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MSc Quantum Fields and Fundamental Forces

Continuous Generalised Symmetries and Higher Groups

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Dedicated to nani.

Abstract

This master's thesis constitutes an in-depth exploration of the realm of generalised symmetries, with a primary focus on continuous higher form symmetries and higher groups. The central objective of this study is to offer a comprehensive and systematic examination of the construction of higher form symmetries and to illuminate their action on extended operators. This construction is elucidated through the lens of a practical example—a $U(1)$ gauge theory—within the framework of generalised symmetries.

Additionally, this research delves into the intricate domain of continuous higher groups, with a particular emphasis on the understanding of 2-groups. Through detailed analysis and exploration, we unveil the underlying structure of 2-groups, exemplified by a model like of quantum electrodynamics (QED). This practical illustration sheds light on the emergence of a 2-group structure within the theoretical framework.

In essence, this thesis offers a holistic and insightful journey into the world of higher form symmetries and higher groups, providing valuable contributions to the understanding of these intricate theoretical constructs.

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Contents

Chapter 1

Introduction

> "Nothing in physics seems so hopeful to as the idea that it is possible for a theory to have a high degree of symmetry was hidden from us in everyday life. The physicist's task is to find this deeper symmetry."
>
> \- Steven Weinberg

From the earliest days of scientific inquiry, our innate inclination to discern patterns in the world around us has guided our exploration. It is a journey that traces back to Euclid's foundational work in geometry, marking the inception of our enduring quest to recognize and comprehend the symmetries inherent in the natural world. Symmetries have served as our guiding light, illuminating the intricate fabric of the universe.

Amidst the aftermath of World War I, Emmy Noether's pioneering efforts formalized the profound connection between continuous symmetries and the conservation laws that govern our physical reality. This pivotal work not only propelled us to unprecedented heights in theoretical physics but also continues to inspire us to break free from the constraints of seemingly counter-intuitive phenomena, unveiling the profound interconnections that underpin our reality.

In the realm of Quantum Field Theory, the foundation of our understanding once again rests upon our ability to grasp the symmetries deeply rooted within the mathematical structure of group theory. Building upon the insights from [1], this dissertation introduces a broader symmetrical formalism grounded in topology, known as Generalised symmetries or higher form symmetries. These symmetries seamlessly fall into the broader framework of category theory, ushering in a novel perspective on the very nature of symmetries.

Within the pages that follow, we embark on an exploration of the construction of this formalism, unraveling its intricacies and shedding light on gauge theories within this innovative context. This dissertation is a journey through the landscapes of higher form symmetries and higher groups, where abstract mathematical concepts converge with the empirical mysteries of the physical universe.

In Chapter 2 [Higher Form Symmetries], we embark on a captivating exploration into the realm of higher form symmetries, focusing exclusively on continuous higher form symmetries. This chapter serves as a foundational stepping stone, where we redefine our conventional understanding of symmetries through the language of generalized symmetries. We begin by reformulation of ordinary symmetries into 0-form symmetries, gradually ascending the ladder to delve into the intricacies of higher form symmetries. Our journey unfolds with a comprehensive examination of 1-form symmetries, which is then illustrated using the $4dU(1)$ gauge theory as a guiding example. As we near the conclusion of this chapter, we extend our insights to encompass the general framework of p-form symmetries, and understanding their action on p-dimensional operators.

In Chapter 3 [Anomalies], our focus shifts to the intriguing concept of anomalies. Here, we undertake a thorough review to build a solid foundation for calculating anomalies, harnessing the power of anomaly inflow and anomaly polynomials. Our journey leads us to a detailed understanding of the construction and utilization of anomaly polynomials and inflow, for anomaly calculation using descent equations. These concepts, explored in this chapter, serve as essential cornerstones for the chapter to follow.

Chapter 4 [Higher Groups] guides us into the realm of higher groups, with a dedicated focus on continuous higher groups. We commence with an introductory overview, gradually narrowing our scope to the fascinating domain of continuous 2-groups. Our exploration unfolds by elucidating the construction of 2-group structures and investigating the types of theories that can incorporate this intricate framework using our earlier discussions on anomalies. We conclude this chapter by offering a tangible example—a theory like QED—where we derive its 2-group structure.

Lastly, in Chapter 5 [Summary and Future Work], we draw the curtains with a comprehensive summary of the dissertation's key insights. Furthermore, we engage in a forward-looking discourse, shedding light on recent developments and charting potential avenues for future research endeavors.

In essence, this dissertation embarks on a captivating journey, navigating through the realms of higher form symmetries, anomalies, and higher groups. It is our hope that this exploration enriches the reader's understanding of these concepts.

Chapter 2

Higher Form Symmetries

Generalized global symmetries, also referred to as **higher form global symmetries**, have become a focal point of extensive research in recent years. Their initial exploration can be traced back to a seminal paper published in 2014 [1]. These symmetries hold a pivotal role in advancing our understanding of gauge theories, offering a profound and an innovative perspective on the dynamics of these theories. Consequently, they contribute to the formulation of hypotheses of theoretical structures that have yet to be explored in depth.

This chapter embarks on an exploration of this intriguing concept, leveraging our existing understanding of symmetries. We will embark on the construction of higher form symmetries through the utilization of differential forms, thereby giving them a geometric interpretation. Initially, our focus will be on redefining conventional symmetries in the framework of generalized symmetries. We will delve into the intricacies of their actions and other properties (gauging), all articulated through the language of higher differential forms. Subsequently, we will proceed to construct more encompassing higher form symmetries, shedding light on what they act on and the mechanisms through which they act on higher dimensional operators.
The culmination of this chapter will revolve around an exploration of the well-studied Maxwell theory in four dimensions. Here, we will emphasize how this theory can be constructed using the novel formalism of generalized symmetries, thereby understanding the practical applications and relevance of this emerging field of study.

2.1 Ordinary Symmetries as 0-Form Symmetries

In this section, we will construct conventional symmetries in the framework of differential forms. We shall delve into the intricacies of how these symmetries are reformulated, all while redefining their action on local operators through the language of generalized symmetries.

2.1.1 Symmetries in Quantum Field Theory

A fundamental principle, established by Wigner's Theorem [2], asserts that symmetries within a quantum theory necessitate their implementation through unitary operators (applicable to both continuous and discrete symmetries) or anti-unitary operators (for discrete symmetries). To be more precise, to comprehend the manner in which the symmetry group G operates on fields within the Hilbert space $\mathcal{H}$ of any given quantum field theory, along with an examination of their corresponding transformations, it is important to construct unitary irreducible representations of G.

If we denote an element of the symmetry group $G(g,\cdot)$ using g, it has the following properties:

Group Axioms

- Closure : $g \cdot g' \in G \qquad \forall g, g' \in G$
- Associativity : $g \cdot (g' \cdot g'') = (g \cdot g') \cdot g'' \qquad \forall g, g', g'' \in G$
- Identity : $\exists e \in G \quad | \; e \cdot g = g \in G \qquad \forall g \in G$
- Inverse : $\exists g' \equiv g^{-1} \in G \quad | \; g \cdot g' = g' \cdot g = e \in G \qquad \forall g, g' \in G$

We can denote a unitary irreducible representation of $g \in G$ as $R(g)$.

Let us consider a quantum field theory with fields $\varphi(x)$ in Euclidean spacetime, as states in a quantum field theory in Lorentzian spacetime can be constructed from the Euclidean path integral using Wick Rotation (Euclidean continuation) as shown in the figure:

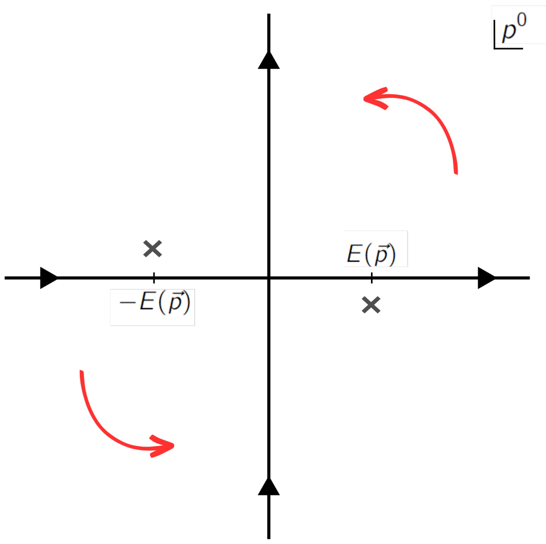

Figure 2.1: Given the 4-momemtum p^μ of a QFT, the contour of the p^0 part of the intergal of the 1-loop contribution of self energy, i.e $\mathcal{H}(\vec{p}) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - (\vec{p}^2 + m_0^2) + i\epsilon}$ of $\mathcal{I} = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \mathcal{H}(\vec{p})$, can be rotated due to the positions of the poles shown above and hence we get $ik_E^0 = p^0$ and hence the Euclidean 4-momentum is now $k_E^\mu = (-ip^0, \vec{p})$.

To see more details on Wick Rotation refer [3]. The rigorous treatment of generating correlation functions of a QFT in Minkowski spacetime in Euclidean spacetime using analytic continuation is done using *Osterwalder-Schrader theorem* [4, 5]

Action

The unitary irreducible representations of g acts on local operators[a] transforming in a representation R present at any point in spacetime. They act as follows:

$$g \cdot \varphi_R(x) = R(g)^i{}_j \, \varphi_R^j(x) \tag{2.1}$$

[a]local operators are 0-dimensional operators that exist at points in spacetime

2.1.2 Noether's Theorem

Noether's theorem deeply emphasises the connection between continuous symmetries and conservation laws. We will be stating it in brief as a reference to redefine it in topological framework later on in this chapter. An explicit derivation of Noether's theorem can be found in [3].

Theorem 2.1.1: Noether's Theorem

According to Noether's Theorem, if the action S of a field theory is invariant with respect to some continuous transformation of a symmetry group G, then there exists a conserved current $j^\mu(x)$ that gives us a conserved charge, which is obtained by integrating the current over all of space Σ_{d-1} i.e.

$$Q = \int_{\Sigma_{d-1}} d^{d-1}x \;\; j^\mu(x) \tag{2.2}$$

The conservation of current is mathematically expressed as:

$$\partial_\mu j^\mu(x) = 0 \tag{2.3}$$

Background Gauge Fields and Ward Identities : Conservation laws in quantum field theories can be formulated as Ward Identities. These identities can be derived from looking at the variation of the background gauge field. Consider the conserved current $j^\mu(x)$ coupled to a background gauge field $A_\mu(x)$. This can be expressed as adding the following term to our action (S) [6]:

$$S_\lambda = +i \int d^dx \;\; A_\mu(x) j^\mu(x) \tag{2.4}$$

Now under a gauge transformation,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \tag{2.5}$$

the action varies as [b]:

$$\delta S_\lambda = +i \int d^dx \;\; \partial_\mu \lambda(x) J^\mu(x) = -i \int d^dx \;\; \lambda(x) \partial_\mu J^\mu(x) \tag{2.6}$$

Since we have a conserved current $\partial_\mu j^\mu = 0$, $\delta S_\lambda = 0$ i.e, the action S is gauge invariant.

[b] using intergation by parts

Thus, using this method we can understand if we have conserved current in our theory by computing if our action (specifically the path integral) is gauge invariant.

In the path integral formalism, the partition function is expressed as:

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{iS}$$

and the correlation functions are expressed as

$$\langle \Psi \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\varphi \; \Psi e^{iS} \tag{2.7}$$

where $\Psi = \prod_a \varphi(x_a)$. Ψ also transforms in the same representation (R) as that of φ.

$$g \cdot \Psi_R(x) = R(g)^i{}_j \, \Psi^j_R(x) \tag{2.8}$$

Now using the fact that G is the symmetry group and the variation of background gauge field $A_\mu(x)$ (2.6) we obtain

$$\delta_\lambda \langle \Psi_R(x) \rangle = 0, \qquad \langle \delta_\lambda \Psi_R(x) - i\lambda(y)\partial_\mu j^\mu(y)\Psi_R(x) \rangle = 0 \tag{2.9}$$

then substituting the transformation of Ψ_R (2.1) the **Ward Identity**:

$$\partial_\mu j^\mu(x)\Psi_R(y) = \delta^{(d)}(x-y)T^a\Psi^a_R(x) \tag{2.10}$$

Details of the formulation of the ward identities using the path integral formalism can be found in [3].

2.1.3 Formulation Of Ordinary Symmetries in Topological Terms

From the previous sections on Unitary Operators and Noether's Theorem, we reviewed the role or ordinary symmetries in quantum field theory. Now we will formulate these sections in topological terms to understand the language of generalised symmetries.

Ordinary symmetries are called 0-form symmetries because they act on 0-dimensional operators in spacetime, which are local operators present at different points in spacetime and excite particles when they act on vacuum. **From now on we will refer to ordinary symmetries as 0-form symmetries.**

To formulate the above understanding of symmetries in a quantum theory in terms of topology, we emphasise the philosophy of [1] using the statement:

Statement 2.1.1

0-form symmetry is a topological codimension 1 operator which is invertible [a].

[a] codimension x= dimension $(d-x)$, where d is the dimension of our manifold

Lets break this statement and understand it further using insights from [1] and [7].

Unitary Operators as Topological Operators

As we know, we can use unitary irreducible representations of the symmetry group G to express symmetries in quantum field theory. Now to express these operators as topological operators, we think of it as associating it to a $(d-1)$ dimensional submanifold Σ_{d-1} [1]. These submanifolds are not necessarily compact. We denote the operators as:

$$U_g(\Sigma_{d-1}) \qquad\qquad , g \in G$$

where G is the symmetry group of quantum theory. To be precise, this $g \in G$ is expressed the unitary irreducible representation $R(g)$.

These 0-form symmetries for the 0-form symmetry group $G^{(0)}$.

Let us rewrite the group axioms for these topological operators to give us an insight as to how they act.

- Closure :

$$U_g(\Sigma_{d-1}) \otimes U_{g'}(\Sigma_{d-1}) = U_{gg'}(\Sigma_{d-1}) \tag{2.11}$$

  This is called the fusion rule[c] [7].

- Associativity :

$$U_g(\Sigma_{d-1}) \otimes \Big(U_{g'}(\Sigma_{d-1}) \otimes U_{g''}(\Sigma_{d-1})\Big) = \Big(U_g(\Sigma_{d-1}) \otimes U_{g'}(\Sigma_{d-1})\Big) \otimes U_{g''}(\Sigma_{d-1}) \tag{2.12}$$

[c] In the language of category theory, this is the fusion of objects [8]

- Identity :

$$\exists U_e(\Sigma_{d-1}) \qquad \Big| U_e(\Sigma_{d-1}) \otimes U_g(\Sigma_{d-1}) = U_g(\Sigma_{d-1}) \otimes U_e(\Sigma_{d-1}) = U_g(\Sigma_{d-1})$$
$$,e \text{ is the identity element of } G \tag{2.13}$$

- Inverse :

$$\exists U_g^{-1}(\Sigma_{d-1}) \equiv U_{g'}(\Sigma_{d-1}) \quad \Big| U_g(\Sigma_{d-1}) \otimes U_{g'}(\Sigma_{d-1}) = \mathbb{1} \ \forall R(g) \cdot R(g') = R(e) \tag{2.14}$$

  The association of the inverse operator along with $U_g(\Sigma_{d-1})$ with the same submanifold Σ_{d-1} implies associating no operator or the identity operator $U_e(\Sigma_{d-1}) \equiv \mathbb{1}$ with Σ_{d-1} [7].

Now before understanding how these operators act on local operators, first lets delve briefly into the topological nature of these operators.
$U_g(\Sigma_{d-1})$ being topological simply implies that the topological unitary operator is independent of the submanifold we associate it to given that we can obtain the new submanifold Σ'_{d-1} by topologically deforming [d] Σ_{d-1}, i.e,

$$\Sigma'_{d-1} - \Sigma_{d-1} = \partial\Sigma_d \tag{2.15}$$

where $\partial\Sigma_d$ is the boundary of Σ_d [7]. Since, these operators are topological we can write the following expression:

$$U_g(\Sigma_{d-1}) = U_g(\Sigma'_{d-1}) \tag{2.16}$$

Action of Topological Operators:

Since $U_g(\Sigma_{d-1})$ are topological, they can change only when the deformation of Σ_{d-1} crosses a local operator $\varphi_R(x)$ charged under the symmetry group G of our theory [1]. The way to think about the action of these topological operators on local operators can be understood in Euclidean picture via linking [Figure: 2.2], which intuitively can be thought of as wrapping the unitary topological operator which is associated to a $(d-1)$ dimensional

[d] continuously transforming which suggest that both the manifolds are homeomorphic

sphere S^{d-1} and then deforming it to a point so that it crosses the local operator located at x and acts on it.

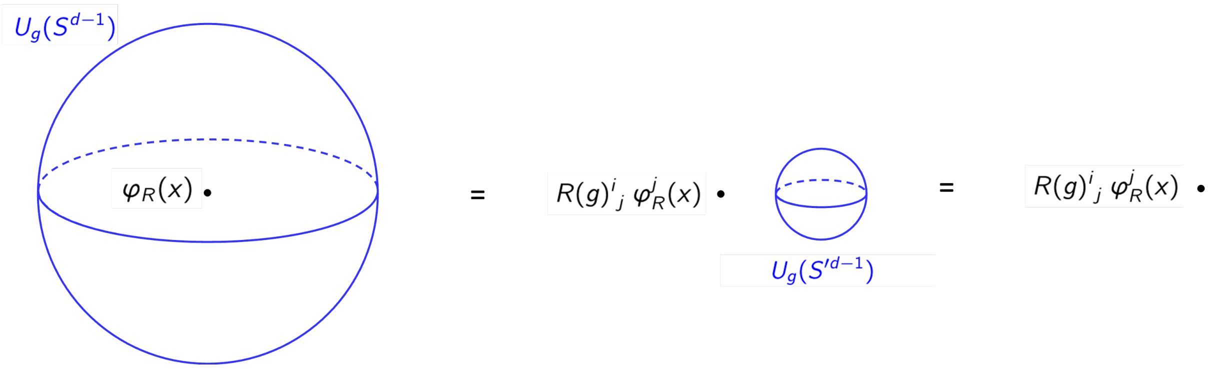

Figure 2.2: In the above figure we illustrate the action of the topological operator $U_g(S^{d-1})$ on the local operator $\varphi^i_R(x)$. *[Left]* Wrapping of $\varphi^i_R(x)$ by $U_g(S^{d-1})$; *[Centre]* Action of $U_g(S^{d-1})$ on $\varphi^i_R(x)$ by contracting it so that it crosses the local operator. Note that $U_g(S'^{d-1})$ does not link the transformed $\varphi^i_R(x)$; *[Right]* $U_g(S'^{d-1})$ can be topologically deformed and shrunk to a point giving us the desired result of the action $R(g)^i{}_j\, \varphi^j_R(x)$

We can mathematically express it as :

$$U_g(S^{d-1})\varphi^i_R(x) = R(g)^i{}_j\, \varphi^j_R(x) \tag{2.17}$$

which is the same transformation as seen in equation (2.1).
In Minkowski spacetime, we can think of the action,

$$U_g(\Sigma_{d-1})\varphi_R(x) = R(g)^i{}_j\, \varphi^j_R(x) U_g(\Sigma'_{d-1}) \tag{2.18}$$

as it giving us the equal time operator commutator relation:
The above equation can be seen from the Ward Identity stated later in (2.22).

2.1.4 Formulation of Noether's Theorem as Generalised Symmetry

Lets rewrite Noether's theorem 2.1.1 in the language of differential forms.
To do this we note that the conserved current as the components a one form over a differential manifold $J = j^\mu dx_\mu$. We can write the Hodge-dual of J as:

$$\star J = \frac{\sqrt{g}}{(d-1)!} J_{\mu_1} \epsilon^{\mu_1}{}_{\mu_2 \ldots \mu_d} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_d}$$

which is a $(d-1)$ form where g is the metric on our d-dimensional manifold [e]. The action

[e] for Euclidean manifold $g_{ij} = \delta_{ij} \implies \sqrt{g} = \sqrt{\det g} = 1$

of the exterior derivative operator d on the Hodge -dual of the one-form current can be expressed as:

$$d \star J = \partial^{\mu} J_{\mu} \Omega_g$$

where Ω_g is a volume form on our manifold [f].
Thus, the conservation law $\partial_{\mu} j^{\mu} = 0$ can be expressed as

$$d \star J = 0 \tag{2.19}$$

This implies that the Hodge-dual of the conserved current ($\star J$) is **closed** and hence the conserved current (J) is **co-closed**.
In this language, the conserved charge (2.2) can be expressed as,

$$Q = \int_{\Sigma_{d-1}} \star J \tag{2.20}$$

where Σ_{d-1} is a $(d-1)$-dimensional submanifold of a d-dimensional submanifold Σ_d of spacetime.
The topological unitary operator can be constructed by exponentiating the charge, we can express the topological unitary operator as :

$$U_g(\Sigma_{d-1}) = \exp\left(i\alpha^a T^a \int_{\Sigma_{d-1}} \star J \right) \tag{2.21}$$

where $R(g) = e^{i\alpha^a T^a} \in G^{(0)}$ which is the global 0-form symmetry group of given quantum field theory, and T^a are the generators of the group.

In presence of a local charged operator φ_R, transforming under the representation R of the symmetry group $G^{(0)}$, the Ward Identity in (2.10) can be expressed as [7]:

$$d \star J \;\; \varphi_R(x) = \delta^d(x) R(g)^i{}_j \, \varphi^j_R(x) \tag{2.22}$$

where $\delta^d(x)$ is Poincaré dual 1-form [g] to the delta function $\delta(x-y)$.
From (2.19), we know that $\star J$ is closed. This in fact implies the topological nature of operator as given in (2.36) [7].

[f] orientable manifold/topform
[g] We obtain this using Poincaré duality. For details refer [9]

Proof 2.1.1

Consider $U_g(\Sigma'_{d-1})$ obtained by smooth deformation of $U_g(\Sigma'_{d-1})$ as in (2.35).

Thus,
$$
\begin{aligned}
U_g(\Sigma'_{d-1}) &= \exp\left(i\alpha^a T^a \int_{\Sigma_{d-1}} \star J + i\alpha^a T^a \int_{\partial\Sigma_d} \star J\right) && \text{using (2.35)} \\
&\text{Now using Stokes Theorem} \\
&= \exp\left(i\alpha^a T^a \int_{\Sigma_{d-1}} \star J + i\alpha^a T^a \int_{\Sigma_d} d\star J\right) \\
&= \exp\left(i\alpha^a T^a \int_{\Sigma_{d-1}} \star J\right) && \text{using (2.19)} \\
&= U_g(\Sigma_{d-1})
\end{aligned}
\tag{2.23}
$$

Now From the Ward Identities stated in (2.22) we can write down the action of these topological operators on a local charged operator as we stated before in (2.18).

$$U_g(\Sigma_{d-1})\varphi^i_R(x) = R(g)^i{}_j\, \varphi^j_R(x) U_g(\Sigma'_{d-1})$$

Here Σ'_{d-1} does not link with the local operator [h].
The action via linking is :

$$U_g(S^{d-1})\varphi^i_R(x) = R(g)^i{}_j\, \varphi^j_R(x)$$

We can prove the above equation by explicitly computing it.

Proof 2.1.2

To do that lets consider a d-dimensional disk $\mathcal{M}^d$ with the boundary S^{d-1}.

$$\partial\mathcal{M}^d = S^{d-1}$$

Now lets compute the action,

$$U_g(S^{d-1})\varphi^i_R(x) = \exp\left(i\alpha^a T^a \int_{S^{d-1}} \star J\right)\varphi^i_R(x)$$

[h] We will look at what linking means in the next section.

Now using (2.24) and Stokes theorem,

$$= \exp\left(i\alpha^a \int_{\mathcal{M}^d} d \star J\right)\varphi^i_R(x)$$

Now using (2.22)

$$= \exp\left(i\alpha^a \int_{\mathcal{M}^d} \delta^d(x) T^a\right)\varphi^i_R(x)$$

$$= \exp(i\alpha^a T^a)\varphi^i_R(x)$$

Now using the fact that $R(g) = \exp(i\alpha^a T^a)$, we can write:

$$U_g(S^{d-1})\varphi^i_R(x) = R(g)^i{}_j\, \varphi^j_R(x) \tag{2.24}$$

■

2.2 1-Form Symmetries

Now that we understand how to construct ordinary symmetries as 0-form symmetries, we can start constructing higher form symmetries or p -form symmetries and delve into their action on higher dimensional operators. Lets first look at the case of p =1.

2.2.1 Construction of 1-Form Symmetries

A 1-form symmetry acts on 1-dimensional operators (eg: Wilson lines, 't Hooft lines). Using the statement 2.1.1 we can say that:

Statement 2.2.1

1-form symmetry is a topological codimension 2 operator which is invertible.

A 1-form symmetry is associated to a $(d-2)$-dimensional submanifold Σ_{d-2} and is denoted as:

$$U_g(\Sigma_{d-2}) \qquad ,\text{where } g \in G^{(1)} \tag{2.25}$$

$G^{(1)}$ is the symmetry group formed by 1-form symmetries. Consider $G^{(1)}$ as a continuous symmetry group of a quantum field theory. In the case of 1-form symmetry, we can associate a 2-form conserved current:

$$J_{(2)} = \frac{1}{2} j_{\mu\nu}\, dx^\mu \wedge dx^\mu \tag{2.26}$$

We can write the conservation law as :

$$\partial_\mu j^{[\mu\nu]}(x) = 0 \qquad \Leftrightarrow \qquad d \star J_{(2)} = 0 \tag{2.27}$$

The conserved charge is now defined as an integral of the hodge dual of the 2-form current over Σ_{d-2} :

$$Q = \int_{\Sigma_{d-2}} \star J_{(2)} \tag{2.28}$$

and now we define the topological operator by exponentiating the conserved charge:

$$U_g(\Sigma_{d-2}) = \exp\left(i\alpha^a T^a \int_{\Sigma_{d-2}} \star J_{(2)} \right) \tag{2.29}$$

Here T^a are the generators of $g \in G^{(1)}$.

Group Multiplication: For $G^{(1)}$ to be a group, topological operators need to satisfy the group multiplication law among other group axioms. The fusion rule (group multiplication law) states:

$$U_g(\Sigma_{d-2}) \otimes U_{g'}(\Sigma_{d-2}) = U_{gg'}(\Sigma_{d-2}) \qquad \forall g, g', gg' \in G^{(1)} \tag{2.30}$$

Remark : Group Multiplication Law

The above group multiplication law is also a result of the operator product expansion:

$$\partial_\mu \, j_a^{\mu\nu}(x) \, j_b^{\rho\sigma}(y) = f_{ab}{}^c \, j_c^{\rho\sigma}(x) \, \delta^{(d)}(x-y) \tag{2.31}$$

Lets simplify our calculations by using the following statement:

Statement 2.2.2

The 1-form global symmetry groups are always Abelian. [6].

We will emphasize and prove this statement later on.

From the above statement, we can say that for continuous case $G^{(1)} \cong \prod_i U(1)_i$ where $i \in \mathbb{N}$, and for simplicity we consider $G^{(1)} = U(1)$.

Since $G^{(1)} = U(1)$, The topological operator (2.104) can be expressed as:

$$U_g(\Sigma_{d-2}) = \exp\left(i\alpha \int_{\Sigma_{d-2}} \star J_{(2)}\right) \qquad , g = e^{i\alpha} \qquad \alpha \in \mathbb{R} \tag{2.32}$$

also (2.31) becomes:

$$\partial_\mu \, j_a^{\mu\nu}(x) \, j_b^{\rho\sigma}(y) = 0 \tag{2.33}$$

and hence we can see the group multiplication (2.30) gives us:

$$U_g(\Sigma_{d-2}) \otimes U_{g'}(\Sigma_{d-2}) = \exp\left(i(\alpha_1 + \alpha_2) \int_{\Sigma_{d-2}} \star J_{(2)}\right) \tag{2.34}$$

Topological Property: Now the requirement of being toplogical is analogous to the 0-form symmetry case.
$U_g(\Sigma_{d-2})$ being topological simply implies that the topological unitary operator is independent of the submanifold we associate it to given that we can obtain the new submanifold Σ'_{d-2} by topologically deforming Σ_{d-2}, i.e,

$$\Sigma'_{d-2} - \Sigma_{d-2} = \partial\Sigma_{d-1} \tag{2.35}$$

where $\partial\Sigma_{d-1}$ is the boundary of Σ_{d-1} Since, these operators are topological we can write the following expression:

$$U_g(\Sigma_{d-2}) = U_g(\Sigma'_{d-2}) \tag{2.36}$$

2.2.2 Action of 1-Form Symmetries

Now that we have understood the construction of 1-form symmetries, lets understand their action on 1-dimensional operators. These 1-dimensional opertors, often called line opertors can be denoted at:

$$L_q(\gamma) \tag{2.37}$$

where q is the charge of the line operator under $G^{(1)} \cong U(1)$, and γ is the curve that parameterises the line operator. Now before we understand the action of the 1-form symmetry on line operators lets step aside and discuss some concepts that will help us in our discussion later.

Remark : Transversality and Linking

Lets consider a d-dimensional manifold $\mathcal{M}_d$. Let $\mathcal{N}_s$ and $\mathcal{X}_r$ be s-dimensional and r-dimensional submanifolds of $\mathcal{M}_d$ respectively.
Definition: *$\mathcal{N}_s$ and $\mathcal{X}_r$ are said to be transverse (or intersect transversally) if, $\forall p \in \mathcal{N}_s \cap \mathcal{X}_r$,*

$$T_p\mathcal{N}_s \oplus T_p\mathcal{X}_r \subseteq T_p\mathcal{M}_d \qquad , s + r \leq d \tag{2.38}$$

where $T_p\mathcal{N}_s$ denotes the tangent space of $\mathcal{N}_s$ at a point p [10].

We can denote it as:

$$\mathcal{N}_s \pitchfork \mathcal{X}_r \tag{2.39}$$

Lets understand linking number using insights from [6]. Now consider $\mathcal{M}_d$ to be a compact, orientable and closed ($\partial\mathcal{M}_d = 0$). Now let $\mathcal{N}_s$ and $\mathcal{X}_r$ be orientable and non-intersecting with the condition, $s + r = d - 1$ and we assume they are trivial homotopically [a]. Now introducing an $(r+1)$-dimesnional submanifold of $\mathcal{M}_d$: $\mathcal{Y}_{r+1}$ such that $\partial\mathcal{Y}_{r+1} = \mathcal{X}_r$. Let $\mathcal{Y}_{r+1} \pitchfork \mathcal{N}_s$ and $\mathcal{Y}_{r+1} \cap \mathcal{N}_s = \{p_i\}$, then

$$T_{p_i}\mathcal{N}_s \oplus T_{p_i}\mathcal{Y}_{r+1} = T_{p_i}\mathcal{M}_d \tag{2.40}$$

Using this we can define an orientation on $\mathcal{M}_d$ using orientations of $\mathcal{N}_s$ and $\mathcal{Y}_{r+1}$. Define $\text{sign}(p_i)$ such that :

$$\text{sign}(p_i) = \begin{cases} +1 & \text{induced orientation of } \mathcal{M}_d \text{ at } p_i \textbf{ same} \text{ as original} \\ -1 & \text{induced orientation of } \mathcal{M}_d \text{ at } p_i \textbf{ opposite} \text{ to original} \end{cases}$$

Definition: *Linking Number is a topological invariant. The linking number of* $\mathcal{N}_s$ *and* $\mathcal{X}_r$ *can be defined as:*

$$\text{Link}(\mathcal{N}_s, \mathcal{X}_r) = \sum_i \text{sign}(p_i) \tag{2.41}$$

where $r = d - s - 1$.

Using differential form notation, we can define the linking number as follows:

$$\begin{aligned} \text{Link}(\mathcal{N}_s, \mathcal{X}_r) &= \int_{\mathcal{M}_d} \delta^{(d-r-1)}(p \in \mathcal{Y}_{r+1}) \wedge \delta^{(d-s)}(p \in \mathcal{N}_s) \\ &= \int_{\mathcal{Y}_{r+1}} \delta^{(d-s)}(p \in \mathcal{N}_s) \\ &= \int_{\mathcal{N}_s} \delta^{(d-r-1)}(p \in \mathcal{Y}_{r+1}) \end{aligned} \tag{2.42}$$

where $\delta^{(r)}$ is a Poincaré dual r-form.

[a] To under homotopy in detail, refer [9].

Now that we have the required mathematical notions ready, lets write down the action of a 1-form symmetry on a line operator.

Since these line operators (2.37) are charged under 1-form symmetries (2.32) they obey the **Ward Identity** [6]:

$$d \star J_{(2)}(x) L_q(\gamma) = q\delta^{(d-1)}(x \in \gamma) L_q(\gamma) \tag{2.43}$$

where $\delta^{(d-1)}(x \in \gamma)$ is a $(d-1)$-form that integrates to 1 on any manifold that is transverse to γ and is 0 otherwise.

Now lets we can link the curve γ on Σ_{d-2}. Thus in a similar manner as the action of 0-form symmetries, as we topologically deform $\Sigma_{d-2} \to \Sigma'_{d-2}$, it crosses γ and due to the Ward Identity (2.43), when $U_g(\Sigma_{d-2})$ intersects $L_q(\gamma)$, it generates a phase :

$$\langle U_g(\Sigma_{d-2}) L_q(\gamma) \rangle = e^{i\alpha q \mathrm{Link}(\Sigma_{d-2}, \gamma)} \langle L_q(\gamma) U_g(\Sigma'_{d-2}) \rangle \tag{2.44}$$

We can prove this as follows:

Proof 2.2.1

Using (2.32), we can rewrite the above action as follows:

$$\left\langle \exp\left(i\alpha \int_{\Sigma_{d-2}} \star J_{(2)} \right) L_q(\gamma) \right\rangle \tag{2.45}$$

Now we can define a $(d-1)$ dimensional manifold $\mathcal{N}_{d-1}$ such that,

$$\partial \mathcal{N}_{d-1} = \Sigma_{d-2}$$

Thus (2.45) can be written as:

$$\left\langle \exp\left(i\alpha \int_{\partial \mathcal{N}_{d-1}} \star J_{(2)} \right) L_q(\gamma) \right\rangle$$

Using Stokes Theorem,

$$= \left\langle \exp\left(i\alpha \int_{\mathcal{N}_{d-1}} d \star J_{(2)} \right) L_q(\gamma) \right\rangle$$

Using the ward identity (2.43),

$$= \left\langle \exp\left(i\alpha \int_{\mathcal{N}_{d-1}} q\delta^{(d-1)}(x \in \gamma) \right) L_q(\gamma) \right\rangle$$

Now we homotopically deform Σ_{d-2} to Σ'_{d-2} such that (2.35) is valid. We also deform it such that Σ'_{d-2} and γ do not link. Now we know from (2.42) and the remark on transversality and linking, that if $\mathcal{N}_{d-1} \pitchfork \gamma$ we can define the linking between Σ_{d-2} and γ as :

$$\text{Link}(\Sigma_{d-2}, \gamma) = \int_{\mathcal{N}_{d-1}} \delta^{(d-1)}(x \in \gamma) \tag{2.46}$$

Thus we can now write the action as:

$$e^{i\alpha q \text{Link}(\Sigma_{d-2}, \gamma)} \langle L_q(\gamma) U_g(\Sigma'_{d-2}) \rangle \tag{2.47}$$

■

By considering the action, using wrapping of $U_g(S^{d-2})$ around the line operator $L_q(\gamma)$, we can write the action as:

$$U_g(S^{d-2}) L_q(\gamma) = e^{i\alpha q \text{Link}(S^{d-2}, \gamma)} L_q(\gamma) \tag{2.48}$$

We can illustrate this action using figure:

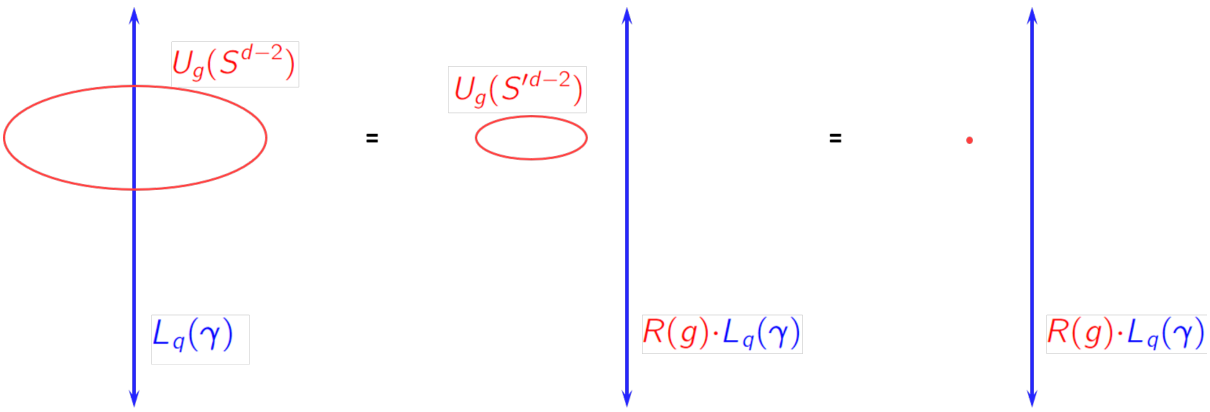

Figure 2.3: *[Left:]* Here we see how the topolgical operator $U_g(S^{d-2})$ wraps around a line operator $L_q(\gamma)$. *[Center:]* We homotopically deform $U_g(S^{d-2})$ till it crosses the line operator to get the transformed operator $R(g) \cdot L_q(\gamma)$ and $U_g(S'^{d-2})$. *[Right:]* Since S'^{d-2} and γ do not link, we can homotopically deform $U_g(S'^{d-2})$ to point giving us the action of the topological operator $U_g(S^{d-2})$ wraps on the line operator $L_q(\gamma)$.

Now lets prove the statement 2.2.2 which says that the 1-form symmetries are Abelian. Since the 1-form symmetry operators are associated to a $(d-2)$-dimensional manifold, there exists a locally transverse plane due to which we can shift the symmetry operators by smooth topological deformations as shown in the figure below.

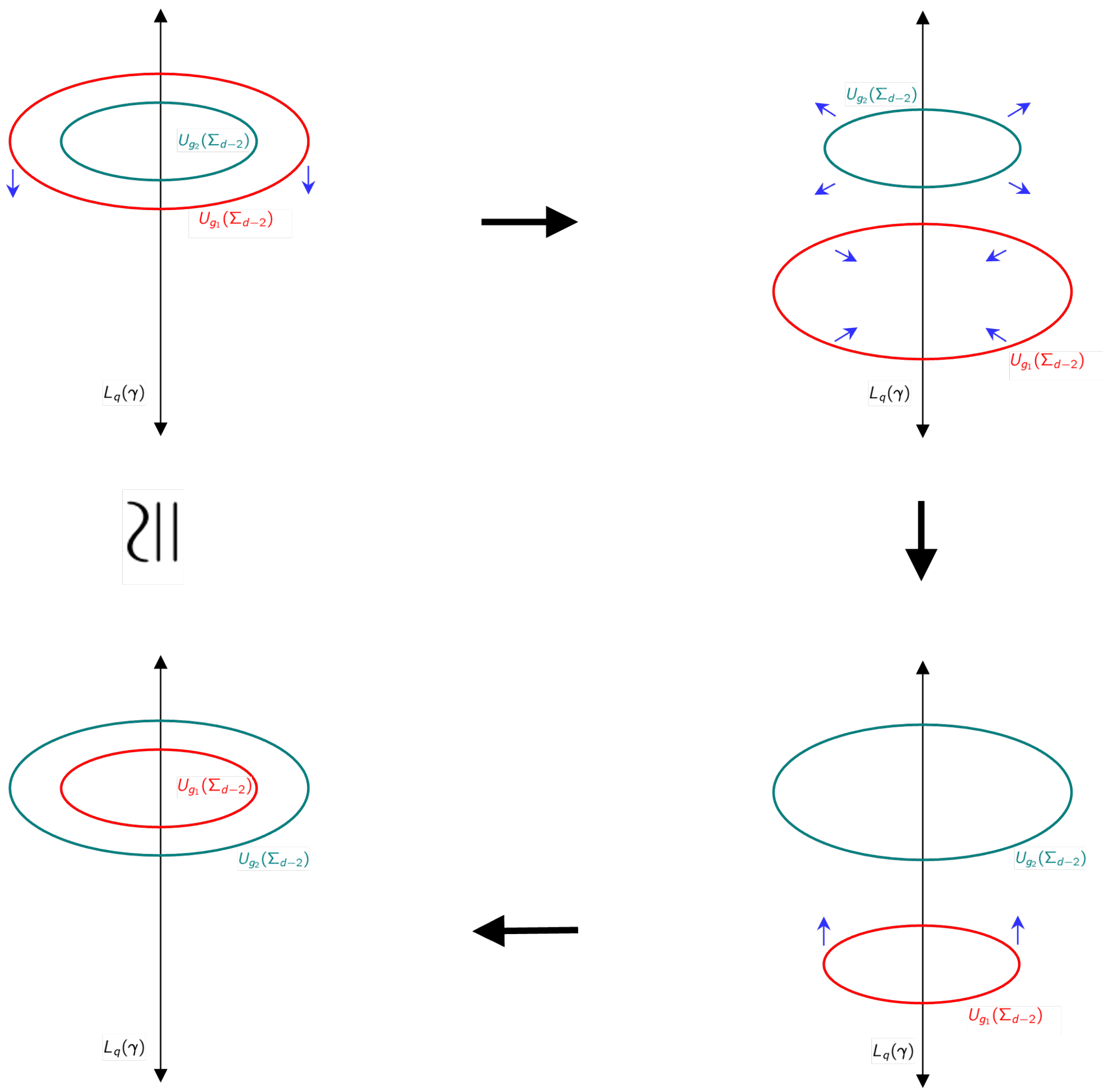

Figure 2.4: The first figure depicts a configuration of both the topological operators $U_{g_1}(\Sigma_{d-2})$ and $U_{g_2}(\Sigma_{d-2})$ wrapping a charged line operator $L_q(\gamma)$. We see that how we can topologically deform these operators such that we can exchange the configuration of these operators.

Thus there is no well defined notion of ordering [6] due to which the 1-form symmetry groups are abelian.

$$U_{g_1}(\Sigma_{d-2}) \otimes U_{g_2}(\Sigma_{d-2}) \cong U_{g_2}(\Sigma_{d-2}) \otimes U_{g_1}(\Sigma_{d-2}) \tag{2.49}$$

2.2.3 Gauging 1-Form Symmetries

Gauging simply implies coupling background fields to our theory. As in the case of ordinary symmetries, this is done because it's an excellent way to understand which currents are conserved in our theory by looking at the variation of the action coupled with the background fields under gauge transformations.

In the case of 1-form symmetries, we couple a 2-form background gauge field $B_{(2)}$ to our 2-form conserved current $\star J_{(2)}$ and add the following term to our action:

$$S_{\text{gauge}} = +i \int B_{(2)} \wedge \star J_{(2)} \tag{2.50}$$

where $B_{(2)}$ has the following gauge transformation:

$$B_{(2)} \to B_{(2)} + d\Lambda_{(1)} \tag{2.51}$$

Under the above mentioned gauge transform, (2.50) varies as follows:

$$\delta S_{\text{gauge}} = +i \int d\Lambda_{(1)} \wedge \star J_{(2)} = +i \int \Lambda_{(1)} \wedge d \star J_{(2)} = 0 \tag{2.52}$$

which is true due to the conservation law (2.27) which implies that $\star J_{(2)}$ is a closed 2-form, which hence implies that the action is invariant under the background gauge transformations.

2.3 4d Maxwell Theory

Lets now look at the example of $U(1)$ gauge theory in 4d ($\mathcal{M}_4$), i.e Maxwell Theory. We'll look at the theory in the light of generalised symmetries. Consider pure Maxwell theory with gauge field $A_\mu(x)$, coupling e, and field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We note that the gauge field can be written as a 1-form:

$$A_{(1)} = A_\mu dx^\mu \tag{2.53}$$

and the field strength tensor can be written as a 2-form

$$F_{(2)} = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{2.54}$$

We also note that F is a locally exact 2-form, i.e

$$F_{(2)} = dA_{(1)} \tag{2.55}$$

The action is given as:

$$S = \frac{1}{2e^2} \int_{\mathcal{M}_4} F_{(2)} \wedge \star F_{(2)} \tag{2.56}$$

Here, A is a standard $U(1)$ connection [i]. It is identified as

$$A_{(1)} \sim A + d\lambda_{(0)} \tag{2.57}$$

where $\lambda_{(0)}$ is a 0-form on our manifold and is periodic due to the fact that $A_{(1)}$ is a standard $U(1)$ connection, thus $\lambda_{(0)} \sim \lambda_{(0)} + 2\pi$. $\lambda_{(0)}$ may not be well-defined.
Due to this the electric charged is quantised.

$$\int_{\Sigma_2} F_{(2)} \in 2\pi\mathbb{Z} \tag{2.58}$$

where Σ_2 is a submanifold of $\mathcal{M}_4$ and $\mathbb{Z}$ denotes integers.
The equation of motion can be derived by varying S with respect to $A_{(1)}$. The equation of motion is:

$$d \star F_{(2)} = 0 \tag{2.59}$$

[i] add something about gauge theory geometry

2.3.1 Construction of 1-Form Symmetries in Maxwell Theory

Now this equation of motion can be alternatively viewed as conservation of a 2-form current:

$$J_2^{\text{elc.}} = \frac{1}{e^2} F_{(2)} \tag{2.60}$$

Thus, using the discussion in the previous section, we can say that $4d$ Pure Maxwell theory has a $U(1)^{(1)}_{\text{elc.}}$ 1-form symmetry called the **electric 1-form symmetry**.

Now from (2.55) and the fact that the exterior derivative operator is nil-potent, i.e $d^2 = 0$, we get the **Bianchi Identity**:

$$dF_{(2)} = 0 \tag{2.61}$$

This equation can also be alternatively viewed as conservation of a dual 2-form current:

$$J_2^{\text{mag.}} = \frac{1}{2\pi} \star F_{(2)} \tag{2.62}$$

thus along with the electric 1-form symmetry, $4d$ Pure Maxwell theory has a $U(1)^{(1)}_{\text{mag.}}$ 1-form symmetry called the **magnetic 1-form symmetry**.

This $U(1)^{(1)}_{\text{mag.}}$ symmetry is dual to the $U(1)^{(1)}_{\text{elc.}}$ symmetry and this has interesting implications on their action on line operators.
Before we construct the electric and magnetic 1-form symmetries and look at their actions on line operators, lets delve briefly in the electric-magnetic duality of Maxwell theory.

Remark : S-Duality

S-Duality in Maxwell theory is the invariance of Maxwell equations under the interchange of electric and magnetic fields. We can phrase this in terms of field strength $F_{\mu\nu}$ and dual field strength $\tilde{F}_{\mu\nu}$ where,

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{2.63}$$

To write the action in terms of the dual field strength, we need to change the reason of $dF_{(2)} = 0$ being the constraint $F_{(2)} = dA_{(1)}$, i.e $F_{(2)}$ is exact. To do this add a

Lagrange multiplier term to the action as follows [11]:

$$\tilde{S} = \frac{1}{2e^2}\int_{M_4} F_{(2)} \wedge \star F_{(2)} + \frac{1}{2\pi}\int_{M_4} F_{(2)} \wedge d\tilde{A}_{(1)} \tag{2.64}$$

where $\tilde{A}_{(1)}$ is the Lagrange multiplier term.
Now we get the condition $dF_{(2)} = 0$ using the equation of motion of $\tilde{A}_{(1)}$ instead of the constraint. Now substituting $\tilde{A}_{(1)}$ in $\tilde{S}$ gives us the original action (2.56).
Now looking at the equation of motion of $F_{(2)}$ we get:

$$\delta_{F_{(2)}}\tilde{S} = \frac{1}{e^2} \star F_{(2)} - \frac{1}{2\pi} d\tilde{A}_{(1)} = 0 \tag{2.65}$$

From this we can see:

$$\star F_{(2)} = \frac{e^2}{2\pi} d\tilde{A}_{(1)} \qquad F_{(2)} = -\frac{e^2}{2\pi} \star d\tilde{A}_{(1)} \tag{2.66}$$

Substituting this back to $\tilde{S}$ gives us:

$$\begin{aligned}
\tilde{S} &= -\frac{e^2}{8\pi^2}\int_{M_4} \star d\tilde{A}_{(1)} \wedge d\tilde{A}_{(1)} - \frac{e^2}{4\pi^2}\int_{M_4} \star d\tilde{A}_{(1)} \wedge d\tilde{A}_{(1)} \\
&= -\frac{e^2}{4\pi^2}\int_{M_4} \star d\tilde{A}_{(1)} \wedge d\tilde{A}_{(1)} \\
&= \frac{e^2}{4\pi^2}\int_{M_4} d\tilde{A}_{(1)} \wedge \star d\tilde{A}_{(1)}
\end{aligned} \tag{2.67}$$

Now considering $\tilde{A}_{(1)}$ as a dual $U(1)$ gauge field (connection) we can define the dual field strength as:

$$\tilde{F}_{(2)} = d\tilde{A}_{(1)} \tag{2.68}$$

and hence write the dual action as:

$$\tilde{S} = \frac{1}{\tilde{e}^2}\int_{M_4} \tilde{F}_{(2)} \wedge \star \tilde{F}_{(2)} \qquad \tilde{e}^2 = \frac{4\pi^2}{e^2} \tag{2.69}$$

where $\tilde{e}$ is the dual gauge coupling.
$\tilde{A}$ has the gauge transform;

$$\tilde{A}_{(1)} \sim \tilde{A}_{(1)} + d\tilde{\lambda}_{(0)} \tag{2.70}$$

where $\tilde{\lambda}_{(0)}$ is a 0-form on our manifold and is periodic due to the fact that $\tilde{A}_{(1)}$ is a

dual $U(1)$ connection, thus $\tilde{\lambda}_{(0)} \sim \tilde{\lambda}_{(0)} + 2\pi$.
Due to this the magnetic charged is quantised.

$$\int_{\Sigma_2} \tilde{F}_{(2)} \in 2\pi\mathbb{Z} \tag{2.71}$$

Hence we can also represent the entire Maxwell theory and it's higher form symmetries in the dual representation.

Coming back to the discussion of 1-form symmetries of Maxwell in 4d, from (2.60) we can construct the topological operator that enacts this symmetry:

$$U_g^{\text{elc.}}(\Sigma_2) = \exp\left(\frac{i\alpha}{e^2}\oint_{\Sigma_2} \star F_{(2)}\right) \tag{2.72}$$

where Σ_2 is a closed 2-dimensional submanifold of $\mathcal{M}_4$.
Now from (2.62) we can construct the corresponding dual magnetic 1-form symmetry :

$$U_g^{\text{mag.}}(\Sigma_2) = \exp\left(i\alpha\oint_{\Sigma_2} \frac{F_{(2)}}{2\pi}\right) \tag{2.73}$$

2.3.2 Action of $U_g^{\text{mag.}}(\Sigma_2)$ and $U_g^{\text{elc.}}(\Sigma_2)$ on Line Operators

We need line operators which are gauge invariant. One such operator is the Wilson Line which is defined as:

$$W(q_e, \gamma) = \exp\left(iq_e \oint_\gamma A\right) \tag{2.74}$$

Under a gauge transform,

$$\begin{aligned} W(q_e, \gamma) \to &\exp\left(iq_e \oint_\gamma A + d\lambda\right) \\ &= \exp\left(iq_e(\oint_\gamma A + \oint_\gamma d\lambda)\right) \\ \text{Using Stokes' Theorem,}& \\ &= \exp\left(iq_e(\oint_\gamma A + \oint_{\partial\gamma} \lambda)\right) \\ &= W(q_e, \gamma) \end{aligned} \tag{2.75}$$

if γ is closed or infinite.

Interpretation of a Wilson Line: It can be interpreted as a world line of a non-dynamical massive charged particle. Such a particle is the source of the electric flux. Using insights from [6] we can compute the correlator of the Wilson Line:

$$\langle W(q_e,\gamma)\rangle = \int \mathcal{D}A \ \exp\left(iq_e \oint_\gamma A\right) e^{iS[A]} \tag{2.76}$$

Now inserting (2.56) in the above equation:

$$\begin{aligned} &= \int \mathcal{D}A \ \exp\left(iq_e \oint_\gamma A\right) \exp\left(\frac{1}{2e^2}\int_{\mathcal{M}_4} F_{(2)} \wedge \star F_{(2)}\right) \\ &= \int \mathcal{D}A \ \exp\left(iq_e \int_{\mathcal{M}_4} \delta^3(\gamma) \wedge A + \frac{1}{2e^2} F_{(2)} \wedge \star F_{(2)}\right) \end{aligned} \tag{2.77}$$

where

$$\int_{\mathcal{M}_3^T} \delta^3(\gamma) = 1 \tag{2.78}$$

such that $\mathcal{M}_3^T \pitchfork \gamma$ once.
The equation of motion for A can be calculated by variation of action with respect to A [show derivation] and it is expressed as [6]:

$$d \star F_{(2)} = q_e e^2 \delta^3(x \in \gamma) \tag{2.79}$$

Now integrating the above equation on $\mathcal{M}_3$ with a boundary such that $\partial\mathcal{M}_3 = \Sigma_2$ and using the remark on transversality and linking we find [6] :

$$\oint_{\Sigma_2} \star F_{(2)} = q_e e^2 \ \mathrm{Link}(\Sigma_2, \gamma) \tag{2.80}$$

Using the above arguments we can see that Wilson Line $W(q_e,\gamma)$ acts as an electric source and modifies Maxwell's equations.
Now lets write the action of $U_g^{\text{elc.}}(\Sigma_2)$ on $W(q_e,\gamma)$ using (2.44) :

$$\langle U_g^{\text{elc.}}(\Sigma_2) W(q_e,\gamma)\rangle = e^{i\alpha q_e \mathrm{Link}(\Sigma_2,\gamma)} \langle W(q_e,\gamma) U_g^{\text{elc.}}(\Sigma_2')\rangle \tag{2.81}$$

We can derive the above action using an analogous proof as Proof 2.2.1.
Alternatively we can see that exponentiating (2.80) also gives the action of $U_g^{\text{elc.}}(\Sigma_2)$ on $W(q_e,\gamma)$.

Now lets see what the symmetry $U_g^{\text{mag.}}(\Sigma_2)$ acts on.
We can define a gauge invariant line operator that is the magnetic dual to the wilson line. We do so because we explored S-duality in which we undertstood that $U_g^{\text{mag.}}(\Sigma_2)$ is

constructed using the dual (magnetic) field strength $\tilde{F}$.
Such an operator is called the **'t Hooft line operator** and is expressed as:

$$T(q_m, \gamma) = \exp\left(i q_m \oint_\gamma \tilde{A}\right) \tag{2.82}$$

where $\tilde{A}$ is the dual gauge field defined as :

$$\star F_{(2)} = d\tilde{A}_{(1)} \tag{2.83}$$

Using the remark on S-Duality in Maxwell theory, we write the dual field strength as:

$$\star F_{(2)} = \star dA_{(1)} = \frac{e^2}{2\pi} d\tilde{A}_{(1)} = \frac{e^2}{2\pi} d\tilde{F}_{(2)} \tag{2.84}$$

This is the reason why we choose the normalization of currents as in (2.60) and (2.62).
The action of $U_g^{\text{mag.}}(\Sigma_2)$ on the 't Hooft line operator $T(q_m, \gamma)$ is same as (2.81), i.e the electric case:

$$\langle U_g^{\text{mag.}}(\Sigma_2) T(q_m, \gamma)\rangle = e^{i\alpha q_m \text{Link}(\Sigma_2, \gamma)} \langle T(q_m, \gamma) U_g^{\text{mag.}}(\Sigma_2')\rangle \tag{2.85}$$

Thus, we conclude that 4d Maxwell Theory has a $U(1)^{(1)}_{\text{elc.}} \times U(1)^{(1)}_{\text{mag.}}$ global symmetry.

2.3.3 Gauging the Higher Form Symmetries of 4d Maxwell

Gauging means introducing background fields corresponding to each conserved current in our action. Following the discussion in section [Gauging 1-Form Symmetries], in the case of Maxwell with $U(1)^{(1)}_{\text{elc.}} \times U(1)^{(1)}_{\text{mag.}}$ symmetry we introduce the background fields $B^e_{(2)}$ and $B^m_{(2)}$ with the corresponding gauge transformations:

$$B^e_{(2)} \to B^e_{(2)} + d\Lambda^e_{(1)} \tag{2.86}$$

$$B^m_{(2)} \to B^m_{(2)} + d\Lambda^m_{(1)} \tag{2.87}$$

where $\Lambda^e_{(1)} = d\lambda^e_{(0)}$ We also have the usual transformation of the gauge field $A_{(1)}$:

$$A_{(1)} \to A_{(1)} + d\lambda^e_{(0)} = A_{(1)} + \Lambda^e_{(1)} \tag{2.88}$$

Now we can couple the fields $B^e_{(2)}$ and $B^m_{(2)}$ to the action S (2.56) by adding the following terms:

$$S_e = -\frac{1}{e^2}\int_{\mathcal{M}_4} B^e_{(2)} \wedge \star F_{(2)} \qquad S_m = \frac{i}{2\pi}\int_{\mathcal{M}_4} B^m_{(2)} \wedge F_{(2)} \tag{2.89}$$

Thus the gauged action is of the form $S_g = S + S_m + S_e$, i.e:

$$S_g = \int_{\mathcal{M}_4} \frac{1}{2e^2}(F_{(2)} - B^e_{(2)}) \wedge \star(F_{(2)} - B^e_{(2)}) + \frac{i}{2\pi}(B^m_{(2)} \wedge F_{(2)}) \tag{2.90}$$

We have added an additional counter term made of only background gauge fields to make the kinetic term invariant under gauged transformations (2.87) and (2.86) due to the fact that $A_{(1)}$ is also shifted under these transformations.

$$S_{ct} = \frac{1}{2e^2}\int_{\mathcal{M}_4} B^e_{(2)} \wedge \star B^e_{(2)} \tag{2.91}$$

Lets look at the transformation of S_g under (2.86):

$$\begin{aligned} S_g \to \int_{\mathcal{M}_4} \frac{1}{2e^2}(F_{(2)} - B^e_{(2)}) \wedge \star (F_{(2)} - B^e_{(2)}) + \frac{i}{2\pi} B^m_{(2)} \wedge (F_{(2)} + d\Lambda^e_{(1)}) \\ = S_g + \frac{i}{2\pi}\int_{\mathcal{M}_4} B^m_{(2)} \wedge d\Lambda^e_{(1)} \end{aligned} \tag{2.92}$$

Thus, S_g is not invariant under the gauging of the electric symmetry.
Now lets look at the transformation of S_g under (2.87):

$$\begin{aligned} S_g &\to S_g + \frac{i}{2\pi}\int_{\mathcal{M}_4} d\Lambda^m_{(1)} \wedge F_{(2)} \\ &= S_g - \frac{i}{2\pi}\int_{\mathcal{M}_4} \Lambda^m_{(1)} \wedge dF_{(2)} = S_g \quad \cdots \text{ignoring boundary terms} \end{aligned} \tag{2.93}$$

Thus, we see that we cannot gauge both electric and magnetic symmetry simultaneously even though gauging the magnetic symmetry leaves the action invariant.
Lets add another counter term to make the gauging of electric symmetry non-anomalous:

$$S'_{ct} = -\frac{i}{2\pi}\int_{\mathcal{M}_4} B^m_{(2)} \wedge B^e_{(2)} \tag{2.94}$$

Now our action is:

$$S'_g = \int_{\mathcal{M}_4} \frac{1}{2e^2}(F_{(2)} - B^e_{(2)}) \wedge \star(F_{(2)} - B^e_{(2)}) + \frac{i}{2\pi} B^m_{(2)} \wedge (F_{(2)} - B^e_{(2)}) \tag{2.95}$$

Gauging of the electric symmetry (2.86), the action S'_g is invariant since we can see that $(F_{(2)} - B^e_{(2)})$ terms are gauge invariant.

Gauging of the magnetic symmetry (2.87), the action is not invariant,

$$S'_g \to S'_g - \frac{i}{2\pi}\int_{\mathcal{M}_4} d\Lambda^m_{(1)} \wedge B^e_{(2)} \tag{2.96}$$

We can hence say that no counter terms in the action can help us gauge the electric and the magnetic symmetry simultaneously. This situation arises when we have a **mixed 't Hooft anomaly** between our two symmetries. We can define the mixed 't Hooft anomaly in terms of anomaly inflow action in 5 dimensions as:

$$S_{\text{inflow}} = -\frac{i}{2\pi}\int_{\mathcal{N}_5} B^e_{(2)} \wedge dB^m_{(2)} \tag{2.97}$$

where $\partial\mathcal{N}_5 = \mathcal{M}_4$.
We have used the idea of anomaly inflow which we will discuss in detail in the next chapter. For now lets compute the background gauge transformation of the inflow action under both the symmetries:

$$\begin{aligned} S_{\text{inflow}} \to & S_{\text{inflow}} - \frac{i}{2\pi}\int_{\mathcal{N}_5} d\Lambda^e_{(1)} \wedge dB^m_{(2)} \\ &= S_{\text{inflow}} + \frac{i}{2\pi}\int_{\mathcal{N}_5} d(d\Lambda^e_{(1)} \wedge B^m_{(2)}) \end{aligned} \tag{2.98}$$

Now using Stokes' Theorem,

$$\begin{aligned} &= S_{\text{inflow}} + \frac{i}{2\pi}\int_{\partial\mathcal{N}_5} d\Lambda^e_{(1)} \wedge B^m_{(2)} \\ &= S_{\text{inflow}} - \frac{i}{2\pi}\int_{\mathcal{M}_4} B^m_{(2)} \wedge d\Lambda^e_{(1)} \end{aligned} \tag{2.99}$$

It's extremely fascinating that (2.92) and (2.99) are the same except their signs. Thus, we can add S_{inflow} to S_g to cancel out the anomaly and hence get a non-anomalous gauged Maxwell theory. We can write the final action as :

$$\mathcal{S} = \int_{\mathcal{M}_4} \frac{1}{2e^2}(F_{(2)} - B^e_{(2)}) \wedge \star(F_{(2)} - B^e_{(2)}) + \frac{i}{2\pi}(B^m_{(2)} \wedge F_{(2)}) - \int_{\mathcal{N}_5} \frac{i}{2\pi}(B^e_{(2)} \wedge dB^m_{(2)}) \tag{2.100}$$

Instead of adding S_{inflow} to S_g, we change also add $S_{\text{inflow}}(e \leftrightarrow m)$ to S'_g to get an anomaly free action.

2.4 p-form Symmetries

After exploring 0-form and 1-form symmetries, we can now generalise our discussion to understand p-form symmetries.

2.4.1 Construction of p-Form Symmetries

We can generalise the statement 2.1.1 for p-form symmetries as:

Statement 2.4.1

p-form symmetry is a topological codimension $(p+1)$ operator which is invertible.

Lets break this statement and understand it further using insights from [1] and [7].

A p-form symmetry is associated to a $(d-p-1)$-dimensional submanifold Σ_{d-p-1} and is denoted as:

$$U_g(\Sigma_{d-p-1}) \qquad ,\text{where } \ g \in G^{(p)} \tag{2.101}$$

$G^{(p)}$ is the symmetry group formed by p-form symmetries.
The fusion rule shows the composition of the p-form topological operators.

$$U_{g_1}(\Sigma_{d-p-1}) \otimes U_{g_2}(\Sigma_{d-p-1}) = U_{g_1 g_2}(\Sigma_{d-p-1}) \tag{2.102}$$

In the case of p-form symmetry, we can associate a $p+1$-form conserved current. We can write the conservation law as :

$$d \star J_{p+1}(x) = 0 \tag{2.103}$$

Thus, $\star J_{p+1}$ is a closed form which implies the topological nature of p-form symmetry operators same as 0-form and 1-form symmetries. We define the topological operator by exponentiating the conserved charge:

$$U_g(\Sigma_{d-p-1}) = \exp\left(i\alpha \int_{\Sigma_{d-p-1}} \star J_{p+1}\right) \qquad , g = \exp(i\lambda) \tag{2.104}$$

2.4.2 Action of Topological Operators of p-Form Symmetry:

As proven before in [Figure 2.4], 1 form symmetry groups are abelian. Similarly we can use topological deformations to exchange the configuration of 2 p-form symmetry topological operators since there exists a locally transverse plane.
p-form symmetries act on k-dimensional operators where, $k \geq p$. [7].
Lets first discuss the action of p-form symmetries on p-dimensional operators.
Let us consider a $p \geq 1$-dimensional operator $\mathcal{W}(\mathcal{M}_p)$ associated to a submanifold $\mathcal{M}_p$ of spacetime. Let $\mathcal{W}(\mathcal{M}_p)$ be a **simple/irreducible** [j] operator. Toplogically deforming $U_g(\Sigma_{d-p-1})$ such that it crosses the operator $\mathcal{W}(\mathcal{M}_p)$ gives us a local operator $\mathcal{W}(x)$ at the point of intersection x of $\mathcal{M}_p$ and Σ_{d-p}. Thus we can write the action as [7]:

$$U_g(\Sigma_{d-p-1})\mathcal{W}(\mathcal{M}_p)) = \mathcal{W}(x)\mathcal{W}(\mathcal{M}_p)U_g(\Sigma'_{d-p-1}) \tag{2.105}$$

These $\mathcal{W}(x)$ are actually the charges of the p-dimensional operators and they are actually numbers such that:

$$\mathcal{W}(x) \leftrightarrow \phi(g) \qquad\qquad , \phi(g) \in \mathbb{C} - \{0\} \tag{2.106}$$

Thus we can rewrite the action as:

$$U_g(\Sigma_{d-p-1})\mathcal{W}(\mathcal{M}_p) = \phi(g) \times \mathcal{W}(\mathcal{M}_p)U_g(\Sigma'_{d-p-1}) \tag{2.107}$$

Since $G^{(p)}$ is an abelian group, it's 1-dimensional representations are it's irreducible representations and due to the fusion rule (2.102) we have:

$$\phi(g_1)\phi(g_2) = \phi(g_1 g_2) \tag{2.108}$$

which implies that the charges $\phi(g)$ form a 1-dimensional irreducible representation of $G^{(p)}$.
We can also define the action by associating the topological operator to a $(d-p-1)$-dimensional sphere S^{d-p-1} and then deforming it to a point so that it crosses $\mathcal{W}(\mathcal{M}_p)$.

$$U_g(S^{d-p-1})\mathcal{W}(\mathcal{M}_p) = \phi(g) \times \mathcal{W}(\mathcal{M}_p) \tag{2.109}$$

Lets look at a special case where $G^{(p)} \cong U(1)$ where $\phi(g) = q$ where q are just numbers and $\mathcal{W}_q(\Gamma_p)$ is a p-dimensional operator with charge q defined on the curve Γ_p.

[j] $\mathcal{W}(\mathcal{M}_p)$ cannot be expressed as a sum of other p-dimensional operators

We express the ward identity in this case as [6]:

$$d \star J_{p+1}(x)\mathcal{W}_q(\mathcal{M}_p) = q\delta^{(d-p)}(x \in \mathcal{M}_p)\mathcal{W}_q(\mathcal{M}_p) \tag{2.110}$$

where $\delta^{(d-p)}$ is the Poincaré dual $(d-p)$ form associated to the delta function on $\mathcal{M}_p$.
Now lets assume a Σ_{d-p} such that it links to $\mathcal{M}_p$ and another Σ'_{d-p} which is homotopic to Σ_{d-p} and does not link to $\mathcal{M}_p$, then we can define the action of the p-form symmetry operator in an analogous way as done previously for 1-form symmetry operator as:

$$\langle U_g(\Sigma_{d-p-1})\mathcal{W}_q(\mathcal{M}_p)\rangle = e^{i\alpha q \mathrm{Link}(\Sigma_{d-p-1},\mathcal{M}_p)}\langle \mathcal{W}_q(\mathcal{M}_p)U_g(\Sigma'_{d-p-1})\rangle \tag{2.111}$$

where $g = \exp(i\alpha)$.

Remark : Characters and Pontryagin Dual Group

Consider the abelian group $G^{(p)}$ and a torus $\mathbb{T}$ which is just a unit circle in $\mathbb{C}$.
The *character* χ is defined as the homomorphism $\chi : G^{(p)} \to \mathbb{T}$ [12] [13].
The group formed by such homomorphisms is called the **Pontryagin Dual Group** $\hat{G}^{(p)}$ of $G^{(p)}$.
Now for our context, the iireducible representations (charges) of $G^{(p)}$ are homomorphisms :

$$\phi : G^{(p)} \to U(1) \tag{2.112}$$

Thus, using above terminology, the charges of a simple p-dimensional operator under a p-form symmetry group $G^{(p)}$ are elements of the Pontryagin Dual Group $\hat{G}^{(p)}$.

Now lets understand the action of p-form symmetries on k-dimensional operators with $k > p$.
We are going to briefly discuss this using insights from [14] and [15]. k-dimensional operators transform in k-**charges** under $G^{(p)}$. We define k-charges such that:

$$k - \text{charges of } G^{(p)} = (k+1) \text{ representations of the (p+1) group } \mathbb{G}^{p+1}_{G^{(p)}} \tag{2.113}$$

where $\mathbb{G}^{p+1}_{G^{(p)}}$ is a higher group. We are going to talk about higher groups in a later chapter but for now just remember that they are 'categorical' generalisations of ordinary groups. For a more detailed undertstanding look at [14] and [15] for higher representations of invertible symmetries and for higher representations of non-invertible symmetries look at [16, 17, 18, 19, 20].

2.4.3 Gauging p-Form Symmetries

In the case of p-form symmetries, we couple a $(p+1)$-form background gauge field $B_{(p+1)}$ to our $(p+1)$-form conserved current $\star J_{(p+1)}$ and add the following term to our action:

$$S_{\text{gauge}} = +i \int B_{(p+1)} \wedge \star J_{(p+1)} \tag{2.114}$$

where $B_{(p+1)}$ has the following gauge transformation:

$$B_{(p+1)} \to B_{(p+1)} + d\Lambda_{(p)} \tag{2.115}$$

Under the above mentioned gauge transform, (2.114) varies as follows:

$$\delta S_{\text{gauge}} = +i \int d\Lambda_{(p)} \wedge \star J_{(p+1)} = +i \int \Lambda_{(p)} \wedge d \star J_{(p+1)} = 0 \tag{2.116}$$

which is true due to the conservation law (2.103) which implies that $\star J_{(p+1)}$ is a closed $(p+1)$-form, which hence implies that the action is invariant under the background gauge transformations.

Chapter 3

Anomalies

Anomalies constitute a pivotal aspect in the understanding of quantum field theories They offer profound insights into the dynamics of gauge theories. Understanding these anomalies, particularly how they cancel, serves as a key to unlocking a more nuanced understanding of the underlying dynamics of a gauge theory.

In this chapter, we embark on an exploration of anomalies, encompassing an introductory overview and the classification of different anomaly types. Subsequently, we delve into 't Hooft anomalies, and understanding them within the context of higher form symmetries. In the last section of this chapter, we delve into the crucial concept of anomaly polynomials and the notion of anomaly inflow, providing us with a powerful toolkit for the determination of anomalies. This toolkit will serve as a vital bridge between the concepts established in our earlier chapter on higher form symmetries and our forthcoming exploration of higher groups in the next chapter.

3.1 Introduction to Anomalies

Anomalies are critical to understand Quantum field theories. They generally arise when symmetry of the Classical Field theories the are invalid in the Quantum Field Theory.
As we mentioned before in the first section, symmetries in the path integral formalism, they are expressed as Ward identities of the correlation functions [21]. If the integral measure of the partition function is not invaraint under the action of symmetries then our theory has an anomaly. This method of finding anomalies in the measure is called the Fujikawa method [a](refer [22] for details). Another way an anomaly arises is while renormalization, no regularization scheme preserves all the symmetries.
The most effective and equivalent way to know if there exists an anomaly on our theory is by checking if the classically conserved current is still conserved in Quantum Field Theory, if it isn't then we have an anomaly. Anomalies are computed using one loop Feynman diagrams called *'triangle diagrams'*. Evaluating triangle diagrams and computing anomalies using them is introduced and explained in detail in [23].

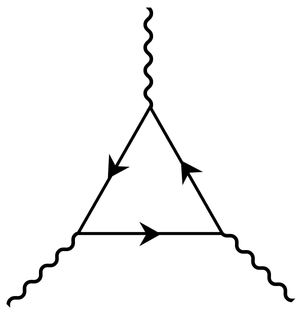

Figure 3.1: This diagram illustrates the one loop Feynmann diagram which calculates the anomaly. The wiggly lines represent gauge fields and each vertex we assign a current. The arrows represent the running of fermions along the triangle loop.

If we have a theory on a manifold $\mathcal{M}_d$ with a partition function $\mathcal{Z}[\{A\}]$ where $\{A\}$ is a set of background gauge fields in our theory with the corresponding transformations:

$$\{A\} \to \{A\} + d\{\lambda\}$$

then we after gauging the partition function we can write the anomaly $\mathcal{A}_d$ as:

$$\mathcal{Z}[\{A\} + d\{\lambda\}] = e^{i\int_{\mathcal{M}_d} \mathcal{A}_{(d)}[\{A\},\{\lambda\}]}\mathcal{Z}[\{A\}] \tag{3.1}$$

[a]We will be using the Fujikawa method to calculate the anomaly of a QED like theory later on in the dissertation.

Different types of anomalies

1. **Chiral anomaly:**
   Also known as *ABJ anomaly* after Adler [24], Bell and Jackiw [25] discovered it in 1969. These kind of anomalies exist when the symmetry of the classical field theory doesn't translate to being 4 symmetry of the Quantum Field Theory. In 4d gauge theories with continuous gauge groups, they are classified by placing currents due to gauge symmetries on 2 vertices and the current due to 0-form global symmetry on the 3rd vertex of the triangle diagram. The ABJ anomaly is an anomaly in the global symmetry.

2. **Gauge anomalies:**
   Gauge symmetry isn't a true symmetry of the theory but a redundancy in our description of the Quantum Field Theory. Anomaly in the gauge symmetry is terrible as it removes then redundancy thus giving us an inconsistent theory. These kind of anomalies exist when the partition function isn't invariant under the gauge fields. For 4d gauge theories with continuous gauge groups we can compute the anomalies using triangle diagrams with currents due to gauge groups on all 3 vertices.

3. **'t Hooft anomalies:**
   These are the anomalies we will be interested in. Consider an anomaly free theory with a non-anomalous gauge group, if we attempt to gauge the symmetry , it turns out it has an quantum anomaly. Such anomalies are called 't Hooft anomalies. They were first discovered by Prof.Gerard 't Hooft in 1980 in [26]. They are famously described as obstruction to gauging a global symmetry. In 4d gauge theories , 't Hooft anomalies are computed using triangle diagrams with currents due to global symmetry on all 3 vertices. We will explain them in detail in the next section

For a more detailed discussion on anomalies and their origin and how they behave we refer the reader to [21, 23, 27, 3, 28].

3.2 't Hooft Anomalies

Lets delve deeper into t'Hooft anomalies. These kind of anomalies are particulary interesting as they can be tracked along the renormalization group(RG) flow [29]. This means that along flowing from UV to IR phase of the theory, these anomalies stay invariant even if the fields in the Lagrangian of our theory change, for example: In quantum chromodynamics at high energies(UV) has quarks which are confined to bound states (baryons and mesons) at low energies (IR).

As suggested by 't Hooft, one of the ways to see the anomaly matching and cancellation is to add decoupled fermions called *spectator fermions* that transform under the symmetry group of the theory to our quantum theory so that it cancels the anomaly. [6] Consider a theory with a 't Hooft anomaly in the UV - $\mathcal{A}_{UV}$, to which we add the anomaly due to the *spectator fermions* $\mathcal{A}_{SF}$ such that:

$$\mathcal{A}_{UV} + \mathcal{A}_{SF} = 0 \tag{3.2}$$

If the anomaly persists in the IR phase of the theory, the *spectator fermions* (which mostly will be different than ones in UV) also provide an anomaly contribution here such that:

$$\mathcal{A}_{IR} + \mathcal{A}_{SF} = 0 \tag{3.3}$$

which hence implies:

$$\mathcal{A}_{IR} = \mathcal{A}_{UV} \tag{3.4}$$

3.3 't Hooft Anomalies for Higher Form Symmetries

We can generalise the discussion of 't Hooft anomalies to higher form symmetries. Consider a p-form symmetry with a background gauge field $B_{(p+1)}$ coupled to the $(p+1)$ form symmetry current with the gauge transformation (2.115).
There are two types of t'Hooft anomlies [7]:

1. **Pure 't Hooft Anomalies:** A pure 't Hooft anomaly $\mathcal{A}_{(d)}[B_{(p+1)}, \Lambda_{(p)}]$ is of the form:

$$\mathcal{Z}[B_{(p+1)} + d\Lambda_{(p)}] = e^{i\int_{\mathcal{M}_d} \mathcal{A}_{(d)}[B_{(p+1)}, \Lambda_{(p)}]} \mathcal{Z}[B_{(p+1)}] \tag{3.5}$$

   No counter terms fix this anomaly.

2. **Mixed 't Hooft Anomalies:** We have theories where in we have a p-form and a q-form symmetry with corresponding background gauge fields $B_{(p+1)}$, $B_{(q+1)}$ and their respective transformations:

$$\begin{aligned} B_{(p+1)} &\to B_{(p+1)} + d\Lambda_{(p)} \\ B_{(q+1)} &\to B_{(q+1)} + d\Lambda_{(q)} \end{aligned}$$

   It is possible that the partition function under gauging one of the symmetries remains invariant but under simultaneous gauge transformations of both the higher form symmetries, it generates an anomalous phase which cannot be fixed by adding more counter terms.

$$\mathcal{Z}[B_{(p+1)} + d\Lambda_{(p)}, B_{(q+1)} + d\Lambda_{(q)}] = e^{i\int_{\mathcal{M}_d} \mathcal{A}_{(d)}[B_{(p+1)}, B_{(q+1)}, \Lambda_{(p)}, \Lambda_{(q)}]} \mathcal{Z}[B_{(p+1)}, B_{(q+1)}] \tag{3.6}$$

   We have seen an example of a mixed 't Hooft anomaly in 4d Maxwell Theory in the section Gauging the Higher Form Symmetries of 4d Maxwell. We will now discuss the idea of anomaly inflow action so that we can clearly understand the fascinating thing we did earlier in that section.

3.4 Anomaly Polynomials and Inflow

In recent times, anomalies have been viewed in the perspective of topology. A neat way of calculating anomalies is using the idea of Anomaly Polynomial and descent equations [27, 30, 31]. An anomaly polynomial is a $(d+2)$-form, $\mathcal{I}_{(d+2)}[B_{(p+1)}]$ that characterizes the anomaly of the theory [b]. It is made up of gauge invariant operators, (as we will explore later) and hence is itself gauge invariant. Lets now understand how it produces the anomaly of our quantum field theory in d-dimensions [c]. In this section we will be using insights from [6, 27, 32].

Consider the anomaly of the theory (boundary theory) written in the form:

$$\mathcal{Z}[B_{(p+1)} + d\Lambda_{(p)}] = e^{i\int_{\mathcal{M}_d} \mathcal{A}_{(d)}[B_{(p+1)},\Lambda_{(p)}]} \mathcal{Z}[B_{(p+1)}] \tag{3.7}$$

Let $S_{\text{anom}} = \int_{\mathcal{M}_d} \mathcal{A}_d[B_{(p+1)}, \lambda_{(p)}]$.

We can write the exterior derivative (total derivative) of $\mathcal{A}_{(d)}[B_{(p+1)}, \lambda_{(p)}]$ as the variation of a $(d+1)$-form dependent on $B_{(p+1)}$ (2.115) [d]:

$$d\mathcal{A}_d[B_{(p+1)}, \Lambda_{(p)}] = 2\pi\delta_{\Lambda_{(p)}}\mathcal{I}_{(d+1)}[B_{(p+1)}] \tag{3.8}$$

Consider a manifold $\mathcal{M}_{d+1}$ such that it's boundary is $\mathcal{M}_d$, i.e

$$\partial\mathcal{M}_{d+1} = \mathcal{M}_d \tag{3.9}$$

Now lets define a theory (bulk theory) in $\mathcal{N}_{d+1}$ with the action:

$$S_{\text{inflow}} = -2\pi \int_{\mathcal{M}_{d+1}} \mathcal{I}_{(d+1)}[B_{(p+1)}] \tag{3.10}$$

Now look at the variation of the action of the bulk theory under gauge transformations and using (3.8) and (3.9) we get :

$$\begin{aligned} \delta_{\Lambda_{(p)}} S_{\text{inflow}} &= -2\pi \int_{\mathcal{M}_{d+1}} \delta_{\Lambda_{(p)}}\mathcal{I}_{(d+1)}[B_{(p+1)}] \\ &= -\int_{\mathcal{M}_{d+1}} d\mathcal{A}_d[B_{(p+1)}, \Lambda_{(p)}] \\ &\overset{\text{Stokes' Thm}}{=} -\int_{\mathcal{M}_d} \mathcal{A}_{(d)}[B_{(p+1)}, \Lambda_{(p)}] = -S_{\text{anom}} \end{aligned} \tag{3.11}$$

[b] for simplicity we consider one p-form symmetry with a background gauge field $B_{(p+1)}$ and it transforms like in (2.115).

[c] We will mainly be keeping $d = 4$ in our consideration when writing this section.

[d] The normalization has been assigned for simplicity of computations.

This seems extremely interesting as now if we combine the bulk theory with the boundary theory, we can define an anomaly free partition function :

$$\tilde{\mathcal{Z}}[B_{(p+1)}] := \mathcal{Z}[B_{(p+1)}] \times \exp(iS_{\text{inflow}}) \tag{3.12}$$

$$= \mathcal{Z}[B_{(p+1)}] \times \exp\Big(-2\pi i \int_{\mathcal{M}_{d+1}} \mathcal{I}_{(d+1)}[B_{(p+1)}]\Big) \tag{3.13}$$

Now the S_{inflow} is called the **anomaly inflow action** as the anomaly flows from the bulk to the boundary and gets cancelled.

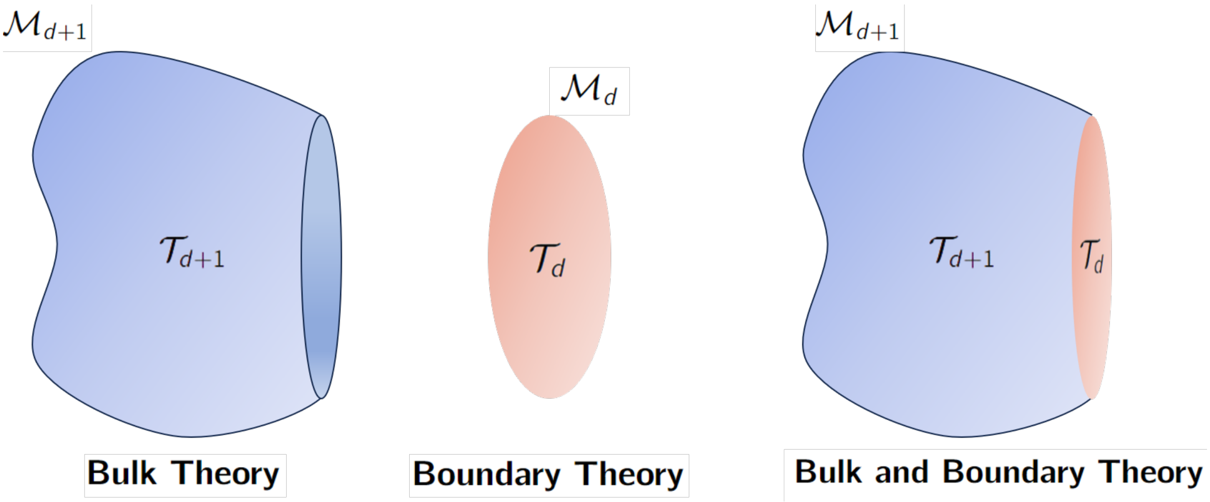

Figure 3.2: *[Left]* This is the theory $\mathcal{T}_{d+1}$ which has the action S_{inflow}; *[Center]* This is the boundary theory $\mathcal{T}_d$ with the anomaly $\mathcal{A}_d[B_{(p+1)}, \lambda_{(p)}]$; *[Right]* This is where we have combined the bulk and boundary theory to get a non anomalous theory with the partition function $\tilde{\mathcal{Z}}[B_{(p+1)}]$ as in (3.12)

We define the **anomaly polynomial** as the exterior derivative of the inflow Lagrangian $\mathcal{I}_{(d+1)}[B_{(p+1)}]$:

$$\mathcal{I}_{(d+2)}[B_{(p+1)}] = d\mathcal{I}_{(d+1)}[B_{(p+1)}] \tag{3.14}$$

Now that we understand the above procedure of deriving the anomaly polynomial, we can use it to calculate the anomaly of our theory using the descent equations [33, 30, 27] :

$$\begin{aligned} \mathcal{I}_{(d+2)}[B_{(p+1)}] &= d\mathcal{I}_{(d+1)}[B_{(p+1)}] \\ 2\pi\delta_{\Lambda_{(p)}}\mathcal{I}_{(d+1)}[B_{(p+1)}] &= d\mathcal{A}_d[B_{(p+1)}, \Lambda_{(p)}] \\ \int_{\mathcal{M}_d} \mathcal{A}_d[B_{(p+1)}, \Lambda_{(p)}] &= S_{\text{anom}} \end{aligned} \tag{3.15}$$

Remark : SPT Phases

In the literature on condensed matter physics, the inflow action is said to describe an invertible topological quantum field theory often referred as *Symmetry Protected Topological Phases* or *SPT Phases*. This is explored in great detail in [34]
We define a d-dimensional SPT phase protected by a set of $p \geq 0$-form symmetries [7]

$$\mathbb{S} = \{G^{(p_1)}, G^{(p_2)}, G^{(p_3)}, ..., G^{(p_n)}\},$$

as an invertible d-dimensional *invertible* topological quantum field theory admitting the symmetries $\mathbb{S}$, such that the partition functions are trivial when all the background fields are turned off

$$\mathcal{Z}[B_{(p_1+1)} = 0, B_{(p_2+1)} = 0, ..., B_{(p_n+1)} = 0] = 1 \tag{3.16}$$

and for general background gauge fields the partition functions are phase factors

$$\mathcal{Z}[B_{(p_1+1)}, B_{(p_2+1)}, ..., B_{(p_n+1)}] \in U(1) \tag{3.17}$$

For a more detailed understanding of the anomaly inflow, polynomials, and SPT phases in terms of categories and cohomology please refer [35, 36, 37, 38, 39].

So now that we understand how to calculate anomalies using anomaly polynomials, we can ask how do we actually construct or build the anomaly polynomial of a theory.

Construction of Anomaly Polynomials

We know that the anomaly polynomial of a theory has to be gauge invariant or it completely defeats the purpose of this discussion. We can construct them using gauge invariant polynomials of the theory. We will use insights from [32] to construct anomaly polynomials. Lets look at different types of symmetries our theory can have and what these symmetries contribute to the anomaly polynomial. Lets consider $d = 4$ dimensional spacetime.

1. For Abelian 0-form symmetries like $U(1)^{(0)}$, the field strength(curvature) itself is a gauge invariant quantity and hence we can have a contribution consisting wedge

product of terms like:

$$\kappa_A \frac{1}{2\pi} F_{(2)} \qquad , F_{(2)} = dA_{(1)} \tag{3.18}$$

where $A_{(1)}$ is the background field [e] and κ_A is a constant [f]. The contribution of an abelian 0-form symmetry can be a mixed anomaly with the of the symmetries $U(1)^{(0)}_A - U(1)^{(0)}_B - U(1)^{(0)}_C$, where $A_{(1)}, B_{(1)}, C_{(1)}$ are respective background gauge fields of the above symmetries. The contribution to $\mathcal{I}_6$ using (3.18) can be expressed as:

$$\frac{1}{(2\pi)^3} \sum_{A,B,C} \frac{\kappa_{ABC}}{\text{Symmetry Factor}} F^A_{(2)} \wedge F^B_{(2)} \wedge F^C_{(2)} \tag{3.19}$$

Here κ_{ABC} can actually be evaluated using the triangle diagrams or Fujikawa's method of derivation of an anomaly. We will see an example in the next chapter. The indices A, B, C can be equal to each other giving us anomalies like the ABJ anomaly and the symmetry factors are calculated based on the which symmetries are being used to calculate the anomaly.

2. For non abelian 0-form symmetries [32], like $SU(N)_C$ with field strength $F_{(2)} = dC_{(1)}$, the contribution consists of terms like [g]:

$$\frac{1}{(2\pi)^r} tr\Big((F_{(2)})^r\Big) \qquad , r \geq 2 \tag{3.20}$$

3. For higher form symmetries, like $U(1)^{(p)}_{B^i}$ the contribution to $\mathcal{I}_{(6)}$ is using terms made up of wedge products of the field strengths of the symmetries in our theory

$$dB^1_{(p+1)} \wedge \cdots \wedge dB^n_{(p+1)} \tag{3.21}$$

where $B^i_{(p+1)}$ is the background gauge field associated with a particular symmetry.

Lets now understand an important concept of *reducible anomaly polynomials* [32]. We will see how we have already used this concept when we fixed the mixed 't Hooft anomaly of 4d-Maxwell Theory previously.

[e] the normalization of the field strength actually comes from the fact that this is the first *chern class*, $c_1(F_{(2)}) = \frac{1}{2\pi} F_{(2)}$. To understand this in detail please refer chapter 11 of [9]

[f] more specifically it's the anomaly constant and we will discuss more on it in the next chapter

[g] They actually contribute by chern classes $c_r(F_{(2)}) = \frac{1}{(2\pi)^k} tr\Big((F_{(2)})^r\Big)$

Statement 3.4.1

An anomaly polynomial $\mathcal{I}_{(d+2)}$ is called a reducible anomaly polynomial if it can be factorized or is the wedge product of gauge invariant, closed anomaly polynomials $\mathcal{U}_{(q)}$ and $\mathcal{V}_{(d+2-q)}$, i.e

$$\mathcal{I}_{(d+2)} = \mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)} \tag{3.22}$$

and

$$d\mathcal{U}_{(q)} = d\mathcal{V}_{(d+2-q)} = 0 \tag{3.23}$$

Now to calculate the anomaly inflow and hence the anomaly we use the descent equations (3.15). If we have a reducible anomaly polynomial we can take an exterior derivative out of either $\mathcal{U}_{(q)}$ or $\mathcal{V}_{(d+2-q)}$. Let us write

$$\mathcal{I}_{(d+2)} = d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) \tag{3.24}$$

Using the Leibnitz property of the exterior derivative operator [h] and (3.23), we can write:

$$d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) = d\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)} + (-1)^{(q-1)}\mathcal{U}_{(q-1)} \wedge d\mathcal{V}_{(d+2-q)} = \mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)} \tag{3.26}$$

which looks exactly like $\mathcal{I}_{(d+2)}$, but we could have removed a degree of power from $\mathcal{V}_{(d+2-p)}$ too. Due to this confusion, we can introduce a real constant term $s \in \mathbb{R}$ in $\mathcal{I}_{(d+2)}$ in the following way [32]:

$$\begin{aligned}\mathcal{I}_{(d+2)} &= \mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)}) + s(-1)^q\Big(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)}\Big) - s(-1)^q\Big(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)}\Big) \\ &= \mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)} + s(-1)^q\Big(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)}\Big) + s(-1)^{(q-1)}\Big(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)}\Big)\end{aligned} \tag{3.27}$$

[h] d-operator obeys the Leibnitz property. If we consider $\omega \in \Omega^r$ and $\xi \in \Omega^s$, where Ω^r and Ω^s is the vector space of r and s forms on our manifold, then [9]:

$$d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^r \omega \wedge d\xi \tag{3.25}$$

Now using (3.26) and (3.25) we can write:

$$\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)} = d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) \tag{3.28}$$

and

$$(-1)^q \mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+2-q)} = d(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+1-q)}) \tag{3.29}$$

Now putting (3.28) and (3.29) in (3.27):

$$\begin{aligned}\mathcal{I}_{(d+2)} &= d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) + sd(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+1-q)}) + s(-1)^{(q-1)} d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) \\ &= d\Big((\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) + s(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+1-q)}) + s(-1)^{(q-1)}(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)})\Big)\end{aligned} \tag{3.30}$$

Thus using (3.15), we can write:

$$\mathcal{I}_{(d+1)} = \Big((\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) + s(\mathcal{U}_{(q)} \wedge \mathcal{V}_{(d+1-q)}) + s(-1)^{(q-1)}(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)})\Big) \tag{3.31}$$

Now using (3.25), we can write:

$$d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+1-q)}) = \mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)} + (-1)^{(q-1)} \mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)} \tag{3.32}$$

Now putting (3.32) in (3.31), we can write the inflow Lagrangian (up to normalization constant 2π) as:

$$\mathcal{I}_{(d+1)} = (\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+2-q)}) + sd(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+1-q)}) \tag{3.33}$$

Now as we can see above in $\mathcal{I}_{(d+1)}$, the real constant s is multiplied to an exact term. When we write the inflow action we can interpret this term as a counterterm in 4 dimensions, of our original action of the theory which can be used to modify, cancel, or absorb any anomaly which can be eliminated using counter terms. [32].

$$S_{ct} = is \int_{\mathcal{M}_{d+1}} d(\mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+1-q)}) = is \int_{\mathcal{M}_d} \mathcal{U}_{(q-1)} \wedge \mathcal{V}_{(d+1-q)} \tag{3.34}$$

where we used Stokes' Theorem, and the fact that $\partial \mathcal{M}_{d+1} = \mathcal{M}_d$.
Lets consider the previous example of $4d$-Maxwell with $U(1)^{(1)}_{\text{elc.}} \times U(1)^{(1)}_{\text{mag.}}$ symmetry and coupling the background fields $B^e_{(2)}$ and $B^m_{(2)}$ with the corresponding gauge transformations

as in (2.86) and (2.87). The action of Maxwell theory can be written as:

$$S = \frac{1}{2e^2} \int_{\mathcal{M}_4} F_{(2)} \wedge \star F_{(2)} \tag{3.35}$$

and under gauging both $U(1)^{(1)}_{\text{elc.}} \times U(1)^{(1)}_{\text{mag.}}$ simultaneously we need to couple the background fields to the action adding the terms (2.89). Following the discussion above, we can write the anomaly polynomial:

$$\mathcal{I}_{(6)}[B^e_{(2)}, B^m_{(2)}] = \kappa_{B^e_{(2)} B^m_{(2)}} dB^e_{(2)} \wedge dB^m_{(2)} \tag{3.36}$$

where $\kappa_{B^e_{(2)} B^m_{(2)}} dB^e_{(2)}$ is a real constant. Now this is a reducible anomaly polynomial and hence we can write the anomaly inflow (up to the normalization 2π) using (3.33) as:

$$\mathcal{I}_{(5)}[B^e_{(2)}, B^m_{(2)}] = \kappa_{B^e_{(2)} B^m_{(2)}} (B^e_{(2)} \wedge dB^m_{(2)}) + sd(B^e_{(2)} \wedge B^m_{(2)}) \tag{3.37}$$

Now using (3.34) the counter term action looks like:

$$S_{ct} = is \int_{\mathcal{M}_4} B^e_{(2)} \wedge B^m_{(2)} \tag{3.38}$$

Now choosing $s = \frac{1}{2\pi}$, we can see that:

$$S_{ct} = \frac{i}{2\pi} \int_{\mathcal{M}_4} B^e_{(2)} \wedge B^m_{(2)} = -\frac{i}{2\pi} \int_{\mathcal{M}_4} B^m_{(2)} \wedge B^e_{(2)} = S'_{ct} \tag{3.39}$$

where S'_{ct} is exactly the counter term (2.94) we added earlier in the Maxwell action to make the electric symmetry non anomalous but this left the magnetic symmetry anomalous as we saw in the previous section.

Now lets make a choice of $s = 0$, such that now write the anomaly inflow as:

$$2\pi \mathcal{I}_{(5)}[B^e_{(2)}, B^m_{(2)}] = 2\pi \kappa_{B^e_{(2)} B^m_{(2)}} \left(B^e_{(2)} \wedge dB^m_{(2)} \right) \tag{3.40}$$

Now fix $\kappa_{B^e_{(2)} B^m_{(2)}} = \frac{1}{4\pi^2}$ and we write the anomaly inflow action as:

$$S_{\text{inflow}} = -\frac{i}{2\pi} \left(B^e_{(2)} \wedge dB^m_{(2)} \right) \tag{3.41}$$

which is exactly the same as the S_{inflow} action we defined earlier (2.97) in Maxwell theory. Now we can see that earlier we just followed the procedure above by actually combining the bulk theory in 5-dimensions with the action:

$$S_{\text{bulk}} = S_{\text{inflow}} = -\frac{i}{2\pi}\int_{\mathcal{M}_5}\left(B^e_{(2)} \wedge dB^m_{(2)}\right) \tag{3.42}$$

to the boundary theory in 4-dimensions with the action [i]:

$$S_{\text{boundary}} = \int_{\mathcal{M}_4} \frac{1}{2e^2}(F_{(2)} - B^e_{(2)}) \wedge \star(F_{(2)} - B^e_{(2)}) + \frac{i}{2\pi}(B^m_{(2)} \wedge F_{(2)}) \tag{3.43}$$

to have a non anomalous theory with the action:

$$\mathcal{S} = \int_{\mathcal{M}_4} \frac{1}{2e^2}(F_{(2)} - B^e_{(2)}) \wedge \star(F_{(2)} - B^e_{(2)}) + \frac{i}{2\pi}(B^m_{(2)} \wedge F_{(2)}) - \int_{\mathcal{M}_5}\frac{i}{2\pi}(B^e_{(2)} \wedge dB^m_{(2)}) \tag{3.44}$$

where $\partial\mathcal{M}_5 = \mathcal{M}_4$

[i] here we have also added a 'seagull' term $\frac{1}{2e^2}\int_{\mathcal{M}_4} B^e_{(2)} \wedge \star B^e_{(2)}$ to make the kinetic terms of the field strength invariant under background gauge transformations

Chapter 4

Higher Groups

The concept of higher groups emerges from the framework of *n-categories* within category theory in mathematics. Credit for its inception is rightly attributed to Hoàng Xuân Sính, who discovered the categorical structure of 2-groups in her groundbreaking Ph.D. thesis on *Gr-Category* [40]. In the historical context of this thesis by John C. Baez [41] he writes,

> "The story of Hoàng Xuân Sính is remarkable because it combines dramatic historical events with revolutionary mathematics. Some mathematicians make exciting discoveries while living peaceful lives. Many have their work disrupted or prematurely cut off by wars and revolutions. But some manage to carry out profound research on the fiery background of history. In war-torn Hanoi, Hoàng Xuân Sính met the visionary mathematician Alexander Grothendieck, who had visited to give a series of lectures—in part as a protest against American aggression. After he returned to France, she did her thesis with him by correspondence, writing it by hand under the light of a kerosene lamp as the bombing of Hanoi reached its peak. In her thesis she established the most fundamental properties of a novel mathematical structure that takes the concept of symmetry and pushes it to new heights, making precise the concept of *symmetries of symmetries*."

In the context of hep-th literature, 2-groups have been explored previously in physics [42, 43] etc. In this chapter, our aim is to delve into the emergence of such structures within the framework of higher form symmetries. We begin our exploration by introducing the concept of higher groups. Subsequently, we dive into the specifics of continuous 2-groups, using the insights gained from our earlier discussion on anomaly polynomials and inflow. We will explore abelian continuous 2-groups in detail, followed by a brief look at their non-abelian counterparts. Finally, we wrap up our exploration by showcasing how 2-group symmetry can be applied in a QED like model to enhance our understanding of continuous 2-groups and their relevance in the context of higher form symmetries.

4.1 Introduction to Higher Group Symmetries

Statement 4.1.1

When a quantum field theory has multiple higher form symmetries, $G^{(0)}, G^{(1)}, \cdots, G^{(p)}$, they can mix together in some fashion and we can have a theory with a higher group symmetry $\mathbb{G}^{(p+1)}$.

They are studied using **category theory** as these higher groups form the structure of an *n-category*. A nice introduction to see this can be found in [35] and gauge theory in terms of differential geometry and category theory is explored in detail in [42, 44].
We will understand higher group symmetries using background field transformations [6]. Lets consider a theory with a set of higher form symmetries $\{G^{(p)}\}$ with their corresponding background fields $\{A_{(p+1)}\}$ which transform as:

$$\{A_{(p+1)}\} \rightarrow \{A_{(p+1)}\} + d\{\Lambda_p\} \tag{4.1}$$

If the partition function of the theory is invariant under the above background field transformations then we say that we have a theory with the product of higher form symmetries. Although if our theory is not invariant under the above background field transformations but rather under transformations of the form:

$$\{A_{(p+1)}\} \rightarrow \{A_{(p+1)}\} + d\{\Lambda_p\} + \sum_{q\leq p} \Lambda_q \wedge \zeta_q(\{A_{(q)}\}) + \{\text{Schwinger Terms}\} \tag{4.2}$$

then we say that our theory has a higher group symmetry $\{G^{(n+1)}\}$, where n is the rank of the *highest* higher form symmetry in our theory.
Here $\zeta_q(\{A_{(q)}\})$ is a $(p-q+1)$-form which is dependent of the background fields $\{A_{(q)}\}$ for $q < p$ and $\{\text{Schwinger Terms}\}$ terms which are non linear under the background field transformations and depend on $\{A_{(q)}, \Lambda_q\}$ for $q < p$.

In this dissertation we are gong to understand continuous higher groups using the example of 2-groups.

4.2 Continuous 2-Groups

2-Group symmetry exists when a 0-form symmetry group $G^{(0)}$ mixes with a 1-form symmetry group $G^{(1)}$. We will try to undertand how the mixing of these groups take place and hence understand what kind of theories can have a 2-group like structure. We will mainly talk about the case where both $G^{(0)}$ and $G^{(1)}$ are continuous. The concept of 't Hooft anomalies, anomaly polynomail and inflow will play a central role in undertanding how these groups mix.

Lets consider the simplest case where a continuous abelian 0-form symmetry $U(1)_A^{(0)}$ and a continuous 1-form symmetry $U(1)_B^{(1)}$ combine to form a continuous 2-group symmetry $\mathbb{G}^{(2)}$. Let $U(1)_A^{(0)}$ have a corresponding background field $A_{(1)}$ and $U(1)_B^{(1)}$ have a corresponding background field $B_{(2)}$ with the following transformations [32]:

$$A_{(1)} \to A_{(1)} + d\lambda^A_{(0)} \tag{4.3}$$

$$B_{(2)} \to B_{(2)} + d\Lambda^B_{(1)} + \frac{\kappa_A}{2\pi}\lambda^A_{(0)} F^A_{(2)} \tag{4.4}$$

where κ_A are real constants which are quantized, i.e $\kappa_A \in \mathbb{Z}$ [Proof 4.2.1]. The transformation of the background field $B_{(2)}$ seems unconventional. This is because we want the 0-form and 1-form symmetry groups to mix and hence we write the transformation of $B_{(2)}$ following (4.2) and choosing the normalization for our convenience.

We can define the field strength(curvature) associated with the background gauge field $A_{(1)}$ as:

$$F^A_{(2)} = dA_{(1)} \tag{4.5}$$

We will define the field strength(curvature) of the background field $B_{(2)}$ in an unconventional manner below after some discussion.

Proof 4.2.1

We know that $A_{(1)}$ and $B_{(2)}$ are standard $U(1)$ connections and hence their gauge parameters $\lambda^A_{(0)}$ and $\Lambda^B_{(1)}$ are quantized and are periodic over closed 1-cycle Σ_1 and 2-cycle Σ_2 respectively. This leads to ambiguities :

$$\lambda^A_{(0)} \sim \lambda^A_{(0)} + 2\pi \qquad\qquad \Lambda^B_{(1)} \sim \Lambda^B_{(1)} + 2\pi \tag{4.6}$$

Due to this, the curvature $F^A_{(2)}$ has the quantization condition:

$$\int_{\Sigma_2} F^A_{(2)} = 2\pi\mathbb{Z} \tag{4.7}$$

which has a similar reasoning as seen previously in (2.58).
These ambiguities can be absorbed in the background transformation (4.3) for the background field $A_{(1)}$ but since $B_{(2)}$ depends on $\lambda^A_{(0)}$, the ambiguity of $\lambda^A_{(0)}$ adds an ambiguity in $B_{(2)}$:

$$B_{(2)} \xrightarrow{\lambda^A_{(0)}\sim\lambda^A_{(0)}+2\pi} B_{(2)} + \kappa_A F^A_{(2)} \tag{4.8}$$

Now from (4.6) and (4.7) we know that $\Lambda^B_{(1)}$ and $F^A_{(2)}$ satisfy the same quantization condition, so to absorb the above ambiguity in (4.4) we need the quantization condition :

$$\kappa_A \in \mathbb{Z} \tag{4.9}$$

■

We can now write a statement:

Statement 4.2.1

A quantum field theory which can be coupled to 2-form background gauge field with a background transformation involving terms dependent on the 1-form background gauge field and it's gauge parameter of the 0-form symmetry of the theory, is said to have a 2-group symmetry.

Now lets define the field strength(curvature) of the background field $B_{(2)}$:

$$H^B_{(3)} = dB_{(2)} - \frac{\kappa_A}{2\pi} A_{(1)} \wedge F^A_{(2)} \tag{4.10}$$

It obeys a modified Bianchi Identity:

$$dH^B_{(3)} = d(dB_{(2)} - \frac{\kappa_A}{2\pi} A_{(1)} \wedge F^A_{(2)}) = -\frac{\kappa_A}{2\pi} d(A_{(1)} \wedge F^A_{(2)}) = -\frac{\kappa_A}{2\pi} F^A_{(2)} \wedge F^A_{(2)} \tag{4.11}$$

where we used the fact that d is a nilpotent operator and $F^A_{(2)}$ is closed.
It is quite straightforward to see that $H^B_{(3)}$ is invariant under the background gauge trans-

formation of $B_{(2)}$ (4.4):

$$\begin{aligned} H^B_{(3)} &\mapsto d\Big(B_{(2)} + d\Lambda^B_{(1)} + \frac{\kappa_A}{2\pi}\lambda^A_{(0)}F^A_{(2)}\Big) - \frac{\kappa_A}{2\pi}A_{(1)} \wedge F^A_{(2)} \\ &= dB_{(2)} + d(d\Lambda^B_{(1)}) + \frac{\kappa_A}{2\pi}\lambda^A_{(0)}dF^A_{(2)} - \frac{\kappa_A}{2\pi}A_{(1)} \wedge F^A_{(2)} \\ &= dB_{(2)} - \frac{\kappa_A}{2\pi}A_{(1)} \wedge F^A_{(2)} = H^B_{(3)} \end{aligned} \tag{4.12}$$

where again we used that d is a nilpotent operator and $F^A_{(2)}$ is closed.
The background field transformation of $B_{(2)}$, (4.4) is the same form as the **Green-Schwarz Mechanism** [45] [a].
Thus we can say that the quantum field theory example discussed above has a 2-group symmetry:

$$\mathbb{G}^{(2)} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B \tag{4.13}$$

where κ_A characterizes the mixing of the groups and the 2-group background field transformations of the theory.
If $\kappa_A = 0$, then we have the regular background field transformation:

$$B_{(2)} \to B_{(2)} + d\Lambda^B_{(1)} \tag{4.14}$$

and then the global symmetry of the theory will just be a product of the 2 (or more) symmetries:

$$U(1)^{(0)}_A \times U(1)^{(1)}_B \tag{4.15}$$

4.2.1 Constructing $\mathbb{G}^{(2)} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B$

Now lets understand how using our discussion of 't Hooft anomalies and anomaly polynomials we can deduce what kind of theories have a 2-group structure. We will be constructing the same example of $\mathbb{G}^{(2)} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B$ to build our intuition on this topic.
Lets consider a theory with the ordinary symmetry group:

$$U(1)^{(0)}_A \times U(1)^{(0)}_C \tag{4.16}$$

[a] Green-Schwarz Mechanism was first used to cancel the anomaly with the gauge group $SO(32)$ in Type I String Theory. We will not explore the original Green-Schwarz Mechanism further but use the ideology of it here to understand higher groups, in particular 2-groups. The relation between the background transformations involving the Green-Schwarz mechanism and symmetries was explored in detail in [46].

where $U(1)^{(0)}_A$ and $U(1)^{(0)}_C$ are 0-form symmetries with the corresponding 1-form background fields $A_{(1)}$ and $C_{(1)}$. This kind of theory can have a 2-group structure and we will see how. This theory is usually called a *'parent theory'* [32].
The background gauge transformations of $A_{(1)}$ and $C_{(1)}$ are:

$$A_{(1)} \mapsto A_{(1)} + d\lambda^A_{(0)} \tag{4.17}$$
$$C_{(1)} \mapsto C_{(1)} + d\lambda^C_{(0)} \tag{4.18}$$

and we define the associated field strengths (curvatures) as:

$$dA_{(1)} = F^A_{(2)} \qquad dC_{(1)} = F^C_{(2)} \tag{4.19}$$

We can write the most general anomaly polynomial in 4-dimensions that this theory can have using the discussion we did on Construction of Anomaly Polynomials in the chapter above. For a theory with symmetry group $U(1)^{(0)}_A \times U(1)^{(0)}_C$,

$$\mathcal{I}_6 = \frac{1}{(2\pi)^3} \sum_{I,J,K} \frac{\kappa_{IJK}}{\text{Symmetry Factor}} F^I_{(2)} \wedge F^J_{(2)} \wedge F^K_{(2)} \tag{4.20}$$

where $I, J, K = A, C$. Thus,

$$\begin{aligned} \mathcal{I}_6[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^3}\Big(&\frac{\kappa_{A^3}}{3!} F^A_{(2)} \wedge F^A_{(2)} \wedge F^A_{(2)} + \frac{\kappa_{A^2C}}{2!} F^A_{(2)} \wedge F^A_{(2)} \wedge F^C_{(2)} \\ &+ \frac{\kappa_{AC^2}}{2!} F^A_{(2)} \wedge F^C_{(2)} \wedge F^C_{(2)} + \frac{\kappa_{C^3}}{3!} F^C_{(2)} \wedge F^C_{(2)} \wedge F^C_{(2)} \Big) \end{aligned} \tag{4.21}$$

where $F^C_{(2)} = dC_{(1)}$.
Now lets use descent equations (3.15) to find the anomaly inflow and the anomaly in the parent theory.
Lets take the first term in (4.21) and apply descent equations to it:

$$\mathcal{I}_6[A_{(1)}, C_{(1)}] = d\mathcal{I}_5[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\left(\frac{\kappa_{A^3}}{3!} F^A_{(2)} \wedge F^A_{(2)} \wedge F^A_{(2)} \right) \tag{4.22}$$

Thus,

$$\mathcal{I}_5[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\left(\frac{\kappa_{A^3}}{3!} A_{(1)} \wedge F^A_{(2)} \wedge F^A_{(2)} \right) \tag{4.23}$$

where we used the Leibniz property of differential forms and the fact that $dF^A_{(2)} = 0$,

$$\begin{aligned} d\Big(A_{(1)} \wedge F^A_{(2)} \wedge F^A_{(2)}\Big) &= \Big(dA_{(1)} \wedge F^A_{(2)} \wedge F^A_{(2)}\Big) + (-1)^1\Big(d(F^A_{(2)} \wedge F^A_{(2)})\Big) \\ &= F^A_{(2)} \wedge F^A_{(2)} \wedge F^A_{(2)} \end{aligned} \tag{4.24}$$

Now lets consider the second term in the anomaly polynomial:

$$\mathcal{I}_6[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{A^2C}}{2!}F^A_{(2)} \wedge F^A_{(2)} \wedge F^C_{(2)}\Big) \tag{4.25}$$

Now this is a reducible terms and hence using (3.33) we can write the anomaly inflow contribution of this term:

$$\mathcal{I}_5[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{A^2C}}{2!}A_{(1)} \wedge F^A_{(2)} \wedge F^C_{(2)} + sd(A_{(1)} \wedge F^A_{(2)} \wedge C_{(1)})\Big) \tag{4.26}$$

where we used,

$$F^A_{(2)} \wedge F^C_{(2)} = d\Big(F^A_{(2)} \wedge C_{(1)}\Big) \tag{4.27}$$

and $s \in \mathbb{R}$.
Following the same procedure as above for the third term in the anomaly polynomial,

$$\mathcal{I}_6[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{AC^2}}{2!}F^A_{(2)} \wedge F^C_{(2)} \wedge F^C_{(2)}\Big) \tag{4.28}$$

we can write the anomaly inflow contribution as:

$$\mathcal{I}_5[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{AC^2}}{2!}A_{(1)} \wedge F^C_{(2)} \wedge F^C_{(2)} + td(A_{(1)} \wedge C_{(1)} \wedge F^C_{(2)})\Big) \tag{4.29}$$

where $t \in \mathbb{R}$.
For the final term in $\mathcal{I}_6[A_{(1)}, C_{(1)}]$,

$$\mathcal{I}_6[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{C^3}}{3!}F^C_{(2)} \wedge F^C_{(2)} \wedge F^C_{(2)}\Big) \tag{4.30}$$

we can write:

$$\mathcal{I}_5[A_{(1)}, C_{(1)}] \supset \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{C^3}}{3!}C_{(1)} \wedge F^C_{(2)} \wedge F^C_{(2)}\Big) \tag{4.31}$$

Now combining (4.23), (4.26), (4.29) and (4.31), we can write the anomaly inflow(up to the normalization of 2π) as:

$$\begin{aligned}\mathcal{I}_5[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^3}\Big(&\frac{\kappa_{A^3}}{3!}A_{(1)} \wedge F^A_{(2)} \wedge F^A_{(2)} + \frac{\kappa_{A^2C}}{2!}A_{(1)} \wedge F^A_{(2)} \wedge F^C_{(2)} \\ &+\frac{\kappa_{AC^2}}{2!}A_{(1)} \wedge F^C_{(2)} \wedge F^C_{(2)} + \frac{\kappa_{C^3}}{3!}C_{(1)} \wedge F^C_{(2)} \wedge F^C_{(2)} \\ &+sd(A_{(1)} \wedge F^A_{(2)} \wedge C_{(1)}) + td(A_{(1)} \wedge C_{(1)} \wedge F^C_{(2)})\Big)\end{aligned} \tag{4.32}$$

If we have the partition function of our theory under $U(1)^{(0)}_A \times U(1)^{(0)}_C$ symmetry as $\mathcal{Z}[A_{(1)}, C_{(1)}]$, then under the background gauge transformations of $A_{(1)}$ and $C_{(1)}$ in (4.17) and (4.18), the partition function transforms as:

$$\mathcal{Z}[A_{(1)} + d\lambda^A_{(0)}, C_{(1)} + d\lambda^C_{(0)}] = e^{i\int_{M_4}\mathcal{A}^A_{(4)}[A_{(1)},\lambda^A_{(0)}]}e^{i\int_{M_4}\mathcal{A}^C_{(4)}[C_{(1)},\lambda^C_{(0)}]}\mathcal{Z}[A_{(1)}, C_{(1)}] \tag{4.33}$$

where $\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}]$ and $\mathcal{A}^C_{(4)}[C_{(1)}, \lambda^C_{(0)}]$ are the anomalous phases generated by $A_{(1)}$ and $C_{(1)}$ respectively.
Now we can use descent equations again on $\mathcal{I}_5[A_{(1)}, C_{(1)}]$, to calculate $\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}]$ and $\mathcal{A}^C_{(4)}[C_{(1)}, \lambda^C_{(0)}]$.

Lets first calculate $\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}]$.
Using the descent equation $2\pi\delta_{\Lambda_{(p)}}\mathcal{I}_{(d+1)}[B_{(p+1)}] = d\mathcal{A}_d[B_{(p+1)}, \lambda_{(p)}]$ we can write,

$$2\pi\delta_{\lambda^A_{(0)}}\mathcal{I}_5[A_{(1)}, C_{(1)}] = d\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}] \tag{4.34}$$

Now LHS =

$$\begin{aligned}2\pi\delta_{\lambda^A_{(0)}}\mathcal{I}_5[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^2}\Big(&\frac{\kappa_{A^3}}{3!}d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^A_{(2)} + \frac{\kappa_{A^2C}}{2!}d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)} \\ &+\frac{\kappa_{AC^2}}{2!}d\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)} + sd(d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge C_{(1)}) \\ &+td(d\lambda^A_{(0)} \wedge C_{(1)} \wedge F^C_{(2)})\Big)\end{aligned} \tag{4.35}$$

For the term multiplied to parameter s, we can write it as:

$$\begin{aligned}d(d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge C_{(1)}) &= (-1)^1\Big(d\lambda^A_{(0)} \wedge d(F^A_{(2)} \wedge C_{(1)})\Big) \\ &= -(d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge dC_{(1)}) \\ &= -(d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)}) \\ &= -d(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)})\end{aligned} \tag{4.36}$$

where we used $d^2 = 0$, $dF^A_{(2)} = 0$ and $dC_{(1)} = F^C_{(2)}$.
Similarly for the term multiplied to parameter t, we can write it as:

$$\begin{aligned} d(d\lambda^A_{(0)} \wedge C_{(1)} \wedge F^C_{(2)}) &= (-1)^1\Big(d\lambda^A_{(0)} \wedge d(C_{(1)} \wedge F^C_{(2)})\Big) \\ &= -(d\lambda^A_{(0)} \wedge dC_{(1)} \wedge F^C_{(2)}) \\ &= -(d\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)}) \\ &= -d(\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)}) \end{aligned} \tag{4.37}$$

Now substituting (4.36) and (4.37) in (4.35) we get:

$$\begin{aligned} 2\pi\delta_{\lambda^A_{(0)}}\mathcal{I}_5[A_{(1)}, C_{(1)}] &= \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^A_{(2)} + \frac{\kappa_{A^2C}}{2!}d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)} \\ &\quad + \frac{\kappa_{AC^2}}{2!}d\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)} - sd(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)}) \\ &\quad - td(\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)})\Big) \\ &= \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}d(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^A_{(2)}) + \frac{\kappa_{A^2C}}{2!}d(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)}) \\ &\quad + \frac{\kappa_{AC^2}}{2!}d(\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)}) - sd(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)}) \\ &\quad - td(\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)})\Big) \\ &= \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}d(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^A_{(2)}) + \Big(\frac{\kappa_{A^2C}}{2!} - s\Big)d(\lambda^A_{(0)} \wedge F^A_{(2)} \wedge F^C_{(2)}) \\ &\quad + \Big(\frac{\kappa_{AC^2}}{2!} - t\Big)d(\lambda^A_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)})\Big) \end{aligned} \tag{4.38}$$

Now since $\lambda^A_{(0)} \in \mathbb{R}$, we can write:

$$\begin{aligned} 2\pi\delta_{\lambda^A_{(0)}}\mathcal{I}_5[A_{(1)}, C_{(1)}] &= \frac{1}{(2\pi)^2}d\Big(\frac{\kappa_{A^3}}{3!}(\lambda^A_{(0)}F^A_{(2)} \wedge F^A_{(2)}) + \Big(\frac{\kappa_{A^2C}}{2!} - s\Big)(\lambda^A_{(0)}F^A_{(2)} \wedge F^C_{(2)}) \\ &\quad + \Big(\frac{\kappa_{AC^2}}{2!} - t\Big)(\lambda^A_{(0)}F^C_{(2)} \wedge F^C_{(2)})\Big) \\ &= d\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}] \end{aligned} \tag{4.39}$$

Thus we can finally write the anomalous phase due to the background field $A_{(1)}$ as:

$$\begin{aligned} \mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}] &= \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}(\lambda^A_{(0)}F^A_{(2)} \wedge F^A_{(2)}) + \Big(\frac{\kappa_{A^2C}}{2!} - s\Big)(\lambda^A_{(0)}F^A_{(2)} \wedge F^C_{(2)}) \\ &\quad + \Big(\frac{\kappa_{AC^2}}{2!} - t\Big)(\lambda^A_{(0)}F^C_{(2)} \wedge F^C_{(2)})\Big) \end{aligned} \tag{4.40}$$

Now lets calculate $\mathcal{A}^C_{(4)}[C_{(1)}, \lambda^C_{(0)}]$.

Using the descent equation $2\pi\delta_{\Lambda_{(p)}}\mathcal{I}_{(d+1)}[B_{(p+1)}] = d\mathcal{A}_d[B_{(p+1)}, \lambda_{(p)}]$ we can write,

$$2\pi\delta_{\lambda^C_{(0)}}\mathcal{I}_5[A_{(1)}, C_{(1)}] = d\mathcal{A}^C_{(4)}[C_{(1)}, \lambda^C_{(0)}] \tag{4.41}$$

Now LHS =

$$\begin{aligned} 2\pi\delta_{\lambda^C_{(0)}}\mathcal{I}_5[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{C^3}}{3!}d\lambda^C_{(0)} \wedge F^C_{(2)} \wedge F^C_{(2)} + sd(A_{(1)} \wedge F^A_{(2)} \wedge d\lambda^C_{(0)}) \\ + td(A_{(1)} \wedge d\lambda^C_{(0)} \wedge F^C_{(2)})\Big) \end{aligned} \tag{4.42}$$

Now lets manipulate the term multiplied to the parameter s as before for ease of calculation [b] :

$$\begin{aligned} d(A_{(1)} \wedge F^A_{(2)} \wedge d\lambda^C_{(0)}) &= dA_{(1)} \wedge F^A_{(2)} \wedge d\lambda^C_{(0)} + (-1)^1 A_{(1)} \wedge d(F^A_{(2)} \wedge d\lambda^C_{(0)}) \\ &= F^A_{(2)} \wedge F^A_{(2)} \wedge d\lambda^C_{(0)} \\ &= d(F^A_{(2)} \wedge F^A_{(2)} \wedge \lambda^C_{(0)}) \end{aligned} \tag{4.44}$$

and similarly we manipulate the term multiplied to the parameter t as:

$$\begin{aligned} d(A_{(1)} \wedge d\lambda^C_{(0)} \wedge F^C_{(2)}) &= dA_{(1)} \wedge d\lambda^C_{(0)} \wedge F^C_{(2)} + (-1)^1 A_{(1)} \wedge d(d\lambda^C_{(0)} \wedge F^C_{(2)}) \\ &= F^A_{(2)} \wedge d\lambda^C_{(0)} \wedge F^C_{(2)} \\ &= d(F^A_{(2)} \wedge \lambda^C_{(0)} \wedge F^C_{(2)}) \end{aligned} \tag{4.45}$$

[b] We can write the last step of this calculation since:

$$\begin{aligned} d(F^A_{(2)} \wedge F^A_{(2)} \wedge \lambda^C_{(0)}) &= dF^A_{(2)} \wedge F^A_{(2)} \wedge \lambda^C_{(0)}) + (-1)^2 F^A_{(2)} \wedge d(F^A_{(2)} \wedge \lambda^C_{(0)}) \\ &= F^A_{(2)} \wedge (dF^A_{(2)} \wedge \lambda^C_{(0)} + (-1)^2 F^A_{(2)} \wedge d\lambda^C_{(0)}) \\ &= F^A_{(2)} \wedge F^A_{(2)} \wedge d\lambda^C_{(0)} \qquad \text{which is step 2} \end{aligned} \tag{4.43}$$

Now substituting (4.44) and (4.45) in (4.42) we get:

$$\begin{aligned}
2\pi\delta_{\lambda^C_{(0)}}\mathcal{I}_5[A_{(1)},C_{(1)}] &= \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{C^3}}{3!}d\lambda^C_{(0)}\wedge F^C_{(2)}\wedge F^C_{(2)} + sd(F^A_{(2)}\wedge F^A_{(2)}\wedge\lambda^C_{(0)}) \\
&\qquad + td(F^A_{(2)}\wedge\lambda^C_{(0)}\wedge F^C_{(2)})\Big) \\
&= \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{C^3}}{3!}d(\lambda^C_{(0)}F^C_{(2)}\wedge F^C_{(2)}) + sd(\lambda^C_{(0)}F^A_{(2)}\wedge F^A_{(2)}) \\
&\qquad + td(\lambda^C_{(0)}F^A_{(2)}\wedge F^C_{(2)})\Big) \qquad \cdots\cdots \text{since } \lambda^C_{(0)}\in\mathbb{R} \\
&= \frac{1}{(2\pi)^2}d\Big(\frac{\kappa_{C^3}}{3!}(\lambda^C_{(0)}F^C_{(2)}\wedge F^C_{(2)}) + s(\lambda^C_{(0)}F^A_{(2)}\wedge F^A_{(2)}) \\
&\qquad + t(\lambda^C_{(0)}F^A_{(2)}\wedge F^C_{(2)})\Big) \\
&= d\mathcal{A}^C_{(4)}[C_{(1)},\lambda^C_{(0)}]
\end{aligned} \tag{4.46}$$

Thus we can write the anomalous phase due to the background field $C_{(1)}$ as:

$$\begin{aligned}
\mathcal{A}^C_{(4)}[C_{(1)},\lambda^C_{(0)}] = \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{C^3}}{3!}(\lambda^C_{(0)}F^C_{(2)}\wedge F^C_{(2)}) + s(\lambda^C_{(0)}F^A_{(2)}\wedge F^A_{(2)}) \\
+ t(\lambda^C_{(0)}F^A_{(2)}\wedge F^C_{(2)})\Big)
\end{aligned} \tag{4.47}$$

Now we know that we couple the background fields to the action of our theory by the following terms :

$$\tilde{S} = i\int_{\mathcal{M}_4} A_{(1)}\wedge\star J^A_{(1)} + i\int_{\mathcal{M}_4} C_{(1)}\wedge\star J^C_{(1)} \tag{4.48}$$

Now under the gauge transformations of $A_{(1)}$ and $C_{(1)}$,

$$\begin{aligned}
&\delta_{\lambda^A_{(0)},\lambda^C_{(0)}}\tilde{S} = i\int_{\mathcal{M}_4} d\lambda^A_{(0)}\wedge\star J^A_{(1)} + i\int_{\mathcal{M}_4} d\lambda^C_{(0)}\wedge\star J^C_{(1)} \\
&= -i\int_{\mathcal{M}_4}\lambda^A_{(0)}\wedge d\star J^A_{(1)} - i\int_{\mathcal{M}_4}\lambda^C_{(0)}\wedge d\star J^C_{(1)} \qquad \cdots \text{Taking boundary terms to 0}
\end{aligned} \tag{4.49}$$

Now comparing this to (4.40) and (4.47), we see that the anomalous phases $\mathcal{A}^A_{(4)}[A_{(1)},\lambda^A_{(0)}]$ and $\mathcal{A}^C_{(4)}[C_{(1)},\lambda^C_{(0)}]$ lead to non conservation equations [32]:

$$\begin{aligned}
d\star J^A_{(1)} = -\frac{i}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}(F^A_{(2)}\wedge F^A_{(2)}) + \Big(\frac{\kappa_{A^2C}}{2!}-s\Big)(F^A_{(2)}\wedge F^C_{(2)}) \\
+\Big(\frac{\kappa_{AC^2}}{2!}-t\Big)(F^C_{(2)}\wedge F^C_{(2)})\Big)
\end{aligned} \tag{4.50}$$

$$d \star J^C_{(1)} = -\frac{i}{(2\pi)^2}\Big(\frac{\kappa_{C^3}}{3!}(F^C_{(2)} \wedge F^C_{(2)}) + s(F^A_{(2)} \wedge F^A_{(2)}) + t(F^A_{(2)} \wedge F^C_{(2)})\Big) \tag{4.51}$$

Now to derive the 2-group structure out of the parent theory $U(1)^{(0)}_A \times U(1)^{(0)}_C$ we would like to gauge the subgroup $U(1)^{(0)}_C$ of the parent theory.
Gauging the subgroup $U(1)^{(0)}_C$ requires the background field $C_{(1)}$ to become dynamical:

$$U(1)^{(0)}_C \mapsto U(1)^{(0)}_c \tag{4.52}$$

$$C_{(1)} \mapsto c_{(1)} \tag{4.53}$$

$$F^C_{(2)} \mapsto f^c_{(2)} \qquad f^c_{(2)} = dc_{(1)} \tag{4.54}$$

Since we are gauging $U(1)^{(0)}_C$, we can add a counter term to make the kinetic term of our action gauge invariant just like we did in the section on [Gauging the Higher Form Symmetries of 4d Maxwell],

$$+\frac{1}{2e^2}\int_{\mathcal{M}_4} f^c_{(2)} \wedge \star f^c_{(2)} \tag{4.55}$$

where e is the gauge coupling [c].
Now since we want to gauge $U(1)^{(0)}_C$, we need it to be anomaly free. This requires setting the anomalous phase:

$$\mathcal{A}^C_{(4)}[C_{(1)}, \lambda^C_{(0)}] = 0 \tag{4.56}$$

Looking at (4.47), this means we need:

$$\kappa_{C^3} = 0 \qquad s, t = 0 \tag{4.57}$$

This lead to the conservation equation:

$$d \star J^C_{(1)} = 0 \tag{4.58}$$

[c] We can also add a topological action term involving θ to our theory to express some ABJ anomalies as shift in θ-terms. For more details on this refer [32].

Now setting $s, t = 0$, sets the anomalous phase $\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}]$ to:

$$\begin{aligned}\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}] = \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}(\lambda^A_{(0)} F^A_{(2)} \wedge F^A_{(2)}) + \frac{\kappa_{A^2C}}{2!}(\lambda^A_{(0)} F^A_{(2)} \wedge F^C_{(2)}) \\ + \frac{\kappa_{AC^2}}{2!}(\lambda^A_{(0)} F^C_{(2)} \wedge F^C_{(2)})\Big)\end{aligned} \tag{4.59}$$

and now the non-conservation equation is:

$$\begin{aligned}d \star J^A_{(1)} = -\frac{i}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}(F^A_{(2)} \wedge F^A_{(2)}) + \frac{\kappa_{A^2C}}{2!}(F^A_{(2)} \wedge F^C_{(2)}) \\ + \frac{\kappa_{AC^2}}{2!}(F^C_{(2)} \wedge F^C_{(2)})\Big)\end{aligned} \tag{4.60}$$

Lets first look at the anomaly with κ_{AC^2} term.
Under gauging $U(1)^{(0)}_C$, the anomalous phase associated to the κ_{AC^2} term becomes:

$$\frac{\kappa_{AC^2}}{8\pi^2}(\lambda^A_{(0)} f^c_{(2)} \wedge f^c_{(2)}) \tag{4.61}$$

This is an ABJ anomaly term associated to the $U(1)^{(0)}_A$ symmetry. We will assume that $U(1)^{(0)}_A$ has no ABJ anomaly for simplicity of calculations and thus we set:

$$\kappa_{AC^2} = 0 \tag{4.62}$$

Thus we have the anomalous phase as:

$$\mathcal{A}^A_{(4)}[A_{(1)}, \lambda^A_{(0)}] = \frac{1}{(2\pi)^2}\Big(\frac{\kappa_{A^3}}{3!}(\lambda^A_{(0)} F^A_{(2)} \wedge F^A_{(2)}) + \frac{\kappa_{A^2C}}{2!}(\lambda^A_{(0)} F^A_{(2)} \wedge F^C_{(2)})\Big) \tag{4.63}$$

Now lets examine the anomalous phase associated to the κ_{A^2C} term.
Under gauging $U(1)^{(0)}_C$, the anomalous phase associated to the κ_{A^2C} term becomes:

$$\frac{\kappa_{A^2C}}{8\pi^2}(\lambda^A_{(0)} F^A_{(2)} \wedge f^C_{(2)}) \tag{4.64}$$

and thus the action varies as:

$$\delta S = i\frac{\kappa_{A^2C}}{8\pi^2}\int_{\mathcal{M}_4} \lambda^A_{(0)} F^A_{(2)} \wedge f^C_{(2)} \tag{4.65}$$

and from (4.50) we can see that this leads to the non-conservation equation:

$$d \star J^A_{(1)} \supset -\frac{i\kappa_{A^2C}}{8\pi^2}(F^A_{(2)} \wedge f^C_{(2)}) \tag{4.66}$$

Now we will try to convert this non-conservation equation into a conservation equation and in doing so will obtain the 2-group structure of the parent theory.
Gauging $U(1)^{(0)}_C$ we can construct a new 2-form current using the dynamical (magnetic) field strength $f^c_{(2)}$:

$$J^B_{(2)} = \frac{i}{2\pi} \star f^c_{(2)} \tag{4.67}$$

Now this 2-form current is conserved:

$$d \star J^B_{(2)} = \frac{i}{2\pi} d \star (\star f^c_{(2)}) = \frac{i}{2\pi} df^c_{(2)} = 0 \tag{4.68}$$

since $df^c_{(2)} = 0$.
Now we can think of the 2-form conserved current $J^B_{(2)}$ as arising from a 2-form background gauge field $B_{(2)}$ and thus we can couple our theory with the term:

$$S_B = \int_{\mathcal{M}_4} B_{(2)} \wedge \star J^B_{(2)} = \frac{i}{2\pi} \int_{\mathcal{M}_4} B_{(2)} \wedge f^c_{(2)} \tag{4.69}$$

Now since we have a 2-form background field $B_{(2)}$, we can think of this as being associated to a $U(1)^{(1)}_B$ 1-form symmetry, where $B_{(2)}$ has the gauge transformation :

$$B_{(2)} \mapsto B_{(2)} + d\Lambda^B_{(1)} \tag{4.70}$$

To cancel the anomalous phase in (4.66), we can define the background field $B_{(2)}$ undergoes a particular shift under the background gauge transformations of $U(1)^{(0)}_A$:

$$B_{(2)} \mapsto B_{(2)} - \frac{\kappa_{A^2C}}{4\pi} \lambda^A_{(0)} F^A_{(2)} \tag{4.71}$$

Lets look at the variation of the coupled term S_B under the gauge transform of $U(1)^{(0)}_A$:

$$\begin{aligned} \delta_{\lambda^A_{(0)}} S_B &= \frac{i}{2\pi} \int_{\mathcal{M}_4} (-\frac{\kappa_{A^2C}}{4\pi} \lambda^A_{(0)} F^A_{(2)}) \wedge f^c_{(2)} \\ &= -i \frac{\kappa_{A^2C}}{8\pi^2} \int_{\mathcal{M}_4} \lambda^A_{(0)} F^A_{(2)} \wedge f^c_{(2)} \end{aligned} \tag{4.72}$$

which is exactly of the same form (with opposite sign) as (4.65).
Hence adding S_B to our action and then undergoing the gauge transform under $U(1)^{(0)}_A$ after gauging $U(1)^{(0)}_C$, we can cancel the anomalous phase due to $U(1)^{(0)}_A$ by having the

background field $B_{(2)}$ transform as:

$$B_{(2)} \to B_{(2)} + d\Lambda^{B}_{(1)} + \frac{\kappa_A}{2\pi}\lambda^{A}_{(0)}F^{A}_{(2)} \tag{4.73}$$

which is exactly of the form (4.4) as we mentioned above. Here we can set:

$$\kappa_A = -\frac{1}{2}\kappa_{A^2C} \tag{4.74}$$

This particular shift of $B_{(2)}$ is called a 2-group shift and is analogous to the **Green-Schwarz Mechanism** we talked about earlier.

Lets summarize the above discussion in a statement:

Statement 4.2.2

We can say that the parent theory $U(1)^{(0)}_A \times U(1)^{(0)}_C$ has a 2-group structure :

$$\mathbb{G}^{(2)} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B \qquad , \kappa_A = -\frac{1}{2}\kappa_{A^2C} \tag{4.75}$$

given that we have

1. The anomaly polynomial of the parent theory as:

$$\mathcal{I}_6[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^3}\left(\frac{\kappa_{A^3}}{3!}F^{A}_{(2)} \wedge F^{A}_{(2)} \wedge F^{A}_{(2)} + \frac{\kappa_{A^2C}}{2!}F^{A}_{(2)} \wedge F^{A}_{(2)} \wedge F^{C}_{(2)}\right) \tag{4.76}$$

2. The anomaly inflow of the parent theory as:

$$\mathcal{I}_5[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^3}\left(\frac{\kappa_{A^3}}{3!}A_{(1)} \wedge F^{A}_{(2)} \wedge F^{A}_{(2)} + \frac{\kappa_{A^2C}}{2!}A_{(1)} \wedge F^{A}_{(2)} \wedge F^{C}_{(2)}\right) \tag{4.77}$$

3. The anomalous phase of the parent theory as:

$$\mathcal{A}^{A}_{(4)}[A_{(1)}, \lambda^{A}_{(0)}] = \frac{1}{(2\pi)^2}\left(\frac{\kappa_{A^3}}{3!}(\lambda^{A}_{(0)}F^{A}_{(2)} \wedge F^{A}_{(2)}) + \frac{\kappa_{A^2C}}{2!}(\lambda^{A}_{(0)}F^{A}_{(2)} \wedge F^{C}_{(2)})\right) \tag{4.78}$$

The anomaly coefficients κ_{A^3} and κ_{A^2C} can be evaluated using triangle diagrams or Fujikawa's method of derivation of anomaly. The anomaly associated κ_{A^3} can be reevaluated and some part of it can be called by using a counter term formed by the shift due Green Schwarz Mechanism. We will not be discussing this in detail. For readers interested in the rigorous treatment of this term, refer [32].

We know that 't Hooft anomalies can be tracked along the renormalization group flow and hence the anomaly in UV matches the anomaly in IR [29]. The 2-group structure constants κ_A stay the same along the RG flow. We can summarize the structure of the above example of the 2-group $\mathbb{G}^{(2)} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B$ in a neat diagram inspired from [32].

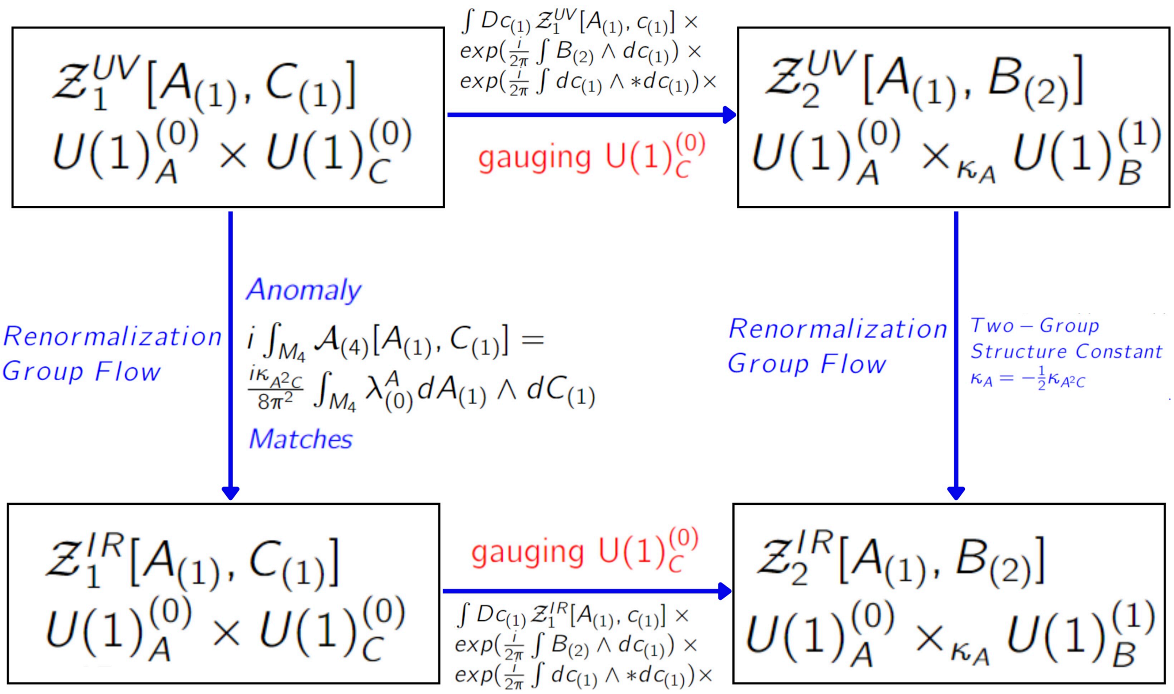

Figure 4.1: This diagram illustrates the above procedure of deriving a 2-group structure. The vertical arrows on both the sides represent 't Hooft anomaly matching[29] along the RG flow and the horizontal arrows on both the top and the bottom, represent the derivation of the 2-group structure of parent theory $U(1)^{(0)}_A \times U(1)^{(0)}_C$ with a mixed κ_{A^2C} 't Hooft anomaly by gauging the subgroup $U(1)^{(0)}_C$ to generate a theory with $U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B$ 2-group symmetry. Here the partition functions of the parent theory are labelled with the subscript [1], and those of the 2-group theory are labelled with the subscript [2].

4.2.2 Some General Continuous 2-groups

In the previous section, we only discussed a specific example of 2-group symmetry $\mathbb{G}^{(2)} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B$. We will now briefly discuss some other general cases of continuous 2-group structure.

Following the above discussion, we know how to construct 2-group structure of a parent theory now. In summary, if we have parent theories of the form $G^{(0)} \times U(1)^{(0)}_C$ with some mixed 't Hooft anomaly, where $G^{(0)}$ can be an abelian or non-abelian subgroup of the parent theory, then we gauge the $U(1)^{(0)}_C$ subgroup of the parent theory. Gauging the $U(1)^{(0)}_C$ subgroup gives us a conserved 2-form $J^B_{(2)}$ current formed by the dynamical field strength of $U(1)^{(0)}_c$ which can be thought of as sourced by a background field $B_{(2)}$ of $U(1)^{(1)}_B$ 1-form group. The anomalous phase of the theory is cancelled by term coupled to our theory due to the background field $B_{(2)}$ of $U(1)^{(1)}_B$, which looks like $\int_{\mathcal{M}_4} B_{(2)} \wedge \star J^B_{(2)}$. The anomalous phase is cancelled upon gauge transformation of $B_{(2)}$ which is analogous to the Green-Schwarz mechanism, and is determined by the anomalous phase due to $G^{(0)}$. Lets look at some general cases [32]:

Abelian 2 group symmetry:

We can have a general parent theory $(\prod_N U(1)^{(0)}_N) \times U(1)^{(0)}_C$ which has a 2-group structure:

$$(\prod_N U(1)^{(0)}_N) \times_{\kappa_{NM}} U(1)^{(1)}_B \qquad , \kappa_{NM} \in \mathbb{Z} \tag{4.79}$$

Here we have the background fields $A^N_{(1)}$ associated to the $\prod_N U(1)^{(0)}_N$ subgroup with the gauge transformations:

$$A^N_{(1)} \mapsto A^N_{(1)} + d\lambda^N_{(0)} \tag{4.80}$$

and the 2-form gauge field $B_{(2)}$ has the gauge transformation:

$$B_{(2)} \mapsto B_{(2)} + d\Lambda^B_{(1)} + \sum_{N,M} \frac{\kappa_{NM}}{2\pi} \lambda^N_{(0)} dA^N_{(1)} \tag{4.81}$$

Non-Abelian 2 group symmetry:

We can also have a parent theory with the symmetry group $G^{(0)} \times U(1)^{(0)}_C$ where $G^{(0)}$ is a non-abelian group. For calculation purposes lets consider $G^{(0)} = SU(N)^{(0)}_A$.

Hence we have a parent theory $SU(N)^{(0)}_A \times U(1)^{(0)}_C$ which has a 2-group structure:

$$SU(N)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B \qquad , \kappa_A \in \mathbb{Z} \tag{4.82}$$

Here the 2-form background field $B_{(2)}$ has the gauge transformation:

$$B_{(2)} \mapsto B_{(2)} + d\Lambda^B_{(1)} + \frac{\kappa_A}{4\pi} tr(\lambda^N_{(0)} dA_{(1)}) \tag{4.83}$$

We have the above 2-group shift of $B_{(2)}$, since to construct the anomaly polynomial of a theory with $SU(N)^{(0)}_A$ subgroup, we have the contributions of terms like:

$$\frac{1}{(2\pi)^r} tr\Big((dA_{(1)})^r\Big) \qquad , r \geq 2 \tag{4.84}$$

In general we can also have a Poincaré 2-group symmetry, $\mathcal{P} \times_{\kappa_\mathcal{P}} U(1)^{(1)}_B$ which have the background fields $B_{(2)}$ with the transformation:

$$B_{(2)} \mapsto B_{(2)} + d\Lambda^B_{(1)} + \frac{\kappa_\mathcal{P}}{16\pi} tr(\theta_{(0)} d\omega_{(1)}) \qquad , \kappa_\mathcal{P} \in \mathbb{Z} \tag{4.85}$$

Here $\theta_{(0)}$ is a local $SO(4)$ frame rotation and $\omega^a_{(1)b}$ is the matrix valued 1-form spin connection:

$$\omega^a_{(1)b} = \Gamma_{bc}{}^a \hat{\theta}^b \tag{4.86}$$

where $\Gamma_{bc}{}^a$ are the connection components in non-coordinate basis and $\hat{\theta}^b$ are the dual non-coordinate basis [9].

For more details of the general cases including the Poincaré 2-group symmetry please refer [32].

4.3 Example: QED like Model

Lets understand 2-group symmetry further using a simple example of a QED like model with 4 Weyl fermions. Lets consider the parent theory with the symmetry group:

$$U(1)^{(0)}_A \times U(1)^{(0)}_C \tag{4.87}$$

where $U(1)^{(0)}_A$ is the gauge group and $U(1)^{(0)}_C$ is the global symmetry flavour group with the background fields $A_{(1)}$ and $C_{(1)}$ with the gauge transformations:

$$A_{(1)} \mapsto A_{(1)} + d\lambda^A_{(0)} \tag{4.88}$$

$$C_{(1)} \mapsto C_{(1)} + d\lambda^C_{(0)} \tag{4.89}$$

Lets consider the theory with the following action:

$$\mathcal{S} = \int d^4x \left(i\bar{\psi}^+_L \bar{\sigma}^\mu \partial_\mu \psi^+_L + i\bar{\psi}^-_L \bar{\sigma}^\mu \partial_\mu \psi^-_L + i\bar{\psi}^+_R \sigma^\mu \partial_\mu \psi^+_R + i\bar{\psi}^-_R \sigma^\mu \partial_\mu \psi^-_R \right) \tag{4.90}$$

where $\sigma^\mu = (\mathbb{1}, \sigma^i)$ [d].
Lets assign some arbitrary charges to the Weyl fermions $\psi^+_L, \psi^-_L, \psi^+_R, \psi^-_R$ under the symmetry $U(1)^{(0)}_A \times U(1)^{(0)}_C$:

Fields	$U(1)^{(0)}_A$	$U(1)^{(0)}_C$
ψ^+_L	$+1$	$+1$
ψ^+_R	-1	$+1$
ψ^-_L	$+q$	-1
ψ^-_R	$-q$	-1

Table 4.1: Charge table for Weyl Fermions under $U(1)^{(0)}_A \times U(1)^{(0)}_C$

We can clearly see that the above action is invariant under the $U(1)^{(0)}_A \times U(1)^{(0)}_C$ symmetry group as:

$$\bar{\psi}^+_L \bar{\sigma}^\mu \partial_\mu \psi^+_L \mapsto e^{-i\alpha} e^{-i\beta} e^{i\alpha} e^{i\beta} \bar{\psi}^+_L \bar{\sigma}^\mu \partial_\mu \psi^+_L = \bar{\psi}^+_L \bar{\sigma}^\mu \partial_\mu \psi^+_L \tag{4.91}$$

$$\bar{\psi}^+_R \sigma^\mu \partial_\mu \psi^+_R \mapsto e^{i\alpha} e^{-i\beta} e^{-i\alpha} e^{i\beta} \bar{\psi}^+_R \sigma^\mu \partial_\mu \psi^+_R = \bar{\psi}^+_R \sigma^\mu \partial_\mu \psi^+_R \tag{4.92}$$

$$\bar{\psi}^-_L \bar{\sigma}^\mu \partial_\mu \psi^-_L \mapsto e^{-iq\alpha} e^{+i\beta} e^{iq\alpha} e^{-i\beta} \bar{\psi}^-_L \bar{\sigma}^\mu \partial_\mu \psi^-_L = \bar{\psi}^-_L \bar{\sigma}^\mu \partial_\mu \psi^-_L \tag{4.93}$$

$$\bar{\psi}^-_R \sigma^\mu \partial_\mu \psi^-_R \mapsto e^{iq\alpha} e^{i\beta} e^{-iq\alpha} e^{-i\beta} \bar{\psi}^-_R \sigma^\mu \partial_\mu \psi^-_R = \bar{\psi}^-_R \sigma^\mu \partial_\mu \psi^-_R \tag{4.94}$$

[d] σ^i are the Pauli Matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

where $e^{i\alpha} \in U(1)^{(0)}_A \quad , \alpha \in S^1$ and $e^{i\beta} \in U(1)^{(0)}_C \quad , \beta \in S^1$ Now from the above section on [Construction of Anomaly Polynomials], we know that the most general anomaly polynomial for this model can be written as:

$$\begin{aligned}\mathcal{I}_6[A_{(1)}, C_{(1)}] = \frac{1}{(2\pi)^3}\Big(&\frac{\kappa_{A^3}}{3!} F^A_{(2)} \wedge F^A_{(2)} \wedge F^A_{(2)} + \frac{\kappa_{A^2C}}{2!} F^A_{(2)} \wedge F^A_{(2)} \wedge F^C_{(2)} \\ &+ \frac{\kappa_{AC^2}}{2!} F^A_{(2)} \wedge F^C_{(2)} \wedge F^C_{(2)} + \frac{\kappa_{C^3}}{3!} F^C_{(2)} \wedge F^C_{(2)} \wedge F^C_{(2)}\Big)\end{aligned} \tag{4.95}$$

where $F^A_{(2)} = dA_{(1)}$ and $F^C_{(2)} = dC_{(1)}$.

Now lets gauge the $U(1)^{(0)}_C$ subgroup of the parent theory which leads to the background field $C_{(1)}$ being dynamical,

$$U(1)^{(0)}_C \mapsto U(1)^{(0)}_c \tag{4.96}$$

$$C_{(1)} \mapsto c_{(1)} \tag{4.97}$$

$$F^C_{(2)} \mapsto f^c_{(2)} \qquad\qquad f^c_{(2)} = dc_{(1)} \tag{4.98}$$

This leads to the anomaly polynomial be written as:

$$\begin{aligned}\mathcal{I}_6[A_{(1)}, c_{(1)}] = \frac{1}{(2\pi)^3}\Big(&\frac{\kappa_{A^3}}{3!} F^A_{(2)} \wedge F^A_{(2)} \wedge F^A_{(2)} + \frac{\kappa_{A^2C}}{2!} F^A_{(2)} \wedge F^A_{(2)} \wedge f^C_{(2)} \\ &+ \frac{\kappa_{AC^2}}{2!} F^A_{(2)} \wedge f^C_{(2)} \wedge f^C_{(2)} + \frac{\kappa_{C^3}}{3!} f^C_{(2)} \wedge f^C_{(2)} \wedge f^C_{(2)}\Big)\end{aligned} \tag{4.99}$$

Now from Appendix A, we can conclude that the anomaly coefficients for the parent theory $U(1)^{(0)}_A \times U(1)^{(0)}_C$ are:

$$\kappa_{A^3} = \sum_{\psi^{\mathcal{C}}_{\mathcal{N}}} \left(r_{\psi^{\mathcal{C}}_{\mathcal{N}}}\right)^3 \tag{4.100}$$

$$\kappa_{A^2C} = \sum_{\psi^{\mathcal{C}}_{\mathcal{N}}} \left(r_{\psi^{\mathcal{C}}_{\mathcal{N}}}\right)^2 \left(p_{\psi^{\mathcal{C}}_{\mathcal{N}}}\right) \tag{4.101}$$

$$\kappa_{AC^2} = \sum_{\psi^{\mathcal{C}}_{\mathcal{N}}} \left(r_{\psi^{\mathcal{C}}_{\mathcal{N}}}\right) \left(p_{\psi^{\mathcal{C}}_{\mathcal{N}}}\right)^2 \tag{4.102}$$

$$\kappa_{C^3} = \sum_{\psi^{\mathcal{C}}_{\mathcal{N}}} \left(p_{\psi^{\mathcal{C}}_{\mathcal{N}}}\right)^3 \tag{4.103}$$

where $\mathcal{N} = \{L, R\}$, $\mathcal{C} = \{+, -\}$.
$r_{\psi^{\mathcal{C}}_{\mathcal{N}}}$ are the charges of the Weyl fermions under $U(1)^{(0)}_A$ symmetry and $p_{\psi^{\mathcal{C}}_{\mathcal{N}}}$ are the charges of the Weyl fermions under $U(1)^{(0)}_C$.

Now by using Table [4.1], we can compute these anomaly coefficients as:

$$\kappa_{A^3} = \sum_{\psi^C_{\mathcal{N}}} \left(r_{\psi^C_{\mathcal{N}}} \right)^3 = (+1)^3 + (-1)^3 + (+q)^3 + (-q)^3 = 0 \tag{4.104}$$

$$\kappa_{A^2C} = \sum_{\psi^C_{\mathcal{N}}} \left(q_{\psi^C_{\mathcal{N}}} \right)^2 \left(p_{\psi^C_{\mathcal{N}}} \right) = (+1)(+1)^2 + (+1)(-1)^2 + (-1)(+q)^2 + (-1)(-q)^2 = 2(1-q^2) \tag{4.105}$$

$$\kappa_{AC^2} = \sum_{\psi^C_{\mathcal{N}}} \left(q_{\psi^C_{\mathcal{N}}} \right) \left(p_{\psi^C_{\mathcal{N}}} \right)^2 = (+1)^2(+1) + (+1)^2(-1) + (-1)^2(+q) + (-1)^2(-q) = 0 \tag{4.106}$$

$$\kappa_{C^3} = \sum_{\psi^C_{\mathcal{N}}} \left(p_{\psi^C_{\mathcal{N}}} \right)^3 = (+1)^3 + (+1)^3 + (-1)^3 + (-1)^3 = 0 \tag{4.107}$$

Now we know that to successfully gauge $U(1)^{(0)}_C$ subgroup, we need that there be no 't Hooft anomaly and no ABJ anomaly in the $U(1)^{(0)}_A$ current, i.e we need;

$$\kappa_{A^3} = 0 \qquad \kappa_{AC^2} = 0 \tag{4.108}$$

which we have in our theory as per our calculations above. Now substituting (4.104), (4.105), (4.106) and (4.107) in (4.99), we get:

$$\mathcal{I}_6[A_{(1)}, c_{(1)}] = \frac{1}{(2\pi)^3} \left(\frac{\kappa_{A^2C}}{2!} F^A_{(2)} \wedge F^A_{(2)} \wedge f^C_{(2)} \right) = \frac{2(1-q^2)}{16\pi^3} F^A_{(2)} \wedge F^A_{(2)} \wedge f^C_{(2)} \tag{4.109}$$

Now using the descent equations (3.15), we can write the anomaly polynomial as :

$$\mathcal{I}_6[A_{(1)}, c_{(1)}] = \frac{2(1-q^2)}{16\pi^3} d(A_{(1)} \wedge F^A_{(2)} \wedge f^C_{(2)}) = d\mathcal{I}_5[A_{(1)}, c_{(1)}] \tag{4.110}$$

Thus the anomaly inflow is:

$$\mathcal{I}_5[A_{(1)}, c_{(1)}] = \frac{2(1-q^2)}{16\pi^3} A_{(1)} \wedge F^A_{(2)} \wedge f^C_{(2)} \tag{4.111}$$

Lets use descent equations, again to calculate the anomalous phase:

$$\begin{aligned} d\mathcal{A}_{(4)}[A_{(1)}, c_{(1)}] &= 2\pi\delta_{\lambda^A_{(0)}}\mathcal{I}_5[A_{(1)}, c_{(1)}] \\ &= \frac{2(1-q^2)}{8\pi^2} d\lambda^A_{(0)} \wedge F^A_{(2)} \wedge f^C_{(2)} \\ &= \frac{2(1-q^2)}{8\pi^2} d(\lambda^A_{(0)} F^A_{(2)} \wedge f^C_{(2)}) \end{aligned} \tag{4.112}$$

Thus we can write:

$$\mathcal{A}_{(4)}[A_{(1)}, c_{(1)}] = \frac{(1-q^2)}{4\pi^2}\lambda^A_{(0)} F^A_{(2)} \wedge f^C_{(2)} \tag{4.113}$$

Now under gauging $U(1)^{(0)}_C$, we generate an anomaly:

$$\mathcal{S}_{\text{anom}} = i\frac{(1-q^2)}{4\pi^2}\int_{\mathcal{M}_4} \lambda^A_{(0)} F^A_{(2)} \wedge f^C_{(2)} \tag{4.114}$$

Now as before, we can cancel the anomalous phase by coupling 2-form background field $B_{(2)}$ to our theory by adding the term:

$$\mathcal{S}_B = \int_{\mathcal{M}_4} B_{(2)} \wedge \star J^B_{(2)} \tag{4.115}$$

where we can think of $J^B_{(2)}$ as the 2-form current built of the dynamical (magnetic)field strength $f^c_{(2)}$:

$$J^B_{(2)} = \frac{i}{2\pi} \star f^c_{(2)} \qquad , d \star J^B_{(2)} = 0 \tag{4.116}$$

Thus we write $\mathcal{S}_B$ as:

$$\mathcal{S}_B = \frac{i}{2\pi}\int_{\mathcal{M}_4} B_{(2)} \wedge f^c_{(2)} \tag{4.117}$$

Now we assign the following shift to $B_{(2)}$ under gauge transformations of $U(1)^{(0)}_A$:

$$B_{(2)} \mapsto B_{(2)} - \frac{\kappa_{A^2C}}{4\pi}\lambda^A_{(0)} F^A_{(2)} \tag{4.118}$$

Substituting (4.105) in the above equation, we get the shift in $B_{(2)}$ as:

$$B_{(2)} \mapsto B_{(2)} - \frac{2(1-q^2)}{4\pi}\lambda^A_{(0)} F^A_{(2)} \tag{4.119}$$

Now we can see that under gauge transformations of $U(1)^{(0)}_A$,

$$\begin{aligned}\delta_{\lambda^A_{(0)}}\mathcal{S}_B &= -\frac{i}{2\pi}\int_{\mathcal{M}_4}\frac{2(1-q^2)}{4\pi}\lambda^A_{(0)}F^A_{(2)}\wedge f^c_{(2)}\\ &= -i\frac{(1-q^2)}{4\pi^2}\int_{\mathcal{M}_4}\lambda^A_{(0)}F^A_{(2)}\wedge f^c_{(2)}\\ &= -\mathcal{S}_{\text{anom}}\end{aligned} \tag{4.120}$$

Hence adding $\mathcal{S}_B$ to our action cancels the anomaly and we have a non-anomalous theory. Now since we have a 2-form background field $B_{(2)}$, we can think of this as being associated to a $U(1)^{(1)}_B$ 1-form symmetry, where $B_{(2)}$ has the gauge transformation :

$$B_{(2)} \mapsto B_{(2)} + d\Lambda^B_{(1)} - \frac{(1-q^2)}{2\pi}\lambda^A_{(0)}F^A_{(2)} \tag{4.121}$$

Thus, we can conclude that this particular example of a QED like model has the 2-group structure :

$$\mathbb{G}^{(2)}_{\text{QED}} = U(1)^{(0)}_A \times_{\kappa_A} U(1)^{(1)}_B \qquad ,\kappa_A = -(1-q^2) \tag{4.122}$$

where κ_A is the 2-group structure constant.

Interesting we can also observe using (4.105) that:

$$\kappa_A = -\frac{1}{2}\kappa_{A^2C} = -(1-q^2) \tag{4.123}$$

Chapter 5

Summary and Future Work

Summary

Beginning with our exploration of higher form symmetries in the first chapter [Higher Form Symmetries], we embarked on a journey into a novel topological approach to understanding symmetries. As we constructed the formalism, we understood the action of higher form symmetries on higher dimensional operators. In passing we note that p-form symmetries can act on q-dimensional extended operators ($q \geq p$) using higher representations. This falls under a really interesting area of research on generalised charges [14, 15, 18]. We also looked at the example of 4D Maxwell theory and proved that it possesses a $U(1)^{(1)}_{\text{elc.}} \times U(1)^{(1)}_{\text{mag.}}$ global symmetry. We also had a sneak peek into the concept of anomaly inflow and how it helped to cancel the anomaly of 4D Maxwell theory under gauging both $U(1)^{(1)}_{\text{elc.}} \times U(1)^{(1)}_{\text{mag.}}$ simultaneously. For undertstanding on Maxwell theory in $d = 2, 3$ please refer [47]. Although our primary emphasis was on the formalism for continuous symmetries yet it's worth noting that even discrete symmetries can be considered within the context of topological symmetries *despite lacking a conserved current*. For an in-depth exploration of discrete symmetries as generalized symmetries, a comprehensive resource can be found in [1, 6, 7].

In the subsequent chapter on [Anomalies], we delved into the intricate art of calculating anomalies through the application of anomaly polynomials and inflow. These concepts are pivotal and serve as the building blocks for our next chapter. To delve deeper into the realm of anomalies, please refer [21, 27, 48].

In our final chapter on [Higher Groups], we extended our discussion to encompass higher group symmetries within the framework of higher form symmetries. Here, we gained insights into the types of theories that can exhibit such structures and learned how to derive the higher group structure of a theory using anomaly polynomials. We culminated our exploration by examining a QED-like model, shedding light on why it possesses a 2-group structure. While we extensively covered the continuous abelian case, we also provided a brief overview of non-abelian continuous cases. Furthermore, for those intrigued by the fascinating realm of higher-group structures, a deep dive into the 3-group structure in Axionic Yang-Mills can be found in [49]. We can also form discrete higher groups using discrete 0-form symmetries. Although we did not explore it extensively here, the mathematics of category theory also provides insights in our discussion of continuous higher groups [50, 51].

The concept of spontaneous symmetry breaking also plays a very important role in the context of higher form symmetries. Extensive literature on this can be found in [6, 52, 47].

Future Work

There are a lot of areas of research that have been explored or are still under work that I want to explore further.

- One of captivating avenue of research lies in contemplating higher form symmetries and extended operators within the domain of gravity. This is done by constructing generalized symmetries for linearised gravity. This has extensively been dealt with in [53, 54, 55].

- While this thesis focused exclusively on *Invertible* higher form symmetries, it's noteworthy that there exists another formalism of *Non-Invertible* symmetries [16, 17, 18, 19, 56, 57], encompassing the construction of symmetries from higher categorical symmetries. The fusion rule for *Non-Invertible* symmetries deviates from the group associativity law and takes the form [35]:

$$U_{g_1}(\Sigma) \otimes U_{g_2}(\Sigma) = \sum_i U_{g_i}(\Sigma_i) \tag{5.1}$$

  Example: The Ising Model studied in Conformal Field Theories exhibits a *Non-Invertible* symmetry [58].

- Symmetries in String Theory and Holography are quite an exciting area of research. They are explored using the tool of SymTFT - Symmetry Topological Field Theory (used to separate the physical theory from its symmetries) and geometric engineering on invertible and non-invertible symmetries. Research on this can be found in [59, 60, 61, 62, 57] and many more.

In conclusion, I would like to express my fascination with this formalism and my hope to continue working on it in the future. It is a captivating field with vast potential for further exploration and discovery.

Appendix A

QED Anomaly Calculation

This calculation is based on the highly influential paper by Kazuo Fujikawa in 1979 [22] [*Erratum* [63]], on anomaly calculation using the path integral measure for theories with gauge-invariant fermions. We will be using some results from this paper and from [23]. The calculation has also been explained in detail in [64, 23, 27, 65].

We will be calculating the anomaly for a theory with a Dirac fermion. The anomaly for a theory with the Weyl fermion can be analogously calculated by shifting from Dirac to Weyl Representation.

Lets consider a theory with a massless Dirac fermion coupled to the electromagnetic field with the partition function:

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[\psi]} \tag{A.1}$$

where the action,

$$S[\psi] = \int d^4x \; i\bar{\psi}\partial\!\!\!/\psi \tag{A.2}$$

where $\partial\!\!\!/ := \gamma^\mu \partial_\mu$. [a]

Here ψ is a Dirac fermion which is a 2-component Weyl Fermion:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{A.3}$$

[a]Here γ^μ are the Clifford algebra matrices in Dirac representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

In terms of the Weyl fermions we write the action as:

$$S[\psi] = \int d^4x \; i\bar{\psi}_L\bar{\sigma}^\mu\partial_\mu\psi_L + i\bar{\psi}_R\sigma^\mu\partial_\mu\psi_R \tag{A.4}$$

We have the action (A.2) invariant under a $U(1)$ vector symmetry under which the Dirac fermion transforms as:

$$\psi \mapsto e^{iq\alpha}\psi \tag{A.5}$$

and a $U(1)$ axial symmetry under which the Dirac fermion transforms as [23]:

$$\psi \mapsto e^{iq\alpha\gamma^5}\psi \tag{A.6}$$

Here $e^{i\alpha} \in U(1)$, α is a constant.
q is the charge of the Dirac fermion under $U(1)$ symmetry rotation. Now we can write the infinitesimal shift due to the axial symmetry as:

$$\delta\psi = iq\alpha\gamma^5\psi \quad , \quad \delta\bar{\psi} = iq\alpha\bar{\psi}\gamma^5 \tag{A.7}$$

Now under gauging the $U(1)$ axial symmetry, we see have the following transformations:

$$C_\mu \mapsto c_\mu \tag{A.8}$$

$$\not{\partial} \mapsto \not{D} = \not{\partial} - iqc_\mu \tag{A.9}$$

$$c_\mu(x) \mapsto c_\mu(x) + \partial_\mu\alpha(x) \tag{A.10}$$

where C_μ was the background gauge field associated to $U(1)$ axial symmetry. Hence we see that now α is position dependent.
The above map from $\not{\partial} \mapsto \not{D}$ takes place due to the coupling of the dynamical background field c_μ in the action $\mathcal{S}$. Now our actions looks like:

$$\mathcal{S}[\psi, c_\mu] = \int d^4x \; i\bar{\psi}\not{D}\psi \tag{A.11}$$

and the partition function can be written as:

$$\mathcal{Z}[c_\mu] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\mathcal{S}[\psi, c_\mu]} \tag{A.12}$$

Now according to [22], the anomaly in the axial symmetry can be calculated using the anomaly in the measure of the path integral, which now is invariant under the infinitesimal shifts due to the axial rotations even though the action is still invariant.

Fujikwa Method

Lets calculate the anomaly in the measure:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \tag{A.13}$$

From [22, 27], we know that we can redefine the Dirac fermion as:

$$\psi' = \psi + iq\alpha\gamma^5\psi \quad , \quad \bar{\psi}' = \bar{\psi} + iq\alpha\bar{\psi}\gamma^5 \tag{A.14}$$

This change of variables is due to transformation of coordinates under the infinitesimal axial rotations which in-turn generates a Jacobian which will be the reason for the anomaly. Lets first write the measure (A.13) in a form that is better suited for the calculation of the Jacobian.
The following method of writing the measure in terms of it's eigenfunctions is famously what is mentioned in [22].
The Dirac operator $\not{D}$ has eigenfunctions ϕ_n, which are 4-component spinors:

$$i \not{D}\, \phi_n = \lambda_n\phi_n \tag{A.15}$$

where $-i\lambda_n$ are the eigenvalues.
This eigenfunction basis has an orthonormality condition:

$$\int d^4x\bar{\phi}_n\phi_m = \delta_{nm} \tag{A.16}$$

Now we expand the Dirac spinor ψ in terms of these eigenfunction basis as:

$$\psi(x) = \sum_n a_n\phi_n(x) \tag{A.17}$$

where a_n are Grassmann-valued numbers [23].
We can also express $\bar{\psi}$ in terms of eigenfunction basis as:

$$\bar{\psi}(x) = \sum_n \bar{b}_n\bar{\phi}_n(x) \tag{A.18}$$

Now substituting (A.16), (A.17), (A.18) in (A.13), we get:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_{nm}\int d\bar{b}_n da_m \delta_{nm} = \prod_n \int d\bar{b}_n da_n \tag{A.19}$$

and substituting (A.16), (A.17), (A.18) in (A.11), we get:

$$\mathcal{S}[\psi, c_\mu] = \int d^4x\, i\bar{\psi}\not{D}\psi = \int d^4\, i\sum_n \bar{b}_n\bar{\phi}_n(x)(-i)\lambda_n \sum_m a_m\phi_m(x) = \sum_n \lambda_n\bar{b}_n a_n \tag{A.20}$$

Thus now we can write the partition function (A.12) as:

$$\begin{aligned}\mathcal{Z}[c_\mu] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\mathcal{S}[\psi,c_\mu]} = \prod_n \int d\bar{b}_n da_n \exp\left(\sum_n \lambda_n\bar{b}_n a_n\right)\\ &= \prod_n \int d\bar{b}_n da_n\left(1 + \sum_n \lambda_n \bar{b}_n a_n + \cdots\right)\\ &= \prod_n \lambda_n = i\not{D} \quad \cdots\left(\text{Since} \int d\bar{b}_n\ \bar{b}_n = 1, \int da_n\ a_n = 1, \int d\bar{b}_n = \int da_n = 0\right)\end{aligned} \tag{A.21}$$

where to write the last step we used the rules of Grassmann integration [b].
Now lets look at how the integral measure transforms under the infinitesimal axial rotations (A.14).
These axial rotations vary the the Grassmann parameters a_n and $\bar{b}_n$:

$$\sum_n a'_n\phi_n(x) = \sum_n a_n\phi_n(x) + iq\alpha\gamma^5\sum_n a_n\phi_n(x) \tag{A.22}$$

Thus,

$$\sum_n \delta a_n\phi_n = iq\alpha\sum_m a_m\gamma^5\phi_n \tag{A.23}$$

$$\implies \delta a_n = iq\alpha a_m\gamma^5 \tag{A.24}$$

Now using the orthonormality relation (A.16) we can insert a complete set of states in between which gives us:

$$\delta a_n = i\delta_{nm}q\alpha a_m\gamma^5 = iqa_m\int d^4x\alpha\bar{\phi}_n\gamma^5\phi_m \tag{A.25}$$

[b] details can be found in [66] or any QFT textbook. For reference if η is grassmann variable then $d\eta = \frac{\partial}{\partial\eta}$

since α is now position dependent as discussed earlier.
Now lets relabel δa_n as:

$$\delta a_n = \mathcal{C}_{nm} a_m \tag{A.26}$$

where,

$$\mathcal{C}_{nm} = iq \int d^4x \alpha \bar{\phi}_n \gamma^5 \phi_m \tag{A.27}$$

Now since the shift in Dirac fermion due to the axial symmetry is same on both ψ and $\bar{\psi}$, the Jacobian for the transformation of b_n is same as that of a_n. Thus the integral measure under the axial rotation is:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} = \prod_n \int d\bar{b}_n da_n = \prod_n \int d\bar{b}'_n da'_n \mathcal{J}^2 \tag{A.28}$$

where we the Jacobian $\mathcal{J}$ is inverse of determinant of the transformation due to the axial rotation from:

$$a_n \mapsto a'_n = a_n + \mathcal{C}_{nm} a_m \quad \text{or} \quad b_n \mapsto b'_n = b_n + \mathcal{C}_{nm} b_m \tag{A.29}$$

Hence,

$$\mathcal{J} = \det{}^{-1} \left(\delta_{nm} + \mathcal{C}_{nm} \right) \tag{A.30}$$

wherein we have written the above result from [23].
Now using binomial theorem, we can write(upto leading order):

$$\mathcal{J} = \det{}^{-1} \left(\delta_{nm} + \mathcal{C}_{nm} \right) = \det \left(\delta_{nm} - \mathcal{C}_{nm} \right) \tag{A.31}$$

and now for $n = m$, using taylor expansion(upto leading order), and the property of determinant and traces, we can express the jacobian as:

$$\mathcal{J} = \det \left(\delta_{nm} - \mathcal{C}_{nm} \right) = \det e^{-\mathcal{C}_{nn}} = \exp\Big(- tr(\mathcal{C}_{nn}) \Big) \tag{A.32}$$

Thus, our final expression for the Jacobian is:

$$\mathcal{J} = \exp\left(-iq \int d^4x \alpha(x) \sum_n \bar{\phi}_n(x) \gamma^5 \phi_n(x) \right) \tag{A.33}$$

This Jacobian $\mathcal{J}$ is now the anomaly of the theory !! Lets evaluate it further.
Now we use the result from [22, 23] and directly write:

$$\left(\sum_n \bar{\phi}_n(x)\gamma^5\phi_n(x)\right) = \frac{q^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{A.34}$$

Now we use the equation for the 2-form field strength:

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \tag{A.35}$$

and we can write the Jacobian as:

$$\mathcal{J} = \exp\left(i\frac{q^3}{8\pi^2}\int_{\mathcal{M}_4}\alpha(x)F\wedge F\right) \tag{A.36}$$

If we have multiple Dirac fermions with $U(1)$ axial symmetry, then the Jacobian can be written as:

$$\mathcal{J} = \exp\left(i\frac{\sum_n q_n^3}{8\pi^2}\int_{\mathcal{M}_4}\alpha(x)F\wedge F\right) \tag{A.37}$$

where q_n is the charge of each Dirac fermion under the $U(1)$ axial symmetry. In Weyl representation, we can write the the Jacobian as:

$$\mathcal{J} = \exp\left(i\frac{\sum_n (q_n^{L,R})^3}{8\pi^2}\int_{\mathcal{M}_4}\alpha(x)F\wedge F\right) \tag{A.38}$$

where $q_n^{L,R}$ is the charge of each Weyl fermion under the $U(1)$ axial symmetry.
Finally we can see that the above equation matches the first term of anomaly that we derived using the anomaly polynomial in the previous chapter. Hence:

$$\kappa_{A^3} = \sum_n (q_n^{L,R})^3 \tag{A.39}$$

Now lets consider we have another symmetry $U(1)_T$ with the background field $T_{(1)}$ under which the measure is not invariant under gauging $U(1)_T$. This will result in the Jacobian:

$$\mathcal{J} = \exp\left(i\frac{\sum_n (q_n^{L,R})(p_n^{L,R})^2}{8\pi^2}\int_{\mathcal{M}_4}\alpha(x)F_A\wedge f_T\right) \tag{A.40}$$

where $p_n^{L,R}$ is the charge of each Weyl fermion under the $U(1)_T$ symmetry.
Thus,

$$\kappa_{AT^2} = \sum_n (q_n^{L,R})(p_n^{L,R})^2 \tag{A.41}$$

We can similarly write:

$$\kappa_{A^2T} = \sum_n (q_n^{L,R})^2(p_n^{L,R}) \qquad \kappa_{T^3} = \sum_n \left(p_n^{L,R}\right)^3 \tag{A.42}$$

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