} D_y &= \partial_y + \frac{i}{3}\gamma_z F_{yz} g^{zz} - \frac{1}{2}\gamma_y \\ D_z &= \partial_z + \frac{1}{2}(\omega^Z_{zY}\gamma_Z{}^Y - i2A_z) - \frac{i}{3}\gamma_y F_{yz} g^{yy} - \frac{1}{2}\gamma_z \\ D_A &= \tilde{\nabla}_A - \Big(\frac{1}{2} + \frac{i}{6}\gamma^{yz}F_{yz}\Big)\gamma_A \end{aligned} \tag{5.45}$$

By requiring a projection on the spinor

$$\gamma_{ZY}\epsilon = -i\epsilon, \qquad \text{or,} \qquad \gamma_Y\epsilon = -i\gamma_Z\epsilon, \tag{5.46}$$

one finds the spin connection cancelled by the gauge field as required by the topological twist, which simplifies the D_y and D_z condition, and the Killing spinor condition is reduced to require

$$\partial_y\epsilon = \partial_z\epsilon = 0, \qquad \tilde{\nabla}_A\epsilon = -a\gamma_A\epsilon. \tag{5.47}$$

The first condition is simply to require a constant spinor on the cycle, due to the topological twist. The second condition is the Killing spinor condition on AdS space which can be solved similar to the sphere case with all supersymmetry preserved on it.[6] Due to the projection of the Killing spinor, the solution breaks further half of the supersymmetry, leaving 4 supercharges. The solution has the geometry structure of AdS3$\times H^2$. To have a compact cycle, we can quotient H^2 by a discrete subgroup, H^2/Γ. Since this does not change the local geometry, the supersymmetry condition is held.

One can uplift a solution of 5D minimal gauged supergravity back to 10D via Eq.(5.39), then we get the AdS3$\times Y^7$ solution of type IIB supergravity given by

$$dS_{10}^2 = \frac{4}{9}dS^2(\text{AdS3}) + \left[\frac{1}{3y^2}(dy^2+dz^2) + \left(\frac{1}{3}d\psi + \sigma + \frac{1}{3y}dz\right)^2 + dS^2(KE4)\right] \tag{5.48}$$

where the Y^7 correspond to the SE5 fibered over H^2. Since this is a solution of type IIB supergravity with the self-dual five-form turned on, we identify the solution as D3 brane wrapped on H^2. With the AdS3 structure, the solution is dual to a 2D superconformal field theory with 4 Majorana-Wyle supercharges. The Y^7 manifold has a Killing vector, which corresponds to the U(1)$\sim$SO(2) R symmetry. Hence the superconformal field theory has (0,2) chiral supersymmetry.

One can also consider the high energy behaviour of the D3-brane wrapped around a 2-cycle. In the limit where the length scale is much smaller than the cycle, the cycle may look uncompact and have dynamical fluctuations, hence is dual to the 4D superconformal field theory in the UV. In the IR limit where the length scale is much larger than the cycle, the fluctuations on the cycle become undynamic, and the dual field theory remains to be 2D superconformal field theory. Hence under the RG flow, a 4D SCFT can flow to a 2D SCFT, with the supergravity interpolating the two field theories. To see this, we can construct the more general solution of the type IIB supergravity in the form of

$$dS_5^2 = e^{2f(r)}(-dt^2+dx^2+dr^2) + e^{2g(r)}\frac{1}{y^2}(dy^2+dz^2). \tag{5.49}$$

Substituting the metric in the Killing spinor equation will give a set of differential equations of the function $f(r)$ and $g(r)$. These differential equations describe the flow of two functions under the flow of r. Then the flow may take the AdS3$\times H^2$ solution to the AdS5 solution in the UV, where the AdS5 metric is given by

$$dS_5^2 \sim \frac{1}{r^2}\Big[-dt^2+dz^2+\frac{1}{y^2}(dy^2+dx^2)+dr^2\Big]. \tag{5.50}$$

For example, we can consider these BPS equations in the 5D minimal gauged supergravity. The spin connections of the metric are given by

$$\begin{gathered}\omega^Z{}_Y = -\frac{1}{y}dx, \qquad \omega^Z{}_R = \frac{1}{y}e^{g-f}g'dz, \qquad \omega^Y{}_R = \frac{1}{y}e^{g-f}g'dy \\ \omega^T{}_R = f'dt \qquad \omega^X{}_R = f'dx\end{gathered} \tag{5.51}$$

and again with $A = -\omega^Z{}_Y/2$ as required by the topological twist. Then the Killing spinor condition become

$$
\begin{aligned}
D_y &= \partial_y + \frac{1}{2} y e^{-g} \omega^{Y}{}_{R} \gamma_y \gamma^R + \frac{i}{3}\gamma_z F_{yz} g^{zz} - \frac{1}{2}\gamma_y \\
D_z &= \partial_z + \frac{1}{2}(\omega^{Z}_{zY}\gamma_Z{}^Y - iA_z) + \frac{1}{2} y e^{-g}\omega^{Z}{}_{R}\gamma_z\gamma^R - \frac{i}{3}\gamma_y F_{yz} g^{yy} - \frac{1}{2}\gamma_z \\
D_x &= \partial_x + \gamma_x\Big(\frac{1}{2}e^{-f}\omega^{X}{}_{R}\gamma^R - \frac{1}{2} - \frac{i}{6}\gamma^{yz}F_{yz}\Big) \\
D_t &= \partial_t + \gamma_t\Big(\frac{1}{2}e^{-f}\omega^{T}{}_{R}\gamma^R - \frac{1}{2} - \frac{i}{6}\gamma^{yz}F_{yz}\Big) \\
D_r &= \partial_r - \Big(\frac{1}{2} + \frac{i}{6}\gamma^{yz}F_{yz}\Big)\gamma_R e^f
\end{aligned}
\tag{5.52}
$$

again by requiring a projection on the spinor

$$
\gamma_{ZY}\epsilon = -i\epsilon, \qquad \text{or,} \qquad \gamma_Y\epsilon = -i\gamma_Z\epsilon, \tag{5.53}
$$

and

$$
\gamma_R\epsilon = -\epsilon, \tag{5.54}
$$

the Killing spinor condition is solved by an r-dependent spinor with the differential equations

$$
\begin{aligned}
g' &= -e^f + \frac{1}{3}e^{f-2g} \\
f' &= -e^f - \frac{1}{6}e^{f-2g}.
\end{aligned}
\tag{5.55}
$$

The solution Eq.(5.58) can be recovered from the differential equation. It can also be seen that a solution that is asymptotic to

$$
e^{2g} \sim e^{2f} \sim \frac{1}{r^2} \tag{5.56}
$$

near $r = 0$ can be found, which corresponds to the solution Eq.(5.50) with the asymptotic AdS5 structure in the UV. Then we are able to identify the UV AdS5 solution and the IR AdS3$\times H^2$ solution.

5.3c D3 Brane Wrapped on S^2

We now consider the D3 brane wrapped on the positively curved Kähler-Einstein manifold S^2 with metric

$$
dS^2(S^2) = \frac{4}{(1+z^2+y^2)^2}(dy^2 + dz^2) \tag{5.57}
$$

in the minimal gauged supergravity.

Again first consider the IR solution to be the form of $AdS3 \times S^2$

$$dS_5^2 = pdS^2(\text{AdS3}) + qdS^2(H^2). \tag{5.58}$$

with spin connections

$$\omega^A{}_B = \tilde{\omega}_{\text{AdS}}{}^A{}_B, \qquad \omega^Z{}_Y = \tilde{\omega}_{\text{S2}}{}^Z{}_Y = 2\frac{zdy - ydz}{1 + x^2 + y^2}. \tag{5.59}$$

Again to ensure the topological twist, a gauge field needs to be turned on with

$$A = -\frac{1}{2}\omega = -\frac{zdy - ydz}{1 + x^2 + y^2}, \qquad F_{[2]} = \frac{2}{(1 + x^2 + y^2)^2}dy \wedge dz. \tag{5.60}$$

Again use the equation of motion Eq.(5.25) to work out the factors p and q. The spheres are positively curved with the Ricci tensor $R_{ij} = g_{ij}/q$. Then the equation of motion become

$$4q^2 + q - \frac{1}{9} = 0, \qquad p = \frac{36q^2}{72q^2 + 1} \tag{5.61}$$

with two roots $q_1 = -\frac{1}{3}$, and $q_2 = \frac{1}{12}$ and the corresponding $p_1 = \frac{4}{9}$, $p_2 = \frac{1}{6}$. Taking the positive roots we get

$$dS_5^2 = \frac{1}{6}dS^2(\text{AdS3}) + \frac{1}{3}\frac{1}{(1 + z^2 + y^2)^2}(dy^2 + dz^2). \tag{5.62}$$

However, this is not the whole story, we still need to check the Killing spinor condition in order to preserve supersymmetry. The Killing spinor equation is again given by

$$\begin{aligned} D_y &= \partial_y + \frac{i}{3}\gamma_z F_{yz} g^{zz} - \frac{1}{2}\gamma_y \\ D_z &= \partial_z + \frac{1}{2}(\omega^Z_{zY}\gamma_Z{}^Y - i2A_z) - \frac{i}{3}\gamma_y F_{yz} g^{yy} - \frac{1}{2}\gamma_z \\ D_A &= \tilde{\nabla}_A - \Big(\frac{1}{2} + \frac{i}{6}\gamma^{yz}F_{yz}\Big)\gamma_A. \end{aligned} \tag{5.63}$$

It turns out the solution Eq.(5.62) does not lead to the constant spinor with a similar projection on the spinor. The negative root, however, can satisfy the Killing spinor condition but leads to a negative metric. Hence there is no supersymmetric solution for the D3 brane wrapped on S^2 in the form of AdS3$\times S^2$ in the minimal gauged supergravity. We will discuss the reason for this in the GK geometry in section(6.2a).

To consider the RG flow, we again consider the more general solution

$$dS_5^2 = e^{2f(r)}(-dt^2 + dx^2 + dr^2) + e^{2g(r)}\frac{4}{(1 + z^2 + y^2)^2}(dy^2 + dz^2), \tag{5.64}$$

with the same gauge connection. With a similar projection on the spinor, the Killing spinor again gives the BPS equation

$$\begin{aligned} g' &= -e^f - \frac{1}{3}e^{f-2g} \\ f' &= -e^f + \frac{1}{6}e^{f-2g} \end{aligned} \tag{5.65}$$

Near $r = 0$, there is again a solution that is asymptotic to

$$e^{2g} \sim e^{2f} \sim \frac{1}{r^2}, \tag{5.66}$$

corresponding to the UV AdS5 solution with the metric

$$dS_5^2 \sim \frac{1}{r^2}\Big[-dt^2 + dz^2 + \frac{4}{(1+z^2+y^2)^2}(dy^2 + dx^2) + dr^2\Big]. \tag{5.67}$$

In the IR, the AdS3$\times S^2$ solution Eq.(5.62) is not recovered. Therefore, although the UV of the dual field theory is described by the 4d superconformal field theory, the IR of the theory does not flow to the 2d conformal field theory in the minimal gauged supergravity. It could be because the solution in the IR with the AdS3 structure is more complicated to analyse. However, in the STU model or the more complicated SO(6) gauged supergravity, the IR AdS3 solution of the D3 brane wrapping S^2 can be found and is dual to a 2D superconformal field theory.

5.4 M2 Branes Wrapped on Kähler 2-Cycles

We now discuss the solutions of M2 brane wrapped on H^2 and S^2 which share many similarities with the D3 brane case. We will first derive the 4D minimal gauged supergravity and then use it to solve the wrapped M2 brane solutions.

5.4a 4D Minimal Gauged Supergravity

In the previous section, we performed the dimension reduction on SE5 from the type IIB supergravity to obtain the 5D minimal gauged supergravity. The same operation can be done to the 11D supergravity. We now perform the dimension reduction on SE7 to derive the 4D $\mathcal{N} = 2$ minimal gauged supergravity.[27] The procedure is similar to the 11D reduction to type IIA, where the main consideration is the action of the theory. The action of the 11D supergravity is given by

$$S_{11} = \int \mathrm{d}x^{11}\sqrt{-g}\Big(R - \frac{1}{48}F_{[4]}^2\Big) \tag{5.68}$$

with the Chern-Simons term vanishes in the discussion. To perform the dimension reduction on SE7, the SE7 is fibered over $\mathbb{R}^{1,3}$ given by

$$dS_{11}^2 = \frac{1}{4}dS_4^2 + (\frac{1}{4}d\psi + \sigma + \frac{1}{2}A)^2 + dS^2(KE6), \tag{5.69}$$

where dS_4^2 is the metric on $\mathbb{R}^{1,3}$. The 4-form field strength is given by

$$F_{[4]} = \frac{3}{8}Vol_4 - \frac{1}{2}(*_4 F_{[2]}) \wedge J, \tag{5.70}$$

where Vol_4 is the volume form and $F_{[2]}$ is the two-form on $\mathbb{R}^{1,3}$.

The vielbeins of the metric are given by

$$e^A = e^A_{\bar{B}} e_4^{\bar{B}} = \frac{1}{2}e_4^{\bar{A}}, \qquad e^\Psi = \frac{1}{4}d\Psi + \sigma + \frac{1}{2}A, \qquad e^M = e^M_{\mathrm{KE6}}. \tag{5.71}$$

Then the spin connection is given by

$$\begin{aligned} \omega^A{}_B &= \tilde{\omega}_5{}^A{}_B - \frac{1}{4}F^A{}_B \cdot e^\Psi & \qquad \omega^\Psi{}_B &= \frac{1}{4}F_{BC} \cdot e^C \\ \omega^M{}_N &= \tilde{\omega}_{\mathrm{KE6}}{}^M{}_N - J^M{}_N \cdot e^\Psi & \qquad \omega^\Psi{}_N &= J_{NK} \cdot e^K \end{aligned} \tag{5.72}$$

Where $\tilde{\omega}_4{}^A{}_B$ is the spin connection on $\mathbb{R}^{1,3}$ and $\tilde{\omega}_{\mathrm{KE6}}{}^M{}_N$ is the spin connection on the transverse Kähler-Einstein space. And the components of the 10D Ricci tensor in the vielbein basis are given by

$$\begin{aligned} R_{AB} &= \tilde{R}_{AB} - \frac{1}{8}F^2_{AB} & \qquad R_{\Psi\Psi} &= \frac{1}{16}F^2 + J^2 = \frac{1}{16}F^2 + 6 \\ R_{A\Psi} &= -\frac{1}{4}\nabla_N F^N{}_A & \qquad R_{M\Psi} &= -\nabla_N J^N{}_M = 0 \\ R_{MN} &= \tilde{R}_{MN} + 2J_{MI}J^I{}_N = 6g_{MN} & \qquad R_{MA} &= 0 \end{aligned} \tag{5.73}$$

Hence the Ricci scalar is given by

$$R = 4\tilde{R} - \tilde{F^2} + 42, \tag{5.74}$$

where $\tilde{R}$ and $\tilde{F^2}$ are contracted with metric on $\mathbb{R}^{1,3}$. The $F_{[4]}^2$ term is given by

$$F_{[4]}^2 = 2 \cdot 4!(18 + 3\tilde{F^2}). \tag{5.75}$$

Substitute into the 11D supergravity action, we get the action of the 4D minimal gauged supergravity

$$S_4 \sim \int dx^4 \sqrt{-g}(R + 6 - F^2). \tag{5.76}$$

with the equations of motion given by

$$\begin{aligned} R_{\mu\nu} &= -3g_{\mu\nu} + 2F^2_{\mu\nu} - \frac{1}{2}F^2 g_{\mu\nu}, \\ d *_4 F &= 0. \end{aligned} \tag{5.77}$$

We now check the supersymmetry condition of the solution in order to preserve some of the supersymmetry in the 11D supergravity. The Killing Spinor of the 11D supergravity is given by

$$\begin{aligned} D_P &= \left[\nabla_P + \frac{1}{288}({\Gamma_P}^{Q_1Q_2Q_3Q_4} - 8\delta_P^{Q_1}\Gamma^{Q_2Q_3Q_4})F_{Q_1Q_2Q_3Q_4}\right]\epsilon = 0 \\ &= \left[e^\mu_P\partial_\mu + \frac{1}{4}\omega^{Q_1}_{PQ_2}{\Gamma_{Q_1}}^{Q_2} + \frac{1}{288}({\Gamma_P}^{Q_1Q_2Q_3Q_4} - 8\delta_P^{Q_1}\Gamma^{Q_2Q_3Q_4})F_{Q_1Q_2Q_3Q_4}\right]\epsilon. \end{aligned} \tag{5.78}$$

To solve the Killing spinor condition, the spinor is decomposed into $\epsilon = \lambda \otimes \eta$ with 4D spinor λ and 7D spinor η. The corresponding gamma matrices are given by

$$\Gamma_A = -i\gamma_A\gamma^5 \otimes \mathbb{I}, \qquad \Gamma_M = \gamma^5 \otimes \Sigma_M \tag{5.79}$$

Then one gets

$$\begin{aligned} \Gamma^{ABCD}\epsilon_{ABCD} = i\gamma^5 \cdot 4!, \qquad \Gamma^{BCD}\epsilon_{ABCD} = \gamma_A \cdot 3!, \\ {\Gamma_A}^{BCMN}\epsilon_{BCDE}F^{DE}J_{MN} = 4\gamma^B F_{AB}J_{MN}\Sigma^{MN}, \\ \Gamma^{ABMN}\epsilon_{ABCD}F^{CD}J_{MN} = -2i\gamma^{CD}F_{CD}\gamma^5 J_{MN}\Sigma^{MN}, \\ \Gamma^{BMN}\epsilon_{ABCD}F^{CD}J_{MN} = -{\gamma_A}^{CD}F_{CD}J_{MN}\Sigma^{MN}, \end{aligned} \tag{5.80}$$

Substitute into the 11D Killing spinor condition, again with $\gamma_\mu\gamma^{\alpha\beta} = {\gamma_\mu}^{\alpha\beta} + 2\delta_\mu^{[\alpha}\gamma^{\beta]}$, one gets

$$\begin{aligned} D_A &= \tilde{\nabla}_A - 2A_A\partial_\psi + \frac{i}{2}F_{AB}\gamma^B\Sigma_\psi + \frac{1}{12}(F_{BC}{\gamma_A}^{BC} + F_{AB}\gamma^B)J_{IJ}\Sigma^{IJ} + \frac{1}{2}\gamma_A \\ D_\Psi &= 4\partial_\psi - \frac{1}{2}F_{BC}\Gamma^{BC} - \frac{i}{12}F_{BC}\gamma^{BC}J_{IJ}\Sigma^{IJ}\Sigma_\psi - \frac{i}{2}\Sigma_\psi - \frac{1}{2}J_{IJ}\Sigma^{IJ} \end{aligned} \tag{5.81}$$

with a Killing spinor condition on KE6. With the projection

$$\begin{aligned} \Sigma_{12}\eta = \Sigma_{34}\eta =& \Sigma_{56}\eta = \frac{1}{6}J_{MN}\Sigma^{MN}\eta = i\eta, \\ \Sigma_\psi\eta =& i\Sigma_{12}\Sigma_{34}\Sigma_{56}\eta = \eta, \end{aligned} \tag{5.82}$$

the Killing spinor condition is solved by

$$\eta = e^{\frac{i\psi}{2}}\eta_0 \tag{5.83}$$

with KE6 preserving a quarter of the supersymmetry, leaving $\mathcal{N} = 2$ on 4D. Then the 4D Killing spinor condition of the theory is given by

$$(\tilde{\nabla}_\mu + \frac{i}{4}F_{\alpha\beta}\gamma^{\ \alpha\beta}\gamma_\mu + \frac{1}{2}\gamma_\mu - iA_\mu)\lambda = 0. \tag{5.84}$$

5.4b M2-Brane Wrapped on H^2 and S^2

We now discuss the M2 brane wrapped on Kähler-Einstein 2-cycle Σ_2, where Σ_2 is taken to be H^2 and S^2. We will focus on the IR AdS2$\times\Sigma_2$ solution in the 4D minimal gauged supergravity which uplifted to the 11d gives an AdS2$\times Y^9$ solution. The geometry structure of the AdS solution in the M2-brane case is very similar to the D3-brane case. To derive a such solution, we again consider the AdS2$\times\Sigma^2$ metric in the form of

$$dS_4^2 = pdS^2(\mathrm{AdS2}) + qe^{h(x,y)}(dy^2 + dz^2), \tag{5.85}$$

where again with

$$H^2: \ e^{h(x,y)} = \frac{1}{y^2}, \qquad\qquad S^2: \ e^{h(x,y)} = \frac{4}{(1+z^2+y^2)^2}. \tag{5.86}$$

The spin connection is given by

$$H^2: \ \omega^Z_{\ Y} = -\frac{1}{y}dx, \qquad\qquad S^2: \ \omega^Z_{\ Y} = 2\frac{zdy - ydz}{1+x^2+y^2}. \tag{5.87}$$

The gauge field is again given by the topological twist

$$A = -\frac{1}{2}\omega. \tag{5.88}$$

Then from the 4D minimal gauged supergravity equation of motion, we get

$$12q^2 \mp 4q - 1 = 0, \qquad p = \frac{4q^2}{12q^2+1}. \tag{5.89}$$

where $+$ for H^2 and $-$ for S^2. Taking the positive solution, we get for H^2, $q = \frac{1}{2}$ and $p = \frac{1}{4}$, and for S^2, $q = \frac{1}{6}$ and $p = \frac{1}{12}$. We then have to check the supersymmetry preserved by these solutions. The Killing spinor condition in the situation is given by

$$\begin{aligned}
D_y &= \partial_y - \frac{i}{2}F_{yz}\gamma^{zy}\gamma_y - \frac{1}{2}\gamma_y,\\
D_z &= \partial_z + \frac{1}{2}(\omega^Z_{zY}\gamma_Z^{\ Y} - i2A_z) + \frac{i}{2}F_{yz}\gamma^{yz}\gamma_z - \frac{1}{2}\gamma_z,\\
D_A &= \tilde{\nabla}_A - \Big(\frac{1}{2} + \frac{i}{2}F_{yz}\gamma^{yz}\Big)\gamma_A.
\end{aligned} \tag{5.90}$$

For the H^2 solution, with the projection on the spinor

$$\gamma_{zy}\epsilon = i\epsilon, \tag{5.91}$$

the 4D Killing spinor condition reduced to the Killing spinor condition on AdS2. Hence the H^2 solution is supersymmetric. The S^2 solution again does not solve the Killing spinor condition.

Thus the solution of M2 brane wrapped on H^2 in the minimal gauged supergravity in the form of AdS2$\times H^2$ is given by

$$dS_4^2 = \frac{1}{4}dS^2(\text{AdS2}) + \frac{1}{2}\frac{1}{y^2}(dy^2 + dz^2), \tag{5.92}$$

The solution can be uplifted to 11D via Eq.(5.69) to obtain the AdS2$\times Y^9$ with Y^9 an SE7 fibered over H^2. The metric of the solution is given by

$$dS_{11}^2 = \frac{1}{16}dS^2(\text{AdS2}) + \frac{1}{16}(d\psi + 4\sigma + 2A)^2 + \frac{1}{8}\frac{1}{y^2}(dy^2 + dz^2) + dS^2(KE6). \tag{5.93}$$

The structure of the solution is very similar to the D3 brane wrapping 2-cycles. In fact, they belong to the same class of geometry known as GK geometry which will be discussed in chapter 6.

Due to the AdS2 structure, the solution is dual to some superconformal quantum mechanics. And under the RG flow, the corresponding supergravity solution could interpolate between UV and IR of the theory which corresponds to the 3D and 1D superconformal field theory.

5.5 Branes Wrapped on Spindles

We have seen how branes are wrapped on cycles in the special holonomy manifolds via calibrations. These cycles preserve supersymmetry via topological twist. We now discuss a new class of supergravity solutions corresponding to the branes wrapping spindles.[28] Different from calibrated cycles, the spindle is an orbifold with two conical singularities. Also, the way that branes wrapping on the spindle preserves supersymmetry is not through the topological twist, and hence, the Killing spinor on the spindle is not constant.

The spindle is a weighted projective space $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$ with two coprime positive integers $n_\pm$. The spindle is topologically a sphere but has two conical singularities with deficit angles $2\pi(1-1/n_{\mp})$. [29] The Euler character of the spindle is given by

$$\chi = \frac{1}{4\pi}\int_\Sigma R = \frac{n_- + n_+}{n_- n_+}. \tag{5.94}$$

With $n_- = n_+ = 1$, the Euler character is just given by 2, which is a 2-sphere.

5.5a D3 Brane Wrapped on Spindle

The dynamic of the D3-brane is again solved in the 5D minimal gauged supergravity with the equation of motion Eq.(5.25) and the Killing spinor condition Eq.(5.36). The solution has the structure of the warped product AdS3×Σ and is given by [29]

$$\begin{aligned} dS_5^2 &= \frac{4y}{9}dS^2(\text{AdS3}) + dS^2(\Sigma) \\ A &= \frac{1}{4}\Big(1-\frac{a}{y}\Big)dz \end{aligned} \tag{5.95}$$

where y and z are the coordinates on the spindle, and $dS^2(\Sigma)$ is the metric on the spindle given by

$$dS^2(\Sigma) = \frac{y}{q(y)}dy^2 + \frac{q(y)}{36y^2}dz^2, \tag{5.96}$$

where $q(y)$ is a cubic function of y

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2. \tag{5.97}$$

For a certain a, the function $q(y)$ has three positive roots. To have a positive metric, the range of y is set to be within two roots of $q(y)$ such that $q(y)$ is positive. Note as y approaches one of the roots, the metric becomes singular, corresponding to the orbifold singularities of the spindle. In addition, by choosing

$$a = \frac{(n_- - n_+)^2(2n_- + n_+)^2(n_- + 2n_+)^2}{4(n_-^2 + n_-n_+ + n_+^2)^3} \tag{5.98}$$

$$\Delta z = \frac{2(n_-^2 + n_-n_+ + n_+^2)}{3n_-n_+(n_- + n_+)}\cdot 2\pi, \tag{5.99}$$

where Δz is the period of z, then the metric of the cycle gives the metric on the spindle. Note that the gauge connection is not cancelled by the gauge connection. Moreover, the magnetic flux of the gauge field through the spindle is given by

$$Q = \frac{1}{2\pi}\int_\Sigma F = \frac{n_- - n_+}{2n_-n_+}, \tag{5.100}$$

which is different from the Euler character of the spindle. Therefore, the way a brane wrapped on a spindle preserves supersymmetry is not via the topological twist.

The Killing spinor condition of the spindle solution is solved by $\epsilon = \theta \otimes \eta(y)$, where θ is the Killing spinor on AdS3 as usual satisfying $\nabla_a \theta = \frac{1}{2}\gamma_a \theta$ and $\eta(y)$ is a spinor on the spindle with components

$$\eta(y) = \Big(\frac{\sqrt{q_1(y)}}{y}, \frac{\sqrt{q_2(y)}}{y}\Big), \tag{5.101}$$

with

$$q_1(y) = -a + 2y^{3/2} + 3y, \qquad q_2(y) = a + 2y^{3/2} - 3y. \tag{5.102}$$

Different from cycles preserving supersymmetry via topological twist, the spindle does not have a constant Killing spinor.

The 5D solution of the minimal gauged supergravity can be uplifted back to 10D via Eq.(5.39) to obtain the solution of type IIB supergravity given by

$$dS_{10}^2 = \frac{4y}{9} dS^2(\text{AdS3}) + \frac{y}{q(y)} dy^2 + \frac{q(y)}{36y^2} dz^2 + \frac{1}{9}\bigg(D\psi + \frac{1}{2}\Big(1 - \frac{a}{y}\Big)dz\bigg)^2 + dS^2(KE4), \tag{5.103}$$

where $D\psi = d\psi + 3\sigma$. The uplifted solution is a warped product of AdS3$\times\mathcal{M}^7$ with the warp factor y and the $\mathcal{M}^7$ the Sasaki-Einstein manifold fibered over the Spindle. $\mathcal{M}^7$ can be written as

$$dS^2(\mathcal{M}^7) = \frac{9}{4y} dS^2(KE4) + \frac{9}{4q(y)} dy^2 + \frac{q(y)}{16y^2(y^2 - 2y + a)} D\psi^2 + \frac{y^2 - 2y + a}{4y^2} Dz^2, \tag{5.104}$$

with

$$Dz = dz - \frac{a - y}{2(y^2 - 2y + a)} D\psi, \tag{5.105}$$

which turns out can be regular.[30] Hence the uplifted solution AdS3$\times\mathcal{M}^7$ is also regular with the conical singularities removed in the uplifting. The warp factor of the warped product is a function on $\mathcal{M}^7$. Therefore the solution AdS3$\times\mathcal{M}^7$ has the AdS3 isometry, hence is dual to a 2D $\mathcal{N} = (0, 2)$ superconformal field theory. The UV solution of D3 branes wrapping spindles in the form of AdS5 is not yet discovered.

5.5b M2 Brane Wrapped on Spindle

The solution of M2 branes wrapped on spindles is solved in the 4D minimal gauged supergravity. The solution is similar to the D3 brane wrapped on the spindle and is given by [31]

$$
\begin{aligned}
dS_4^2 &= \frac{y^2}{4} dS^2(\mathrm{AdS2}) + dS^2(\Sigma) \\
A &= \frac{1}{2}\Big(1 - \frac{a}{y}\Big) dz
\end{aligned}
\tag{5.106}
$$

with

$$
dS^2(\Sigma) = \frac{y^2}{q(y)} dy^2 + \frac{q(y)}{4y^2} dz^2, \tag{5.107}
$$

where $q(y)$ is now a quartic function of y

$$
q(y) = y^4 - 4y^2 + 4ay - a^2. \tag{5.108}
$$

The solution again can be uplifted to the 11D via Eq.(5.69) to obtain the AdS2$\times Y^9$ solution of the 11D supergravity with Y^9 corresponds to SE7 manifold fibered over the spindle. The metric of the solution is given by

$$
dS_{11}^2 = \frac{y^2}{16} dS^2(\mathrm{AdS3}) + \frac{y}{4q(y)} dy^2 + \frac{q(y)}{36y^2} dz^2 + \frac{1}{16}\bigg(D\psi + \Big(1 - \frac{a}{y}\Big) dz\bigg)^2 + dS^2(KE4). \tag{5.109}
$$

which again can be made regular.[33]

Different from the D3 brane case, a UV-completed solution corresponding to M2 branes wrapping spindles is identified. Such a solution corresponds to the accelerating supersymmetric extremal black hole in AdS4.[32] An accelerating black hole has five parameters, acceleration, electric charge, magnetic charge, angular momentum and mass. They are described by the same action as the 4D minimal gauged supergravity Eq.(5.76) which is an Einstein-Maxwell theory with a negative cosmological constant. These black holes are solved by Plebanski–Demianski solutions.[31] The acceleration of the black hole cause two conical deficits on the horizon which gives a spindle, and magnetic flux through the horizon. By requiring the solution to be supersymmetric and extremal, the solution depends on one parameter, the angular momentum or the electric charge. In the near horizon limit, the solution is of the form of AdS2$\times\Sigma_2$ where Σ_2 is the spindle. Such a solution can be uplifted to 11D to obtain a regular solution in the form of AdS2$\times Y^9$. When the rotation parameter is set to zero, the uplifted M2 brane wrapped on the spindle solution described by Eq.(5.109) can be recovered in the near horizon limit. In the near horizon limit with non-zero rotation, one recovers new AdS2$\times Y^9$ solutions. hence we are able to identify the UV of the spindle solution as the form of AdS4, which is dual to a 3-dimensional superconformal field theory. Under renormalization group flow, the solution flow to AdS2$\times\Sigma_2$ which is dual to a 1-dimensional superconformal quantum mechanics.

Chapter 6

GK Geometry

We have discussed in the previous chapter about D3-branes and M2-branes wrapping Kähler-Einstein two cycles and spindle two cycles, leading to solutions in the form of AdS3$\times Y^7$ and AdS2$\times Y^9$ with the manifold Y a fibration of the Sasaki-Einstein manifold on the Kähler-Einstein or the spindle two-cycle. We have found that the geometrical structure of the two types of solutions shares many similarities. In fact, they belong to the same class of geometry called GK geometry, where GK stands for Gauntlett-Kim.[34] The GK geometry describes the AdS3$\times Y^7$ and AdS2$\times Y^9$ solution with non-vanishing self-dual 5-form or 4-form field strength for each case. Though these two cases are of interest in physics, the GK geometry can also be generalised to Y^{2n+1} with arbitrary $n \geqslant 3$. The GK geometry is closely related to but different from the Sasakian geometry mentioned in section(4.2a).

6.1 The General Structure of GK Geometry

The AdS3 solutions of type IIB supergravity in the context of GK geometry are given by [34]

$$\begin{aligned} ds^2_{10} &= e^{-B/2}[ds^2(\mathrm{AdS}_3) + ds^2(Y^7)] \\ F_{[5]} &= -[vol(\mathrm{AdS}_3) \wedge F_{[2]} + *_7 F_{[2]}], \end{aligned} \tag{6.1}$$

where B is a function on Y^7 and $F_{[2]}$ is a 2-form on Y^7. And similarly, AdS2 solutions of the 11D supergravity are given by

$$\begin{aligned} ds^2_{11} &= e^{-2B/3}[ds^2(\mathrm{AdS}_2) + ds^2(Y^9)] \\ F_{[4]} &= -vol(\mathrm{AdS}_2) \wedge F_{[2]}. \end{aligned} \tag{6.2}$$

The vielbein for a general AdS(d)×Y($\tilde{d}$) metric is given by

$$e^A = e^{-B/c}\tilde{e}^A_{\mathrm{AdS}}, \qquad e^B = e^{-B/c}\tilde{e}^B_Y. \tag{6.3}$$

where $c = 3$ for M2-brane and $c = 4$ for D3-brane. Then the spin connections are given by

$$\begin{aligned} \omega^A_{\ B} &= \tilde{\omega}_{\mathrm{AdS}}{}^A{}_B, \qquad \omega^A_{\ M} = -\frac{1}{c}\partial_M B e^A \\ \omega^M_{\ N} &= \tilde{\omega}_Y{}^A{}_B - \frac{1}{c}\partial_N B e^M + \frac{1}{c}\partial^M B e_N, \end{aligned} \tag{6.4}$$

And Ricci tensor

$$\begin{aligned} R_{\mu\nu} &= \Big(-(d-1) + \frac{1}{c}\nabla^2 B - \frac{d+\tilde{d}-2}{c^2}(\nabla B)^2\Big) g_{\mu\nu}(\mathrm{AdS}) \\ R_{mn} &= \tilde{R}_{mn} + \frac{d+\tilde{d}-2}{c}\nabla^2_{mn}B + \frac{d+\tilde{d}-2}{c^2}\nabla_m B\nabla_n B + \frac{1}{c}\nabla^2 B - \frac{d+\tilde{d}-2}{c^2}(\nabla B)^2 \end{aligned} \tag{6.5}$$

Substitute this into the supergravity equation of motion, for general $\tilde{d} = 2n+1$, we get

$$\begin{aligned} -\frac{4(n-1)}{(n-2)^2} + \nabla^2 B - (n-1)(\nabla B)^2 + \frac{1}{2}e^{2B}F^2 &= 0 \\ \tilde{R}_{mn} + (n-1)\nabla^2_{mn}B + \frac{n-2}{2}\nabla_m B\nabla_n B + \frac{2}{n-2}g_{mn}(Y) + \frac{1}{2}e^{2B}F^2_{mn} - \frac{1}{4}g_{mn}(Y)F^2 &= 0 \\ d(e^{(3-n)B} *_{2n+1} F) &= 0 \end{aligned} \tag{6.6}$$

which is described by the action

$$S = \int_{Y^{2n+1}} e^{(1-n)B}\Big[R_{2n+1} - \frac{2n}{(n-2)^2} + \frac{n(2n-3)}{2}(\partial B)^2 + \frac{1}{4}e^{2B}F^2\Big] \cdot vol(Y^{2n+1}). \tag{6.7}$$

We now discuss the GK geometry structure of Y_{2n+1}, which is similar to the Sasakian geometry. For a super symmetric solution, the manifold Y_{2n+1} admits at least one Killing vector $\xi = 1/c\partial_z$ constructed by Killing spinors and is corresponding to the R-symmetry of the dual field. From the Killing vector, we can write the metric as [35] [36]

$$ds^2_{2n+1} = c^2(dz+P)^2 + e^B ds^2_{2n} = \eta^2 + e^B ds^2_{2n} \tag{6.8}$$

where $\eta = c(dz + P)$ is the covector dual to ξ, and $c = (n-2)/2$. Moreover, the supersymmetric condition further require the transverse metric ds^2_{2n} to be Kähler with $dP = \rho$ the Ricci form of the transverse Kähler metric, and

$$e^B = \frac{c^2}{2} R_{2n} \tag{6.9}$$

$$F = -\frac{1}{c} J + cd[e^{-B}(dz + P)] \tag{6.10}$$

where R_{2n} is the Ricci tensor of the transverse Kähler metric. By imposing the equation of motion of the two-form F given by Eq.(6.6), one gets a PDE

$$\Box R = \frac{1}{2} R^2 - R_{ij} R^{ij}. \tag{6.11}$$

Recall that the supersymmetry condition plus $dF = d * F = 0$ solves the supergravity equation of motion. By relaxing the PDE, one gets the off-shell supersymmetric GK geometry.

6.2 Examples

We have briefly introduced the GK geometry which describes the AdS3 and AdS2 solutions with non-vanishing 5-form and 4-form field strength in the type IIB and 11D supergravity. We now use the GK geometry to obtain the AdS3 and AdS2 solutions corresponding to D3 and M2 branes wrapped on Kähler-Einstein and spindle 2-cycles to see how these examples are fitted into the GK geometry. We will show that the same results are obtained as in the last chapter.

6.2a KE2×KE4 Example

We first consider a simple example of the GK geometry where the Y^{2n+1} for $n = 3$ is taken to be a product of M_1 =KE2 and M_2 =KE4, which corresponds to the D3 brane wrapped on the KE 2-cycle.[37] Since both manifolds are Einstein, we take the Ricci tensors of two manifolds to be

$$R^1_{ij} = l_1 g^1_{ij}, \qquad R^2_{ij} = l_2 g^2_{ij}, \tag{6.12}$$

Therefore the Ricci scalar is $R = 2l_1 + 4l_2$. Then the PDE given by Eq.(6.11) simply become

$$(l_1 + 2l_2)^2 = l_1^2 + 2l_2^2, \tag{6.13}$$

and we get $l_2 = -2l_1$ or $l_2 = 0$.

For the $l_2 = -2l_1$ case with positive l_1, this corresponds to the positively curved KE 2-manifold S^2. The corresponding case for positive l_1 is the D3 brane wrapped on S^2 as discussed in the previous sections. However, when l_1 is taken to be positive, l_2 needs to be taken negatively. The consequence of that is the KE4 manifold now needs to be negatively curved which is not what we assumed in the dimension reduction on the SE5. Also for negative l_1, the Ricci tensor of the transverse Kähler metric of Y^7 is negative, therefore the factor $e^B \sim R$ in the metric is also negative, leaving a negative metric just as discussed in the previous section.

For $l_2 = -2l_1$ with negative l_1, this corresponds to the negatively curved KE 2-manifold H^2. Taking the l_2 to be 6 as in the dimension reduction on the SE5. Then $l_1 = -3$ and hence the metric of the H^2 is scaled by 1/3. Then the Ricci scalar of the transverse metic is $R = 18$ and the factor $e^B = c^2/2R = 9/4$. The full metric is therefore given by

$$dS_{10}^2 = \frac{2}{3}\left[dS^2(\mathrm{AdS3}) + \frac{1}{4}(d\psi + P)^2 + \frac{9}{4}\left(\frac{1}{3}dS^2(H^2) + dS^2(KE4)\right)\right]. \tag{6.14}$$

Taking $P = 3\sigma + 2A$, the metric

$$dS_{10}^2 = \frac{3}{2}\left[\frac{4}{9}dS^2(\mathrm{AdS3}) + (\frac{1}{3}d\psi + \sigma + \frac{2}{3}A)^2 + \frac{1}{3}dS^2(H^2) + dS^2(KE4)\right] \tag{6.15}$$

is then exactly the metric given by Eq.(5.48) up to a scale.

6.2b KE2×KE6 Example

We then consider the example of GK geometry with n=4. The Y^9 is taken to be the product of $M1$ =KE2 and $M2$ =KE6 with the Ricci tensor again taken to be

$$R^1_{ij} = l_1 g^1_{ij}, \qquad R^2_{ij} = l_2 g^2_{ij}, \tag{6.16}$$

and the Ricci scalar is $R = 2l_1 + 6l_2$. Then the PDE Eq.(6.11) become

$$(l_1 + 3l_2)^2 = l_1^2 + 3l_2^2, \tag{6.17}$$

and we get $l_2 = -l_1$ or $l_2 = 0$. The former situation again admits negative l_1 which corresponds to the M2-Brane wrapped on H^2. For KE6, $l_1 = -l_2 = -8$. Then the metric of H^2 is scaled by 1/8. The Ricci scalar is given by $R = 32$ and the factor $e^B = c^2/2R = 16$. Hence the metric of AdS2$\times Y^9$ is given by

$$dS_{11}^2 = 16^{-2/3} \cdot \left(dS^2(\text{AdS2}) + (d\psi + P)^2 + 16\Big(\frac{1}{8}dS^2(H^2) + dS^2(KE6)\Big)\right) \tag{6.18}$$

which with rearrangement gives

$$dS_{11}^2 \sim \frac{1}{16}dS^2(\text{AdS2}) + \frac{1}{16}(d\psi + P)^2 + \frac{1}{8}dS^2(H^2) + dS^2(KE6) \tag{6.19}$$

which again up to a scale is the same as the result from the last chapter.

6.2c The Spindle Example

We now briefly explain how to recover the solution of the D3-brane wrapped on a spindle mentioned in the section(5.5a) in the GK geometry which is discussed in [37]. Start with a 6d transverse metric of some 2-cycle fibered over KE4,

$$dS_6^2 = \frac{d\rho^2}{U(\rho)} + U(\rho)\rho^2 D\phi^2 + \rho^2 dS^2(KE4), \tag{6.20}$$

where $D\phi = d\phi + \sigma$ with $d\sigma = 2J_{KE}$ and J_{KE} is Kähler form on KE4. The 6d metric integrable complex structure and a closed Kähler form given by

$$J = \rho D\phi \wedge d\rho + \rho^2 J_{KE}. \tag{6.21}$$

Therefore, the transverse 6d space is Kähler. And the Ricci form of the metric is given by

$$R = dP, \qquad \text{with} \qquad P = \Big(3(1 - U(\rho)) - \frac{\rho}{2}\frac{dU(\rho)}{d\rho}\Big)D\phi. \tag{6.22}$$

It is convenient to transform the coordinate to $x = 1/\rho$ and consider $U(x)$ to be a polynomial of x, and the metric becomes

$$dS_6^2 = \frac{1}{x}\Big(\frac{dx^2}{4x^2U(x)} + U(x)D\phi^2 + dS^2(KE4)\Big). \tag{6.23}$$

The form of $U(x)$ is constrained by the PDE Eq.(6.11). For the spindle case, we take

$$U(x) = 1 - \frac{x(x-\beta)^2}{\alpha}, \tag{6.24}$$

hence P in the Ricci form is given by

$$P = -\frac{2x(x-\beta)}{\alpha}, \tag{6.25}$$

and the Ricci scalar is given by

$$R = \frac{8\beta^2}{\alpha}x^2. \tag{6.26}$$

With this, we can construct the Y^7 of GK geometry with metric given by

$$dS^2(Y^7) = \frac{1}{4}\Big(dz - \frac{2\beta x(x-\beta)}{\alpha}D\phi\Big) + \frac{\beta^2}{\alpha}\Big(\frac{1}{4xU}dx^2 + xUD\phi^2 + xdS^2(KE4)\Big). \tag{6.27}$$

By choosing

$$\beta = \frac{4}{3a}, \qquad \alpha = \frac{256}{729a^2}, \tag{6.28}$$

and changing the coordinate to $y = \frac{4}{9x}$ and $\psi = 3\phi + z$ we get the same expression for M^7 as the section(5.5a),

$$dS^2(\mathcal{M}^7) = \frac{9}{4y}dS^2(KE4) + \frac{9}{4q(y)}dy^2 + \frac{q(y)}{16y^2(y^2-2y+a)}D\psi^2 + \frac{y^2-2y+a}{4y^2}Dz^2. \tag{6.29}$$

And the full AdS3×M^7 solution is given by

$$dS^2_{10} = \frac{4y}{9}\Big(dS^2(\text{AdS3}) + dS^2(\mathcal{M}^7)\Big). \tag{6.30}$$

6.3 More about the GK geometry

6.3a The Cone Geometry

Similar to the Sasakian geometry, we can define a complex cone with complex dimension n over Y_{2n+1} with a radial direction r such that acting the complex structure of the cone gives the Killing vector $\mathcal{I}(r\partial_r) = \xi$.[34] The metric of the cone is given by

$$ds^2_{2n+2} = dr^2 + r^2 ds^2_{2n+1}. \tag{6.31}$$

The complex structure two-form of the complex cone can be constructed by

$$\mathcal{I} = -crdr \wedge (dz + P) + r^2 e^B J \tag{6.32}$$

where J is the complex structure over the transverse Kähler manifold. Note that the two-form is not closed, hence the complex cone is not Kähler. The complex cone also admits a holomorphic volume form

$$\Omega_{(n+1,0)} = e^{iz}(e^{B/2}r)^2[dr - irc(dz + P)] \wedge \Omega_{(n,0)} \tag{6.33}$$

where $\Omega_{(n,0)}$ is the holomorphic volume form on the transverse Kähler manifold. The holomorphic volume form is also not closed but is conformally closed.

6.3b The Action and the C-Extrimization

Substitute the metric Eq.(6.8) and the off-shell 2-form Eq.(6.10) into the Y_{2n+1} action Eq.(6.7), one gets the supergravity action restricted to the off-shell supersymmetric GK geometry which is simply given by

$$S = \int_{Y_{2n+1}} \eta \wedge \rho \wedge e^J. \tag{6.34}$$

To impose the on-shell condition, relax the local condition to global by requiring the integration of the PDE to be satisfied. Via integration by part, one gets

$$\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge e^J = 0. \tag{6.35}$$

Note for the Sasakian geometry, the Ricci is proportional to the Kähler form. Then the above integral gives the volume of the Sasakian manifold. Hence Sasakian geometry is not GK.

To further constrain the solution, we can exert the flux quantizations as in the string and M theory. The flux quantization of the self-dual 5-form field strength is given by

$$\int_{\Sigma_A} F_{[5]} = N_A \tag{6.36}$$

for all 5-cycle Σ_A in Y_7 where N_A is proportional to integers. For the 4-form field strength, the flux quantization condition is given by

$$\int_{\Sigma_A} *F_{[4]} = N_A \tag{6.37}$$

for all 7-cycle Σ_A in Y_9 where N_A is proportional to integers. With Eq.(6.10), these conditions can be reformulated into a simpler form

$$\int \eta \wedge \rho \wedge e^J \tag{6.38}$$

is proportional to some integer. Hence the supersymmetric AdS3$\times$Y^7 and AdS2$\times$Y^9 solutions to the supergravity are described by the action Eq.(6.34) subjects to two constraints the PDE Eq.(6.35) and the flux quantization Eq.(6.38). [34]

There are also some interesting field theory aspects of the geometry that it is dual to the c-extremization.[38] The AdS3$\times$Y^7 solution is dual to a 2-dimensional conformal field theory which admits a central charge. It can be calculated that the extremized action is proportional to the central charge. And for AdS2$\times$Y^9 solutions, they are dual to one-dimensional conformal quantum mechanics, which has the log of the partition function proportional to the action.

There is also a Sasakian analogy, the volume minimization. [39] In the previous chapter we obtained AdS×SE solution for branes, we now relax the Sasaki-Einstein to the general Sasakian geometry. The supergravity action restricted to the Sasakian geometry gives the volume over the Sasakian manifold

$$S = \int_{\mathcal{M}} \eta \wedge e^{J} \tag{6.39}$$

which is a functional of the Reeb vector. The action is minimized by the Sasaki-Einstein manifold, leading to the solutions described in chapter3. The AdS5×SE5 solution is dual to a four-dimensional conformal field theory. The volume minimization, as an analogue of the c-extremization, is dual to the a-extremization. One can also generalize the idea of central charge to three-dimensional conformal field theory in a similar way as the a-extremization.[40]

Chapter 7

Discussion

In this article, we first introduced the supergravity theories and AdS-CFT correspondence. Following that, we found the M2, M5, and D3 brane solutions of 11D and type IIB supergravities on flat backgrounds. Then we discussed the branes on curved backgrounds. The curved backgrounds preserving supersymmetry are classified via their special holonomy. We then calculated the D3 and M2 branes on the apex of the Calabi-Yau manifolds which yield asymptotic AdS$\times SE$ solutions.

After that, we discussed branes wrapped on calibrated cycles which satisfies the supersymmetry condition via topological twist. To study the D3 and M2 brane wrapping 2-cycles, we first derived the 5D and 4D minimal gauged supergravity via dimension reduction from the type IIB and 11D supergravity on the Sasaki-Einstein manifolds. We then used these minimal gauged supergravities to obtain the solution of D3 and M2 branes wrapped on H^2 and S^2. For the D3-brane wrapping H^2 case, we are able to get the solution which flows from an AdS5 solution in the UV to an AdS3$\times H^2$ solution in the IR. The AdS-CFT dual description is given by a 4D superconformal field theory in the UV flowing to a 2D conformal field theory in the IR. For the D3-brane wrapping S^2 case, we are not able to identify the IR AdS3$\times S^2$ solution while the UV AdS5 solution is recovered, in the context of minimal gauged supergravity. For the M2 brane, we get a similar situation but with the AdS4 solution flowing to the AdS2$\times H^2$. These AdS2 and AdS3$\times H^2$ solutions can be uplifted to the 11/10 dimension to obtain the AdS2$\times Y^9$ and AdS3$\times Y^7$ with Y^9 and Y^7 the SE7 and SE5 fibered over H^2.

We then discussed the D3 and M2 branes wrapped on a spindle. We found that brane wrapping spindles do not preserve supersymmetry via topological twist, and hence do not have a constant spinor over the spindle. We presented the AdS3$\times\Sigma_2$ and AdS2$\times\Sigma_2$ solution for the D3 and M2 branes where Σ_2 corresponds to the spindle. These solutions uplifted to the 10/11D have the form of AdS3$\times Y^7$ and AdS2$\times Y^9$ with Y^7 and Y^9 the Sasaki-Einstein manifold fibered over the spindle and is free from the conical singularities on the spindle. For the M2 brane case, we are able to identify the UV solutions as AdS4 supersymmetric and extremal black holes. For the D3 brane case, the UV solution is not yet known. These spindle solutions open a new topic of branes wrapping on cycles. One can also look at branes wrapping higher dimensional spindles.

In the above discussions in D3 and M2 branes wrapping 2-cycles and spindles, we have obtained multiple AdS3$\times Y^7$ and AdS2$\times Y^9$ solutions which can be classified in the Gauntlett-Kim geometry. We briefly introduced the GK geometry and performed calculations in the GK geometry to recover the D3 and M2 branes wrapped on H^2 and spindle solutions.

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