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Helicity and Topological Evolution of Time

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در جستجوی خرد ما

Anonymous, Persian

Translation: "At the discovering of the human virtue".
Traduzione: "Alla scoperta della virtù umana".

गते गते पारगते पारसंगते बोधि स्वाहा

From Prajñāpāramitāhṛdaya, Sanskrit

Translation: "Gone, gone, gone beyond, gone utterly beyond, may enlightenment BE!"

Abstract

In this thesis, we investigate the complex interplay between topological hydrodynamics and topological field theory within the framework of theoretical physics. Our research establishes that the generalization of helicity in fluids is equivalent to the Hopf invariant, while in quantum field theory, it corresponds to the Abelian dual Chern-Simons 3-form. This equivalence serves as the foundation for an original theory that implements a topological clock in the context of Henneaux and Teitelboim's Unimodular gravity. This innovation introduces the concept of 'physical time', which is sensitive to the topological evolution of the magnetic field. The implications of this work extend beyond a novel understanding of coordinate time, offering fresh perspectives in cosmology and quantum field theories, and providing new tools for our approach to gravitational theories. Future research could explore the extension of this framework to non-Abelian gauge groups and its applicability in the study of black hole physics.

Acknowledgments

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Contents

1 Introduction

The study of topological properties in physical systems has long been a cornerstone in the quest to understand the fundamental nature of our universe. From the intricate patterns of fluid flows to the enigmatic properties of quantum fields, topology offers a unique lens through which to view and understand the cosmos. Yet, one of the most elusive and intriguing aspects of physical systems is the concept of time. Traditional approaches often treat time as a mere coordinate, a backdrop against which the drama of physical interactions unfolds. This thesis challenges that paradigm by introducing the concept of 'physical time', a dynamic entity that is intrinsically linked to the topological evolution of the magnetic field. Building on the foundational theories of topological hydrodynamics and topological field theory, we establish a novel framework that unifies these seemingly disparate fields. Our work serves as a stepping stone for implementing a topological clock within the context of Henneaux and Teitelboim's Unimodular gravity.

This thesis prioritizes a physical approach over a purely mathematical one, striving for self-containment to provide a comprehensive introduction to the concepts at hand. Nothing is extraneous or omitted, and updated versions will be available on the cited website [1]. The primary objective is to offer a fundamental physical framework for understanding the original contributions of this work, specifically in building connections between topological hydrodynamics and topological field theory.

We will begin with the fundamentals of knot theory and establish its intriguing connection to electromagnetism through the Gauss linking number. This theme will recur throughout the thesis, demonstrating its re-derivation in hydrodynamics and, ultimately, in three-dimensional Quantum Field Theory. This serves as compelling evidence of the topological physics underpinning our study and raises the possibility that we are dealing with consistent quantities across different contexts.

This thesis is primarily divided into three parts. After a brief introduction to knot theory, the sections are as follows: Helicity in the context of fluids, topological physics and field theory (primarily within the realm of Quantum Field Theory), and finally, the topological evolution of time within a framework of General Relativity. The first two sections are crucial for establishing the vital link between topological hydrodynamics and topological quantum field theory. We introduce helicity as defined by Moffatt in fluid dynamics and demonstrate its invariance in ideal fluids and ideal magnetohydrodynamics. Subsequently, we generalize this concept of helicity, which turns out to be the scalar component of a more extensive four-vector helicity in viscous fluids. Here, we employ the machinery of differential geometry and introduce the Hopf invariant.

We then transition to topological physics, elucidating the topological implications of the Dirac monopole and the Aharonov-Bohm effect. These topics serve as a discursive introduction to de Rham cohomology and homology theory, leading to the definition of the Wilson loop. We then focus on the topological theta term in four dimensions, connecting it to the divergence of the generalized four-vector helicity. This introduces the Chern-Simons term in three dimensions, which corresponds to the scalar helicity as defined by Woltjer and Moffatt. Armed with these arguments, we will showcase the Gauss linking number in a $2+1$ dimensional quantum field theory.

Finally, the third part focuses on a Unimodular formulation of General Relativity, introducing the Henneaux and Teitelboim formulation. Utilizing this framework, we incorporate the Chern-Pontryagin term into the four-dimensional action, which crystallizes into the three-dimensional Chern-Simons term in cosmological time, representing scalar helicity. This establishes that the evolution of time is governed by the topological evolution of scalar helicity. We conclude by discussing the implications of this theory, emphasizing that without any topological evolution, both the 4-Volume and time would cease to exist.

1.1 Mathematical Helicity/Hopf Invariant

There is no doubt that topological solutions in various fields such as Hydrodynamics [2], Electromagnetism [3], the Skyrme-Faddeev model [4], Chern-Simons theory [5], and many more are deeply interconnected in a topological manner. The Hopf invariant, which goes by various names in the literature such as Helicity or the Abelian 3-form Chern-Simons term, serves as a common thread linking these diverse topics. To gain a comprehensive understanding of these interconnected topics in Theoretical Physics and Mathematical Physics, we will follow the mathematical description provided in an established book on Topological Hydrodynamics [6].

Definition 1.1. The Hopf invariant (or Helicity) is defined for a divergence-free vector field ξ in the domain $M \subset \mathbb{R}^3$ as

$$H(\xi) = \langle \xi, \operatorname{curl}^{-1} \xi \rangle = \int_M (\xi, \operatorname{curl}^{-1} \xi) dV, \tag{1.1}$$

where $(\cdot, \cdot)$ is the usual pairing in Euclidean space. Here, $\operatorname{curl}^{-1} \xi = \mathbf{A}$ is the divergence-free vector potential, i.e., $\nabla \times \mathbf{A} = \xi$ and $\operatorname{div} \mathbf{A} = 0$. This quantity measures the degree of linking of the field lines in a given volume [2].

Remark 1. The operator curl^{-1} is non-local and symmetric. In a connected manifold, it maps the space of divergence-free vector fields onto itself.

Hence these should serve as an excellent cross-reference for us looking to delve deeper into these complex topics.

2 Theory of Knots in a Knotshell

Knot theory is a fascinating subfield of topology that studies the properties of simple closed curves embedded in three-dimensional space. The subject has profound implications in various disciplines, including physics, chemistry, and biology.

Here, we are not aiming to provide an extensive introduction to knot theory; rather, our goal is to establish and clarify the essential terminology used in the field. We will primarily adopt the standard terminology and definitions from well-regarded textbooks [7] [8].

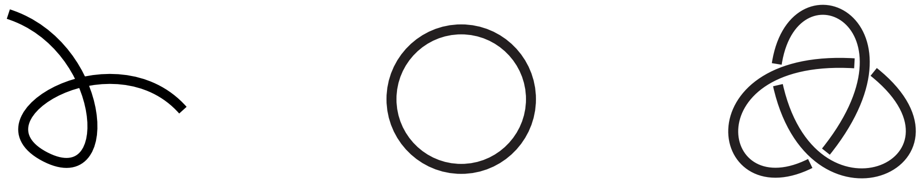

Figure 1: In this set of figures, the first one is not considered a knot because it is not a closed path. Next, we have the simplest knot, known as the unknot. Lastly, we have an example of a more complex knot: the trefoil knot.

Therefore, a *knot* is a closed curve embedded in three-dimensional Euclidean space $\mathbb{R}^3$. It is a loop that does not intersect itself, and its properties remain invariant under continuous deformations, known as isotopies. Two knots are considered the same if one can be deformed into the other. The simplest knot is known as the unknot or trivial knot, as shown in Figure 1.

In knot theory, a *projection* is a two-dimensional representation of a knot obtained by projecting it onto a plane. Every knot can have multiple projections, but they all encode the properties of the same knot. This projection captures the over-under crossing information essential for knot classification. Mathematically, it can be viewed as a mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ that preserves the knot's essential features. The resulting diagram is often used to calculate knot invariants like the Jones polynomial, which we will discuss later.

From the projections, we can observe that the knot crosses itself a number of times; this is an important quantity called the *crossing number*. It is a fundamental knot invariant that counts the number of crossings in a given knot projection. The crossing number is the minimum number of crossings over all possible projections of the knot onto a plane. It serves as a measure of the knot's complexity and is denoted, for a knot K, as $c(K)$. Any non-trivial knot (apart from the unknot) must have a crossing number of at least 3.

To distinguish different types of knot projections, we have two important tools: namely, *tricolorability* and the famous *Reidemeister moves*, which are neatly represented in Figure 2. Tricolorability is a knot invariant that examines whether the segments between crossings in a knot projection can be colored using three colors such that, at each crossing, either all segments have the same color or they have three distinct colors. Reidemeister moves are local transformations consisting of three types: Type I (Twist) involves twisting or untwisting a loop; Type II (Poke) moves one loop over another; and Type III (Slide) slides one strand between two others.

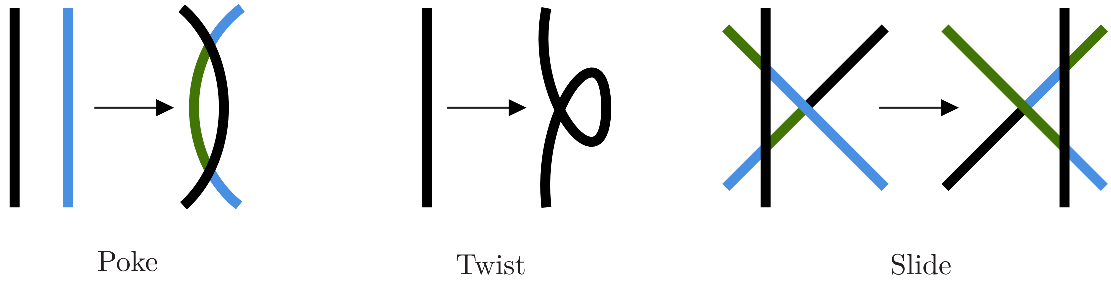

Figure 2: In this illustration, we demonstrate the feature of tricolorability, where each crossing should allow us to identify three different colors. We also show the Reidemeister moves: Type I (Twist), Type II (Poke), and Type III (Slide). It's important to note that these figures represent portions of a knot near a crossing; we assume that the lines will close to form a complete knot K.

Now that we have covered the basics, we can define the linking between different knots; for some examples, refer to Figure 3. Each individual closed curve within a link is itself a knot, and these knots may or may not be entangled with each other. Links extend the study of individual knots to multiple intertwined loops, allowing for a richer set of topological properties and invariants. The study of links employs similar mathematical tools to those used in knot theory, such as link polynomials and Reidemeister moves, to classify and analyze their properties.

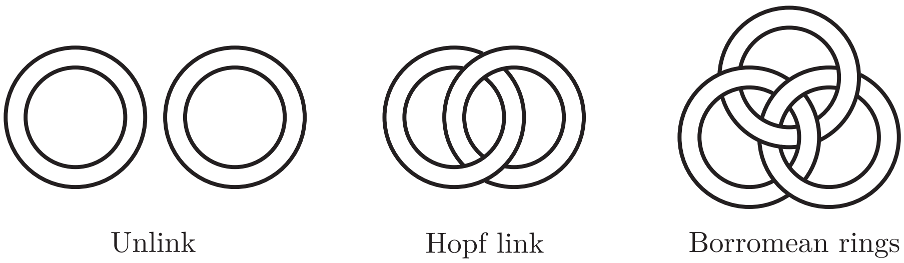

Figure 3: Here, we have depicted various unknots, showing two of them that are not linked. Next, we present the Hopf link, which has two crossings. Finally, we illustrate the Borromean rings, which consist of three links and six crossings.

If you orient all the loops in the link, you can assign either +1 or −1 to each crossing, as shown in Figure 4. The sum of all these crossing numbers, divided by 2, yields the *linking number* of the link, given by:

$$\text{Linking number} = \frac{1}{2}\sum_{ij}\text{sgn}(c_{ij}) \tag{2.1}$$

where c_{ij} represents the crossings and sgn denotes the sign of each crossing. The linking number is an integer that remains invariant under isotopies and Reidemeister moves, serving as a fundamental tool for classifying and understanding links. It is important to note that this is the same linking number that Gauss discovered in electromagnetism, a connection we will explore in the next section.

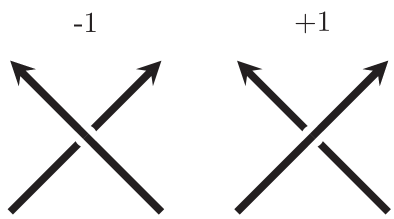

Figure 4: Here, the illustration shows how to distinguish the two sign assignments for a crossing in a given knot.

Finally, the last relevant topic to touch upon is the Jones polynomial [9]. We will see in Section 4.3.1 that it is linked to Quantum Field Theory. To discuss this, we need to introduce the concept of a skein triple.

In knot theory, a *skein triple* refers to a set of three oriented links or knots, denoted as K_+, K_-, and K_0, that differ only at a single crossing. Specifically, these three links are identical except in a small neighborhood where:

- K_+ has a positive crossing.
- K_- has a negative crossing.
- K_0 has the crossing replaced by two parallel strands, effectively "smoothing" the crossing.

The skein triple is crucial for defining *skein relations*, which are recursive relations used to compute knot invariants like the Jones polynomial. The skein relation for the Jones polynomial is often expressed as:

$$tV(K_+) - t^{-1}V(K_-) = (t^{1/2} - t^{-1/2})V(K_0) \tag{2.2}$$

Here, $V(K_+)$, $V(K_-)$, and $V(K_0)$ are the Jones polynomials of the knots K_+, K_-, and K_0, respectively, and t is the variable of the Jones polynomial. The two important properties of the Jones polynomial are its invariance under Reidemeister moves and its normalization, namely $V(\text{unknot}) = 1$.

Hence, now armed with these definitions and terms, we can continue our journey through topological invariants and the topological evolution of knots. Our first encounter will be with the Gauss Linking Number in electromagnetism.

2.1 Knots in Electromagnetic theory

Here, we will provide an explicit demonstration of how knot theory naturally arises in the classical formulation of electromagnetism. In the 19th century, a seminal intersection between physics and topology was observed, specifically in the realm of electromagnetic theory. In this context, we will discuss how Gauss discovered the linking number in 1833 [10], a discovery later revisited by Maxwell [11]. For a more in-depth look into the historical aspects, please refer to [12]. The topological construct under examination is the linking number, a topological invariant, between two loops (forming a Hopf link) in three-dimensional space.

In the framework of electromagnetic theory, the magnetic field vector $\mathbf{B}$ generated by an electric current I flowing through a conductor is described by Maxwell's curl and divergence equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.4}$$

To derive Ampère's circuital law, one evaluates the line integral of $\mathbf{B}$ around a closed curve C that encloses a surface S:

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\mathbf{l} &= \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s} \\ &= \mu_0 I, \end{aligned} \tag{2.5}$$

where here we have used Stokes' theorem. It is fundamental to note that in the last step I represent the exact amount of current $\mathbf{J}$ passing trough the surface S, this can be generalised summing n amount of contribution of $\pm\mu_0 I$, the sign depend if the current is parallel or antiparallel to the surface unit vector. Hence the generalized form of Ampère's law becomes:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 m I, \quad m \in \mathbb{Z}, \tag{2.6}$$

here m is a integer, this is familiar throughout standard textbooks as it is related to the magnetic field due to a solenoid [13]. To see how Gauss [10] wrote and found this property we need an alternative approach involving the introduction of the vector potential $\mathbf{A}$, defined as:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.7}$$

The governing equation for $\mathbf{A}$ becomes:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \tag{2.8}$$

Integrating $\mathbf{B}$ around a second loop C gives:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = -\frac{\mu_0 I}{4\pi} \oint_C \oint_{C'} \frac{(\mathbf{r} - \mathbf{r}') \times d\mathbf{l}' \cdot d\mathbf{l}}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2.9}$$

This leads to the topological invariant m, the linking number of C and C', explicitly we can see from equation (2.5):

$$-\frac{1}{4\pi} \oint_C \oint_{C'} \frac{(\mathbf{r} - \mathbf{r}') \times d\mathbf{l}' \cdot d\mathbf{l}}{|\mathbf{r} - \mathbf{r}'|^3} = m. \tag{2.10}$$

Now we can open this equation and writing out explicitly using:

$$\mathbf{r} = (x, y, z), \quad \text{and} \quad \mathbf{r}' = (x', y', z') \tag{2.11}$$

then this will lead the form that Gauss written on his notes

$$-\frac{1}{4\pi} \iint \frac{(x' - x)(dydz' - dzdy') + (y' - y)(dzdx' - dxdz') + (z - z')(dxdy' - dydx')}{\left[(x' - x)^2 + (y' - y)^2 + (z' - z)^2\right]^{\frac{3}{2}}} = m \tag{2.12}$$

This result originated from his investigations into electromagnetism in 1833 and serves as an early contribution to knot theory and shows how the two subjects are fundamentally interlinked.

3 Helicity

Helicity is a pseudoscalar quantity that serves as an indicator of the "degree of knottedness" within a contractible space filled by a fluid. This measure provides insights into the topological structure of a fluid in a given space, with particular focus on the vorticity field and its self-entanglement. The study of the topological structure of vorticity fields traces its origins to Kelvin's seminal paper "On Vortex Motion" [14] published in 1868. In this work, Kelvin posited that vortex rings should remain stable over time in ideal fluids. He also expressed skepticism about the concept of point particles, proposing instead that vortices in the ether could offer a more satisfactory explanation for the fundamental constituents of matter. By the time of Maxwell's death in 1879, most of the foundational laws concerning vortices in fluid mechanics had been established. This subject is intrinsically linked with knot theory.

We begin by presenting Moffatt's definition of helicity [2] and subsequently explore its relationship with knots via the Gauss linking number. We then examine the conditions under which these vorticity fields are conserved, before delving into the topological features of helicity in viscous fluids. In this context, we introduce a framework that generalizes 4-vector helicity and redefines Moffatt's helicity as scalar helicity. This section serves as the foundation for the subsequent discussions and is crucial for the new interpretation of time presented at the end.

3.1 Mathematical definition by Moffatt for Helicity

The term "Helicity" was coined by Moffat in 1969 [2] and has since been extensively used in astrophysics for computations in magnetohydrodynamics (MHD) and dynamo theory. It is also used in the context of hydrodynamics, often paired with the term Hopf invariant, which will be discussed further in the section on non-invariants. Here, we will follow Moffat's derivation of helicity in the context of fluid dynamics, although the derivation is applicable to any vector field with a non-vanishing vorticity field in the same setup. We start by considering an inviscid fluid with a velocity vector $\mathbf{u}(\mathbf{x}, t)$. For a closed loop C moving with the fluid, its circulation is given by:

$$K = \oint_C \mathbf{u} \cdot d\mathbf{l} \tag{3.1}$$

where K is constant. Now, consider a setup where this vector field has vanishing vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ everywhere except along two curves C_1 and C_2 with strengths κ_1 and κ_2, as shown in Figure 5. Then, by Stoke's theorem, we have:

$$K_1 = \oint_{C_1} \mathbf{u} \cdot d\mathbf{l} = \int_{S_1} \boldsymbol{\omega} \cdot d\mathbf{S}, \tag{3.2}$$

where S_1 is the surface enclosed by the curve C_1. Therefore, K_1 is sensitive to C_2 crossing or touching the area inside of C_1. Nevertheless, we can summarize K_1 as:

$$K_1 = \alpha_{12}\kappa_2, \tag{3.3}$$

where α_{12} is a symmetric positive or negative integer and represents the "winding number" for the curves C_1 and C_2. For example, for the curves that are either not linked or singly linked, we can summarize K_1 as:

$$K_1 = \begin{cases} 0 & \text{if} \quad C_1 \quad \text{and} \quad C_2 \text{ are not linked,} \\ \pm\kappa_2 & \text{if} \quad C_1 \quad \text{and} \quad C_2 \text{ are singly linked.} \end{cases} \tag{3.4}$$

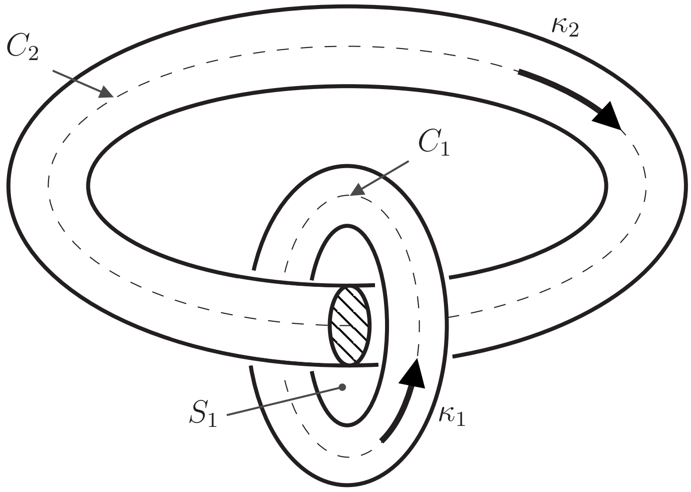

Figure 5: Here, we represent two interlinked circular flux tubes where κ_1 and κ_2 denote the fluxes of vorticity. C_1 and C_2 represent the central curves of the flux tubes. Additionally, the surface S_1 enclosed by the loop C_1 is also depicted.

It is possible to generalize K_1 for n unknotted closed curves $C_1, C_2, \ldots, C_n$ as:

$$K_i = \oint_{C_j} \mathbf{u} \cdot d\mathbf{l} = \sum_j \alpha_{ij} \kappa_j, \tag{3.5}$$

where, as before, α_{ij} is the winding number for curves C_i and C_j. These κ_i can be interpreted as the fluxes carried by the $\boldsymbol{\omega}$ lines that run parallel to the closed curve C. Therefore, since $d\mathbf{l}$ is parallel to $\boldsymbol{\omega}$, we can substitute $\kappa_i d\mathbf{l}$ with $\boldsymbol{\omega} dV$. Now, we can consider the following quantity:

$$\kappa_i K_i = \oint_{C_i} \kappa_i d\mathbf{l} \cdot \mathbf{u} = \int_{V_i} \mathbf{u} \cdot \boldsymbol{\omega} dV, \tag{3.6}$$

where V_i denotes the volume of the flux tube of κ_i around C_1. The last part of this equation can already be defined as the Helicity. We can generalize this quantity to the sum of all the closed $\boldsymbol{\omega}$ lines in a contractible three-dimensional space as:

$$\mathcal{H} = \sum_i \kappa_i K_i = \sum_{i,j} \alpha_{ij} \kappa_i \kappa_j = \int_V \mathbf{u} \cdot \boldsymbol{\omega} dV, \tag{3.7}$$

where V is the volume occupied by all vortices or, equivalently, the volume that the fluid occupies in the manifold. This is the Helicity, and it is now easier to see why this pseudo-scalar quantity gives the "degree of knottedness" for a given vorticity field $\boldsymbol{\omega}$, which is fundamentally linked with the winding number for different closed paths traced by $\boldsymbol{\omega}$. It is also useful to note that this quantity is invariant under gauge transformations of $\mathbf{u}$, again following Moffatt [2]:

$$\mathbf{u}' = \mathbf{u} + \nabla \phi, \tag{3.8}$$

We can see that the Helicity is invariant under this transformation:

$$\int_V \nabla\phi \cdot \boldsymbol{\omega} dV = \int_V \nabla \cdot (\boldsymbol{\omega}\phi) dV = \int_S \mathbf{n} \cdot \boldsymbol{\omega}\phi dS = 0, \tag{3.9}$$

where we assume the important condition that the vorticity field does not cross the boundary of the manifold and consists only of closed lines within the space.

3.1.1 Knots in fluids

Although we have already shown in the previous section how helicity in fluids is related to knots, here we will emphasize how the Gauss linking number again arises naturally in this formulation, further demonstrating how these subjects are interlinked. Hence, following Moffat [2] and [15], we can define the velocity vector $\mathbf{u}$ by the inverse of the curl:

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{(\mathbf{x} - \mathbf{x}') \times \omega(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dV', \tag{3.10}$$

Now, if we substitute this back into the Helicity Equation (3.7), we find:

$$\mathcal{H} = \frac{1}{4\pi} \iint \frac{(\mathbf{x} - \mathbf{x}') \cdot [\boldsymbol{\omega}(\mathbf{x}) \times \boldsymbol{\omega}(\mathbf{x}')]}{|\mathbf{x} - \mathbf{x}'|^3} dV dV', \tag{3.11}$$

Then, we can re-express this quantity in terms of line elements. Therefore, for two curves C_i and C_j with respective $\mathbf{x}_i \in C_i$ and $\mathbf{x}_j \in C_j$, we have:

$$\alpha_{ij} = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\mathbf{x} - \mathbf{x}') \cdot d\mathbf{l}_i \times d\mathbf{l}_j}{|\mathbf{x} - \mathbf{x}'|^3} = -\frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\mathbf{x} - \mathbf{x}') \times d\mathbf{l}_j \cdot d\mathbf{l}_i}{|\mathbf{x} - \mathbf{x}'|^3}, \tag{3.12}$$

where in the last step we have used $\epsilon^{kij} (\mathbf{x} - \mathbf{x}')_k \, d\mathbf{l}_i d\mathbf{l}_j = -\epsilon^{ikj} (\mathbf{x} - \mathbf{x}')_k \, d\mathbf{l}_j d\mathbf{l}_i$ to match the Gauss linking number found in Equation (2.10). Hence, we see that from the formulation of helicity in fluids due to Moffat, the Gauss linking number is present and provides a bridge between apparently different areas of physics.

3.1.2 Conservation and invariance

Here we will follow Moffat [16] to show the important property of Helicity that is conserved in ideal fluids. This result will clarify the setup needed to achieve this and elucidate its topological properties. The vorticity field $\boldsymbol{\omega}$ has associated vortex lines, which are lines that "follow" closely this vorticity field in the sense that they are tangent and parallel to it at every point. In ideal fluids, these lines are thought to be "frozen" in the fluid under Euler evolution. We consider here an inviscid barotropic fluid, meaning that the pressure is solely dependent on the fluid density: $p = p(\rho)$.

Starting by considering the momentum equation for an incompressible fluid with no viscosity:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\phi, \tag{3.13}$$

where $\nabla\phi$ is the body force distribution and $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ is the Lagrangian derivative (or Material derivative). This keeps track of both the local and the convective changes of the velocity field as you follow a fluid particle. We can easily apply this to the vorticity field in this manner:

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\right)\cdot\nabla\mathbf{u}. \tag{3.14}$$

Hence, we can compute their dot product:

$$\begin{aligned}
\frac{D}{Dt}\left(\frac{\mathbf{u}\cdot\boldsymbol{\omega}}{\rho}\right) &= \mathbf{u}\cdot\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) + \frac{D\mathbf{u}}{Dt}\cdot\left(\frac{\boldsymbol{\omega}}{\rho}\right) && (3.15)\\
&= \mathbf{u}\cdot\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\mathbf{u}\right) - \frac{\boldsymbol{\omega}}{\rho}\cdot\left(-\frac{1}{\rho}\nabla p + \nabla\phi\right) && (3.16)\\
&= \frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\left(\frac{1}{2}\mathbf{u}^2 - \int\frac{dp}{\rho} - \phi\right) && (3.17)\\
&= \frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\mathbf{Q}, && (3.18)
\end{aligned}$$

where $\int dp/\rho$ is the enthalpy per unit mass. Now, we can calculate the time derivative of Helicity, keeping in mind that $\frac{D}{Dt}(\rho dV) = 0$ from the properties of perfect fluids:

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \int_V \frac{D}{Dt}\left(\frac{\mathbf{u}\cdot\boldsymbol{\omega}}{\rho}\right)\rho dV && (3.19)\\
&= \int_V (\boldsymbol{\omega}\cdot\nabla)\mathbf{Q}dV && (3.20)\\
&= \int_S (\mathbf{n}\cdot\boldsymbol{\omega})\mathbf{Q}dS && (3.21)\\
&= 0, && (3.22)
\end{aligned}$$

where we used the divergence theorem for the vorticity field. This is true and sufficient only if $\mathbf{n}\cdot\boldsymbol{\omega} = 0$ at the boundary S. Hence, this ensures that Helicity is constant, consistent with our derivation of the Helicity formula considering only closed vortex filaments in their volume. Each of the vortex filaments would have its invariant Helicity.

3.1.3 Ideal Magnetohydrodynamics invariance

Magnetohydrodynamics (MHD) is the study of electrically conducting fluids interacting with magnetic and electric fields. It combines the Navier-Stokes equations for fluid dynamics with Maxwell's equations for electromagnetism. MHD is pivotal in understanding phenomena like solar flares and magnetic confinement in fusion reactors. The field is governed by conservation laws and key invariants such as helicity.

By introducing how Helicity arises in an MHD setup, we can introduce fundamental concepts such as cross-Helicity and new properties. In this example, we will be able to prove that Helicity here is conserved, which was first proved by Woltjer in 1958 [17]. Starting by considering the usual magnetic field and electric field:

$$\mathbf{B} = \nabla \times \mathbf{A} \text{ and } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \tag{3.23}$$

it is already possible to see the similarity between standard electromagnetic fields **A** and **B** with Euler perfect fluids velocity field $\mathbf{u}$ and vorticity field $\boldsymbol{\omega}$. Hence, we could already write the integral for magnetic Helicity as follows:

$$\mathcal{H}_M = \int_V \mathbf{A} \cdot \mathbf{B} dV. \tag{3.24}$$

Here again, as in the example in ideal fluids, we are going to use the same setup for ideal MHD. This theory describes the behavior of a plasma, combining the properties of a fluid and a magnetic field. Therefore, in an MHD setup, we have the induction equation with vanishing resistivity and orthogonality condition as:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \text{ and } \mathbf{E} \cdot \mathbf{B} = 0. \tag{3.25}$$

Hence, we can combine these equations to find:

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times (\nabla \times \mathbf{A}) - \nabla \phi, \tag{3.26}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{3.27}$$

Now, starting from the magnetic Helicity equation, we can take the time derivative:

$$\frac{d\mathcal{H}_M}{dt} = \int_V \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} dV + \int_V \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} dV, \tag{3.28}$$

$$= -2 \int_V \mathbf{E} \cdot \mathbf{B} dV + \int_V \nabla \cdot (\mathbf{A} \times \mathbf{E} - \phi \mathbf{B}) dV, \tag{3.29}$$

$$= -\int_S (\mathbf{A} \times \mathbf{u} \times \mathbf{B} + \phi \mathbf{B}) \cdot \mathbf{n} dS = 0, \tag{3.30}$$

where we used the condition that $\nabla \cdot \mathbf{B} = 0$ and the identity in Equation A.5. The last step is again assuming that $\mathbf{B} \cdot \mathbf{n} = 0$ on the closed surface of the volume. This condition on the **B** field imposes that the magnetic Helicity is conserved and describes the knottedness of the magnetic lines that are invariant through time. It's important to note that this quantity in Equation 3.24 is solely dependent only on the magnetic field and volume as it is gauge invariant in **A** as shown earlier with **u**. Now we can derive the conservation for cross Helicity, which appears in theories of MHD turbulence and explicitly involves both the flow velocity and the magnetic field:

$$\mathcal{H}_C = \int_V \mathbf{u} \cdot \mathbf{B} dV. \tag{3.31}$$

We can begin by stating the equation of motion for **u**:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{j} \times \mathbf{B}, \tag{3.32}$$

where $\mathbf{j} = \nabla \times \mathbf{B}$, and it follows:

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{1}{\rho}\nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\mathbf{u}}{\rho}\nabla\mathbf{B}, \tag{3.33}$$

$$= \frac{\mathbf{B}}{\rho} \cdot \nabla\mathbf{u}. \tag{3.34}$$

Now we can see that the Lagrangian derivative for the cross Helicity density is:

$$\frac{D}{Dt}\left(\frac{\mathbf{B}\cdot\mathbf{u}}{\rho}\right) = \frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right)\cdot\mathbf{u} + \frac{\mathbf{B}}{\rho}\cdot\frac{D\mathbf{u}}{Dt}, \tag{3.35}$$

$$= \frac{\mathbf{B}}{\rho}\cdot\nabla\left(-\int\frac{dp}{\rho} + \frac{1}{2}\mathbf{u}^2\right). \tag{3.36}$$

This is the same case as in fluid dynamics, as shown in Equation 3.18. Here, we are considering that $\mathbf{B}\cdot\mathbf{n} = 0$ at the surface of V, as in the previous cases integrating this will vanish for the dot product of the magnetic field and the surface measure. Hence, the time derivative of the cross Helicity is vanishing, meaning that this cross Helicity for an ideal MHD with no dissipation is constant.

3.2 Topological evolution of Helicity

Although the study of topological properties of the vorticity field are invariant for an inviscid flow, this is usually not the case in nature. In this section, we will explore how Helicity behaves in viscous fluids. This will help to generalize the notion of Helicity from the previous sections and introduce the concept of topological evolution. The generalization consists of discovering a vector part of the usual magnetic helicity, which was first understood by Carter in 1979 [18]. However, here we will follow a fluid setup to show how these subjects are interlinked. To do this, we will need a different formulation of the Navier-Stokes equations (NSEs). Hence, we will introduce geometrofluid dynamics (GFD) in the context of viscous fluids. Using this new formulation of the stress-energy tensor will help to detach from the classical vision of electromagnetism and explore new solutions.

We will use this formulation as a tool to generalize the concept of Helicity and its evolution, making important statements about the results of the previous sections. We will mainly follow Berger and Field [19] and Sconfield and Huq [20].

3.2.1 Geometrofluid Dynamics

Here we will give a short introduction on GFD this will give a solid base to explore Helicity evolution in a viscid fluid. Here we will be dealing with a homogeneous isotropic fluid and we will follow the description of Sconfield and Huq in [21] and [22]. The main quantities that we are going to use in GFD are the vortex field F where its components are composed by the vorticity vector $\boldsymbol{\omega}$ and the swirl vector field $\boldsymbol{\zeta}$, altogether with its vector potential $\mathbf{A}$ and current $\mathbf{J} = \rho\mathbf{u}$.

Its important to emphasise the difference with the classical NSEs for viscid fluids, where instead one have the vorticity defined as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ and the swirl vector as $\boldsymbol{\zeta} = \boldsymbol{\omega} \times \mathbf{u}$. Although the quantities like swirl and vorticity are comparable with the ones found in Euler or NSE the potential $\mathbf{A}$ that allows to the computation of the vortex field $F(\boldsymbol{\omega}, \boldsymbol{\zeta})$ is not the vorticity field as in the previous sections, we will continue to remark these differences being fundamental for understanding the generalization of Helicity for fluids flows.

We need to invoke a different formulation of the Navier Stokes stress balance as we want a scenario with turbulent flows having a vortex field. We will use three different stress energy tensors for the stress balance, respectively: inertial $\tau_e^{\kappa\nu}$, Newtonian viscous $\tau_n^{\kappa\nu}$ and the vortex field $\tau_m^{\kappa\nu}$. Starting from the stress-energy of the mass distribution is defined as:

$$\tau_e^{\mu\nu} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + c_m^{-2} u^\mu u^\nu\right), \tag{3.37}$$

here the c_m stands for maximum transverse wave speed, p and ρ are the usual pressure and density. Then the Newtonian fluid stress energy with viscosity η is given by:

$$\tau_{\mu\nu}^n = 2\eta\left(\widetilde{\sigma}_{\mu\nu} + \delta\theta\mathcal{P}_{\mu\nu}\right), \tag{3.38}$$

where $\widetilde{\sigma}_{\mu\nu} = \frac{1}{2}\left(\mathcal{P}_\nu^\epsilon u_{\mu;\epsilon} + \mathcal{P}_\mu^\epsilon u_{\nu;\epsilon}\right) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}$ and the projector operator defined as $\mathcal{P}_{\mu\nu} = g_{\mu\nu} + \hat{u}_\mu\hat{u}_\nu$, here $\hat{u}_\mu$ is the normalized velocity vector. It is important to note that in this context the 4 velocity u_μ satisfies the normalization condition $u^\mu u_\mu = -1$ with $c_m = 1$. Lastly the stress energy for the vortex field is defined as:

$$4\pi\mathring{\tau}_m^{\mu\nu} = g^{\mu\alpha}\mathring{F}_{\alpha\beta}\mathring{F}^{\beta\nu} - \frac{1}{4}g^{\mu\nu} \star \mathring{F}_{\alpha\beta}\mathring{F}^{\alpha\beta}, \tag{3.39}$$

where the star $\star$ is the Hodge operator [23] (refer to B.5).

Therefore we can now state the equation that governs the vorticity, velocity, swirl fields and energies. Hence by following [22] we can state:

$$\left(\tau_e^{\kappa\nu} - \tau_n^{\kappa\nu} - \tau_m^{\kappa\nu}\right)_{;\nu} = 0, \tag{3.40a}$$

$$\left(\tau_e - \tau_n\right)_{;\nu}^{\mu\nu} = -\mathring{F}^{\mu\nu} j_\nu, \tag{3.40b}$$

$$\mathring{H}_{\kappa\lambda} = C_{\kappa\lambda}^{\mu\nu}\mathring{F}_{\mu\nu}, \tag{3.40c}$$

$$\mathring{F}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right), \tag{3.40d}$$

$$-\Box A_\mu = -\left(\partial_t^2 - \nabla^2\right)A_\mu = \frac{4\pi}{\bar{\eta}} j_\mu. \tag{3.40e}$$

Here the last 3 equation are *vortex field equations*. The parameter $C^{\mu\nu}_{\kappa\lambda}$ for the vortex field can be expressed in term of four quantity $(\bar{\lambda}, \lambda, \kappa, \bar{\kappa})$ therefore we can explicitly state $\mathring{H}$ and $\mathring{F}$ in this manner:

$$-\star\mathring{H}_{\kappa\lambda} = \begin{pmatrix} 0 & -\bar{\lambda}\xi_1 & -\bar{\lambda}\xi_2 & -\bar{\lambda}\xi_3 \\ \bar{\lambda}\xi_1 & 0 & \bar{\kappa}\varpi_3 & -\bar{\kappa}\varpi_2 \\ \bar{\lambda}\xi_2 & -\bar{\kappa}\varpi_3 & 0 & \bar{\kappa}\varpi_1 \\ \bar{\lambda}\xi_3 & \bar{\kappa}\varpi_2 & -\bar{\kappa}\varpi_1 & 0 \end{pmatrix}, \quad \mathring{F}_{\kappa\nu} = \begin{pmatrix} 0 & -\lambda\zeta_1 & -\lambda\zeta_2 & -\lambda\zeta_3 \\ \lambda\zeta_1 & 0 & \kappa\omega_3 & -\kappa\omega_2 \\ \lambda\zeta_2 & -\kappa\omega_3 & 0 & \kappa\omega_1 \\ \lambda\zeta_3 & \kappa\omega_2 & -\kappa\omega_1 & 0 \end{pmatrix}. \tag{3.41}$$

It is easy to see that the components of the vortex field $\mathring{\tau}^{\mu\nu}_m$ can be computed to be:

$$4\pi\mathring{\tau}^{00}_m = \frac{1}{2}\left(\kappa^2\omega^2 + \lambda^2\zeta^2\right), \tag{3.42a}$$

$$4\pi\mathring{\tau}^{0j}_m = 4\pi\mathring{\tau}^{j0}_m = -(\kappa\omega \times \lambda\zeta)^j, \tag{3.42b}$$

$$4\pi\mathring{\tau}^{jk}_m = -\left(\kappa^2\omega^j\omega^k + \lambda^2\zeta^j\zeta^k\right) + \frac{1}{2}\left(\kappa^2\omega^2 + \lambda^2\zeta^2\right)\delta^{jk}, \tag{3.42c}$$

where the parameters κ and λ are chosen to yield a stress energy density, we will assume throughout the next section that $\kappa = 1$ as we are mostly interested to the viscous property that depends on λ. We can also define the stress energy dissipation tensor as

$$\dot{\tau}^{\kappa\lambda}_m \equiv \eta\mathring{\tau}^{\kappa\lambda}_m. \tag{3.43}$$

Here we can see that this configuration of vortex field is indeed a function of $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ (defined before as $F(\boldsymbol{\omega}, \boldsymbol{\zeta})$), it is important to note that these equation differ from the classical NSEs.
Now we can turn up our attention to the fundamental *vortex field equations*, in exterior calculus notation we have $\mathring{H} = \star\mathring{F}$ then by considering the equation above we can rewrite those as:

$$d\mathring{H} = 4\pi J \text{ and } d\mathring{F} = 0. \tag{3.44}$$

It is important to note that although there is a obvious similarity with the electromagnetism of Maxwell equations [23] these are not sufficient here to give the whole dynamics and must be supplied with the continuity equation for a incompressible fluid or either with an equation of state for the pressure.

We conclude by stating the main difference with the classical Euler set up. Here in this formulation in Equations (3.40) the special acoustic relativistic Euler equation for a perfect fluid would correspond to have $\tau^{\mu\nu}_{e;\nu} = 0$. Nevertheless for special acoustic space-time (SAST) theory for the NSEs we would have $(\tau^{\kappa\nu}_e - \tau^{\kappa\nu}_n)_{;\nu} = 0$. Furthermore the equations in (3.40) are not the equation for the vorticity field as previous noted, this is in contrast to the standard NSEs. However the GFD equations up to approximation can be assumed as NSEs.

3.2.2 Evolution and generalization of Helicity

After defining a general framework to work with viscid flows we can apply and focus on the evolution of helicity, this is essentially the manifestation of topological evolution in a flow. Helicity in electromagnetism by Woltjer [17] is defined as $\int \mathbf{A} \cdot \nabla \times \mathbf{A}$ contained in a given volume, here in the context of GFD we will use the Helicity density $h_t = \mathbf{A} \cdot \boldsymbol{\omega}$ which is a scalar and the choice of writing it with the subscript t will be apparent later one. From Equation (3.43) ($dF = 0$) is possible to deduce that we have a similar set of equation to electromagnetism for the vorticity field:

$$\nabla \cdot \boldsymbol{\omega} = 0 \text{ and } -\frac{\partial \boldsymbol{\omega}}{\partial t} + \lambda \nabla \times \boldsymbol{\zeta} = 0, \tag{3.45}$$

where we are setting $c_m = 1$. Therefore we can choose the vorticity to be $\boldsymbol{\omega} = \nabla \times \mathbf{A}$. It is important to punctualise that this is not the same vorticity and vector potential of the previous sections in fact for classical NSE or for perfect fluids one defines $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Therefore the rate of change for h_t is given by:

$$\frac{\partial \mathbf{A} \cdot \boldsymbol{\omega}}{\partial t} = \frac{\partial \mathbf{A}}{\partial t} \cdot \boldsymbol{\omega} + \mathbf{A} \cdot \frac{\partial \boldsymbol{\omega}}{\partial t}. \tag{3.46}$$

Now we can define from Equation (3.45) this quantity:

$$\nabla \times \left(\frac{\partial \mathbf{A}}{\partial t} + \lambda \boldsymbol{\zeta} + \nabla \phi \right) = 0, \tag{3.47}$$

where we associate the scalar field ϕ with the time component of the four vector A^μ. From the last equation we can compute that $\partial \boldsymbol{\omega}/\partial t = \partial(\nabla \times \mathbf{A})/\partial t = -\nabla \times (\lambda \boldsymbol{\zeta} + \nabla \phi)$, then we obtain:

$$\frac{\partial \mathbf{A} \cdot \boldsymbol{\omega}}{\partial t} = -(\lambda \boldsymbol{\zeta} + \nabla \phi) \cdot \boldsymbol{\omega} - \mathbf{A} \cdot \nabla \times (\lambda \boldsymbol{\zeta} + \nabla \phi) \tag{3.48}$$

$$= -(\lambda \boldsymbol{\zeta} + \nabla \phi) \cdot \boldsymbol{\omega} - \mathbf{A} \cdot \nabla \times (\lambda \boldsymbol{\zeta}) \tag{3.49}$$

$$= \nabla \cdot (\mathbf{A} \times \lambda \boldsymbol{\zeta}) - (2\lambda \boldsymbol{\zeta} + \nabla \phi) \cdot \boldsymbol{\omega} \tag{3.50}$$

$$= \nabla \cdot (\mathbf{A} \times \lambda \boldsymbol{\zeta}) - 2\lambda \boldsymbol{\zeta} \cdot \boldsymbol{\omega} - \nabla \cdot (\phi \boldsymbol{\omega}) \tag{3.51}$$

$$= \nabla \cdot (\mathbf{A} \times \lambda \boldsymbol{\zeta} - \phi \boldsymbol{\omega}) - 2\lambda \boldsymbol{\zeta} \cdot \boldsymbol{\omega}, \tag{3.52}$$

where we used the identities (A.5) and (A.7). Hence we can summarize the helicity evolution equation as:

$$-\frac{\partial \mathbf{A} \cdot \boldsymbol{\omega}}{\partial t} + \nabla \cdot (\mathbf{A} \times \lambda \boldsymbol{\zeta} - \phi \boldsymbol{\omega}) = 2\lambda \boldsymbol{\zeta} \cdot \boldsymbol{\omega}, \tag{3.53}$$

it is important to note that if the viscosity parameter λ vanishes there is no production of helicity, this make perfectly sense as if there is no viscosity helicity does not evolve in time, in other words with non viscous fluids Helicity is a topological invariant of the system. Also is worth mentioning that if we take $\lambda = 0$ we have a similar scenario comparable to the MHD case in Equation (3.25). We can already see that the topological change must be related to the quantity $\lambda\boldsymbol{\zeta}\cdot\boldsymbol{\omega}$, this relates the phenomenological viscosity parameter λ to the topological change which is sensitive of how the vorticity $\boldsymbol{\omega}$ and swirl $\boldsymbol{\zeta}$ are aligned. Furthermore the equation in (3.53) it is a hint for the four vector helicity where the spatial components are $(\mathbf{A}\times\lambda\boldsymbol{\zeta}-\phi\boldsymbol{\omega})$, to further delve and understand these concepts we will use differential geometry to show that the full 4 helicity density vector is equivalent to the Hopf invariant and this connection rises naturally in this context.

Hence now we will continue by deriving explicitly the equation (3.53) using differential form. We can introduce the 2-form vortex field F in 4 dimensions in terms of the 2-form vorticity $\omega=\frac{1}{2}\omega_{ij}dx^i dx^j$ and the 1-form swirl $\zeta=\zeta_i dx^i$ both in the spatial 3 dimensions, where here the vorticity is a pseudovector and swirl is a vector. Therefore F is defined as:

$$F=\omega_x dy\wedge dz+\omega_y dz\wedge dx+\omega_z dx\wedge dy+\lambda\left(\zeta_x dx+\zeta_y dy+\zeta_z dz\right)\wedge c_m dt, \tag{3.54}$$

here we have that this 2-form is defined in terms of the 1-form vector potential A as:

$$A=\phi dt+A_x dx+A_y dy+A_z dz, \tag{3.55}$$

from (3.43) we can see that this is true $F=dA$, this is the analog of the electromagnetism field tensor. Hence now we can define the 3-form helicity density as:

$$h=A\wedge dA=A\wedge F. \tag{3.56}$$

In literature the integral of this helicity density is called the Hopf Invariant or the abelian Chern Simon 3 form [5] and is defined as:

$$\mathcal{H}_G=\frac{1}{16\pi^2}\int_{\mathcal{M}^3}A\wedge dA, \tag{3.57}$$

we will embark in a quick detour that would give context and be relevant later on. You can look to this integral over a three dimensional boundary of a four dimensional manifold, say $\partial\mathcal{M}^4$, then applying Stokes theorem we have $\int_{\partial\mathcal{M}^4}A\wedge dA=\int_{\mathcal{M}^4}dA\wedge dA$. Therefore the integral vanishes for $h=A\wedge dA$ being close or exact, however this is a indicator of topology of the boundary as if h is closed but not exact (for example when the manifold have holes) then this integral do not vanish and is equal to a integer this is deeply linked with homotopy and the presence of invariant knots, we will embark in these discussions in the next section. Following the discussion we can open the 3 form helicity density explicitly

$$h=\frac{1}{3!}h_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho \tag{3.58}$$

$$=h_t dx\wedge dy\wedge dz-h_x dy\wedge dz\wedge c_m dt+h_y dz\wedge dx\wedge c_m dt-h_z dx\wedge dy\wedge c_m dt \tag{3.59}$$

here we have associated the components that are $\frac{1}{3!}h_{xyz} = h_t$ and the other follows in the same manner. We will use the Hodge star in 4 dimensions as follows

$$\star(dx^\mu) = \eta^{\mu\lambda}\varepsilon_{\lambda\nu\rho\sigma}\frac{1}{3!}dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{3.60}$$

$$\star(dx^\nu \wedge dx^\rho \wedge dx^\sigma) = \varepsilon^{\mu\rho\sigma}{}_{\lambda}dx^\lambda. \tag{3.61}$$

In equation (3.53) we have found a 4 vector this is actually the Hodge star of the 3 form helicity density as:

$$\star h = -h_t dt + h_i dx^i = (A \cdot \omega, \lambda A \times \zeta - \phi\omega), \tag{3.62}$$

therefore from now on we will call the helicity defined by Moffat in [2] as the scalar helicity and the spacial component as the helicity current. We can check this equation by considering $A \wedge F$ and take its Hodge star, therefore we can see

$$A \wedge F = (\phi dt + A_i dx^i) \wedge (\omega_{ij}dx^i dx^j + \lambda(\zeta_i dx^i) \wedge c_m dt) \tag{3.63}$$

$$= \omega_{ij}\phi dx^i dx^j dt + A_k\lambda\zeta_i c_m dx^k dx^i dt + A_k\omega_{ij}dx^k dx^i dx^j, \tag{3.64}$$

here the term with repeated indices are vanishing due to the antisymmetry of the wedge operator. Now by taking its hodge star we can see

$$\star(A \wedge F) = \omega_{ij}\phi\varepsilon^{ijt}{}_{k}dx^k + A_k\lambda\zeta_i c_m\varepsilon^{kit}{}_{j}dx^j + A_k\omega_{ij}\varepsilon^{kij}{}_{t}dx^t \tag{3.65}$$

here up to a symmetry factor this is exactly the equation (3.62). Therefore we can see that the scalar part of the helicity density is $h_t \equiv A \cdot \omega$ that we where investigating in (3.46), this is the equivalent of what Woltjer found $\mathbf{A} \cdot \mathbf{B}$ [17]. To compute fully the equation (3.53) in terms of the 3 form Helicity density we need

$$dh = d(A \wedge dA) = dA \wedge dA = F \wedge F, \tag{3.66}$$

where we use the fact that the operator exterior derivative is nilpotent $d^2A = 0$. Then we can see explicitly that

$$F \wedge F = (\omega_{ij}dx^i dx^j + \lambda(\zeta_i dx^i) \wedge c_m dt) \wedge (\omega_{kl}dx^k dx^l + \lambda(\zeta_k dx^k) \wedge c_m dt) \tag{3.67}$$

$$= \omega_{ij}\lambda\zeta_k c_m dx^i \wedge dx^j \wedge dx^k \wedge dt + \omega_{kl}\lambda\zeta_i c_m dx^i \wedge dx^k \wedge dx^l \wedge dt \tag{3.68}$$

$$= 2\omega_{ij}\lambda\zeta_i c_m dx^i \wedge dx^j \wedge dx^k \wedge dt \tag{3.69}$$

$$= -2\omega_{ij}\lambda\zeta_i\varepsilon^{ijk}c_m dt \wedge dx \wedge dy \wedge dz \tag{3.70}$$

$$= -2(\lambda\omega \cdot \zeta)\Omega_4, \tag{3.71}$$

where $\Omega_4 = c_m dt \wedge dx \wedge dy \wedge dz$ is the four space volume. Hence we can see that the quantity $-2\lambda\omega \cdot \zeta$ serve as contracting or expanding the differential four volume Ω_4. It is important to note that the quantity $\omega \cdot \zeta$ is the analog of $\mathbf{E} \cdot \mathbf{B}$ in electromagnetism, this quantity is related to dissipation when is not equal to zero and greater than one. In the situation of perfect fluid it is easy to see that this quantity is actually equal to zero: $\zeta = u \times \omega$ and therefore $\omega \cdot \zeta = \omega \cdot u \times \omega = 0$, here there is no dissipation and this results having the vorticity and the swirl fields to be perpendicular. However for this case in viscous fluids there are not perpendicular hence we have a topological evolution of knottedness for the field ω. Hence we can summarise the equation for the evolution of the 4 vector helicity as:

$$\frac{\partial h_t}{\partial t} + \nabla \cdot \mathbf{h} = -2\lambda\omega \cdot \zeta. \tag{3.72}$$

where h_t is the scalar helicity and $\mathbf{h}$ is the current helicity. This quantity as showed is equal to the component of $F \wedge F = dh$ have different values for different region of the manifold which is describing the topology or the singularities of the vector potential. This is due that the vortex field is closed ($dF = 0$) but only exact locally ($F = dA$).

In conclusion, the non-vanishing term in equation (3.72) is characteristic of viscous flow and determines the evolution of helicity. This results in a change of topology, rather than maintaining an invariant as presented in previous sections.

4 Topological physics and field theory

In this section, we delve into the fascinating realm of topological physics and field theory. We aim to provide a heuristic introduction to topological physics, focusing on giving the best physical interpretation to key concepts, particularly those needed in the original section on the helicity of time. We will not be bogged down by signs and conventions but will concentrate on those that are topologically relevant. Algebraic calculations will be deferred to the novel part of this thesis to avoid redundancy.

We will introduce the Dirac monopole to elucidate the Chern class of a closed curve, showcasing a prime example of transitioning from $3 + 1$ dimensions to 3 spatial dimensions and its implications. With a practical example in electromagnetism, we will describe homology and cohomology classes, demonstrating that parallel transport is the dual pairing between cohomology and homology. This will introduce an important observable in Quantum Field Theory: the Wilson Loop.

Shifting our perspective to field theories, we will introduce the Chern-Pontryagin term and its action, known as the theta term. We will draw important connections to the helicity discussed in previous sections. We will also explore how the Abelian Chern-Simons term arises in 3 dimensions, its topological meaning, and its connection to scalar helicity. This will pave the way for a discussion on knots in Quantum Field Theory, where we will employ the previously introduced Wilson Loop to find the Gauss linking number.

4.1 The Dirac monopole

The upshot of the Dirac Paper [24] is that he demonstrated that the existence of magnetic monopoles would naturally quantize electric charge. He introduced a singular string-like configuration in the vector potential to describe the monopole field. This led to a topological argument that reconciled the monopole with Maxwell's equations. The paper laid the groundwork for the theoretical study of magnetic monopoles in quantum field theory. However here we will embark on a topological explanation of the Dirac monopole, we will use some mathematics that was being developed around the publication of Dirac [24].

By Poincaré's lemma in standard Electromagnetism theory we have that the F is globally exact ($F = dA$) and usually one embark using a standard manifold without any discontinuity, however the Dirac monopole arises actually in a less trivial manifold. In mathematical terms we say for a base space $\mathbb{R}^4$ that is trivial will make the Abelian group $U(1)$ bundle (Electromagnetism) trivial. A bundle over a base manifold is like a "layered" structure where each point on the base manifold has an associated "fiber", which is itself a mathematical space.

Hence we will take the standard construction of our base manifold to be three dimensional euclidean without the origin $\mathbb{R}^3 - \{0\}$, it is important to note that this is equivalent to take $S^2 \times \mathbb{R}^+$, where S^2 is the standard 3d sphere and $\mathbb{R}^+$ is the positive real axis. These two spaces have the same homotopy type, two spaces having the same homotopy type means they can be "stretched" or "deformed" into each other without tearing or gluing. In essence, they share the same topological "skeleton".

Here for simplicity we are working without time and using only the three spacial dimensions as done by Dirac's paper. Therefore we can define the one form electric field $\mathbf{E} = E_i dx^i$ and the 2-form magnetic field $\mathbf{B} = \frac{1}{2} F_{ij} dx^i \wedge dx^j$ both in three dimension. However the connection F is defined in four dimensions as a 2-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and in therms of the electric and magnetic fields takes this form:

$$F = dx^0 \wedge \mathbf{E} + \mathbf{B}. \tag{4.1}$$

Hence we can see heuristically that by ignoring the time dimension we would have $F|_{\mathbb{R}^3} = \mathbf{B}$, or more formally that F pulls back to a 2-form which is the magnetic field on $\mathbb{R}^3$. Moreover the magnetic field here in static Electromagnetism actually represents a curvature on $\mathbb{R}^3$ with associated connection one form $A_i dx^i$, here it easy to see that the magnetic field is fundamentally geometrically different from the one form electric field.

To capture and find the magnetic monopole we need to embark in the first Chern class $c_1(P)$ in this case of a $U(1)$ bundle P that is defined as a integral. Therefore in mathematics the first Chern class is a topological invariant that captures the "twistiness" of a complex line bundle over a base manifold. It quantifies how much the bundle deviates from being trivial, this is defined for the connection F as:

$$c_1(P) = \frac{1}{2\pi} \int_{S^2} F, \quad c_1(P) \in \mathbb{Z}. \tag{4.2}$$

Where for this integral expression we have showed that the connection F is the magnetic field B and represents the magnetic field of the monopole. When you integrate this curvature over a closed surface surrounding the monopole, you get an integer multiple of 2π, confirming the quantization of magnetic charge that Dirac's found. So, the first Chern class provides a rigorous, topological framework for understanding why and how magnetic charge would be quantized in the presence of a magnetic monopole, linking abstract mathematical concepts directly to physical observables.
Hence we showed actually that the vector potential and the Electromagnetism construction is sensible to the topology of the manifold, we will continue this abstraction with a practical experimental example to show the concreteness of topological physics while still introducing new concepts.

4.1.1 The Aharonov and Bohm effect and Wilson loop

Three decades after Dirac's paper on monopoles, another intersection between physics and topology emerged with the Aharonov-Bohm (AB) effect in 1959 [25]. This quantum phenomenon alters an electron's phase through the connection potential or gauge field A, even in regions where the field strength is zero. In the magnetic AB effect, a solenoid with magnetic flux influences electrons circling it, despite the absence of a magnetic field outside the solenoid; for the experimental setup, please refer to Figure 6. The effect underscores the non-locality of quantum mechanics and elevates the importance of vector potentials over the electromagnetic field or curvature F. It was experimentally verified in 1960 by Brill and Werner [26] and has significant implications for both quantum interference and topology.

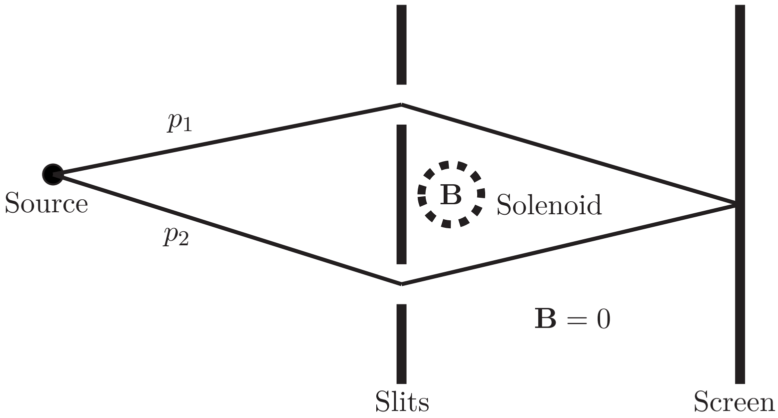

Figure 6: This figure depicts the experimental setup for the Aharonov-Bohm effect. Here, a source emits electrons that pass through a double slit before striking a screen. Positioned behind the double slit is a shielded solenoid where $\mathbf{B} \neq 0$; elsewhere, $\mathbf{B} = 0$. When the solenoid is turned on, the behavior of the electrons changes compared to when it is turned off. We define the curve C as the union of the paths of electrons emitted through slits p_1 and p_2.

To understand the topological implications of the experiment, consider a space manifold Ω where the curvature F is zero everywhere except near the solenoid, as shown in Figure 6. Similar to the situation with the Dirac monopole, we have a non-contractible loop C in Ω. We aim to compute the "circulation" of the potential A for the two emitted electrons, represented as paths p_1 and p_2 in the figure. To accomplish this, we invoke the notion of parallel transport for closed curves. This moves a vector along a curve in a manifold such that the vector remains parallel to itself according to the connection. It is defined as:

$$P_t(C) = \exp\left[\oint_C A\right] \tag{4.3}$$

This is also known as the holonomy operator [23]. This operator is fundamental and turns out to be non-trivial in this context. We will continue by describing what exactly it computes. In this setup we have the curvature $F = dA = 0$ vanishes everywhere except in the solenoid, where $dA \neq 0$. This implies that, except in the solenoid, one must have:

$$A = df, \tag{4.4}$$

where f is a function on Ω. Therefore, A determines a de Rham cohomology class [A], representing the equivalence class (refer to Appendix B.1) of differential forms that are closed but not exact, modulo exact forms. This captures topological invariants of a manifold, such as holes or non-contractible loops. In physics, de Rham cohomology is often used to classify field configurations and topological defects. This equivalence class is represented in this notation:

$$[A] \in H^1(\Omega; \mathbb{R}). \tag{4.5}$$

From the parallel transport operator in equation (4.3) we see that there is a clearly dependence on loop C this is in fact the homology class $[C]$ of this curve. The homology class captures essential topological and geometric features of the space, such as loops that encircle holes or other non-contractible features. We can represent this by:

$$[C] \in H_1(\Omega; \mathbb{R}). \tag{4.6}$$

Hence this means that the integral in equation (4.3) is actually the dual paring between cohomology and homology :

$$([A], [C]) = \oint_C A \tag{4.7}$$

The dual pairing between cohomology and homology is a mathematical framework that allows one to evaluate cohomology classes on homology classes, yielding a scalar. In physical terms, this dual pairing can be thought of as a way to "measure" the flux of a field (captured by the cohomology class) through a cycle (captured by the homology class). This is particularly relevant in gauge theories and topological field theories, where such pairings can correspond to physically measurable quantities like magnetic flux through a loop.

Hence this experiment is similar to the Dirac monopole argument, here we can take the space to be $\Omega = \mathbb{R}^3 - L$ where L is the cylinder represented by the solenoid. Hence we see that the cohomology here becomes $H^1(\Omega; \mathbb{R}) = H^1(\Omega - L; \mathbb{R}) = \mathbb{Z}$ and therefore is not trivial contributing to the experiment.

After discussing the Aharonov-Bohm effect, it is pertinent to introduce the concept of the Wilson loop [27], defined for this case as:

$$W_R(C) = \exp\left[\oint_C A\right] \tag{4.8}$$

where R stands for the irreducible representation of the group G that we are working, in our case is always the Abelian $U(1)$. This is exactly our parallel transport that we have discussed until now but in a Quantum Field Theory (QFT) perspective, It captures the net effect of parallel transporting a state along a closed loop in the presence of a gauge field, serving as a bridge to the dual pairing between cohomology and homology. It is important to note that in Quantum Field Theory, observables must be gauge-invariant to have physical meaning, as they should not depend on the choice of gauge. The Wilson loop satisfies this criterion by being constructed as a path-ordered exponential of the gauge field along a closed loop. This gauge invariance ensures that the Wilson loop is a legitimate observable in QFT, capable of providing physically meaningful information about the system, such as confinement in QCD or topological phases in condensed matter systems.

We will use this concept to find the Gauss linking number in a QFT set up in the last subsection of this section.

4.2 Chern-Pontryagin density and the theta term

Here we will introduce the Chern-Pontryagin density and theta term, It is important to say that these quantities they goes for many names in the literature but we will follow the nomenclature given by Jackiw [5] and the Tasi lecture on Charge conjugation and Parity (CP) symmetry violation [28]. Here we are working special relativistic electromagnetism in four dimensions with the usual field strength tensor (curvature) defined as $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$, this namely are the components of the two form F. Therefore we will define the Chern-Pontryagin density is formulated as follows:

$$\mathcal{A}_4 = -F \wedge F = \frac{1}{2} \star F^{\mu\nu} F_{\mu\nu} = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \tag{4.9}$$

It is important to state that this is a covariant density and upon integration over a manifold gives a pseudo scalar, and most important part is that they do not need the metric upon integration this makes it independent from local properties of the manifold and makes them a topological object as are sensible of the "boundary information" of the manifold. We will return and explain what we mean with densities when we will introduce a General Relativity framework in section 5.1. Therefore the integration of the Chern-Pontryagin density is called or is known as the theta term [28]:

$$S_\theta = \frac{\theta e}{4\pi^2\hbar} \int d^4x \, \frac{1}{2} \star F_{\mu\nu} F^{\mu\nu} \tag{4.10}$$

$$= -\frac{\theta e}{4\pi^2\hbar} \int F \wedge F = \frac{\theta e}{4\pi^2\hbar} \int 2\mathbf{E} \cdot \mathbf{B}. \tag{4.11}$$

Here the θ is a parameter and is dimensionless. To show that the theta term is equal to a constant and effectively that is a Chern number of the $U(1)$ bundle we will follow David Tong in [29] for a simple effective proof. We will embark to use fully euclidean four dimensional torus space $\mathbf{T}^4$, to make computation easier we will be restricting to the special case that the magnetic and electric field are constant and that $\mathbf{E} = (0, 0, E)$ following as well $\mathbf{B} = (0, 0, B)$. Therefore we can see that the relevant part of the theta term becomes:

$$\int_{\mathbf{T}^4} d^4x EB = \int_{\mathbf{T}^2} dx^0 dx^3 E \int_{\mathbf{T}^2} dx^1 dx^2 B. \tag{4.12}$$

To compute the two integrals, we need to consider the gauge field A_μ as it is the fundamental field, as demonstrated in the Aharonov-Bohm effect. Assuming periodicity in the x^1 direction with radius

R, one of the constant components of the gauge field is given as follows:

$$A_1 \equiv A_1 + \frac{\hbar}{eR}. \tag{4.13}$$

This comes from the notion of gauge transformation on a circle $A_1 \to A_1 + \partial_1 \omega$. Now taking in consideration that the magnetic field is:

$$B = \partial_1 A_2 - \partial_2 A_1, \tag{4.14}$$

where we can gauge fix $A_1 = 0$ resulting $B = \partial_1 A_2$, as B here is a constant we find that $A_2 = Bx^1$. Therefore we have that at any rotation of $2\pi R$ the magnetic field must be equal to

$$B\, 2\pi R = \frac{\hbar}{eR} \Rightarrow B = \frac{\hbar n}{2\pi e R^2} \text{ with } n \in \mathbb{Z}, \tag{4.15}$$

where in the last equation we added the integer n that stands for the number of revolution around the torus. Therefore it is easy to see that we can now solve the second part of the equation (4.12) as:

$$\int_{\mathbf{T}^2} dx^1 dx^2 B = \frac{2\pi\hbar n}{e}, \tag{4.16}$$

this is expected, in fact this is the same condition that we derived in the section on Dirac monopoles in equation (4.2).

The same argument applies for the electric field,

$$E = \partial_0 A_3 - \partial_3 A_0, \tag{4.17}$$

as previous we can gauge fix $A_0 = 0$ making $E = \partial_0 A_3$ and working with constant electromagnetic field, this leads to have $A_3 = Ex^0$. To be compatible with the periodicity of A_3 we have $E = \hbar n'/2\pi e R^2$ where $n' \in \mathbb{Z}$, hence we find that the integral is equal to

$$\int_{\mathbf{T}^2} dx^0 dx^3 E = \frac{2\pi c\hbar n'}{e}. \tag{4.18}$$

Hence now we can see that using equation (4.18) and (4.16) we have

$$\int_{\mathbf{T}^4} d^4x \mathbf{E} \cdot \mathbf{B} = \frac{4\pi^2\hbar^2 cN}{e^2} \Rightarrow S_\theta = \hbar\theta N \quad \text{with} \quad N = nn' \in \mathbb{Z}, \tag{4.19}$$

as promised this is a scalar and actually now becomes clear that this is actually a result of the product of two Chern numbers on the $U(1)$ bundle, where have exploited the fact the base manifold is made up of circles then used the fact $c_1(P) \in \mathbb{Z}$ introduced in the Dirac monopole section. It is possible to generalize this for non-constant electric and magnetic fields. However, the primary focus was to demonstrate concretely how the θ-term reacts to the boundary of the manifold, in this case $\mathbf{T}^4$.

We end this section by explaining why this action is called theta term also known as axion Electrodynamics [30] which is relevant in topological insulators. The theta here in a classical theory can take any value however in a quantum theory it becomes periodic namely:

$$\theta \in [0, 2\pi), \tag{4.20}$$

this is because the theta term S_θ contributes to the partition function in the following manner:

$$\exp\left(\frac{iS_\theta}{\hbar}\right) = e^{iN\theta}. \tag{4.21}$$

Hence as N is a integer as shown before, θ in the partition function is relevant only if takes value modulo 2π. This is also the reason why it is accompanied by other constants, to ensure that the periodicity is natural and physical.

However the Chern-Pontryagin therm can be shown to be written as a total derivative of what is called the Chern-Simons term, we will introduce it in the next section.

4.3 Abelian Chern Simon term

As discussed in the previous section, the Chern-Pontryagin entity is topological in nature, providing insights into the behavior at the boundary of the given manifold. Given that these are scalar densities, it should be possible to represent them as the divergence of vector densities. This reformulation allows us to shift our focus from the manifold to its boundary surface, particularly at infinity, thanks to the application of Gauss's theorem.

Here we will start by noting that the Chern-Pontryagin density in equation (4.9) it can be expressed as a total derivative [5]:

$$\begin{aligned} \mathcal{A}_4 &= \frac{1}{2} \star F^{\mu\nu} F_{\mu\nu} = \partial_\mu C_4^\mu \\ C_4^\mu &= \epsilon^{\mu\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma, \end{aligned} \tag{4.22}$$

where the subscript "4" indicates the dimensions of the space in which we are working. Here, C_4^μ represents the Chern-Simons current. This is in the context of the anomaly current j^μ, which is a quantum deviation from classical symmetry, we have the following: Classically, $\partial_\mu j^\mu = 0$, but quantum mechanically, $\partial_\mu j^\mu = \mathcal{A}$. In this equation, $\mathcal{A}$ is the anomaly term and is often field-dependent. For those C^μ that produce an anomaly term, the corresponding anomaly currents $\partial_\mu C_4^\mu$ are referred to as Chern-Simons currents. These are essential for capturing topological properties of gauge fields and are central to theories like topological quantum field theory in four dimension. They play a crucial role for example in condensed matter systems, not only in physics but as well in biology in the context of DNA mutation [31].

We will continue by stating that the Chern-Simon current in four dimensions it is indeed a three form and upon exterior derivative we end up with the Chern-Pontryagin four form ($F \wedge F$) as mention in the last section, explicitly

$$\text{Chern-Pontryagin} = d(\text{Chern-Simons}) \tag{4.23}$$

$$F \wedge F = d(A \wedge dA), \tag{4.24}$$

hence we can see this is exactly what we where discussing for viscid fluids in section 3.2.2 in equation (3.66) where the helicity 3 form was defined as h and in fact it is a Chern-Simons form. This means up to a definition of the curvature and vector potential connection we where dealing with the same topological quantities and hence the arguments that we have exposed here are valid and enrich our discussion on a generalised helicity and its evolution.

Furthermore although this quantities are defined now in four dimensional context we can in a natural way lower down to odd dimensional manifolds. One just need to note that in equation (4.22) the fully antisymmetric Levi-Civita symbol carries an extra free index and if we fix one of this indexes we can see that it will restrict the others in 3 dimension, for example fixing the time component of the Chern-Simons current we have

$$\begin{aligned} C_4^0 &= \epsilon^{0\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \\ &= \epsilon^{0ijk} A_i \partial_j A_k \\ &= \mp \epsilon^{ijk} (A_i \partial_j A_k), \end{aligned} \tag{4.25}$$

where the sign in the last equation is due up to convention and if we are in a Riemannian or Lorentzian geometry. This is known as the Abelian Chern-Simons term [32] and this can be integrated in 3 dimension and give a gauge invariant term suitable for an action for a three dimensional theory, it is important to note the Chern-Simons current in four dimension (C_4^μ) can not be standalone term in a action as is not gauge invariant. Therefore just to make it explicit we can see that the Chern-Simons term can be rewritten using vector calculus as

$$\text{Abelian 3d: } C_3 = \epsilon^{ijk}(A_i \partial_j A_k) \tag{4.26}$$

$$= \mathbf{A} \cdot (\nabla \times \mathbf{A}) \,. \tag{4.27}$$

Now should be easier to see that this is exacly the same topology what we where dealing in the past fluid theory sections, hence this is the helicity for Moffatt and Woltjer in section 3.1 and it is actually the 3d Chern-Simons term.

In the context of Woltjer's theory of closed magnetic lines, the situation is analogous to the Aharonov-Bohm (AB) effect, where the region with a non-zero magnetic field is considered to be a closed line, which could either be a knot or a link between knots. Furthermore, the evolution of the 3-form Chern-Simons term is governed by the four-dimensional Chern-Pontryagin term, which is equal to $\mathbf{E} \cdot \mathbf{B}$. For a $U(1)$ bundle P, this term represents the product of two first-class Chern numbers for a base manifold $\mathbf{T}^4$. We will explore these connections further in the section on the topology of time in the context of general relativity.

Before concluding this section, we will delve into another connection between knots and Quantum Field Theory (QFT) in the next subsection, where we will employ the Wilson loop to rediscover the Gauss linking number.

4.3.1 Knots in Quantum Field Theory

Here we will follow Witten [33] and Polyakov [34] on their work of incorporating knots in QFT, here they are considering the Chern-Simons action in $2+1$ dimensions and we will restrict ourself to the Abelian $U(1)$ group. This is a quick way to see that indeed the Chern-Simons action really leads to topological invariants in the context of QFT, here the action is given as

$$S = \frac{ik}{4\pi} \int_{\mathcal{M}} d^3x \, \epsilon^{ijk} A_i \partial_j A_k, \tag{4.28}$$

where k is a integer, here $\mathcal{M}$ is a closed compact manifold and in Witten [33] is taken to be S^3 the four dimensional sphere. Hence the partition function for this QFT is given as

$$Z(\mathcal{M}) = \int \mathcal{DA} \exp\left[-\frac{ik}{4\pi} \int_{\mathcal{M}} d^3x \, \epsilon^{ijk} A_i \partial_j A_k \right] \tag{4.29}$$

where the partition function is a functional integral over all field configurations, weighted by e^{-S}, where S is our action functional. It serves as a generating functional for correlation functions and encodes the vacuum-to-vacuum amplitude. The partition function encapsulates the quantum and statistical behavior of the field theory. This partition itself actually is an invariant and is called the Witten invariant.

Therefore we want to incorporate a knot K on the manifold which is an embedded closed curve C on $\mathcal{M}$. Here we will be using the Wilson loop introduced in equation (4.8) which parallel transport the vector potential around the curve C and we can formulate the expectation value of this operator which is

$$< W_R(C) >= \frac{1}{Z(M)} \int \mathcal{DA} \, W_R(C) \exp\left[-\frac{ik}{4\pi} \int_{\mathcal{M}} d^3x \, \epsilon^{ijk} A_i \partial_j A_k \right]. \tag{4.30}$$

This quantity is shown by Witten to be the Jones Polynomial for the knot $V_k(t)$ for a function t, discussed in section 2.

We can also generalised this expectation value for number C_p knots rising p Wilson loops we have:

$$
\begin{aligned}
&< W_{R_1}(C_1)\cdots W_{R_p}(C_p) >= \\
&\qquad \frac{1}{Z(M)}\int \mathcal{DA}\, W_{R_1}(C_1)\cdots W_{R_p}(C_p)\exp\left[-\frac{ik}{4\pi}\int_{\mathcal{M}} d^3x\, \epsilon^{ijk}A_i\partial_j A_k\right]. \qquad (4.31)
\end{aligned}
$$

As this action is quadratic with the exponential dependence on A of a Wilson loop it is possible to write the integrand as a Gaussian by completing the square, following the computation of the Green's function is elementary on S^3 then is possible to see that the expectation value is equivalent as

$$
\langle W_{n_1}(C_1)\cdots W_{n_p}(C_p)\rangle = \exp\left[\frac{i}{4k}\epsilon_{ijk}\sum_{l,m=1}^{p} n_l n_m \int_{C_l} dx^i \int_{C_m} dy^j \frac{(x-y)^k}{|x-y|^3}\right] \qquad (4.32)
$$

$$
= \exp\left[-\frac{i}{4k}\sum_{l,m=1}^{p}\int_{C_l}\int_{C_m}\frac{(\mathbf{x}-\mathbf{y})\times d\mathbf{y}\cdot d\mathbf{x}}{|\mathbf{x}-\mathbf{y}|^3}.\right], \qquad (4.33)
$$

where x^i and y^j are the euclidean coordinates for a patch say U on the three sphere evaluated along the respective curves. Hence we can see that this is the same Gauss linking number found in previous sections for C_p curves, this is another context where the this quantity is present. It is important to note that the work done by Witten it is relevant for three dimensional theories and not dependent on the two spatial dimensions for its validation. Also here we can see the knots as a closed paths of charged particles that form the curve C in the manifold.

In conclusion, one may naturally question why Witten's paper [33] does not discuss the Hopf invariant in three dimensions or the potential interpretations related to fluid dynamics that are now apparent to us. In correspondence with Witten, he indicated that he was not aware of the advancements in topological fluid dynamics at that time. Interestingly, Polyakov's paper does mention the Hopf invariant as a Chern-Simons term. This raises the question of how these two fields could mutually benefit from each other's contributions.

5 Topological evolution of Time

In this section, we will first introduce the concept of Unimodular Gravity in the context of General Relativity, followed by a discussion of its variation as proposed by Henneaux and Teitelboim. This will lead to the introduction of a physical or cosmological time that is dependent on the coordinate time. Building upon the knowledge and important connections between topological hydrodynamics and topological field theory established in previous sections, we will then present an original framework that incorporates the concept of topological time. Subsequently, we will offer some remarks on this novel theory, contrasting it with existing topological theories and highlighting the key differences.

5.1 Unimodular Gravity

Unimodular gravity is a modification of General Relativity that has gained attention for its unique approach to the cosmological constant problem, among other issues in gravitational physics. In General Relativity, the metric tensor $g_{\mu\nu}$ is a dynamical variable, and its determinant $g = \det(g_{\mu\nu})$ is not fixed. In contrast, unimodular gravity imposes a constraint on the determinant of the metric, fixing it to a constant value, usually $\sqrt{-g} = 1$ [35] [36]. This seemingly minor modification leads to a profound alteration in the structure of the gravitational field equations and has far-reaching implications for cosmology, particularly in the context of the cosmological constant problem.

The cosmological constant problem is one of the most puzzling conundrums in theoretical physics. It arises from the discrepancy between the observed value of the cosmological constant Λ and its theoretically predicted value, which can differ by many orders of magnitude. Unimodular gravity provides a novel perspective on this problem by treating the cosmological constant not as a fundamental constant but as a dynamical variable that emerges from the theory itself. Here we will explain how the unimodular gravity as a concept arises, in this calculation we will not consider the matter terms as are not the main focus in the novel part presented later one.

In standard general relativity, the Einstein Hilbert action [37] is defined as:

$$S_{\text{EH}} = \frac{1}{2k} \int_{\mathcal{M}} d^4x \sqrt{-g}(R - 2\Lambda), \tag{5.1}$$

where $k = 8\pi G$ is a constant, then varying respect to the metric we get the Einstein equations:

$$\frac{\delta S_{\text{EH}}}{\delta g_{\mu\nu}} = G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{5.2}$$

It is also possible see that the Ricci scalar is related to the cosmological constant as such:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{5.3}$$

$$-R + 4\Lambda = 0 \tag{5.4}$$

$$R = 4\Lambda. \tag{5.5}$$

Therefore, the cosmological constant Λ appears as a constant term when solving Einstein's equations. It is usually treated as a fixed parameter of the theory, and its value must be fine-tuned to match observations. This fine-tuning is considered unnatural, leading researchers to explore alternative approaches. We will continue by exploring the unimodular condition.

The symmetry group for general relativity is defined as the diffeomorphism group *Diff*($\mathcal{M}$) for a manifold $\mathcal{M}$ that is second-countable and Hausdorff. A diffeomorphism is a map $x : \mathcal{M} \to \mathcal{M}$ that maps $\mathcal{M}$ onto itself. For example, an action of this map is given by

$$g_{\mu\nu} \to \bar{g}_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial \bar{x}^{\bar{\mu}}} \frac{\partial x^\nu}{\partial \bar{x}^{\bar{\nu}}} g_{\mu\nu}. \tag{5.6}$$

Now, consider the top form that is invariant under coordinate transformations and volume-preserving diffeomorphisms. We define the top form as:

$$\Omega_g \equiv \sqrt{-g}d^4x. \tag{5.7}$$

This form is coordinate-invariant, as each part transforms as follows:

$$\sqrt{-g} \mapsto \sqrt{-\bar{g}} = \left|\frac{\partial x^\mu}{\partial \bar{x}^{\bar{\mu}}}\right| \sqrt{-g}, \text{ with } w = -1, \tag{5.8}$$

$$d^4x \mapsto d^4\bar{x} = \left|\frac{\partial \bar{x}^{\bar{\mu}}}{\partial x^\mu}\right| d^4x, \text{ with } w = 1. \tag{5.9}$$

Here, we define the Jacobian as $J = \left|\frac{\partial \bar{x}^{\bar{\mu}}}{\partial x^\mu}\right|^w$ and w is the weight. It is easy to see that the top form is indeed invariant, $\Omega_g \mapsto \bar{\Omega}_g = \Omega_g$, where the Jacobian terms cancel perfectly in the top form due to the transformation properties. The unimodular condition imposes that $J = 1$, which means that the measure is invariant under transformations:

$$d^4x \mapsto d^4\bar{x} = d^4x. \tag{5.10}$$

This implies that we can consider this as a top form, as it is invariant under coordinate transformations and therefore suitable for integration. Note that one could in the original formulation take an arbitrary Jacobian in the top form, and it would still be invariant. The unimodular restriction imposes that $\sqrt{-g} = 1$ in the top form, making the top form perfectly volume-preserving under diffeomorphisms. This was originally expressed by Einstein in [36] as a partial gauge-fixing condition used to simplify some calculations in general relativity. We can then try to impose the unimodular condition in the Einstein-Hilbert action and propose to write this action as:

$$S_{\mathrm{U?}} = \frac{1}{2k}\int_{\mathcal{M}} d^4x(R - 2\Lambda), \tag{5.11}$$

which yields the equation of motion for the metric as:

$$\frac{\delta S_{\mathrm{U?}}}{\delta g_{\mu\nu}} = R_{\mu\nu} = 0, \tag{5.12}$$

$$= R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = 0, \tag{5.13}$$

where in the last equation we added the Ricci scalar. This is possible because $R = 0$ from the equation of motion, implying $g^{\mu\nu}R_{\mu\nu} = 0$, and therefore we can include it in the last equation. This equation is called the trace-free equation as it is traceless:

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{4}Rg^{\mu\nu}g_{\mu\nu} = 0, \tag{5.14}$$

$$R - \frac{1}{4} \times 4R = 0. \tag{5.15}$$

Here, we observe that Equation (5.13) does not contain any information regarding the cosmological constant, whereas the original Einstein-Hilbert action (5.2) does. This is advantageous as we are seeking a theory without a fixed cosmological constant.

However, taking the covariant derivative of the trace-free equation, we find:

$$\partial_\mu R = 0, \tag{5.16}$$

This equation can easily be solved by integration, yielding $R = \text{constant}$, where we can set the constant to be 4Λ to match the Einstein equation case.

However, by introducing a cosmological constant, we end up in a recursive argument and essentially do not obtain a dynamical constant in the end. This adds an inconsistency to these equations. Setting $R = 4\Lambda$ does not agree with the equation of motion for this action, creating a clear inconsistency between Equation (5.13) and the original equation of motion in (5.12).

Nevertheless, there is a clever way to formulate an action suitable for the unimodular condition:

$$S_{\text{UG}} = \frac{1}{2k}\int_{\mathcal{M}} d^4x \left(\sqrt{-g}R - 2\Lambda\left(\sqrt{-g} - 1\right)\right), \tag{5.17}$$

$$= \frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R - 2\Lambda\right) + \frac{1}{k}\int_{\mathcal{M}} d^4x\Lambda, \tag{5.18}$$

Here, the cosmological constant becomes a scalar field, serving as a Lagrange multiplier. The equation of motion for the cosmological constant imposes the unimodular condition on-shell:

$$\frac{\delta S_{\text{UG}}}{\delta\Lambda} = 0 \Rightarrow \sqrt{-g} = 1, \tag{5.19}$$

so it is not imposed at the level of the action. Moreover, this action yields the same Einstein equations as the standard Einstein-Hilbert action, but with the crucial difference that the cosmological constant is not a fixed constant but a scalar field in the action. Taking the covariant derivative of this equation of motion, we find:

$$\partial_\mu \Lambda = 0, \tag{5.20}$$

indicating that Λ is a constant on-shell, which can locally be seen as a mere gauge choice [38].

In this theory, following [38] and [39], the action is invariant under a specific subset of diffeomorphisms, known as *transverse diffeomorphisms*.

These diffeomorphisms are generated by a vector field ξ^μ that satisfies the condition

$$\nabla_\mu \xi^\mu = 0. \tag{5.21}$$

This equation stipulates that the divergence of ξ^μ must vanish for the diffeomorphism to be considered transverse. Transverse diffeomorphisms preserve the metric density $\sqrt{-g}$, as demonstrated by

$$\delta_\xi\sqrt{-g} = \mathcal{L}_\xi\sqrt{-g} = \frac{1}{2}\sqrt{-g}\nabla_\mu\xi^\mu = 0. \tag{5.22}$$

Here, $\delta_\xi\sqrt{-g}$ and $\mathcal{L}_\xi\sqrt{-g}$ represent the change in the metric density $\sqrt{-g}$ under the diffeomorphism generated by ξ^μ. This equation confirms that the metric density remains invariant under such diffeomorphisms. This restricted invariance distinguishes the symmetry group of this theory significantly from that of General Relativity.

The promotion of the cosmological constant to a dynamical variable has several implications. First, it alleviates the fine-tuning problem associated with Λ, as it is no longer a fixed parameter that must be adjusted to match observations. Instead, its value is determined dynamically, providing a natural explanation for its observed smallness.

Second, the dynamical nature of Λ opens up new avenues for cosmological models. For instance, in the early universe, the value of Λ could differ, leading to distinct expansion dynamics. This could have implications for inflationary models and the formation of cosmic structures.

Unimodular gravity presents a compelling alternative to General Relativity, offering novel solutions to longstanding problems in gravitational physics and cosmology. By constraining the determinant of the metric tensor and promoting the cosmological constant to a dynamical variable, unimodular gravity provides a fresh perspective on the nature of spacetime and the evolution of the universe.

Next, we will consider an alternative formulation of unimodular gravity that will extend the invariance to match exactly those of General Relativity, while retaining the core properties of unimodular gravity.

5.1.1 Henneaux-Teitelboim formulation

The Henneaux-Teitelboim formulation provides an alternative perspective on unimodular gravity by the inclusion of an auxiliary vector density $\mathcal{V}^\mu$. Initially conceived by Henneaux and Teitelboim [40], the action in this framework is articulated as follows:

$$S_{\mathrm{HT}} = \frac{1}{2k}\int_{\mathcal{M}} d^4x \left(\sqrt{-g}R - 2\Lambda\left(\sqrt{-g} - \partial_\mu\mathcal{V}^\mu\right)\right) \tag{5.23}$$

where Λ as before acts as a Lagrange multiplier that eventually crystallizes into a constant, thereby manifesting as an effective cosmological constant in the Einstein equations. The resultant field equations are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{5.24}$$

where Λ is a constant cosmological term, here also as before on shell we have $\nabla_\mu\Lambda = 0$.

This formulation introduces an additional gauge symmetry, defined by transformations that preserve $g_{\mu\nu}$ and Λ while altering $\mathcal{V}^\mu$ as:

$$\mathcal{V}^\mu \to \mathcal{V}^\mu + \epsilon^\mu, \quad \text{where} \quad \nabla_\mu\epsilon^\mu = 0. \tag{5.25}$$

Of the $\mathcal{V}^\mu$ components, only $\mathcal{V}^0$ possesses physical relevance [40], serving as the zero mode canonically conjugate to the cosmological constant.

Furthermore we can see that we can rewrite the action as the Einstein Hilbert action plus a term as:

$$S_{\text{HT}} = S_{\text{EH}}[g_{\mu\nu}, \Lambda] + \int_{\mathcal{M}} d^4x \ \Lambda \partial_\mu \mathcal{V}^\mu \tag{5.26}$$

$$= S_{\text{EH}}[g_{\mu\nu}, \Lambda] - \int_{\mathcal{M}} d^4x \ \mathcal{V}^\mu \partial_\mu \Lambda. \tag{5.27}$$

In the last equation we have integrated by parts, it is clear that now V^μ acts now as a Lagrange multiplier this ensures that the cosmological constant is zero in every direction hence constant on shell. Here we can note as well that $\mathcal{V}^\mu$ as weight of 1, this is cancelling the weight of d^4x that is -1 as showed in equation (5.9). We will define the added term as $S_{\text{HT-EH}}$ for simplicity to use in the next sections.

Considering the equation of motion for Λ we have:

$$\sqrt{-g} = \partial_\mu \mathcal{V}^\mu, \tag{5.28}$$

then by integrate this equation over full d^4x we can see this leads to the concept of unimodular time [38]:

$$\begin{aligned} \text{4-Volume} \equiv \int_{\mathcal{M}} d^4x \ \sqrt{-g} &= \int_{\mathcal{M}} d^4x \partial_\mu \mathcal{V}^\mu \\ &= \int_{\mathcal{M}} d^4x \nabla_\mu \mathcal{V}^\mu \\ &= \int_{\partial\mathcal{M}} d\Sigma_\mu \mathcal{V}^\mu \qquad (5.29) \\ &= \int_{\Sigma(t_f)} d\Sigma_\mu \mathcal{V}^\mu - \int_{\Sigma(t_i)} d\Sigma_\mu \mathcal{V}^\mu \\ &= \int_{\Sigma(t_f)} d\Sigma_0 \mathcal{V}^0 - \int_{\Sigma(t_i)} d\Sigma_0 \mathcal{V}^0 \qquad (5.30) \\ &= T_\Lambda(t_f) - T_\Lambda(t_i) \equiv \Delta T, \qquad (5.31) \end{aligned}$$

here we have considered that for a vector density we have $\nabla_\mu \mathcal{V}^\mu = \partial_\mu \mathcal{V}^\mu + \Gamma^\mu_{\mu\alpha} - w\Gamma^\beta_{\beta\mu}\mathcal{V}^\mu$ and in our case for $w = 1$ this leads to $\nabla_\mu \mathcal{V}^\mu = \partial_\mu \mathcal{V}^\mu$. In equation (5.29) we used stokes theorem and as there is only time like fluxes the space like ones are taken to be zero in equation (5.30). Furthermore we take the boundary of the manifold as $\partial\mathcal{M} = \Sigma(t_f) \cup \Sigma(t_i)$ where $\Sigma(t)$ is an hypersurface at an instance t. We have defined throughout the last equation the time function $T_\Lambda(t)$ as:

$$T_\Lambda(t) := \int_{\Sigma(t)} d\Sigma_0 \mathcal{V}^0 = \int_{\Sigma(t)} d^3x n_0 \mathcal{V}^0. \tag{5.32}$$

Here we are following the classical Arnowitt-Deser-Misner (ADM) formalism [41] in which one foliate the Lorentzian manifold in spacelike hypersurfaces, see for example figure 7. In this formalism hence

h_{ij} is the metric induced on the hypersurface and n^{μ} is the normalised vector perpendicular to the hypersurface. Here also we have that $\sqrt{-g} = N\sqrt{-h}$ where N is the lapse function and h is the determinant of the induced metric on the hypersurface.

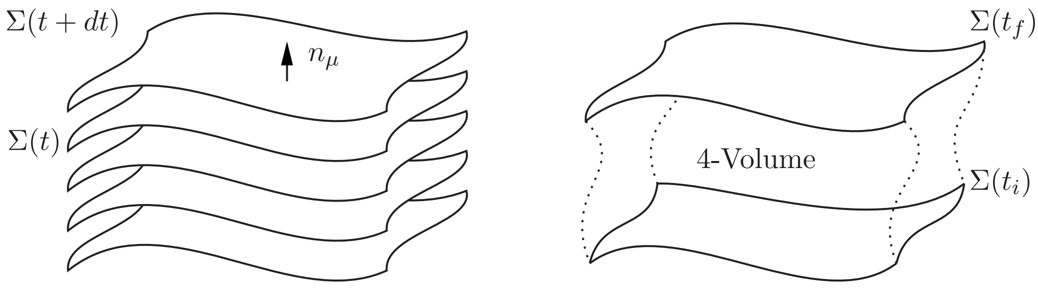

Figure 7: Here is presented the foliation of the $3+1$ dimension manifold $\mathcal{M}$, each three dimensional hypersurface is represented here as a surface that is spacelike with n_μ being orthogonal and timelike. Next to this is depicted the 4-Volume swiped between the two hypersurfaces with $t_i < t < t_f$.

It is important to note that this time is defined in respect of the constant Λ and could be applied the same mechanism for other constant that would get different times, we will keep the subscript with the cosmological constant to keep remind ourself.

$T_\Lambda(t)$ thus acts as a time function conjugate to the cosmological constant and modulates the spacetime volume, serving as a fixed volume form in this formalism. This is very important as it gives a physical interpretation of time based on coordinate time, we will come back in the next section. For completeness it is important to note that we could use another derivation taking directly the 3 space integral and then integrating again in time for the interval that we are interested as:

$$T_\Lambda\left(t_f\right) - T_\Lambda\left(t_i\right) = \int_{t_i}^{t_f} dx^0 \int d^3x \partial_\mu \mathcal{V}^\mu = \int_{t_i}^{t_f} dx^0 \int d^3x \sqrt{-g}. \tag{5.33}$$

The framework can also be equivalently articulated using a fully antisymmetric dual 3-dimensional form, which serve as duals to the tensorial entities in this formulation. This is exactly what we will implement in the next section.

5.2 Helicity of Time

In this section, we discuss a novel interpretation of time within the framework of unimodular gravity [42], as proposed by Henneaux and Teitelboim. This interpretation arises from considering $\partial_\mu \mathcal{V}^\mu$ as the Chern-Pontryagin density, which we introduced in Section 4.2. Before proceeding with this formulation, it is essential to establish the conventions and provide a formal description of the electromagnetic tensor.

We work in a local coordinate system where the Minkowski metric tensor is given by $\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)$. The Levi-Civita symbol is defined such that $\epsilon^{0123} = -\epsilon^{123} = -1$. We employ the standard field strength tensor, denoted as $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, which is equivalent to the components of $F = dA$, where A is a one-form. Therefore we define the electric vector as follows:

$$F_{0i} = -E_i = 2\partial_{[0}A_{i]}. \tag{5.34}$$

Subsequently, we introduce the magnetic pseudo-vector as:

$$B^i = \epsilon^{ijk}\partial_j A_k, \tag{5.35}$$

$$= -\epsilon^{0ijk}\partial_j A_k, \tag{5.36}$$

$$-B^i = \epsilon^{0ijk}\partial_j A_k = \frac{1}{2}\epsilon^{0ijk}F_{jk}. \tag{5.37}$$

Thus, the electromagnetic field tensor takes the form:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{5.38}$$

where we set $c = 1$ for subsequent computations. It is important to note that this convention aligns closely with the one used in Section 3.2.1 for the non-conserved helicity section.

We can reformulate the Chern-Pontryagin density as:

$$\mathcal{P} = \tilde{F}^{\mu\nu}F_{\mu\nu}, \tag{5.39}$$

where $\tilde{F}^{\mu\nu}$ is the dual tensor of $F_{\mu\nu}$, defined as:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{5.40}$$

Here, we introduce the Levi-Civita tensor as $\hat{\epsilon}_{\mu\nu\alpha\beta} = \sqrt{-g}\epsilon_{\mu\nu\alpha\beta}$ and $\hat{\epsilon}^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}$, where $\epsilon^{\mu\nu\alpha\beta}$ is the standard Levi-Civita symbol. Incorporating the Chern-Pontryagin term into the Unimodular

Henneaux-Teitelboim action yields:

$$S_{\text{HT-EH}} = \int d^4x \frac{\Lambda}{2} \tilde{F}^{\mu\nu} F_{\mu\nu} \tag{5.41}$$

$$= \int d^4x \frac{\Lambda}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \partial_\mu A_\nu$$

$$= \int d^4x \frac{\Lambda}{2} \epsilon^{\alpha\beta\mu\nu} \left[\partial_\mu \left(F_{\alpha\beta} A_\nu\right) - \left(\partial_\mu F_{\alpha\beta}\right) A_\nu\right] \tag{5.42}$$

$$= \int d^4x \Lambda \partial_\mu \left(\frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} A_\nu\right)$$

$$= \int d^4x \Lambda \partial_\mu \left(\tilde{F}^{\mu\nu} A_\nu\right) \tag{5.43}$$

In Eq. (5.42), the last term vanishes due to the commutativity of the partial derivatives when multiplied by the fully antisymmetric Levi-Civita symbol. Consequently, the vector density can be expressed as the dual of the field strength tensor contracted with the vector potential, $\mathcal{V}^\mu = \tilde{F}^{\mu\nu} A_\nu$, where both quantities have weight -1, which cancels under integration over d^4x, as previously discussed. The divergence of this term corresponds to the theta term described in Section 4.2, given by:

$$\partial_\mu \mathcal{V}^\mu d^4x = \partial_\mu (\tilde{F}^{\mu\nu} A_\nu) d^4x$$

$$= -F \wedge F = -dA \wedge dA = -d(A \wedge dA), \tag{5.44}$$

where we utilized the nilpotency of the exterior derivative ($d^2 = 0$). This can be explicitly demonstrated as follows:

$$S_{\text{HT-EH}} = -\int \Lambda d(A \wedge dA)$$

$$= -\int \Lambda \partial_\gamma (A_\mu \partial_\alpha A_\beta) dx^\gamma \wedge dx^\mu \wedge dx^\alpha \wedge dx^\beta$$

$$= -\int \Lambda \partial_\gamma (A_\mu \partial_\alpha A_\beta)(-\epsilon^{\gamma\mu\alpha\beta}) d^4x \tag{5.45}$$

$$= \int d^4x \Lambda \partial_\gamma \left(\epsilon^{\gamma\mu\alpha\beta} A_\mu \frac{1}{2} 2 \partial_\alpha A_\beta\right) \tag{5.46}$$

$$= \int d^4x \Lambda \partial_\mu \left(\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} A_\nu\right) \tag{5.47}$$

$$= \int d^4x \Lambda \partial_\mu \left(\tilde{F}^{\mu\nu} A_\nu\right), \tag{5.48}$$

As expected, this theory is fully compatible with the unimodular gravity formulation by Henneaux-Teitelboim.

We can now begin to investigate the properties of this formulation by examining the unimodular time, denoted as $T_\Lambda(t)$, which we described in Eq. (5.32). In this context, it takes the form:

$$\begin{aligned} T_\Lambda(t) &= \int_{\Sigma(t)} d\Sigma_0 \; \mathcal{V}^0, & (5.49)\\ &= \int_{\Sigma(t)} d\Sigma_0 \; \frac{1}{2}\epsilon^{\alpha\beta 0\mu} F_{\alpha\beta} A_\mu, & (5.50)\\ &= \int_{\Sigma(t)} d\Sigma_0 \; \frac{1}{2} A_i \epsilon^{0ijk} F_{jk}, & (5.51)\\ &= -\int_{\Sigma(t)} d\Sigma_0 \; A_i \epsilon^{ijk} \partial_j A_k, & (5.52)\\ &= -\int_{\Sigma(t)} d\Sigma_0 \; \mathbf{A}\cdot(\nabla\times\mathbf{A})\,. & (5.53) \end{aligned}$$

It becomes evident that the unimodular time is the negative of the scalar magnetic helicity, which aligns precisely with Woltjer's findings [17]. Importantly, this helicity resides in a Euclidean three-dimensional hypersurface. This is a significant development, as it links the unimodular time to the topological evolution of the scalar helicity. Specifically, at each coordinate time step, this quantity measures the degree of knottedness in the hypersurface and is an integer.

We can now proceed to the next logical step, which is to explore the implications of equation (5.31). This equation relates the 4-volume spanned between two hypersurfaces at two distinct times to the difference in unimodular time over the same interval of coordinate time. As previously demonstrated, this unimodular time also has another interpretation, namely, as the scalar helicity. Therefore, we can combine these equations to analyze their interrelationship further:

$$\int_{t_i}^{t_f}\int d^3x\sqrt{-g} = -\int_{\Sigma(t_f)} d\Sigma_0 \mathbf{A}\cdot(\nabla\times\mathbf{A}) + \int_{\Sigma(t_i)} d\Sigma_0 \mathbf{A}\cdot(\nabla\times\mathbf{A}) = \int_{\Sigma(t_f)} d\Sigma_0 \; \mathcal{V}^0 - \int_{\Sigma(t_i)} d\Sigma_0 \; \mathcal{V}^0$$

$$\text{4-Volume}(t_i \to t_f) = -\Delta\mathcal{H} = \Delta T. \qquad (5.54)$$

The 4-volume swept between the two hypersurfaces, as depicted in Figure 8, is equal to the negative difference in scalar magnetic helicity. This, in turn, corresponds to the difference in unimodular time between the coordinate times t_i and t_f.

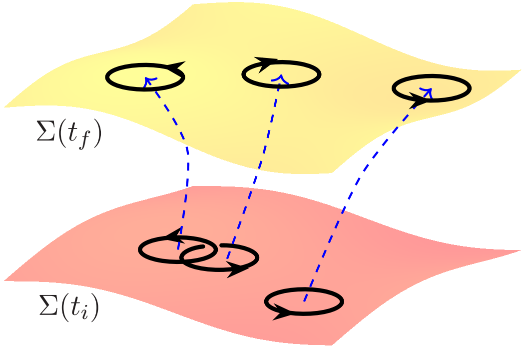

Figure 8: Spacetime volume depiction between two hypersurfaces at final and initial time where there is an evident difference in the configuration of knots made of **B** field. This is a pictorial depiction of topological evolution of the scalar helicity.

This finding is remarkable. If there is no evolution of magnetic helicity, then no 4-volume is generated between t_i and t_f. In other words, if the topological degree of knottedness of the magnetic field **B** remains constant between the two hypersurfaces, then there is no creation of 4-volume or time. This could be interpreted as a manifestation of physical time from the perspective of the gauge field A. This insight raises several questions; we will attempt to address the most pressing ones and offer some reflections.

An important question arises: Is the three-dimensional scalar helicity formally invariant in this context? Given equation (5.54), it should not be invariant at different times; otherwise, there would be neither unimodular time nor 4-volume. To investigate this, we consider the time derivative of unimodular time and examine its evolution using equation (5.28). We start by focusing on the time component of $\partial_\mu \mathcal{V}^\mu$, which is given by:

$$\partial_0 \mathcal{V}^0 = -\partial_0 \left(\mathbf{A} \cdot (\nabla \times \mathbf{A}) \right), \tag{5.55}$$

$$= \mathbf{E} \cdot \mathbf{B} + \mathbf{A} \cdot (\nabla \times \mathbf{E}), \tag{5.56}$$

$$= 2\mathbf{E} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{E}), \tag{5.57}$$

where we used the identity (A.5). This equation is structurally similar to the discussion on helicity invariance in equation (3.52) from the section on viscid fluids. Substituting equation (5.55) into the time derivative of unimodular time from equation (5.32), we obtain:

$$\dot{T}_\Lambda = \int_{\Sigma(t)} d\Sigma_0 \; 2\mathbf{E} \cdot \mathbf{B} - \int_{\partial\Sigma(t)} d\mathbf{S} \; \mathbf{A} \times \mathbf{E}, \tag{5.58}$$

$$= \int_{\Sigma(t)} d\Sigma_0 \; 2\mathbf{E} \cdot \mathbf{B}, \tag{5.59}$$

where we assume that the boundary term vanishes because $\mathbf{A}$ approaches zero at infinity. This is a standard derivation in the literature for the evolution of scalar helicity in electromagnetism [3]. However, we must also consider the other side of the equation of motion, as given by equation (5.33), which satisfies:

$$\dot{T}_\Lambda = \partial_0 \int dx^0 \int_{\Sigma(t)} d^3x\sqrt{-g} = \int_{\Sigma(t)} d^3xN\sqrt{-h}. \tag{5.60}$$

Combining these two equations, we obtain:

$$\int_{\Sigma(t)} d^3xN\sqrt{-h} = \int_{\Sigma(t)} d\Sigma_0\ 2\mathbf{E}\cdot\mathbf{B}. \tag{5.61}$$

From this, it is evident that helicity is not conserved, as $2\mathbf{E}\cdot\mathbf{B} \neq 0$. Hence, we observe a topological evolution in different instances of the scalar helicity.

Next, we turn our attention to the equation of motion for the cosmological constant, which further elucidates the relationship between the metric and the scalar helicity. We find:

$$\sqrt{-g}\ d^4x = -F\wedge F \tag{5.62}$$

$$= -\frac{1}{2}F_{\mu\nu}\frac{1}{2}F_{\alpha\beta}dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \tag{5.63}$$

$$= -\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}(-\epsilon^{\mu\nu\alpha\beta})d^4x \tag{5.64}$$

$$= F_{0i}F_{\alpha\beta}\epsilon^{0i\alpha\beta}d^4x \tag{5.65}$$

$$= F_{0i}2\partial_jA_k\epsilon^{0ijk}d^4x \tag{5.66}$$

$$= -2E_i(-B^i)d^4x \tag{5.67}$$

$$= 2\mathbf{E}\cdot\mathbf{B}\ d^4x, \tag{5.68}$$

This situation bears resemblance to the generalization of helicity in viscid fluids discussed in Section 3.2.2. The key difference here is that we are not dealing with an extra equation for a fluid but rather the standard electromagnetic field tensor. Nevertheless, this framework can be adapted for more general cases, such as fluid dynamics.

We can associate the action with the helicity 3-form $dh = d(A\wedge dA)$ up to a sign, which corresponds to the theta term containing the Abelian Chern-Simons current in four dimensions. Thus, we have:

$$\partial_\mu(\star h)^\mu = \partial_\mu\mathcal{V}^\mu = \partial_\mu(\tilde{F}^{\mu\nu}A_\nu). \tag{5.69}$$

As previously discussed, $\star h$ will yield the four-vector in Section 4.3, where the 0th component gives the Abelian Chern-Simons term in three dimensions and the three-vector helicity. Therefore, the Hodge dual of h is:

$$(\star h)^0 = -\mathbf{A}\cdot\mathbf{B}, \tag{5.70}$$

$$(\star h)^i = \mathbf{E}\times\mathbf{A} + \mathbf{B}\phi. \tag{5.71}$$

An important observation is that the quantity $\mathbf{E}\cdot\mathbf{B}$ serves as the square root of the determinant of the field strength tensor. From equation (5.44), we deduce that, on-shell, the following condition holds:

$$\sqrt{-g} = \frac{1}{2}\tilde{F}^{\mu\nu}F_{\mu\nu} = 2\mathbf{E}\cdot\mathbf{B} = 2\sqrt{F} = \partial_\mu(\star h)^\mu, \tag{5.72}$$

where we define $F = \det(F_{\mu\nu})$. The interpretation of this result is still a subject of ongoing investigation. However, it is intriguing to note that reintroducing the speed of light constant c yields $\sqrt{-g} = \frac{2}{c^2}\sqrt{F}$, suggesting the possibility of a variable speed of light in this framework.
Although we have not delved into the quantization of this theory, the integer-valued scalar helicity, which describes unimodular time, could be indicative of a quantization of time or a ladder process involving changes in the linking number of the magnetic field.
We conclude this section by noting that this formulation arises quite naturally and could be considered more of a discovery than a mere formulation. In the subsequent section, we will provide a general overview of this discovery and discuss potential frameworks.

5.2.1 2+1 versus 3+1 dimensions and a theory of many helicities

In this concluding section, we aim to highlight some key distinctions between the standard Chern-Simons theory in 2+1 dimensions, as employed by Polyakov [34] and Witten [33], and the novel theory presented in the previous section in $3+1$ dimensions [42].

The scalar helicity that appears in unimodular time is actually the 3-dimensional Chern-Simons term discussed in Witten's paper [33]. However, there is a peculiar difference that is physical in nature. In standard Chern-Simons theory employed in Witten and Polyakov, which is described in 2+1 dimensions, because knots in 4 dimensions become trivial. To enforce a physical interpretation there, time is included as one of the three dimensions. This is depicted in Figure 9, where the knot is represented by a close motion of a charged particle in $2+1$ spacetime, thereby focusing on topological invariants of this knots.

In contrast, our case is fundamentally different. Although it is true that each 3-dimensional hypersurface in our framework has topological invariants in the form of closed magnetic field lines, these are merely trivial invariants because by the definition of the spacial hypersurface these are constant in time coordinate. For a more detailed illustration, please refer to the second part of Figure 9.

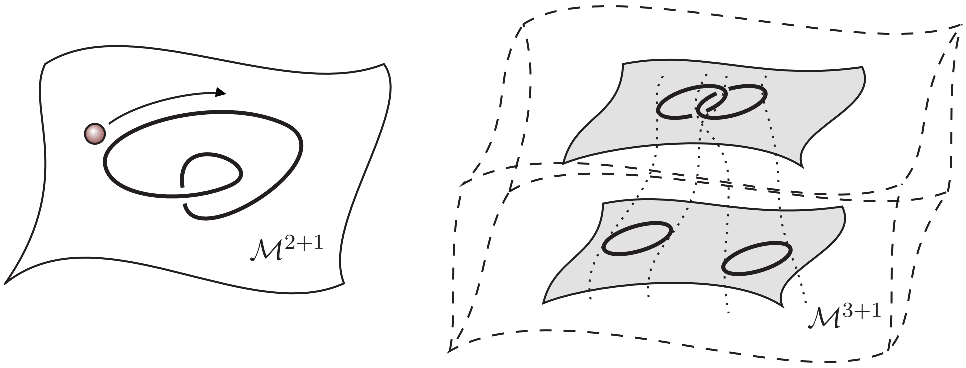

Figure 9: In this figure we present two contrasting scenarios. On the left, we depict the classical picture of $2+1$-dimensional Chern-Simons theory, featuring a closed path traversed by a charged particle in space. Adjacent to it, we illustrate the $3+1$-dimensional manifold, where the hypersurfaces are represented in shades of gray and host closed magnetic field lines. It is crucial to emphasize that these spaces extend infinitely in their respective dimensions.

Upon fixing the fourth dimension, as elaborated in Section 4.1, the magnetic field emerges as the curvature ($F = \mathbf{B}$) on the hypersurfaces, rendering the electric field perpendicular to these surfaces. While the Wilson loop could serve as an observable, it would be more appropriate to employ the 't Hooft loop operator [43], the magnetic analogue of the Wilson loop. This is akin to parallel transport in Equation (4.3), differing mainly in the application of Stokes' theorem over the magnetic flux. Investigating the expectation values of this operator could yield fruitful insights into knot polynomials.

Another noteworthy point is that this theory could serve as a mechanism to implement the 3D Chern-Simons term from a 4D action with a theta term via ADM foliation. One could generalize unimodular time by choosing a different constant, extending from a single variable to multi-variable constants $\boldsymbol{\alpha}$. This introduces a multitude of magnetic fields $\mathbf{B}^i_{\boldsymbol{\alpha}}$ within the new helicity framework. For each gauge field $A^{\boldsymbol{\alpha}}_\mu$, the action is given by

$$S_{\text{HT-EH}} = \int d^4x \; \boldsymbol{\alpha} \cdot \partial_\mu \mathcal{V}^\mu_{\boldsymbol{\alpha}}, \tag{5.73}$$

where $\mathcal{V}^\mu_{\boldsymbol{\alpha}} = \frac{1}{2}\tilde{F}^{\mu\nu}_{\boldsymbol{\alpha}} A^{\boldsymbol{\alpha}}_\nu$. This leads to $\boldsymbol{\alpha}$-dependent magnetic helicities, and similar results to Equation (5.54) are expected in terms of physical time differences.

A more elegant perspective involves considering a diagonal matrix K_{IJ}, with $I, J = 1, ..., d$, where d is the dimension of the vector space formed by each variable $\boldsymbol{\alpha}$. The matrix is defined as $K_{IJ} = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_d)$. A Chern-Pontryagin term with gauge group d-torus is then introduced, with action

$$S_{\text{HT-EH}} = -\int K_{IJ} F^I \wedge F^J. \tag{5.74}$$

The action has a symmetry group $U(1)^d$, the torus group, and each unimodular-like time is associated with its gauge field A^I and corresponding magnetic helicity.

In conclusion, this theory could be compellingly applied to turbulent hydrodynamics, offering an effective way to incorporate a fluid in $3 + 1$ dimensions while accounting for effective topological evolution.

6 Conclusion

In conclusion, we have successfully demonstrated a significant link between topological hydrodynamics and topological field theory. This connection manifests in the form of scalar helicity as defined by Moffatt [2], which aligns with that of Abelian Chern-Simons theory. Both of these theories are essential for exploring topological invariance. Here, we have successfully generalized helicity and identified its additional components, as anticipated by Carter [18]. This generalization provides us with the tools to examine the properties that govern the evolution of this generalized helicity, particularly in viscous fluids. We have also presented an original application of this concept in the context of Henneaux-Teitelboim's unimodular gravity [40], thereby emphasizing the benefits of bridging the gap between fluid dynamics and field theory.

Our innovative formulation offers an alternative perspective on invariants in three dimensions. While the original Chern-Simons theory is formulated in a 3D space with one temporal dimension, we have shown that an alternative setup is feasible. By starting with a 3+1-dimensional theory that includes a theta term, we were able to recover cosmological time as an Abelian Chern-Simons term. The advantage of using ADM foliation [41] is that it allows us to examine the evolution of helicity at any given time coordinate. This would crystalise in a topological physical time, where if there is no change in the scalar helicity between two hypersurfaces then there would not be physical time and the 4-Volume swept between the hypersurfaces.

One of the most significant contributions of this work is the introduction of a concept of 'physical time,' which is sensitive to the topological evolution of the magnetic field. This stands in contrast to the traditional notion of 'coordinate time', which serves as a mere parameter in the equations of motion. Our formulation of physical time offers a dynamic framework that is intrinsically tied to the underlying physics, specifically the topology of the magnetic field. This not only enriches our understanding of time itself but also has potential implications for various domains, from cosmological models to quantum field theories.

We have also proposed a theory involving multiple variable constants and multiple helicities, opening new avenues for future research. This includes the potential involvement of cross helicity, as discussed in Section 3.1.3.

Although our work lays a solid foundation for this novel formulation, it is limited to the Abelian gauge group $U(1)$. This has significant implications, as the curvature on the hypersurface corresponds to the magnetic field. A natural next step would be to explore how the theory behaves with non-Abelian gauge groups. We have also not addressed the quantization of this theory, which could have intriguing implications for black holes, as recent advancements suggest [44].

Our framework also implies a phase transition from laminar to turbulent flows, which could be relevant for cosmology and planetary formation, as discussed by Sconfield [45]. Future work could benefit from computational simulations, potentially using FireDrake [46] for fluid dynamics or GRChombo for general relativity simulations [47].

While our framework raises more questions than it answers, it provides a strong foundation for future research. One open question is the implication of the equation of motion $\sqrt{-g} = 2\sqrt{F}$, where F is the determinant of the curvature field strength tensor. Nevertheless, the connections and generalizations we have introduced offer promising avenues for various applications in both fluid dynamics and theoretical physics.

Appendix A Vector Calculus

Here we give some basic definitions of vector calculus employed in the thesis.

$$\nabla f = grad\, f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \tag{A.1}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{A.2}$$

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{k}} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix} \tag{A.3}$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{A.4}$$

A.1 Identities

Here we give all the identities used in the thesis.

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \ = \ (\nabla\times\mathbf{A}) \cdot \mathbf{B} \ - \ \mathbf{A} \cdot (\nabla\times\mathbf{B}) \tag{A.5}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B} \tag{A.6}$$

$$\nabla \cdot (\psi \mathbf{A}) = \psi\, \nabla\cdot\mathbf{A} + (\nabla\psi)\cdot\mathbf{A} \tag{A.7}$$

Appendix B Differential Geometry

In this appendix, we provide the essential concepts of differential geometry required to understand the key ideas presented in this thesis. For those unfamiliar with exterior calculus and differential geometry, a comprehensive reference is Nakahara [23].

B.1 Equivalence Classes

An *equivalence class* is a fundamental concept in set theory and abstract algebra that partitions a set into disjoint subsets. Given a set S and an equivalence relation $\sim$ on S, an equivalence class $[a]$ is defined as the set of all elements in S that are equivalent to a particular element $a \in S$.

Mathematically, the equivalence class $[a]$ containing a is defined as:

$$[a] = \{x \in S \mid x \sim a\}.$$

The set of all equivalence classes for S under $\sim$ is denoted by $S/\sim$, and it forms a partition of S. That is, $S/\sim$ satisfies the following properties:

1. $\bigcup_{[a]\in S/\sim}[a] = S$
2. $[a] \cap [b] = \emptyset$ for $[a] \neq [b]$
3. $[a] \neq \emptyset$ for all $[a] \in S/\sim$

This partitioning is crucial in various areas of mathematics, including the construction of quotient spaces in topology and the factorization of algebraic structures.

B.2 Differential Forms

A differential k-form ω on a smooth manifold M is a smooth section of the k-th exterior power of the cotangent bundle, denoted as $\omega \in \Omega^k(M)$. Mathematically, it can be expressed locally as:

$$\omega = f(x_1, \dots, x_n)\, dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where $f(x_1, \dots, x_n) = \omega_{x_1 \dots x_n}$ is a smooth function and dx^i are the basis 1-forms.

B.3 Wedge Product

The wedge product $\wedge$ is an associative and anticommutative operation on differential forms. Given two forms $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, their wedge product $\alpha \wedge \beta \in \Omega^{p+q}(M)$ is defined as:

$$\alpha \wedge \beta = \frac{(p+q)!}{p!\,q!}\,\mathrm{Alt}(\alpha \otimes \beta),$$

where Alt denotes the antisymmetrization of the tensor product $\alpha \otimes \beta$.

Properties:

- **Associativity**: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$
- **Anticommutativity**: $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$
- **Linearity**: $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

B.4 Exterior Derivative

The exterior derivative is a linear operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ that acts on differential k-forms to produce $(k+1)$-forms. For a k-form ω, the exterior derivative $d\omega$ is defined as:

$$d\omega = d(\omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = \partial_{i_j} \omega_{i_1 \ldots i_k} dx^{i_j} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where $\omega_{i_1 \ldots i_k}$ are the components of the form and dx^i are the basis 1-forms.

Properties:

- **Linearity**: $d(\alpha + \beta) = d\alpha + d\beta$
- **Leibniz Rule**: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$
- **Nilpotency**: $d^2 = 0$

B.5 Hodge Star Operator

The Hodge star operator, denoted by $\star$, is a map $\star : \Omega^k(M) \to \Omega^{n-k}(M)$, where n is the dimension of the manifold M. Given a k-form ω, the Hodge star of ω is defined as:

$$\star\omega = \sqrt{|\det(g)|}\, \omega_{i_1 \ldots i_k} \epsilon^{i_1 \ldots i_k}{}_{j_1 \ldots j_{n-k}} dx^{j_1} \wedge \ldots \wedge dx^{j_{n-k}},$$

where g is the metric tensor, ϵ is the Levi-Civita symbol, and $\omega_{i_1 \ldots i_k}$ are the components of ω.

Properties:

- **Involution**: $\star(\star\omega) = (-1)^{k(n-k)} \omega$
- **Orthogonality**: $\langle \omega, \star\phi \rangle = \langle \star\omega, \phi \rangle$

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