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Recent Developments in the Theory of Partially Massless Gravity

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Abstract

In the pursuit of finding a quantum field theory for gravity, theories of massive gravity have regained interest. Massive gravity in de Sitter and Anti-de Sitter spaces gives rise to a particularly promising avenue known as the theory of ‘Partially Massless Gravity’. Despite the inconsistencies and challenges found in many non-linear formulations of partially massless gravity, recent research indicates a potential breakthrough by considering non-unitary theories of multiple partially massless fields. This dissertation delves into the construction of partially massless gravity and its most recent progress, a non-unitary theory of multiple partially massless spin-2 fields, which we refer to as ‘Non-unitary Multi-field Partially Massless Gravity’. We present the consistency of the theory along with its implications and derive a simple test for the consistency of the theory for two fields in the mini-superspace ansatz. We highlight that the field is still in its recent stages, and numerous tests, enhancements, and discoveries are actively being made, paving the way to find a consistent description of gravity in the realm of quantum fields.

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Chapter 1

Introduction

Gravity — one of the four fundamental forces in nature — is responsible for governing the universe at large scales [1]. While gravity is the most noticeable force shaping our daily life experiences, from a theoretical perspective, it remains one of the most enigmatic topics, being the sole force not yet consistent with the principles of quantum theories. To date, it is universally acknowledged that Einstein's theory of General Relativity [2] has remained at the forefront in describing gravity. Since its formulation back in 1916, General Relativity has been serving as the primary model for describing gravitational interactions. Its alignment with observed phenomena has been nothing but remarkable. Nevertheless, it does not provide any framework for the interface between gravity and particle physics [3].

Besides mere theoretical interests, it is valuable to investigate other alternative theories that describe gravitational fields in the framework of quantum field theories. Such models might hold the key to unresolved mysteries in cosmology, issues that even General Relativity falls short of addressing, such as the old cosmological constant problem and the phenomenon of late-time cosmic acceleration [4]. In the realm of quantum field theory, fields can be quantized in which their excitations appear as quanta with definite energy and momenta, classified by their mass and spin. This leads to the intriguing potential of finding an alternative theory capable of quantizing gravitational fields and describing its quanta, often referred to as 'Gravitons' [5].

Initially, theoretical physicists agreed that the quantum aspects of gravity should encode gravitons as massless spin-2 particles [6–10]. They also believed that gravitons must be massless to maintain gauge invariance — a fundamental property ensuring that physical laws remain consistent under gauge transformations [11]. However, as highlighted by Schwinger [12], maintaining gauge invariance does not always imply masslessness, meaning that masslessness of the graviton is not a necessity [13]. This idea has led to a renewed interest in theories of massive gravity, where one maintains the notion of spin-2 fields but considers the graviton to acquire a non-zero mass. The notion of massive gravity has existed since the late 1930s with the formalism of Markus Fierz and Wolfgang Pauli [14]. Their formalism suggests a linearized spin-2 field action compatible with describing general relativity when deriving its equations of motion. However, it has been shown that the linear Fierz-Pauli action suffers from continuity issues, known as Van Dam-Veltman-Zakharov (vDVZ) discontinuity [15, 16]. This discontinuity becomes apparent by the difference within the theory when directly setting the mass to zero as opposed to gradually approaching zero mass. Fortunately, the Vainshtein mechanism [17] offers a solution to this issue, which suggests the necessity of non-linear extensions of massive gravity. Although most non-linear extensions give rise to pathologies, particularly the Boulware-Deser (BD) ghost [18], a ghost-free realization of massive gravity has been recently discovered, known as the dRGT theory [19]. This theory could re-sum non-linear terms of an effective field theory of massive gravity. Nowadays, dGRT theory is believed to avoid many inconsistencies that used to be present in the previous constructions of massive gravity. Unfortunately, even with the absence of the BD ghost, dRGT theory still suffers from other pathologies [20–24]. This raises the question of whether one can provide a resolution of these pathologies by a different construction of massive gravity [25].

Another way to modify the linearized Fierz-Pauli massive gravity is by investigating the theory in maximally symmetric spaces [26]. An interesting result can be found in de Sitter or Anti-de Sitter $\big($(A)dS$\big)$ spaces: for a specific value of the graviton's mass relative to the (A)dS curvature, the theory reveals a new gauge symmetry. This symmetry effectively eliminates the pathological terms in the theory, forcing it to propagate with fewer degrees of freedom. This exceptional case of massive gravity has been referred to as **'Partially Massless Gravity'** [27–34]. The existence of the extra symmetry in the linearized theory of partially massless gravity raises the question of whether this symmetry and the fewer degrees of freedom could be extended to a fully non-linear theory. This question is significant as we anticipate that a non-linear extension of partially massless gravity may help avoid the pathologies that dRGT theory still faces [35]. Moreover, a unique characteristic that sets PM gravity apart from other theories is its constraint on the value of the cosmological constant. If there exists a non-linear extension to the theory, then the gauge symmetry of the PM gravity could potentially tune the cosmological constant so that the quantum contributions are canceled up to the small residual value that we observe, offering a natural resolution to the old cosmological constant problem. Hence, if one finds a consistent non-linear extension of partially massless gravity, the non-linear PM gauge symmetry would eliminate the pathological terms at all orders, naturally avoid discontinuities in the massless limit, tackle cosmological problems and capture Einstein's theory as an effective field theory of gravity [36].

Many models of non-linear partially massless gravity have led to inconsistencies at higher orders. This issue is usually referred to as a 'no-go result' [37, 38], as it was thought to be unavoidable. Several works have tried to resolve the no-go results by considering theories of multiple partially massless spin-2 fields [37, 39, 40]. In these analyses, the focus has been on unitary theories, where the relative sign among the multiple field terms is positive. Yet, no success in navigating past the inconsistencies has been reached. However, In 2020, the significant work of Boulanger, Deffayet, Garcia-Saenz, and Traina [41] showed that relaxing the requirement of unitarity leads to a complete and consistent theory of multiple partially massless fields. This intriguing result brings attention to further investigations on the many unexplored aspects of non-linear partially massless gravity.

This dissertation captures the recent development in the theory of partially massless gravity. We start by setting the foundations and reviewing the theory of massive gravity in Chapter 2. This should pave the way to present our main focus, partially massless gravity. In Chapter 3, we highlight the first construction of linear partially massless gravity, the method of testing candidates for non-linear extensions, and the main obstructions shown with an example from a candidate theory. Next, we present in Chapter 4 the recent development and the formulation of the revolutionary non-unitary partially massless theory with multiple fields, stating by the unitary formulation, then enforcing non-unitarity and presenting the principal outcomes from such property. Afterward, we present our test for the consistency of theory in a simple setup: two fields in the mini-superspace ansatz (homogeneous and isotropic universe). Finally, in Chapter 5, we conclude with a summary and outlook for future developments in this theory.

Notations and Conventions

Throughout this dissertation, we maintain the usage of natural units, where the reduced Planck constant $\hbar$ and the speed of light c are both set to unity. We also adopt the plus signature convention $(-,+,+,\cdots,+)$ for the metric tensors. Additionally, we refer to the number of spacetime dimensions by D and the spacetime indices by Greek letters $\mu,\nu,\rho,\cdots\in\{0,1,2\cdots,D-1\}$, while the spacial indices are denoted by Latin letters $i,j,k,\cdots\in\{1,2,\cdots,D-1\}$. Hence the index $\mu=0$ represents the time-like direction such that $x_0=t$. For a given tensor $A_{\mu\nu}$, we write the indices as (μ,ν) to denote symmetrization. i.e., $A_{(\mu\nu)}=\frac{1}{2}(A_{\mu\nu}+A_{\nu\mu})$. Similarly, [$\mu$,$\nu$] denotes anti-symmetrization such that $A_{[\mu\nu]}=\frac{1}{2}(A_{\mu\nu}-A_{\nu\mu})$. We also use the square brackets to express the trace of the tensor with respect to the Minkowski metric $\eta_{\mu\nu}=diag(-1,+1,+1,+1)$. i.e., $[A]=A^{\mu}_{\mu}=\eta^{\mu\nu}A_{\mu\nu}$. For an arbitrary metric $g_{\mu\nu}$ other than Minkowski, we denote the trace of the tensor using angle brackets $\langle A\rangle=A^{\mu}_{\mu}=g^{\mu\nu}A_{\mu\nu}$, unless specified otherwise.

Chapter 2

Review of Massive Gravity

2.1 Fierz-Pauli Action

2.1.1 Linear Fierz-Pauli

The first construction of a theory that suggests a non-zero mass for the graviton was made by Markus Fierz and Wolfgang Pauli in 1939 [14]. The formulation of Fierz and Pauli considers adding a graviton mass term for the kinetic Einstein-Hilbert action, in which the theory returns to GR in the massless limit. Note that the linearized kinetic term is uniquely fixed to maintain a local and Lorentz-invariant theory and to prevent higher derivatives in the longitudinal (helicity-0) mode. The latter is needed because higher than the first-order derivatives leads to an unbounded Hamiltonian. This is the essence of the Ostrogradsky's theorem [42, 43]. We denote these problematic higher derivatives in the helicity-0 mode as 'ghosts' [44] and will always try to avoid them. Moreover, a priori for the mass term is that it contains only two powers of the spin-2 field $h_{\mu\nu}$ and no derivatives. For an arbitrary number of spacetime dimensions d and a generic mass m, the Lagrangian of the Fierz-Pauli action is written as:

$$\mathcal{L}_{FP} = -\frac{M_{\rm Pl}^{D-2}}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{M_{\rm Pl}^{D-2}}{8}m^2(\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu} - \tilde{h}^2) \tag{2.1}$$

where $M_{\rm Pl}$ is Plank's mass, and $\hat{\mathcal{E}}$ is the Lichnerowicz operator

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} = -\frac{1}{2}\big(\Box\tilde{h}_{\mu\nu} - 2\partial_{(\mu}\partial_\alpha\tilde{h}^\alpha_{\nu)} + \partial_\mu\partial_\nu\tilde{h} - \eta_{\mu\nu}(\Box\tilde{h} - \partial_\alpha\partial_\beta\tilde{h}^{\alpha\beta})\big), \tag{2.2}$$

with defining $\tilde{h}_{\mu\nu} = h_{\mu\nu}/M_{Pl}^{(D-2)/2}$, $\tilde{h} = \eta^{\mu\nu}\tilde{h}_{\mu\nu}$, and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. We note that the kinetic term is invariant under the gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}, \tag{2.3}$$

We will make use of this property in a later discussion.

Stückelberg Fields

The form of the mass term presented in (2.1) breaks diffeomorphism invariance. This invariance can be restored by the introduction of the well-known Stückelberg trick, first introduced by Stückelberg in 1938 [45, 46] in the context of electromagnetic theories. In the Stückelberg approach, the key is to introduce d fields ϕ_a that transform under linear diffeomorphism in a specific way to make the mass term invariant. For example, when considering the theory in $D = 4$ dimensions and after including four linearized Stückelberg fields, the resulting diffeomorphism invariant theory will be

$$\mathcal{L}_{FP} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2\big((h_{\mu\nu} + 2\partial_{(\mu}\phi_{\nu)})^2 - (h + 2\partial_\alpha\phi^\alpha)^2\big) \tag{2.4}$$

which is invariant under the simultaneous gauge transformations

$$\begin{aligned} h_{\mu\nu} &\to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} \\ \phi_\mu &\to \phi_\mu - \frac{1}{2}\xi_\mu \end{aligned} \tag{2.5}$$

Propagating degrees of freedom (DoFs)

We will identify the propagating DoF for the four-dimensional linear theory presented in (2.4). For such, we split the Stückelberg field into a transverse mode A^a and a longitudinal mode π,

$$\phi^a = \frac{1}{m}A^a + \frac{1}{m^2}\eta^{ab}\partial_b\pi \tag{2.6}$$

Hence, the linearized Fierz-Pauli action, in terms of $h_{\mu\nu}$ and the Stückelberg fields A_μ and π, is written as

$$\begin{aligned} \mathcal{L}_{FP} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{2}h^{\mu\nu}(\Pi_{\mu\nu} - [\Pi]\eta_{\mu\nu}) - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} \\ &-\frac{1}{8}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{2}m(h^{\mu\nu} - h\eta^{\mu\nu})\partial_{(\mu}A_{\nu)} \end{aligned} \tag{2.7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$, and $[\Pi] = \eta^{\mu\nu}\Pi_{\mu\nu}$.

This form of the action allows us to identify the kinetic term of the different fields, the mass terms, and the mixing terms. Nevertheless, one still misses the kinetic term of the field π as it is hidden in the mixing with $h_{\mu\nu}$. To extract the missing kinetic term, we use the shifting $\bar{h}_{\mu\nu} = h_{\mu\nu} - \pi\eta_{\mu\nu}$. Hence, the linearized Fierz-Pauli action becomes

$$\begin{aligned} \mathcal{L}_{FP} = &-\frac{1}{4}\bar{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\bar{h}_{\alpha\beta} - \frac{3}{4}(\partial\pi)^2 - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} \\ &-\frac{1}{8}m^2(\bar{h}_{\mu\nu}\bar{h}^{\mu\nu} - \bar{h}^2) + \frac{3}{2}m^2\pi^2 + \frac{3}{4}m^2\pi\bar{h} \\ &-\frac{1}{2}m(\bar{h}^{\mu\nu} - \bar{h}\eta_{\mu\nu})\partial_{(\mu}A_{\nu)} + 3m\pi\partial_\alpha A^\alpha \end{aligned} \tag{2.8}$$

This decomposition shows the different DoFs in the linearized massive gravity. One finds that:

• $h_{\mu\nu}$ represents the helicity-2 mode and propagates two DoFs.

• A_μ represents the helicity-1 mode and propagates two DoFs.

• π represents the helicity-0 mode and propagates one DoFs.

Thus, the linearized Fierz-Pauli action propagates a total of five DoF, which is consistent for a massive spin-2 field in four dimensions.

2.1.2 Van Dam-Veltman-Zakharov (vDVZ) Discontinuity

Even though the issue of the diffeomorphism invariance has been resolved by the Stückelberg approach, the linearized Fierz-Pauli action suffers from a critical discontinuity issue, known as the Van Dam-Veltman-Zakharov (vDVZ) Discontinuity [15, 16], which will lead us to conclude that the linear description is no longer appropriate for a theory of massive gravity.

To demonstrate the vDVZ discontinuity, we will study the theory (2.1) with $D = 4$ and with the presence of external sources, expressed by their stress-energy tensors $T_{\mu\nu}$ and $T'_{\mu\nu}$. Note that for the spin-2 field, the helicity-0 mode could couple to the external source, exciting the two helicity-2 modes of the graviton and an additional helicity-0 mode. This excited helicity-0 mode could have dramatic consequences, which will be seen when computing the gravitational exchange amplitude between the two sources in the massive $m \neq 0$ and massless $m = 0$ cases.

Massive spin-2 field

Consider the linear four-dimensional Fierz-Pauli action by adding the response to a conserved external source $T_{\mu\nu}$,

$$\mathcal{L} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2\big(h_{\mu\nu}h^{\mu\nu} - h^2\big) + \frac{1}{2M_{\rm Pl}}h_{\mu\nu}T^{\mu\nu} \tag{2.9}$$

One finds that the linearized Einstein equation is given by

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{2}m^2\big(h_{\mu\nu} - h\eta_{\mu\nu}\big) = \frac{1}{M_{\rm Pl}}T_{\mu\nu} \tag{2.10}$$

To solve this equation for $h_{\mu\nu}$, we consider its trace and divergence

$$h = -\frac{2}{3m^2M_{\rm Pl}}\left(T + \frac{2}{m^2}\partial_\alpha\partial_\beta T^{\alpha\beta}\right) \tag{2.11}$$

$$\partial_\mu h^\mu_\nu = -\frac{2}{3m^2M_{\rm Pl}}\left(- \partial_\mu T^\mu_\nu + \frac{1}{3}\partial_\nu T + \frac{2}{3m^2}\partial_\nu\partial_\alpha\partial_\beta T^{\alpha\beta}\right) \tag{2.12}$$

Plugging them back into the Einstein equation gives

$$\begin{aligned}(\Box - m^2)h_{\mu\nu} = -\frac{2}{M_{\rm Pl}}\Bigg(&T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu} - \frac{2}{m^2}\partial_{(\mu}\partial_\alpha T^\alpha_{\nu)} + \frac{1}{3m^2}\partial_\mu\partial_\nu T \\ &+ \frac{1}{3m^2}\partial_\alpha\partial_\beta T^{\alpha\beta}\eta_{\mu\nu} + \frac{2}{3m^4}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta T^{\alpha\beta}\Bigg)\end{aligned} \tag{2.13}$$

which can be simplified to

$$(\Box - m^2)h_{\mu\nu} = -\frac{2}{M_{\rm Pl}}\left(\tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}\right)T^{\alpha\beta} \tag{2.14}$$

where $\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu$. To avoid confusion, we note that the term $\tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)}$ is symmetrized over only two indices α and β.

Now the propagator for the massive spin-2 field can be written in terms of the polarization tensor $f^{massive}_{\mu\nu\alpha\beta}$ as

$$G^{massive}_{\mu\nu\alpha\beta}(x,x') = \frac{\left(\tilde{\eta}_{\mu(\alpha}\tilde{\eta}_{\nu\beta)} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}\right)}{(\Box - m^2)} = \frac{f^{massive}_{\mu\nu\alpha\beta}}{(\Box - m^2)}. \tag{2.15}$$

hence the amplitude exchange between two sources $T_{\mu\nu}$ and $T'_{\mu\nu}$ via the massive spin-2 field is given by

$$\mathcal{A}^{massive}_{TT'} = \int \mathrm{d}^4x h_{\mu\nu} T'^{\mu\nu} = \int \mathrm{d}^4x T'^{\mu\nu} \frac{f^{\text{massive}}_{\mu\nu\alpha\beta}}{(\Box - m^2)} T^{\alpha\beta}. \tag{2.16}$$

and when taking the massless limit $m \to 0$, the exchange amplitude will become

$$\mathcal{A}^{m\to 0}_{TT'} = -\frac{2}{M_{\mathrm{Pl}}} \int \mathrm{d}^4x T'^{\mu\nu} \frac{1}{\Box} \left(T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu}\right) \tag{2.17}$$

We will compare this result with the exchange amplitude derived from the massless spin-2 field.

Massless spin-2 field

When considering a spin-2 field with $m = 0$, we will have

$$\mathcal{L} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \frac{1}{2M_{\mathrm{Pl}}} h_{\mu\nu} T^{\mu\nu} \tag{2.18}$$

and the equation of motion of this Lagrangian will be simply

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = \frac{1}{M_{\mathrm{Pl}}} T_{\mu\nu}. \tag{2.19}$$

The massless theory has only the kinetic term, so we could use the invariance under the transformation (2.3) and choose any gauge. For our convenience, we will use the de Donder gauge

$$\partial_\mu h^\mu_\nu = \frac{1}{2} p_\nu, \tag{2.20}$$

which will reduce the Einstein equation to

$$\Box h_{\mu\nu} = -\frac{2}{M_{\rm Pl}} \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right). \tag{2.21}$$

Now, the massless propagator in terms of the massless polarization tensor is given by

$$G^{massless}_{\mu\nu\alpha\beta}(x, x') = \frac{(\eta_{\mu(\alpha}\eta_{\nu\beta)} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta})}{\Box} = \frac{f^{massless}_{\mu\nu\alpha\beta}}{\Box}. \tag{2.22}$$

Finally, we get the expression of amplitude exchange between two sources $T_{\mu\nu}$ and $T'_{\mu\nu}$ via the massless spin-2 field

$$\mathcal{A}^{massless}_{TT'} = -\frac{2}{M_{\rm Pl}} \int {\rm d}^4 x T'^{\mu\nu} \frac{1}{\Box} \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right), \tag{2.23}$$

which differs from the result derived from the massless limit (2.17).

The unavoidable difference between (2.17) and (2.23) means that there is a discontinuity between the purely massless propagator and the massless limit of the massive propagator, which is the vDVZ discontinuity. The Vainshtein mechanism [17] has resolved this problem, but the resolution implies the necessity of considering the non-linear interactions. Consequently, massive gravity must be a non-linear theory, and one must promote the linear Fierz-Pauli action to a non-linear theory.

2.1.3 Non-linear Fierz-Pauli

To construct a non-linear version of the Fierz-Pauli action, one must consider promoting the theory to possess a full diffeomorphism invariance (or covariance). i.e., in order for the spin-2 field to interact, the gauge symmetry is forced to include non-linear terms, promoting diffeomorphism invariance to a non-linearly realized gauge symmetry

$$h \to h + \partial\xi + \frac{1}{M_{\mathrm{Pl}}}\partial(h\xi) + \cdots \tag{2.24}$$

This gauge invariance will be automatically satisfied on-shell, order by order in h/M_{Pl}.

Starting from the kinetic term, satisfying this symmetry will uniquely promote it to the well-known fully covariant Einstein–Hilbert term

$$\mathcal{L}^{spin-2}_{kin.\ covariant} = \frac{M_{\mathrm{Pl}}^2}{2}\sqrt{-g}R[g], \tag{2.25}$$

where R is the Ricci scalar associated with a fluctuated metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\mathrm{Pl}}$. Since the kinetic term is non-linear, we conclude that any interacting theory of a spin-2 field must be fully non-linear and possess non-linear diffeomorphism invariance (or covariance). Note that the local gauge symmetry will prevent the presence of any ghost from the kinetic term, making it a consistent choice as a kinetic term for a theory of massive gravity.

non-linear Stückelberg

Achieving full diffeomorphism invariance requires the theory to be built from scalar objects constructed from the fluctuated metric $g_{\mu\nu}$ and other tensors. However, the mass term is instead built out of the fluctuation $h_{\mu\nu} = M_{\mathrm{Pl}}\,(g_{\mu\nu} - \eta_{\mu\nu})$, which does not transform as a tensor under diffeomorphism and hence breaks covariance. To restore it, we will introduce a non-linear modification of the Stückelberg trick [47, 48], which suggests introducing the Stückelberg trick by making the simultaneous replacements of the reference metric of the theory $f_{\mu\nu}$ and the fluctuation $h_{\mu\nu}$ as

$$f_{\mu\nu} \xrightarrow{\text{Replace}} \tilde{f}_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b f_{ab} \tag{2.26}$$

$$h_{\mu\nu} = M_{\text{Pl}} \left(g_{\mu\nu} - \eta_{\mu\nu} \right) \xrightarrow{\text{Replace}} H_{\mu\nu} = M_{\text{Pl}} \left(g_{\mu\nu} - \tilde{f}_{\mu\nu} \right). \tag{2.27}$$

where ϕ^a are the four Stückelberg fields.

We note that $\tilde{f}_{\mu\nu}$ will transform as a tensor under coordinate transformations as long as the Stückelberg fields transform as scalars. In addition, the introduction of the reference metric $f_{\mu\nu}$ allows us to construct a theory of massive gravity being a scalar Lagrangian of the two tensors $\tilde{f}_{\mu\nu}$ and $g_{\mu\nu}$.

Helicity decomposition

Applying the helicity decomposition on $H_{\mu\nu}$ will give

$$\begin{aligned} H_{\mu\nu} &= h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} - \frac{1}{M_{\text{Pl}}} \eta_{ab} \partial_\mu \chi^a \partial_\nu \chi^b \\ &= h_{\mu\nu} + \frac{2}{m} \partial_{(\mu} A_{\nu)} + \frac{2}{m^2} \Pi_{\mu\nu} \\ &\quad - \frac{1}{M_{\text{Pl}} m^2} \partial_\mu A^\alpha \partial_\nu \mathcal{A}_\alpha - \frac{2}{M_{\text{Pl}} m^3} \partial_\mu A^\alpha \Pi_{\nu\alpha} - \frac{1}{M_{\text{Pl}} m^4} \Pi^2_{\mu\nu} \end{aligned} \tag{2.28}$$

We can identify helicity-2 mode of the graviton in $h_{\mu\nu}$, helicity-1 in A_μ, and helicity-0 in π. These are the same propagating DoF in the linear theory.

Non-linear mass term

We can now promote the linear Fierz-Pauli mass term to a non-linear theory. A straightforward extension is constructed by using the tensor quantity

$$\mathbb{X}^{\mu}_{\nu} = g^{\mu\alpha}\tilde{f}_{\alpha\nu} = \partial^{\mu}\phi^{a}\partial_{\nu}\phi^{b}f_{ab}. \tag{2.29}$$

Hence, the non-linear Fierz-Pauli mass term can be written in a manifestly diffeomorphism invariant form

$$\mathcal{L}^{(nl)}_{FP,mass} = -m^2 M_{\rm Pl}^2\sqrt{-g}\left(\left[(\mathbb{I}-\mathbb{X})^2\right] - [\mathbb{I}-\mathbb{X}]^2\right), \tag{2.30}$$

where $\mathbb{I}$ denotes the identity matrix, and the square brackets denote the trace with respect to Minkowski. Note that this formulation has been used in [49], along with other alternative forms such as the one presented in [18]. This is the essence of the fact that the Fierz-Pauli action can be generalized to a non-linear theory in an arbitrary number of ways. Unfortunately, as we shall demonstrate next, most of these generalizations generate a ghost known as the Bouldware-Deser (BD) Ghost.

2.1.4 Bouldware-Deser (BD) Ghost

The BD ghosts arising from (2.30) are observed by the appearance of an Ostrogradsky's instability [42, 50]. For demonstration, we will take the reference metric to be Minkowski $f_{\mu\nu} = \eta_{\mu\nu}$ and focus on the helicity-0 mode. With this setting, we first find that the tensor quantity defined in (2.30) becomes

$$\mathbb{X}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \frac{2}{M_{\rm Pl}m^2}\Pi^{\mu}_{\nu} + \frac{1}{M_{\rm Pl}^2 m^4}\Pi^{\mu}_{\alpha}\Pi^{\alpha}_{\nu}. \tag{2.31}$$

where the indices are all raised and lowered with respect to $\eta_{\mu\nu}$. Plugging this expression in the non-linear Fierz-Pauli mass term gives

$$\begin{aligned}\mathcal{L}^{(nl)}_{FP,mass} \supset &-\frac{4}{m^2}\left([\Pi^2]-[\Pi]^2\right)+\frac{4}{M_{\rm Pl}m^4}\left([\Pi^3]-[\Pi][\Pi^2]\right)\\ &-\frac{1}{M_{\rm Pl}^2m^6}\left([\Pi^4]-[\Pi^2]^2\right).\end{aligned} \tag{2.32}$$

We note that the quadratic term in (2.32) becomes a total derivative when integrating by parts, which does not contribute to the equations of motion. However, this argument does not apply to the higher-order interactions as they are genuine higher-order operators and will lead to equations of motion with cubic or higher derivatives. Thus, the higher order operators $\left([\Pi^3]-[\Pi][\Pi^2]\right)$ and $\left([\Pi^4]-[\Pi^2]^2\right)$ in the non-linear Fierz-Pauli mass term will propagate an additional DoF that result in an Ostrogradsky's instability. Hence, the additional DoF will unavoidably give rise to a ghost, which is the BD ghost.

To avoid the BD ghosts, it has been considered in [18] to formulate the Fierz-Pauli mass term as a functional, given by

$$\mathcal{L}^{F,(nl)}_{FP} = -m^2\sqrt{-g}F\left[g^{\mu\nu}g^{\alpha\beta}\left(H_{\mu\alpha}H_{\nu\beta}-H_{\mu\nu}H_{\alpha\beta}\right)\right]. \tag{2.33}$$

Alas, there does not exist a nonzero analytic choice of the function F that can overcome the non-linear propagation of the BD ghost. Luckily, a different construction of the mass term makes the absence of the BD ghost achievable, which we will highlight in the next section.

2.2 Ghost-Free Massive Gravity: dRGT Theory

2.2.1 Decoupling Limit

Here, we will briefly discuss the most well-known ghost-free massive gravity: the dRGT theory, formulated by de Rham, Gabadadze, and Tolley [19, 51]. The dRGT formalism was originally constructed in the decoupling limit of massive gravity, which aims to detach the relevant interactions of the helicity-0 modes from the standard and well-understood complications of General Relativity [52]. In ghost-free massive gravity, the first irrelevant interactions arise at the scale $\Lambda_3 = (m^3 M_{\rm Pl})^{1/3}$. Hence, Λ_3 is used for the decoupling scale. With that in mind, we proceed to define the canonically normalized variables:

$$\hat{\pi} = \Lambda_3^3 \pi, \quad \Lambda_3^3 = m^2 M_{\rm Pl}, \quad \hat{h}_{\mu\nu} = M_{\rm Pl} h_{\mu\nu}, \tag{2.34}$$

and the decoupling limit is then obtained by taking $M_{\rm Pl} \to \infty$ $m \to 0$, and keeping $\hat{\pi}$, $\hat{h}$, Λ_3 to be fixed.

2.2.2 General Formulation

The idea of this ghost-free model is to prevent the presence of the BD ghost in non-linear theory by constructing the mass (or rather potential) term in a way where all the higher derivative operators involving the helicity-0 mode are total derivatives, which will then not contribute to the equations of motion, leading to a healthy non-linear theory of massive gravity.

We start by defining the tensor $H_{\mu\nu}$ to be the covariant form of the metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = H_{\mu\nu} + \eta_{ab}\partial_\mu \phi^a \partial_\nu \phi^b$, where $\eta_{\mu\nu}$ is Minkowski metric, $h_{\mu\nu}$ is the spin-2 field, and ϕ^a are four Stückelberg fields transforming as scalars. We will also extract the helicity-0 mode of the graviton π by expressing $\phi^a = (x^a - \eta^{a\mu}\partial_\mu \pi)$. Hence we have

$$H_{\mu\nu} = h_{\mu\nu} + 2\Pi_{\mu\nu} - \eta^{\alpha\beta}\Pi_{\mu\alpha}\Pi_{\beta\nu}, \tag{2.35}$$

where $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$. We now introduce the tensor quantity

$$\mathcal{K}^\mu_\nu(g, H) = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} = -\sum_{n=1}^{\infty} d_n \left(H^n\right)^\mu_\nu, \quad d_n = \frac{(2n)!}{(1-2n)(n!)^2 4^n}, \tag{2.36}$$

where in this expression, $H^\mu_\nu = g^{\mu\alpha}H_{\alpha\nu}$, and $(H^n)^\mu_\nu = H^\mu_{\alpha_1}H^{\alpha_1}_{\alpha_2}\cdots H^{\alpha_{n-1}}_\nu$. Moreover, we will use the notion of the square brackets $[\ldots]$ to denote the trace with respect to Minkowski metric, e.g., $[\Pi] = \eta^{\mu\nu}\Pi_{\mu\nu}$ and $[\Pi^2] = \eta^{\mu\nu}\eta^{\alpha\beta}\Pi_{\mu\alpha}\Pi_{\nu\beta}$, along with the angle brackets $\langle\ldots\rangle$ to denote the trace with respect to the physical metric $g_{\mu\nu}$, e.g. $\langle\Pi\rangle = g^{\mu\nu}\Pi_{\mu\nu}$ and $\langle\Pi^2\rangle = g^{\mu\nu}g^{\alpha\beta}\Pi_{\mu\alpha}\Pi_{\nu\beta}$.

Recall that we found from the non-linear Fierz-Pauli mass term that the combination $[\Pi]^2 - [\Pi^2]$ turns out to be a total derivative. Hence, one could make use of this ghost-free combination and extend it by replacing $[\Pi]^2$ and $[\Pi^2]$ into $\langle\mathcal{K}\rangle$ and $\langle\mathcal{K}^2\rangle$, respectively. The total Lagrangian then reads

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}\left(R - \frac{m^2}{4}\mathcal{U}(g, H)\right), \tag{2.37}$$

with the potential $\mathcal{U}$ expressed as

$$\begin{aligned}\mathcal{U}(g,H) &= -4\left(\langle\mathcal{K}\rangle^2 - \left\langle\mathcal{K}^2\right\rangle\right) \\ &= -4\Big(\sum_{n\geq 1} d_n \langle H^n\rangle\Big)^2 - 8\sum_{n\geq 2} d_n \left\langle H^n\right\rangle \\ &= \left(\left\langle H^2\right\rangle - \langle H\rangle^2\right) - \frac{1}{2}\left(\langle H\rangle\left\langle H^2\right\rangle - \left\langle H^3\right\rangle\right) \\ &\quad - \frac{1}{16}\left(\left\langle H^2\right\rangle^2 + 4\langle H\rangle\left\langle H^3\right\rangle - 5\left\langle H^4\right\rangle\right) \\ &\quad - \frac{1}{32}\left(2\left\langle H^2\right\rangle\left\langle H^3\right\rangle + 5\langle H\rangle\left\langle H^4\right\rangle - 7\left\langle H^5\right\rangle\right) + \cdots\end{aligned} \tag{2.38}$$

We can rewrite the potential in a better notion in which any ghost-free theory of massive gravity in the decoupling limit can adapt

$$\mathcal{U}(g,H) = -4\sum_{n\geq 2}\alpha_n \mathcal{L}^{(n)}_{\text{der}}(\mathcal{K}) \tag{2.39}$$

where the expression

$$\mathcal{L}^{(n)}_{\text{der}}(\mathcal{K}) = -\sum_{m=1}^{n}(-1)^m \frac{(n-1)!}{(n-m)!}\langle\mathcal{K}^m\rangle\mathcal{L}^{(n-m)}_{\text{der}}(\mathcal{K}) \tag{2.40}$$

represent the total derivative contributions. As demonstrated in [51], the contributions vanish beyond the quartic order. i.e. $\mathcal{L}^{(n)}_{\text{der}}(\mathcal{K}) = 0$ for any $n \geq 5$, and the decoupling limit therefore stops at that order. Hence, we can set the upper limit of the sum to be $n = 4$. The Lagrangian of the dRGT theory then reads

$$\mathcal{L}_{\text{dRGT}} = \frac{M_{\text{Pl}}^2}{2}\sqrt{-g}\bigg(R + m^2\left(\mathcal{L}^{(2)}_{\text{der}}(\mathcal{K}) + \alpha_3\mathcal{L}^{(3)}_{\text{der}}(\mathcal{K}) + \alpha_4\mathcal{L}^{(4)}_{\text{der}}(\mathcal{K})\right)\bigg) \tag{2.41}$$

Alternatively, one can write the dRGT theory with a more compact form and with two dynamical metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, given by the action

$$S_{\mathrm{dRGT}} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\left(R + \frac{m^2}{2}\sum_{n=0}^{4}\alpha_n\mathcal{L}_n[\mathcal{K}[g,f]]\right) \tag{2.42}$$

where the tensor $\mathcal{K}^\mu_\nu[g,f]$ given by

$$\mathcal{K}^\mu{}_\nu[g,f] = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu{}_\nu \tag{2.43}$$

and the overall potential $\mathcal{U}$ written as

$$\mathcal{U} = -\frac{M_{\mathrm{Pl}}^2}{4}\sqrt{-g}\sum_{n=0}^{4}\alpha_n\mathcal{L}_n[\mathcal{K}[g,f]] \tag{2.44}$$

with the Lagrangians $\mathcal{L}_n$ defined as

$$\begin{aligned}
\mathcal{L}_0[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta} \\
\mathcal{L}_1[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu\alpha\beta}Q^{\mu'}_{\mu} \\
\mathcal{L}_2[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha\beta}Q^{\mu'}_{\nu}Q^{\nu'}_{\nu} \\
\mathcal{L}_3[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta}Q^{\mu'}_{\nu}Q^{\nu'}_{\nu}Q^{\alpha'}_{\alpha} \\
\mathcal{L}_4[Q] &= \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu'\nu'\alpha'\beta'}Q^{\mu'}_{\nu}Q^{\nu'}_{\nu}Q^{\alpha'}_{\alpha}Q^{\beta'}_{\beta}
\end{aligned} \tag{2.45}$$

The theory has proven to be indeed absent of BD ghosts. We will leave the further details for the reader to examine; see [4, 19, 53].

2.2.3 Absence of Cosmological Solutions

Regardless of the absence of ghosts, a significant obstruction of dRGT theory becomes evident when trying to construct useful solutions in the context of cosmology, that is, the Friedmann–Robertson–Walker (FRW) ansatz, which describes a homogeneous and isotropic universe. The formalism of the dRGT theory indeed eliminates the BD ghost in massive gravity. However, it will also prohibit the existence of homogeneous and isotropic cosmological solutions, which was demonstrated by the authors in [54]. We will present their proof by first setting the metric $g_{\mu\nu}$ to the FRW ansatz:

$$g_{\mu\nu} = -N(t)^2 dt^2 + a(t)^2 d\boldsymbol{x}^2, \tag{2.46}$$

where $a(t)$ is the dynamical scale factor, and $N(t)$ is the lapse function. In addition, homogeneity and isotropy force the time and space components of the Stückelberg fields to become

$$\phi^0 = f(t), \quad \phi^i = x^i. \tag{2.47}$$

For the sake of simplicity, we will narrow our focus to three spatial dimensions, as it should be sufficient to show the absence of FRW solutions. Plugging these expressions into (2.41) and setting $\alpha_3 = \alpha_4 = 0$, one obtains the following Lagrangian

$$\mathcal{L} = 3M_{\rm Pl}^2 \Big(- a\dot{a}^2 - m^2 |\dot{f}| \left(a^3 - a^2\right) + m^2 \left(2a^3 - 3a^2 + a\right) \Big) \tag{2.48}$$

where the dot denotes the time derivative. By taking variation with respect to f, we find

$$m^2 \partial_t \left(a^3 - a^2\right) = 0 \tag{2.49}$$

This particular constraint presents a significant limitation, effectively making the evolution of the scale factor a impossible. Even if we did set α_3 and α_4 to be non-zero, the only change we find is the polynomial function $(a^3 - a^2)$, which will still be acted on by ∂_t, hence preventing the time evolution of the scale factor. This outcome is unavoidable, meaning that any alternative setting within the dRGT framework would converge to the same result. Therefore, there exist no significant homogeneous and isotropic solutions for the theory (2.41). This finding underlines a notable limitation of the theoretical framework in question, which points towards the necessity of exploring refinements, enhancements, or potential alternative approaches for a consistent theory of gravity.

2.3 Massive Gravity on de Sitter Space

2.3.1 Fierz-Pauli on de Sitter

Up to this point, our discussion of massive gravity has revolved within the context of the Minkowski background. Transitioning forward, We now demonstrate the construction of massive gravity on de sitter (dS) spacetime. This discussion holds significant importance since the first construction of partially massless gravity is derived from massive gravity on the dS background.

We will start our discussion with the linear Fierz-Pauli formalism of massive gravity on four-dimensional dS spacetime. Consider the Fierz-Pauli action (2.1) expanded around dS, such that $g_{\mu\nu} = \gamma_{\mu\nu} + \tilde{h}_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}/M_{\rm Pl}$, where $\gamma_{\mu\nu}$ is the dS reference metric. Hence, the linearized Fierz-Pauli action around dS reads

$$\mathcal{L}_{FP,dS} = -\frac{1}{4} h^{\mu\nu} (\hat{\mathcal{E}}_{dS})^{\alpha\beta}_{\mu\nu}\, h_{\alpha\beta} - \frac{m^2}{8} \gamma^{\mu\nu}\gamma^{\alpha\beta} \left(H_{\mu\alpha}H_{\nu\beta} - H_{\mu\nu}H_{\alpha\beta}\right) \tag{2.50}$$

where $H_{\mu\nu}$ is the tensor fluctuation defined in (2.27) with setting $f_{\mu\nu} = \gamma_{\mu\nu}$, and $\hat{\mathcal{E}}_{dS}$ is the Lichnerowicz operator on de Sitter, given by

$$
\begin{aligned}
(\hat{\mathcal{E}}_{dS})^{\alpha\beta}_{\mu\nu}\, h_{\alpha\beta} = -\frac{1}{2}\Big[&\Box h_{\mu\nu} - 2\nabla_{(\mu}\nabla_{\alpha}h^{\alpha}_{\nu)} + \nabla_{\mu}\nabla_{\nu}h - \gamma_{\mu\nu}\left(\Box h - \nabla_{\alpha}\nabla_{\beta}h^{\alpha\beta}\right) \\
&+ 6H_0^2\left(h_{\mu\nu} - \frac{1}{2}h\gamma_{\mu\nu}\right)\Big]
\end{aligned}
\tag{2.51}
$$

Where H_0 here is the Hubble parameter, and all the covariant derivatives are with respect to the dS metric, and the indices are raised and lowered with respect to the same metric.

If we consider the helicity decomposition on $H_{\mu\nu}$, we find

$$
\begin{aligned}
H_{\mu\nu} = h_{\mu\nu} &+ 2\frac{\nabla_{(\mu}A_{\nu)}}{m} + 2\frac{\Pi_{\mu\nu}}{m^2} \\
&- \frac{1}{M_{\rm Pl}}\left[\frac{\nabla_{\mu}A_{\alpha}}{m} + \frac{\Pi_{\mu\alpha}}{m^2}\right]\left[\frac{\nabla_{\nu}A_{\beta}}{m} + \frac{\Pi_{\nu\beta}}{m^2}\right]\gamma^{\alpha\beta},
\end{aligned}
\tag{2.52}
$$

with $\Pi_{\mu\nu} = \nabla_{\mu}\nabla_{\nu}\pi$. So, at the linearized level and by neglecting the vector fields, we see that the helicity-0 and -2 modes in the Lagrangian behave as

$$
\begin{aligned}
\mathcal{L}_{FP,dS} = &-\frac{1}{4}h^{\mu\nu}\left(\hat{\mathcal{E}}_{\rm dS}\right)^{\alpha\beta}_{\mu\nu}\, h_{\alpha\beta} - \frac{m^2}{8}\left(h^2_{\mu\nu} - h^2\right) - \frac{1}{8}F^2_{\mu\nu} \\
&- \frac{1}{2}h^{\mu\nu}\left(\Pi_{\mu\nu} - [\Pi]\gamma_{\mu\nu}\right) - \frac{1}{2m^2}\left([\Pi^2] - [\Pi]^2\right).
\end{aligned}
\tag{2.53}
$$

After integration by parts, we can diagonalize the helicity-2 and -0 modes are by setting $h_{\mu\nu} = \bar{h}_{\mu\nu} + \pi\gamma_{\mu\nu}$, which then gives

$$
\begin{aligned}
\mathcal{L}_{FP,dS} = &-\frac{1}{4}\bar{h}^{\mu\nu}\left(\hat{\mathcal{E}}_{\rm dS}\right)^{\alpha\beta}_{\mu\nu}\bar{h}_{\alpha\beta} - \frac{m^2}{8}\left(\bar{h}^2_{\mu\nu} - \bar{h}^2\right) - \frac{1}{8}F^2_{\mu\nu} \\
&- \frac{3}{4}\left(1 - 2\left(\frac{H_0^2}{m^2}\right)\right)\left((\partial\pi)^2 - m^2\bar{h}\pi - 2m^2\pi^2\right).
\end{aligned}
\tag{2.54}
$$

We note that the helicity-0 mode terms now differ from the one on Minkowski (2.8) by a factor $\big(1 - 2(H_0^2/m^2)\big)$.

2.3.2 The Higuchi Bound

By closely inspecting the linearized Fierz-Pauli action on dS as presented in (2.54), we would be able to detect cases where ghosts and tachyons are avoided, but only under specific constraints on the graviton mass m. This is the well-known Higuchi bound [28]. To elaborate further, on dS there exists a defined range of the graviton mass that is not permitted. For instance, in $0 < m^2 < 2H_0^2$ or $m^2 < 0$, the theory always excites at least one ghost (distinct from the BD ghost) that corresponds to a sixth degree of freedom. On the other hand, when $m^2 > 2H_0^2$, we obtain a healthy theory of massive gravity of four-dimensional dS, which propagates five DoF. While if we consider $m^2 = 0$, the theory gracefully reduces to GR. Moreover, for the case when $0 < m^2 < 2H_0^2$, helicity-0 mode propagates a ghost. Also, in the scenario of $m^2 < 0$, the ghost propagates from the helicity-1 mode, and the helicity-2 and -0 modes are tachyonic. Finally and most importantly, the case where the graviton mass lies on the bound $m^2 = 2H_0^2$. In this unique case, the helicity-0 mode loses its kinetic and mass term and hence disappears, resulting in a theory that propagates in only four DoF. This corresponds to the theory of Partially massless gravity [30–32].

Extending these results to an arbitrary number of dimensions, the discussion of the Higuchi bound is then summarized in the following key highlights:

• $m^2 = 0$: General Relativity, with 2 DoF in 4d-spacetime.

• $m^2 > (d-2)H_0^2$: Massive gravity on dS, with 5 DoF in 4d-spacetime

• $m^2 < 0$: Ghostly Helicity-1 modes and tachyonic helicity-2 and -0 modes.

• $m^2 = (d-2)H_0^2$: **Partially massless gravity**, with 4 DoF in 4d-spacetime.

We may now delve deeper and pay more attention to partially massless gravity, starting from the next chapter onward.

Chapter 3

Partially Massless (PM) Gravity

3.1 PM Gravity in Fierz-Pauli

As a simple definition, PM gravity is a theory of massive gravity in dS space with a relation between the graviton mass m and the cosmological constant Λ (or equivalently H_0)

$$m^2 = \frac{2\Lambda}{(D-1)}, \quad \Lambda = \frac{(D-1)(D-2)}{2} H_0^2 \tag{3.1}$$

PM gravity was first proposed in 2001 by Deser and Waldron [30–32] to avoid the issues coming from the linear Fierz-Pauli action in Minkowski background (2.1.2). We have demonstrated the existence of PM gravity from linearized Fierz-Pauli on the dS background in section (2.3). In that linearized PM gravity, by substituting (3.1) in (2.54), the resulting action would be

$$\mathcal{L}^{\text{PM}}_{\text{FP,dS}} = -\frac{1}{4}\bar{h}^{\mu\nu}\left(\hat{\mathcal{E}}_{\text{dS}}\right)^{\alpha\beta}_{\mu\nu}\bar{h}_{\alpha\beta} - \frac{m^2}{8}\left(\bar{h}^2_{\mu\nu} - \bar{h}^2\right) - \frac{1}{8}F^2_{\mu\nu}, \tag{3.2}$$

where the condition of the mass eliminates the helicity-0 mode. One would also find a new scalar gauge symmetry [27]

$$h_{\mu\nu} \to h_{\mu\nu} + \delta h_{\mu\nu} = h_{\mu\nu} + \left(\nabla_\mu \nabla_\nu + \frac{m^2}{D-2}\gamma_{\mu\nu}\right)\alpha, \tag{3.3}$$

with α as a scalar gauge parameter. This result is generalizable to Anti-de Sitter (AdS) [29, 32, 34, 52, 55]. We also highlight that the condition of PM gravity (3.1) forbids the existence of a bare cosmological constant. This raises the interest in investigating the capability of PM gravity to solve the old cosmological constant problem [56, 57]. Indeed, a linear theory is insufficient, and we still need to look for a non-linear extension of PM gravity.

If one analyzes the non-linear Fierz-Pauli mass term presented in section (2.1.3) in dS space, the helicity-0 mode will reappear and be infinitely strongly coupled with the helicity-1 mode. This is inconsistent with the number of DoFs of the linear theory and is thus a sick theory [4]. Consequently, one would look for an alternative non-linear PM gravity theory with a gauge symmetry like the linear theory at all orders without propagating any ghosts [36].

3.2 Looking for Non-linear PM Gravity

3.2.1 Motivation

The desire to achieve a consistent non-linear completion of PM gravity emerges from its potential theoretical refinement and profound implications for understanding cosmological phenomena. One primary motivation is the suggestive resolution of the old cosmological constant problem [56, 57]. To elaborate further, the cosmological constant Λ is a parameter viewed as the curvature of empty spacetime. As highlighted by Zel'dovich in 1968 [58], we can express the energy-momentum tensor of the vacuum as

$$T_{\text{vac}}^{\mu\nu} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}, \tag{3.4}$$

where G is Newton's gravitational constant. Note that the equation (3.4) gives us an interpretation that the cosmological constant corresponds to the vacuum energy density ρ_{vac}; thus, we can calculate a theoretical value for ρ_{vac} and then compare it with observations. Now, assuming the accuracy of quantum field theory up to the Planck scale, then the scale of 10^{19} GeV would serve as an ultraviolet cutoff in all field theory processes, such that the energy density can be calculated by

$$\langle \rho \rangle = \int_0^{10^{19}\text{GeV}} \frac{d^3k}{2(2\pi)^3} \sqrt{k^2 + m^2}. \tag{3.5}$$

This established cutoff results in a vacuum energy density $\rho_{\text{vac}} \approx 10^{76}$ GeV. Comparing to the currently observed value ($\rho_{\text{vac}} \approx 10^{-47}$ GeV), this is a disaster. It means that if ρ_{vac} was expressed in Planck units, then we need to fine-tune the cosmological constant Λ to around 120 decimal places in order to match the observed result. Even using the lowest possible cutoff, such as the quantum chromodynamics scale ($\sim$ 200 MeV), would still give a mismatch of around 40 orders of magnitude [59]. Hence, we must approach this problem from a different theory, and non-linear PM gravity is a promising candidate. This is because the presence of the PM gauge symmetry (provided there are no anomalies) guarantees that the quantum corrections must maintain the relation (3.1), and the old cosmological constant problem could be tackled in a more manageable way by the PM gauge symmetry [36].

Another motivation to seek a non-linear completion of PM gravity comes from the properties of the linearized theory that could be extended non-linearly. For instance, there would be no vDVZ discontinuity, so the theory would not face any of the issues presented in section (2.1.2) in the limit $m \to 0$, provided that we maintain the PM relation (3.1), and thus there is no need for the Vainshtein mechanism presented in section (2.1.3). This would allow us to avoid the complexities of the strong coupling effects [4, 36]. In addition, ghost-free theories of massive gravity have a hidden strong interaction scale at low energies, which makes their consistency in the high-energy regime problematic [20–22]. Moreover, The cosmological solutions of dRGT theory (2.42) are generically unstable [23, 24], and have no cosmological solutions in the FRW regime, which was demonstrated previously in section (2.2.3), meaning that we still need to seek an alternative theory that possesses consistent and stable cosmological solutions. Hence, a consistent non-linear theory of PM gravity has important theoretical and physical implications [25].

3.2.2 General Approach

To search for a consistent theory of non-linear PM gravity, the initial step would be to consider the most general form of the action

$$\mathcal{S} = \int d^d x \mathcal{L} = \int d^d x (\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{PM}}), \tag{3.6}$$

where $\mathcal{L}_{\text{EH}}$ is the Einstein-Hilbert kinetic term

$$\mathcal{L}_{\text{EH}} = \frac{M_{Pl}^{(D-2)}}{2} \sqrt{-g}(R[g] - 2\Lambda), \tag{3.7}$$

and $\mathcal{L}_{\mathrm{PM}}$ is the PM Lagrangian of the theory that requires to be diffeomorphism invariant. In addition, the full action (3.6) must be invariant under the non-linear extension of the PM gauge symmetry

$$\delta_\alpha \mathcal{S} = 0 \tag{3.8}$$

where δ_α is the non-linearly deformed PM gauge transformation extended from (3.3). Both the action $\mathcal{S}$ and the transformation δ_α can be expanded in orders of the field $h_{\mu\nu}$

$$\begin{aligned} \delta_\alpha &= \delta_\alpha^{(0)} + \delta_\alpha^{(1)} + \delta_\alpha^{(2)} + \cdots \\ \mathcal{S} &= \mathcal{S}_{(0)} + \mathcal{S}_{(1)} + \mathcal{S}_{(2)} + \cdots \end{aligned} \tag{3.9}$$

The aim is to formulate $\mathcal{L}_{\mathrm{PM}}$ together with a consistent deformed PM gauge transformation δ_α so that if a PM theory of gravity exists, it must be formulated in a way that the non-linear action respects the gauge invariance (3.8) at all orders [41].

3.2.3 Candidate from dRGT Theory

One approach to look for PM gravity is by taking a ghost-free theory of massive gravity and then modifying it into PM gravity. To check if the resulting theory consistently describes PM gravity, one can focus on the resulting gauge symmetry, and the consistent theory should be able to extend the linear PM gauge symmetry (3.3) to the interacting level [37]. The recent developments in ghost-free massive gravity theories allow us to test them in the partially massless case. For example, the dRGT theory presented in section (2.2) has been considered a candidate for non-linear PM gravity [36], and its non-linear PM action was formulated as

$$\mathcal{L}_{\rm PM} = \frac{M_{\rm Pl}^{D-2}}{2}\sqrt{-g}\left(R[g] - 2\Lambda - \frac{m^2}{4}\sum_{n=0}^{d}\beta_n S_n\left(\sqrt{g^{-1}\bar{g}}\right)\right) \tag{3.10}$$

where $g_{\mu\nu}$ is the dynamical metric, $\bar{g}_{\mu\nu}$ is the reference dS metric, and $S(M)_n$ are the symmetric polynomials of a matrix M^μ_ν, given by

$$S_n(M) = M^{[\mu_1}_{\mu_1}\cdots M^{\mu_n]}_{\mu_n}, \tag{3.11}$$

with $S_0 \equiv 1$. Note that when setting the PM condition (3.1) in the Lagrangian, a unique choice of the constants β

$$\beta_0 = -4(D-1), \quad \beta_2 = \frac{8}{D-2}, \quad \beta_{n\neq 0,2} = 0 \tag{3.12}$$

eliminates the helicity-0 mode in the decoupling limit, making the theory a candidate for non-linear PM gravity [52]. The Lagrangian (3.10) becomes

$$\mathcal{L}_{\rm PM} = \frac{M_{\rm Pl}^{D-2}}{2}\sqrt{-g}\left(R[g] - 2H_0^2 S_2\left(\sqrt{g^{-1}\bar{g}}\right)\right) \tag{3.13}$$

3.2.4 Non-linear Gauge Symmetry in Mini-superspace

Indeed, it is challenging to test the non-linear PM candidate for an arbitrary metric. However, the analysis in [36] simplifies the approach by choosing a particular ansatz for the dynamic metric and verifying the presence of PM symmetry. Thus, we can introduce the theory in a simplified case of mini-superspace that is also useful in the context of cosmology i.e., the Friedmann–Robertson–Walker (FRW) ansatz: homogeneous and isotropic universe [60–63].

We start from the Lagrangian (3.13) and replace H_0^2 with an arbitrary coefficient λ. This replacement will help us later to investigate the case $\lambda \to H_0^2$. We enforce the flat FRW ansatz in the flat slicing for the reference dS metric. The two metric tensors $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are now given by

$$g_{\mu\nu} = \begin{pmatrix} -N^2(t) & 0 \\ 0 & a^2(t)\delta_{ij} \end{pmatrix} \quad \bar{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & e^{2H_0 t}\delta_{ij} \end{pmatrix} \tag{3.14}$$

where $a(t)$ is the dynamical scale factor, and $N(t)$ is the lapse function. Plugging all of this in (3.13), and with the mini-superspace ansatz, the action — up to a total derivative — reads

$$\begin{aligned} \mathcal{S} = \frac{M_{\rm Pl}^{D-2}}{2} \int dt \Big[&- \frac{(D-1)(D-2)}{N}(a^{D-3}\dot{a}^2) \\ &- 2\lambda(D-1)a^{D-3}e^{H_0 t}\left(a + \frac{D-2}{2}Ne^{H_0 t}\right)\Big]. \end{aligned} \tag{3.15}$$

We can remove $N(t)$ using its equation of motion. i.e., substituting

$$N = \frac{1}{\sqrt{\lambda}} e^{-H_0 t}\dot{a} \tag{3.16}$$

and up to a total derivative, the action will become

$$\mathcal{S} = M_{\rm Pl}^{D-2} \int \mathrm{d}^d x \left[(D-1)\sqrt{\lambda}e^{H_0 t}a^{D-2}(H_0 - \sqrt{\lambda})\right]. \tag{3.17}$$

Note that for $\lambda \neq H_0^2$, the equations of motion for a indicate that $a = 0$, and the action will be inconsistent. This is a result of the fact that dRGT theory does not have consistent FRW solutions [54]. Nevertheless, for the special case $\lambda = H_0^2$, the Lagrangian in (3.17) becomes empty. Meaning that the theory possesses a gauge symmetry with gauge parameter $\epsilon(t)$ that allows us to determine the value of the scale factor arbitrarily

$$\delta a = \epsilon, \quad \delta N = \frac{1}{H_0} e^{-H_0 t} \dot{\epsilon}, \tag{3.18}$$

implying that any arbitrary function is a solution for the scale factor, with the lapse function determined by (3.16). Thus, the transformation (3.18) should be the full non-linear PM gauge symmetry in the mini-superspace form. The action (3.16) also implies that the PM Lagrangian is invariant under the metric transformation that is first order in derivatives

$$\delta g_{\mu\nu} = \mathrm{diag}\left(-2N \frac{1}{H_0} e^{-H_0 t} \dot{\epsilon}, 2a\epsilon, \cdots\right). \tag{3.19}$$

By performing the change of variables

$$\epsilon = \frac{1}{2} e^{H_0 t} \left(-H_0 \dot{\alpha} + H_0^2 \alpha\right) \tag{3.20}$$

The transformation (3.19) can be written as

$$\delta g_{\mu\nu} = \left(\bar{\nabla}_\mu \bar{\nabla}_\nu + \frac{m^2}{D-2} \bar{g}_{\mu\nu}\right) \alpha, \tag{3.21}$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$. Remarkably, this transformation is consistent with the linear PM gauge symmetry (3.3). This is a shred of evidence that supports the theory (3.10) to be considered as a theory of non-linear PM gravity. But we still need to test the theory further.

3.2.5 Obstructions

The authors in [36] applied a systematic analysis of the candidate away from the decoupling limit and for an arbitrary dynamical metric $g_{\mu\nu}$. This should determine whether the theory can be considered a non-linear completion of PM gravity. The investigation starts by expanding the Lagrangian (3.10) in orders of h

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm m} \\ &= \frac{M_{\rm Pl}^{D-2}}{2}\left[\sqrt{-g}(R[g]-2\Lambda) - \frac{m^2}{4}\sqrt{-\bar{g}}\left(\mathcal{L}_{\rm m}^{(2)} + \mathcal{L}_{\rm m}^{(3)} + \mathcal{L}_{\rm m}^{(4)} + \cdots\right)\right],\end{aligned} \tag{3.22}$$

with

$$\begin{aligned}\mathcal{L}_{\rm m}^{(2)} &= b_1\left\langle h^2\right\rangle + b_2\left\langle h\right\rangle^2, \\ \mathcal{L}_{\rm m}^{(3)} &= c_1\left\langle h^3\right\rangle + c_2\left\langle h^2\right\rangle\left\langle h\right\rangle + c_3\left\langle h\right\rangle^3, \\ \mathcal{L}_{\rm m}^{(4)} &= d_1\left\langle h^4\right\rangle + d_2\left\langle h^3\right\rangle\left\langle h\right\rangle + d_3\left\langle h^2\right\rangle^2 + d_4\left\langle h^2\right\rangle\left\langle h\right\rangle^2 + d_5\left\langle h\right\rangle^4, \\ &\cdots\end{aligned} \tag{3.23}$$

where $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ and the indices raised and lowered with respect to the dS metric $\bar{g}_{\mu\nu}$. i.e., $\langle h\rangle = \bar{g}^{\mu\nu}h_{\mu\nu}$. Note that we have applied the interchange $g_{\mu\nu} \leftrightarrow \bar{g}_{\mu\nu}$ on the mass term in (3.10) as the Lagrangian enjoys a Z_2 symmetry under the interchange of $g_{\mu\nu} \leftrightarrow \bar{g}_{\mu\nu}$ [64]. Now, if a PM theory of gravity exists, it must have a scalar gauge symmetry and be invariant under a transformation

$$\delta h_{\mu\nu} = \hat{L}_{\mu\nu}\alpha, \tag{3.24}$$

where α is the gauge parameter and L is some operator that can be expanded in powers of h,

$$\hat{L} = \hat{L}^{(0)} + \hat{L}^{(1)} + \hat{L}^{(2)} + \cdots \tag{3.25}$$

The gauge invariance will give us the Bianchi identity

$$\hat{L}_{\mu\nu}\frac{\delta\mathcal{L}}{\delta h_{\mu\nu}} = \hat{L}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}\left(G^{\mu\nu}+\Lambda g^{\mu\nu}\right)+\frac{\delta\mathcal{L}_{\rm m}}{\delta h_{\mu\nu}}\right] = 0. \tag{3.26}$$

The idea is to solve order-by-order and set the right coefficients of the mass term and for the operator $\hat{L}$ to satisfy the identity. This is a brute force calculation, which has been done by [36]. We summarize the results by the following:

• At quadratic order, the gauge invariance requirement (3.26) reads

$$\hat{L}^{(0)}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}\left(G^{\mu\nu}+\Lambda g^{\mu\nu}\right)\bigg|_{(1)}+\frac{\delta\mathcal{L}^{(2)}_{\rm m}}{\delta h_{\mu\nu}}\right] = 0. \tag{3.27}$$

where $\mathcal{L}^{(2)}_{\rm m}$ is given by (3.23) and the subscript $|_{(n)}$ indicates the expansion to nth in $h_{\mu\nu}$. We consider the most general form of $\hat{L}^{(0)}$ (up to two derivatives) is

$$\hat{L}^{(0)}_{\mu\nu}\alpha = B_1\bar{\nabla}_\mu\bar{\nabla}_\nu\alpha + B_2\bar{g}_{\mu\nu}\alpha + B_3\bar{g}_{\mu\nu}\bar{\Box}\alpha, \tag{3.28}$$

with B_i are constants determined to satisfy (3.27). By choosing the coefficients for both $\mathcal{L}^{(2)}_{\rm m}$ and $\hat{L}^{(0)}$ as

$$b_1 = -b_2 = \frac{2\Lambda}{D-1}, \quad B_2 = \frac{2\Lambda}{(D-1)(D-2)}B_1, \quad B_3 = 0, \tag{3.29}$$

the theory succeeds in being consistent at the quadratic order of $h_{\mu\nu}$. One also finds that the transformation law at the lowest order becomes

$$\delta^{(0)}h_{\mu\nu} = \bar{\nabla}_\mu\bar{\nabla}_\nu\alpha + \frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}\alpha, \tag{3.30}$$

which is the same transformation law found from linear Fierz-Pauli in dS (3.3) with $m^2 = 2\Lambda/(D-1)$.

• For the cubic order, gauge invariance statement (3.26) becomes

$$\begin{aligned}&\hat{L}^{(0)}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}\,(G^{\mu\nu}+\Lambda g^{\mu\nu})\Big|_{(2)}+\frac{\delta\mathcal{L}^{(3)}_{\rm m}}{\delta h_{\mu\nu}}\right]\\ +&\hat{L}^{(1)}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}\,(G^{\mu\nu}+\Lambda g^{\mu\nu})\Big|_{(1)}+\frac{\delta\mathcal{L}^{(2)}_{\rm m}}{\delta h_{\mu\nu}}\right]=0.\end{aligned} \tag{3.31}$$

We have already found the right form of $\hat{L}^{(0)}$ (3.28) and its coefficients (3.29). Here, we need to work out $\hat{L}^{(1)}$. The most general form of $\hat{L}^{(1)}$ (also up to two derivatives) has 18 terms

$$\begin{aligned}&\hat{L}^{(1)}_{\mu\nu}\alpha = C_1 h_{\mu\nu}\bar{\Box}\alpha + C_2 h_{(\mu}{}^{\lambda}\bar{\nabla}_{\nu)}\bar{\nabla}_{\lambda}\alpha + C_3 h\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\alpha\\ &+\cdots + C_{17}h_{\mu\nu}\alpha + C_{18}\bar{g}_{\mu\nu}h\alpha,\end{aligned} \tag{3.32}$$

where $h_{\mu\nu}$ is raised and lowered with respect to $\bar{g}_{\mu\nu}$ Plugging this expression in (3.31) would fix some of the coefficients but not all. We find that the gauge invariance would be satisfied, and the right coefficients would work only when $D=4$. This agrees with the result found in [65, 66], which states that conformally invariant theories of non-linear PM gravity can only exist in four dimensions. We Also find that the transformation law in that order reads

$$\begin{aligned}\delta^{(1)}h_{\mu\nu} = &\frac{1}{2}\,h_{(\mu}{}^{\lambda}\bar{\nabla}_{\nu)}\bar{\nabla}_{\lambda}\alpha - \frac{1}{2}\bar{\nabla}_{(\mu}h_{\nu)\lambda}\bar{\nabla}^{\lambda}\alpha\\ &+\frac{1}{2}\bar{\nabla}_{\lambda}h_{\mu\nu}\bar{\nabla}^{\lambda}\alpha - \frac{\Lambda}{2}\frac{D-6}{(D-1)(D-2)}h_{\mu\nu}\alpha\end{aligned} \tag{3.33}$$

• In the quartic order, the gauge invariance requirement (3.26) becomes

$$\hat{L}^{(0)}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}(G^{\mu\nu}+\Lambda g^{\mu\nu})\Big|_{(3)}+\frac{\delta\mathcal{L}^{(4)}_{\rm m}}{\delta h_{\mu\nu}}\right]+\hat{L}^{(1)}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}(G^{\mu\nu}+\Lambda g^{\mu\nu})\Big|_{(2)}+\frac{\delta\mathcal{L}^{(3)}_{\rm m}}{\delta h_{\mu\nu}}\right]$$
$$+\hat{L}^{(2)}_{\mu\nu}\left[-\frac{1}{2}\sqrt{-g}\,(G^{\mu\nu}+\Lambda g^{\mu\nu})\Big|_{(1)}+\frac{\delta\mathcal{L}^{(2)}_{\rm m}}{\delta h_{\mu\nu}}\right]=0, \tag{3.34}$$

where the most general form of $\hat{L}^{(2)}$ has 72 terms. Unfortunately, with all possible choices of these coefficients and even with $D=4$, it is impossible to satisfy the gauge invariance at this level. Hence, the candidate fails the test.

Various theories have been tested as candidates for non-linear PM gravity. But they also fail to realize a non-linear PM gauge symmetry [65, 67–69]. Thus, a consistent theory of non-linear PM gravity is yet to be found.

Chapter 4

Non-unitary Multi-field PM Gravity

4.1 Indications of a Consistent Theory

The consecutive fails in finding a consistent non-linear theory of PM gravity emphasize the necessity of considering different approaches. One method suggests that the higher-order inconsistencies could be avoided by considering theories with multiple PM spin-2 fields. Prior literature, specifically [37, 65], considered using models with unitary interacting PM fields, where the theory is described as 'unitary' when there are positive signs between the kinetic terms of fields, ensuring the conservation of total probability in processes [70]. Yet, even when introducing multiple spin-2 fields, we run into no-go results. Nevertheless, the authors in [37] highlighted an exception: the case of 'non-unitarity', where kinetic terms possess a relatively negative sign. This scenario brings us into the realm of Conformal Gravity, which is known to be a non-unitary theory due to the wrong relative sign between the Einstein-Hilbert kinetic term and the PM term. Conformal gravity has proven to provide a natural arena for studying the PM spin-2 field when expanded on dS backgrounds [71, 72], making non-unitarity a considerable approach. The authors in [38] also found that the consistency of the gauge symmetry is not possible unless we give up unitarity. One would now think: is it plausible to relax the condition of unitarity in our theories? Lee and Wick's work [73] provides an affirmation, illustrating that non-unitary theories can still lead to a unitary scattering matrix, hence preserving probability amplitudes and avoiding pathologies. This perspective has gained more support in later studies [74–76]. Nowadays, we believe that non-unitarity is not necessarily a pathology. Thus, relaxing the condition of unitarity might pave the way to a consistent non-linear theory of PM gravity that can generalize the PM gauge symmetry non-linearly.

4.2 Candidate Theory

4.2.1 Formalism

In a groundbreaking development, Boulanger, Deffayet, Garcia-Saenz, and Traina have recently constructed a non-unitary Yang-Mills-like theory of multiple PM spin-2 fields [41]. We will refer to their work as 'Non-unitary multi-field PM gravity'. To clarify the construction of this theory, We will first present the formalism of the unitary version of the theory. Next, we relax the unitary condition, and finally, we display a proof of consistency for the non-unitary theory.

Yang-Mills-like spin-2 theory

The foundations of the promising candidate trace back to the formalism of the unitary Partially Massless Spin-2 Yang–Mills [40]. The motivation behind this method lies within the similarities observed between the linearized PM theory (3.2) and Electrodynamics (EM), such as their propagation at the speed of light (often called null propagation) [32], exhibiting scalar gauge symmetry, duality invariance [77, 78], monopole solutions [79], and most importantly, the difficulties in constructing self-interactions, where one cannot find non-trivial multi-point interaction terms without manifestly breaking the gauge invariance [80]. Notably, the issue in EM has been resolved by Yang-Mills theory [81] with having multiple photon fields A^a with the index $a = 1, 2, \cdots, N$ raised and lowered by the Kronecker-delta δ_{ab}. The Lagrangian of the theory is

$$\begin{aligned}\mathcal{L}_{\rm YM} = &-\frac{1}{4}\left(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)\\ &- g f_{abc}\partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{g^2}{4} f_{abc} f^a_{de} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e},\end{aligned} \tag{4.1}$$

where g is the coupling constant and f_{abc} is the structure constant. This construction deforms the abelian gauge symmetry to a non-abelian gauge symmetry. The transformation of the photon field is then

$$\delta A^a_\mu = \partial_\mu \alpha^a + f_{bc}{}^a A^b_\mu \alpha^c. \tag{4.2}$$

Note that for the Lagrangian (4.1) to be consistent with this gauge symmetry, it requires the structure constant to be fully antisymmetric and to satisfy the Jacobi identity $f_{ac}{}^d f_{be}{}^c + f_{bc}{}^d f_{ea}{}^c + f_{ec}{}^d f_{ab}{}^c = 0$. We also note that Yang-Mills theory could be constructed by defining a non-abelian version of the field strength tensor $F^a_{\mu\nu}$, which contains quadratic powers of the field

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu, \tag{4.3}$$

and transforms covariantly under the gauge symmetry

$$\delta F^a_{\mu\nu} = f^a_{bc}\alpha^b F^c_{\mu\nu}, \tag{4.4}$$

where α is the scalar gauge parameter.

With the formalism above, we may construct a unitary Yang-Mills-like theory for multiple PM fields. Starting by defining the gauge invariant PM field strength tensor $F_{\mu\nu}$ [82]

$$F_{\mu\nu\rho} = \nabla_\mu h_{\nu\rho} - \nabla_\nu h_{\mu\rho}, \tag{4.5}$$

and promoting it to a non-abelian version for multiple fields

$$F^a_{\mu\nu\rho} = \nabla_\mu h^a_{\nu\rho} - \nabla_\nu h^a_{\mu\rho} + \mathcal{O}\left(h^2\right), \tag{4.6}$$

where $\mathcal{O}\left(h^2\right)$ are higher order terms of h, whose exact forms remain to be specified. Now, the non-linear action for multiple PM spin-2 fields is considered to be

$$\mathcal{S} = -\frac{1}{4}\int \mathrm{d}^D x\sqrt{-g}\left(F_a^{\lambda\mu\nu}F^a_{\lambda\mu\nu} - 2F_a^{\lambda\mu}{}_\mu F^{a\ \nu}_{\lambda\nu}\right). \tag{4.7}$$

with the greek indices, $\mu, \nu ...$ are raised and lowered by the dS metric $g_{\mu\nu}$ The idea is to find a non-abelian PM gauge symmetry that is consistent with the theory at all orders. Unfortunately, even by trying all possible combinations of higher-order terms in (4.6), such symmetry could not be found (see [40] for details). Hence, the Yang-Mills-like theory was considered a no-go. Nevertheless, the results differ after relaxing the condition of unitarity, as we shall see next.

Enforcing non-unitarity

The work of [41] starts by writing the action (4.7) with a tensor k_{ab}: a diagonal N by N matrix with entries +1 and -1. This would be enough to enforce non-unitarity on the Lagrangian terms. The undeformed action for a collection of N PM fields is then

$$\mathcal{S}_0 = -\frac{1}{4}\int d^D x\sqrt{-g}k_{ab}\left[F^{a\mu\nu\rho}F^b_{\mu\nu\rho} - 2F^{a\mu}F^b_{\mu}\right], \tag{4.8}$$

where

$$F^a_{\mu\nu\rho} = \nabla_\mu h^a_{\nu\rho} - \nabla_\nu h^a_{\mu\rho}, \quad F^a_\mu = g^{\nu\rho}F^a_{\mu\nu\rho}. \tag{4.9}$$

Noticing that the unitary theory (4.7) had $k_{ab} = \delta_{ab}$, although we see that this choice leads to a no-go. The objective of this analysis is to extend the action with nontrivial interactions while preserving the gauge symmetry. This requires using the general approach presented in section (3.2.1). Schematically, we modify the linear PM gauge transformation with field-dependent terms, such that

$$\delta_\alpha = \delta^{(0)}_\alpha + \delta^{(1)}_\alpha + \delta^{(2)}_\alpha + \cdots, \tag{4.10}$$

where (n) denotes the order in a gauge parameter ϵ, and $\delta^{(0)}_\alpha$ is required to be kept as the linear PM gauge transformation (3.3). i.e.,

$$\begin{aligned}\delta^{(0)}_\alpha h^a_{\mu\nu} &= \nabla_\mu\nabla_\nu\alpha^a + \frac{2\Lambda}{(D-1)(D-2)}g_{\mu\nu}\alpha^a \\ &= \nabla_\mu\nabla_\nu\alpha^a + (H_0)^2\, g_{\mu\nu}\alpha^a\end{aligned} \tag{4.11}$$

here, $\frac{\sigma}{L^2}$ is defined as the curvature scalar background. We are required to form (4.10) in such a way that the non-linear action $\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(1)} + \mathcal{S}_{(2)} + \cdots$ will respect the gauge invariance $\delta_\alpha \mathcal{S} = 0$.

We have already found $\mathcal{S}_/$ in (4.7), and we can observe that it is quadratic in the field h. Hence, we expect $\mathcal{S}_\infty$ to be cubic in h (which encodes the cubic interactions), and so on. We may proceed with a systematic analysis just like we did with the dRGT candidate in section (3.2.5). However, we would like to present a different (and more convenient) way to prove the consistency, known as the Closure Condition.

Proof of consistency: Closure condition

One method to confirm the consistency of this theory is the so-called 'Closure Condition': A powerful tool to search for the nonlinear deformations of the PM gauge symmetry. It is the requirement that the gauge symmetries must form an algebra up to on-shell trivial symmetries. This provides a set of non-trivial relations that for any two gauge parameters α and β, the equation

$$\left[\delta_\alpha, \delta_\beta\right] h^a_{\mu\nu} = \delta_\chi h^a_{\mu\nu} + \text{on shell trivial} \tag{4.12}$$

holds for some functional χ that depends on the gauge parameters α and β, where 'on shell trivial' refers to gauge transformations that vanish on shell, maintaining the invariance of the action. This would be more convenient in practice than trying to find the gauge invariance on the action at each order. The power of the closure condition lies in its generality, as one does not need to make any assumptions regarding the action of the nonlinear theory. The only assumptions used are that the full transformation involves terms with up to two derivatives, and the gauge transformation must reduce to the linear PM symmetry (3.3) at zeroth order in powers of the field. Here, we will use the closure condition to extend the gauge algebra, gauge symmetry, and the action to non-trivial order in the fields. We note that χ, and consequently the gauge algebra, can be constrained based on algebraic considerations only, even without preliminary information about the possible deformations of the gauge symmetry itself.

Let us consider our deformations in orders of a parameter ϵ. At zeroth order of ϵ, we would have

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\, h^a_{\mu\nu} = 0 + O(\alpha), \tag{4.13}$$

meaning that the algebra of the free theory is abelian, which is the same result of the unitary theory (4.7).

At first order of ϵ, the unique candidate extension to the closure condition would be given by

$$[\delta_{\alpha_1}, \delta_{\alpha_2}]\, h^a_{\mu\nu} = \delta^{(0)}_\chi h^a_{\mu\nu} + O\left(\epsilon^2\right) \tag{4.14}$$

with $\delta^{(0)}_\chi$ specified by (4.11), and χ is given as

$$\chi = \epsilon\left(m^a{}_{bc}\alpha_1^b\alpha_2^c + n^a{}_{bc}\nabla^\mu\alpha_1^b\nabla_\mu\alpha_2^c\right) + O\left(\epsilon^2\right), \tag{4.15}$$

where $m^a_{bc} = m^a_{[bc]}$ and $n^a_{bc} = n^a_{[bc]}$ correspond to the structure constants of the gauge algebra. Further constraints will arise on the structure constants by demanding that the algebra is represented in the fields through an infinitesimal gauge symmetry. We discover that this condition is quite strong and will lead to

$$m^a{}_{bc} = n^a{}_{bc} = 0, \tag{4.16}$$

meaning that the gauge algebra cannot accommodate any non-Abelian extensions. This circles back to the no-go result from the unitary theory (4.12) and implies the necessity to extend the gauge algebra $\delta_\alpha^{(1)} + \delta_\alpha^{(2)} + \cdots$ and the action to higher order terms $\mathcal{S}_{(2)} + \cdots$.

We examine the first-order deformation $\delta_\alpha^{(1)}$ by restricting ourselves to contractions that are linear in field-strength tensor $F^a_{\mu\nu\rho}$ and containing no more than two derivatives. The most general form of the transformation $\delta_\alpha^{(1)}$ is then

$$\begin{aligned}\delta_\alpha^{(1)} h^a_{\mu\nu} =& \epsilon \left(u^a_{(1)bc} \nabla^\rho F^b_{\rho(\mu\nu)} \alpha^c + u^a_{(2)bc} \nabla_{(\mu} F^b_{\nu)} \alpha^c \right. \\ & + u^a_{(3)bc} g_{\mu\nu} \nabla^\rho F^b_\rho \alpha^c + v^a_{(1)bc} F^b_{\rho(\mu\nu)} \nabla^\rho \alpha^c \\ & \left. + v^a_{(2)bc} F^b_{(\mu} \nabla_{\nu)} \alpha^c + v^a_{(3)bc} g_{\mu\nu} F^b_\rho \nabla^\rho \alpha^c \right) ,\end{aligned} \tag{4.17}$$

where the constants $u^a_{(i)bc}$ and $v^a_{(i)bc}$ are arbitrary for now. By forcing the closure condition with the existence of non-trivial cubic terms with no more than two derivatives, we demand that the constants are then

$$v^a_{(1)bc} \equiv f^a_{b,c}, \quad u^a_{(i)bc} = v^a_{(i \neq 1)bc} = 0. \tag{4.18}$$

Hence, the gauge transformation in that order is

$$\delta_\alpha^{(1)} h^a_{\mu\nu} = \epsilon f^a{}_{b,c} F^b_{\rho(\mu\nu)} \nabla^\rho \alpha^c, \tag{4.19}$$

with forcing the structure constants $f^a{}_{b,c}$ to be symmetric under the exchange of the first two indices. All these non-trivial conditions are only satisfied when the spacetime dimension is $D = 4$. The cubic term of the action is then given by

$$\mathcal{S}_1 = \int d^4x \sqrt{-g} h^a_{\mu\nu} J^{\mu\nu}_a, \tag{4.20}$$

where

$$J_a^{\mu\nu} = f_{bc,a}\bigg[F^{b\mu}{}_{\rho\sigma}F^{c\nu\rho\sigma} - F^{b\mu}F^{c\nu} - F^{b\rho(\mu\nu)}F^c_\rho \\ - \frac{1}{4}g^{\mu\nu}F^{b\rho\sigma\lambda}F^c_{\rho\sigma\lambda} + \frac{1}{2}g^{\mu\nu}F^{b\rho}F^c_\rho\bigg]. \tag{4.21}$$

We can easily check that this cubic action satisfies the PM gauge invariance up to the expansion $\delta^{(0)}$. i.e., one can easily check that

$$(\delta^{(0)} + \delta^{(1)})\mathcal{S}_1 = 0. \tag{4.22}$$

We observe that $J_a^{\mu\nu}$ is manifestly invariant under the undeformed PM symmetry $\delta^{(0)}$, which defines a conserved current in the following sense

$$\nabla_\mu\nabla_\nu J_a^{\mu\nu} + (H_0)^2 g_{\mu\nu}J_a^{\mu\nu} \approx 0 \tag{4.23}$$

where the notion $\approx$ is an equality modulo the equations of motion of the free theory.

4.2.2 Consistency of the Gauge Symmetry

We have identified the most general first-order transformation of the PM spin-2 gauge symmetry and classical action in that order. Now, we will examine higher orders. To do so, We must recall that the statement that a gauge symmetry must be consistent with an algebra leads to further constraints at higher orders. For example, we have discussed in Yang-Mills theory (4.1) that the consistency of the gauge symmetry in higher orders implies that the structure constants $f^a{}_{bc}$ must satisfy the Jacobi identity. This means that there must be a similar constraint to the structure constants in the first-order PM gauge transformation (4.19) and the action (4.20) where the symmetry is preserved.

This ‘consistency requirement’ is found to be:

$$f_{ae,b}f^{e}_{c,d} \equiv k^{ef} f_{ea,b} f_{fc,d} = 0. \tag{4.24}$$

We can readily see two simple, yet important outcomes:

• In the case of a single field (N = 1), we immediately find that the only solution is $f_{11,1} = 0$, meaning that for a single PM spin-2 field, it is impossible to extend the PM gauge symmetry beyond the lowest order $\delta^{(0)}$. This is the no-go result we encountered in section (3.2.5).

• For multiple PM spin-2 fields, setting $k_{ab} = \delta_{ab} = \mathrm{diag}(+1, +1, \cdots, +1)$ will lead to the result that all structure constants must be zero. i.e., $f_{ab,e} = 0, \ \ \forall \ a, b, e \in \{1, 2 \cdots, N\}$. Which takes us back to the no-go result of the unitary theory (4.7). The consistency requirement of the gauge symmetry can only be achieved if at least one or more of the fields enter the action with a negative sign on its kinetic term, forcing non-unitarity to avoid the no-go results.

Hence, the requirement (4.24) shows great rigor by the ability to re-establish the previous no-go results and provides a rigorous resolution to no-go results of PM gravity.

Applying the consistency requirement

We now have the ability to use the consistency requirement (4.24) to maintain the gauge invariance of the action at higher orders. We will demonstrate this by taking the statement of the gauge invariance

$$\delta_\alpha \mathcal{S} = (\delta^{(0)}_\alpha + \delta^{(1)}_\alpha)(\mathcal{S}_0 + \mathcal{S}_1) = 0. \tag{4.25}$$

This statement would be second order in ϵ. Meaning it extends the deformation to quadratic order. By plugging in (4.12) and (4.20), along with the transformations (4.11) and (4.19), we find that

$$(\delta_{\alpha}^{(0)} + \delta_{\alpha}^{(1)})(\mathcal{S}_0 + \mathcal{S}_1) = 2\epsilon^2 f_{ab,e} f^{e}{}_{c,d} \int d^4x \sqrt{-g} \left[F^{a\mu}{}_{\rho\sigma} F^{b\nu\rho\sigma} F^{c}_{\lambda\mu\nu} \nabla^{\lambda} \alpha^{d} + \cdots \right] \quad (4.26)$$

will satisfy the gauge invariance by just imposing the constraint

$$f_{ab,e} f^{e}_{c,d} \equiv k^{ef} f_{ab,e} f_{fc,d} = 0, \quad (4.27)$$

and solving for the structure constants $f_{ab,c}$. Hence, the unique non-linear extension to the PM gauge symmetry up to and PM action up to the first order in α remains consistent as a non-linear extension to PM gravity with multiple fields up to the second order in α, provided that no further deformations are introduced and that the constants $f_{ab,c}$ satisfy the consistency requirements (4.24) and (4.27).

To simplify the approach to solve for the structure constants, one would first determine the number of fields N, the matrix k_{ab}, and then determine the structure constants $f_{ab,c}$ which forces the consistency of the gauge transformations. One simple case would be $N = 2$. This number of fields forces the choice of $k_{ab} = \mathrm{diag}(+1, -1)$ in order to achieve non-unitarity. In this case, the structure constants will become totally symmetric, and the solution is unique, which is simply $f_{ab,c} = 1, \forall a, b, c \in \{1, 2\}$.

To conclude this section, we proceed to highlight the critical findings within the work of the Non-unitary multi-field PM gravity, presented by the following key insights:

With enforcing non-unitarity, a theory of multiple PM spin-2 fields can possess cubic-order interaction terms with two derivatives allowed only in four dimensions, which demands introducing a field-dependent deformation of the PM gauge symmetry, yet remaining consistent in the full non-linear level.

The gauge algebra is always Abelian to all orders in perturbations, and any attempt to deform the gauge algebra in a non-Abelian manner is unfeasible. Consequently, a unitary theory of multiple PM spin-2 fields leads to no-go results.

Any non-trivial consistent theory of multi-field PM gravity must be non-unitary.

4.2.3 Candidate Theory in mini-superspace

Given its recent establishment, examinations into the consistent non-unitary multi-field theory of PM gravity [41] are still lacking. Hence, we attempt to explore this candidate theory in mini-superspace, i.e., in the FRW ansatz, by presenting novel calculations on the actions (4.8) and (4.20) with the gauge transformations (4.11) and (4.19).

Setup

We proceed by examining a particular case of $N = 2$ PM spin-2 fields in four dimensions $D = 4$. For this system, the internal metric k_{ab} and the structure constants $f_{ab,c}$ are given by

$$k_{ab} = \text{diag}(1,\text{-}1), \quad f_{ab,c} = 1 \; \forall a, b, c \in \{1, 2\} \tag{4.28}$$

Having two fields, we may introduce the FRW metrics $f^a_{\mu\nu}$ as follows

$$f^1_{\mu\nu} = \begin{pmatrix} -N^2(t) & 0 \\ 0 & a^2(t)\delta_{ij} \end{pmatrix} \quad f^2_{\mu\nu} = \begin{pmatrix} -M^2(t) & 0 \\ 0 & b^2(t)\delta_{ij} \end{pmatrix} \tag{4.29}$$

Now we can apply the FRW setting by expressing the PM spin-2 fields $h^a_{\mu\nu}$ as a perturbation about the de Sitter metric $g_{\mu\nu}$ such that

$$h^a_{\mu\nu} = f^a_{\mu\nu} - g_{\mu\nu} \tag{4.30}$$

Subsequently, by substituting (4.30) in the expression of the tensor $F^a_{\mu\nu\rho}$ defined in (4.9), We find that the non-zero components of $F^a_{\mu\nu\rho}$ are

$$\begin{aligned} F^1_{tij} &= \big(2a\dot{a} - H_0 a^2 - H_0 e^{2H_0 t} N^2\big)\delta_{ij} = -F^1_{itj} \\ F^2_{tij} &= \big(2b\dot{b} - H_0 b^2 - H_0 e^{2H_0 t} b^2\big)\delta_{ij} = -F^2_{itj} \end{aligned} \tag{4.31}$$

For simplicity, we will denote these tensors in the following discussion with F_1 for F^1_{tij}, and F_2 for F^2_{tij}.

Results

Utilizing the presented setup, we directly substitute (4.30) in the expression of the action (4.8), which then gives the form of the undeformed action $\mathcal{S}_0$ in the FRW ansatz in our special case

$$\mathcal{S}_0 = \int d^4x\,\big(-3e^{-H_0 t}\big)\,\big((F_1)^2 - (F_2)^2\big). \tag{4.32}$$

We proceed to find the undeformed gauge transformation laws for the scale factors and the lapse functions on which the action must be invariant. Using the expression (4.30), the undeformed gauge transformations (4.11) are given by

$$\delta^{(0)}_{\alpha} = \delta^{(0)}_{\alpha}(f^{a}_{\mu\nu} - g_{\mu\nu}) = \big(\nabla_{\mu}\nabla_{\nu} + (H_0)^2\, g_{\mu\nu}\big)\alpha^a \tag{4.33}$$

Solving for the time component yields

$$\begin{aligned} \delta^{(0)}_{\alpha} f^{a}_{tt} &= \big(\nabla_t\nabla_t + (H_0)^2\, g_{tt}\big)\,\alpha^a \\ &= \ddot{\alpha}^a - (H_0)^2\,\alpha^a \end{aligned} \tag{4.34}$$

Note that α^a is the scalar gauge parameter for each field. To avoid confusion, we will use the notion $\alpha^{a=1} = \alpha_1$ and $\alpha^{a=2} = \alpha_2$. The undeformed gauge transformation laws for the lapse functions $N(t)$ and $M(t)$ are then

$$\delta^{(0)}_{\alpha} N(t) = -\frac{1}{2N}\bigg(\ddot{\alpha}_1 - (H_0)^2\,\alpha_1\bigg) \tag{4.35}$$

$$\delta^{(0)}_{\alpha} M(t) = -\frac{1}{2M}\bigg(\ddot{\alpha}_2 - (H_0)^2\,\alpha_2\bigg) \tag{4.36}$$

Now for the spacial components,

$$\begin{aligned} \delta^{(0)}_{\alpha} f^{a}_{ii} &= \bigg(\nabla_i\nabla_i + (H_0)^2\, g_{ii}\bigg)\alpha^a \\ &= \bigg(\partial_i\partial_i\alpha^a - \Gamma^{\rho}_{ii}\partial_{\rho}\alpha^a + (H_0)^2\, g_{ii}\alpha^a\bigg) \\ &= \bigg(- H_0 e^{2H_0 t}\dot{\alpha}^a + (H_0)^2 e^{2H_0 t}\alpha^a\bigg) \end{aligned} \tag{4.37}$$

which then gives the undeformed gauge transformation laws for the scale factors $a(t)$ and $b(t)$ as

$$\delta^{(0)}_{\alpha} a(t) = \frac{1}{2a}\bigg(- H_0 e^{2H_0 t}\dot{\alpha}_1 + (H_0)^2 e^{2H_0 t}\alpha_1\bigg) \tag{4.38}$$

$$\delta_\alpha^{(0)}b(t) = \frac{1}{2b}\bigg(- H_0 e^{2H_0 t}\dot{\alpha}_2 + (H_0)^2 e^{2H_0 t}\alpha_2\bigg) \tag{4.39}$$

Applying these gauge transformations on the action $\mathcal{S}_0$ gives

$$\delta_\alpha^{(0)}\mathcal{S}_0 = \int d^4x \left(-3e^{-H_0 t}\right)\left\{2\left(F_1\right)\delta_\alpha^{(0)}\left(F_1\right) - 2\left(F_2\right)\delta_\alpha^{(0)}\left(F_2\right)\right\}. \tag{4.40}$$

One then finds that

$$\begin{aligned}\delta_\alpha^{(0)}(F_1) &= 2a\delta_\alpha^{(0)}\dot{a} + 2\dot{a}\delta_\alpha^{(0)}a - 2H_0\delta_\alpha^{(0)}a - 2H_0 e^{2H_0 t}\delta_\alpha^{(0)}N = 0\\ \delta_\alpha^{(0)}(F_2) &= 2a\delta_\alpha^{(0)}\dot{b} + 2\dot{b}\delta_\alpha^{(0)}b - 2H_0\delta_\alpha^{(0)}b - 2H_0 e^{2H_0 t}\delta_\alpha^{(0)}M = 0,\end{aligned} \tag{4.41}$$

hence proving the consistency of the gauge symmetry for the undeformed action in our FRW setting.

Moving forward, we express the action in the 1st order of deformations by again substituting (4.30) into the action $\mathcal{S}_1$ presented in (4.20). The resulting expression is

$$\mathcal{S}_1 = \int d^4x \left(-3e^{-H_0 t}\right)\left(F_1 + F_2\right)^2\bigg\{e^{-2H_0 t}(a^2 + b^2) - (N^2 + M^2)\bigg\}, \tag{4.42}$$

and the gauge transformation law in the 1st order of deformations is then

$$\delta_\alpha^{(1)}(f^a_{\mu\nu} - g_{\mu\nu}) = \epsilon\big(k^{ad} f_{db,c} F^b_{\rho(\mu\nu)} g^{\rho\sigma}\nabla_\sigma \alpha^c\big) \tag{4.43}$$

$$\begin{aligned}\delta_\alpha^{(1)}(f^a_{\mu\nu}) = \epsilon k^{ad}\bigg(& f_{d1,1}F^1_{\rho(\mu\nu)}g^{\rho\sigma}\nabla_\sigma\alpha^1 + f_{d1,2}F^1_{\rho(\mu\nu)}g^{\rho\sigma}\nabla_\sigma\alpha^2\\ & + f_{d2,1}F^2_{\rho(\mu\nu)}g^{\rho\sigma}\nabla_\sigma\alpha^1 + f_{d2,2}F^2_{\rho(\mu\nu)}g^{\rho\sigma}\nabla_\sigma\alpha^2\bigg)\end{aligned} \tag{4.44}$$

which results in the gauge transformations for the lapse functions and the scale factors in the form

$$\delta_\alpha^{(1)} N(t) = \delta_\alpha^{(1)} M(t) = 0 \tag{4.45}$$

$$\delta_\alpha^{(1)} a(t) = \frac{\epsilon}{2a}\left(F_1 + F_2\right)\left(\dot{\alpha}_1 + \dot{\alpha}_2\right) \tag{4.46}$$

$$\delta_\alpha^{(1)} b(t) = -\frac{\epsilon}{2b}\left(F_1 + F_2\right)\left(\dot{\alpha}_1 + \dot{\alpha}_2\right) = -\frac{a}{b}\delta_\alpha^{(1)} a(t) \tag{4.47}$$

Now if we apply these gauge transformations into the action (4.42), we find the following

$$\begin{aligned}
\delta_\alpha^{(1)} \mathcal{S}_1 =& \epsilon \int d^4x \left(-3e^{-H_0 t}\right) 2\big(F_1 + F_2\big)\big(\delta_\alpha^{(1)} F_1 + \delta_\alpha^{(1)} F_2\big)\Big\{e^{-2H_0 t}(a^2 + b^2) - (N^2 + M^2)\Big\} \\
&+ \int d^4x \left(-3e^{-H_0 t}\right)\left(F_1 + F_2\right)^2 \Big\{e^{-2H_0 t}(2a\delta_\alpha^{(1)} a + 2b\delta_\alpha^{(1)} b) - (\delta_\alpha^{(1)} N^2 + \delta_\alpha^{(1)} M^2)\Big\} \\
&= 0,
\end{aligned} \tag{4.48}$$

which again proves consistency.

We can also proceed to check the consistency of the extended action $(\mathcal{S}_0 + \mathcal{S}_1)$ by proving that

$$(\delta_\alpha^{(0)} + \delta_\alpha^{(1)})(\mathcal{S}_0 + \mathcal{S}_1) \tag{4.49}$$

satisfies the gauge invariance. We already found that $\delta_\alpha^{(0)} \mathcal{S}_0 = \delta_\alpha^{(1)} \mathcal{S}_1 = 0$. But for the cross terms, The result

$$\begin{aligned}
\delta_\alpha^{(0)} \mathcal{S}_1 + \delta_\alpha^{(1)} \mathcal{S}_0 = 3\epsilon \Bigg\{ \int & d^4x \left(-He^{-H_0 t}\right)\left(F_1 + F_2\right)^2\left(\dot{\alpha}_1 + \dot{\alpha}_2\right) \\
& \left(+2e^{-H_0 t}\right)\left(F_1 + F_2\right)\left(\dot{F}_1 + \dot{F}_2\right)\left(\dot{\alpha}_1 + \dot{\alpha}_2\right) \\
& \left(+e^{-H_0 t}\right)\left(F_1 + F_2\right)^2\left(\ddot{\alpha}_1 + \ddot{\alpha}_2\right) \Bigg\}
\end{aligned} \tag{4.50}$$

is a total derivative term that vanishes when taking the equations of motion, proving the gauge invariance. These results show that the non-unitary multi-field theory of PM gravity in our two-fields FRW setup is consistent from the point of view of the gauge structure, as the free action $\mathcal{S}_0$, the 1st order deformed action $\mathcal{S}_0$, and the extended action $(\mathcal{S}_0 + \mathcal{S}_1)$ are invariant under the gauge symmetries $\delta_\alpha^{(0)}$, $\delta_\alpha^{(1)}$, and $(\delta_\alpha^{(0)} + \delta_\alpha^{(1)})$, respectively.

Chapter 5

Conclusion

This dissertation reviews and examines Partially Massless (PM) Gravity as a candidate theory of gravity, focusing on its most recent and promising development, Non-unitary Multi-field PM gravity. We start our discussion by introducing the fundamental principles from the theory of massive gravity as elaborated in chapter 2. In section 2.1, we deliver the formalism and the primary properties of linear massive gravity — linear Fierz-Pauli theory — along with the challenges posed by the vDVZ discontinuity. This discontinuity is manifest when investigating the exchange amplitude between the two sources in the massive case approaching zero $m \to 0$ and massless case $m = 0$. vDVZ discontinuity is solved through the Vainshtein mechanism, which forces the extension of massive gravity to the non-linear regime. However, this is a non-trivial task; as we detail, some non-linear extensions of massive gravity face a significant challenge due to a problem known as the BD ghost: terms with higher-order partial derivatives acting on the longitudinal (helicity-0) mode of the field, which pose pathologies as they give rise to energies with no lower bound. Fortunately, ghost-free non-linear theories of massive gravity exist. We introduce in section 2.2 one favored ghost-free theory —

the dRGT theory — detailing its formulation from the decoupling limit and highlighting the absence of BD ghost. We comment that despite its promising structure that avoids the BD ghost, the dRGT theory suffers from a critical limitation: the absence of FRW cosmological solutions. Subsequently, in section 2.3, we revisit linear massive gravity, but instead of flat spacetime, we consider the de Sitter (dS) spacetime. In that notion, we discuss the idea of the Higuchi bound, which shows forbidden ranges on the mass of the graviton m expressed in terms of the spacetime dimensions D and the Hubble parameter H_0. We highlight that when the mass of the graviton lies on the bound $m^2 = (D-2)H_0^2$, massive gravity on dS becomes PM gravity. This was the first existence of the theory of PM gravity. Moving to Chapter 3, we analyze linear PM gravity in section 3.1 and highlight its main properties, such as the relation between the mass and the Hubble parameter/cosmological constant, propagation in four degrees of freedom in four-dimensional spacetime, and the existence of a gauge symmetry. Next, in section 3.2, we emphasize the motivation for progressing towards a non-linear completion for such a theory by discussing the potential resolution for the old cosmological constant problem by the gauge symmetry and avoiding the pathologies inherent in massive gravity. Nevertheless, the journey to find a consistent non-linear PM gravity is a challenge, as higher orders of the theory lead to inconsistencies with its gauge symmetry, known as the no-go result. This was demonstrated by the construction and obstruction for a candidate derived from the ghost-free dRGT theory of massive gravity. Progressing to chapter 4, we turn our attention to the Non-unitary Multi-field PM gravity, a promising candidate for a non-linear theory of PM gravity. In section 4.1, we first discuss some arguments for avoiding the no-go result by including multiple PM fields and relaxing the unitarity condition in the theory. We elaborate on how non-unitarity does not necessarily result in pathologies, and relaxing the condition of classical unitarity is a valid option for constructing our theories. In section 4.2, we systematically present the architecture of the newly proposed theory, beginning with its unitary counterpart, then enforcing non-unitary on the theory, and presenting the formalism of the action with the gauge symmetry and its 1st order deformations. This candidate theory is a non-unitary Yang-Mills-like theory for PM gravity, and its consistency is obtained not only by multiple fields and non-unitarity but also with a constraint on the structure constants. We show examples of special cases for the theory and its solution for the structure constants chosen to maintain the gauge symmetry. Most importantly, we highlight the ground-breaking result: any non-trivial consistent multiple-fields theory of PM gravity must be non-unitary. Finally, in section 4.2.3, we perform a simple test for the theory up to the 1st order in deformations by deriving the candidate Theory in mini-superspace, i.e., in a homogeneous and isotropic universe, known as the FRW ansatz. Our test considers two fields $N = 2$ and four spacetime dimensions $D = 4$, provided the internal metric k_{ab} and the structure constants $f_{ab,c}$ as given in literature [41]. Our test has shown that this theory is indeed consistent as the gauge symmetry is maintained in our setup. We may say that the non-unitary construction of PM gravity is indeed an outstanding establishment, which overcomes the problematic no-go results and provides the first example of an interacting theory consisting of only PM fields.

The examination of the recent development of PM Gravity presented in this dissertation provides fertile ground for further exploration and refinement. We believe our consistency result presented in section 4.2.3 is worth validating for an arbitrary number of fields. In addition, we encourage testing the theory under perturbations of FRW, along with searching for cosmological solutions to the theory, which will then be a critical test for the reliability and applicability of the non-unitary multi-field PM gravity. Moreover, there are many aspects and directions beyond this dissertation that non-unitary multi-field PM gravity could cover, such as its link to conformal gravity and the structure of higher-spin gauge theories for which the irreducible representations of the ((A)dS) group play a crucial role. We warmly encourage readers seeking additional insights to delve into the listed bibliography for further reading.

In conclusion, the theoretical framework of the non-unitary multi-field PM gravity unfolds vast opportunities for promising future research endeavors. given that the primary theoretical obstacle of deriving a consistent non-linear completion of PM gravity has now been overcome, broader horizons of the theory of PM gravity would open, bringing forth new challenges. Therefore, further examinations are needed to ascertain validity at this level. The field is relatively young, and many developments are still in progress. Nonetheless, we are eager to see more revolutionary findings in the near future.

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