A review on D=11 supergravity and M2-brane

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ABSTRACT

In this dissertation, we review some aspects of D=11 supergravity and M2-brane. We start from the construction of D=11 supergravity theory using Noether's method. The superspace formulation of D=11 supergravity is briefly discussed. We then look at the M2-solutions of the bosonic sector D=11 supergravity, which shows an interpolation between Minkowski spacetime and an $AdS_4 \times S^7$ spacetime. The analytic continuation of the metric encounters a timelike singularity, suggesting a source term in the form of a bosonic membrane can be added to the bosonic sector of the supergravity action. This leads us to look at the actions for M2-brane. We construct the M2-brane actions in different target superspace backgrounds by generalizations of the Green-Schwarz action, and we discuss aspects of gauge choices and semiclassical quantization for the flat superspace background case. We then look at the relation between the supermembrane action and the superstring action by double dimensional reduction from D=11 to D=10 together with worldvolume reducing from d=3 to d=2.

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CHAPTER 1

Introduction

M-theory is a proposed quantum theory of interacting fields and extended branes in eleven spacetime dimensions. At low energies, it reduces to the D=11 supergravity action. M-theory possesses two types of stable branes; these are M2-branes (also referred to as supermembranes) and M5-branes.

Supergravity [1] is a theory of massless bosons and fermions of spins less than or equal to 2 that satisfies a local supersymmetry. D=11 is the maximum dimension to satisfy local supersymmetry without including particles of spin higher than 2 [2]. The D=10 type II supergravity theories have two sets of supersymmetry charges. There are two types of such theories, the D=10 type IIA and D=10 type IIB supergravity. D=11 supergravity is related to D=10 type IIA supergravity by dimension reduction. Supergravity theory originally arose with the hope of solving some of the difficulties in quantum gravity. One of these is the non-renormalizability of gravity in dimensions greater than or equal to 4. However, the local supersymmetry in supergravity does not avoid this non-renormalizability, and these theories themselves do not define an ultraviolet-complete quantum theory.

The ultraviolet completions of D=10 type II supergravities are known. D=10 type IIA and type IIB supergravities are the low-energy limits of corresponding type IIA and type IIB superstring theories. With considerable evidence of string theory being ultraviolet finite, superstring theories can be thought of as the ultraviolet completion of the corresponding D=10 supergravity theories. The σ-model action for string is classically invariant under a Weyl symmetry on the worldvolume. The requirement of cancellation of Weyl anomalies at the quantum level results in constraints on the background fields. These constraints can be viewed as the effective equations of motion for these fields, and an effective action can be constructed. The effective action of D=10 superstring theories corresponds to the D=10 supergravity theories [3]. In fact, type IIB supergravity was originally discovered [4] and constructed by the low energy limit of type IIB superstring theory [5]. The relation between superstring theory and D=10 supergravity leads to the conjecture of the existence of M-theory as the ultraviolet completion of D=11 supergravity. This is further motivated by the double dimension reduction relating membranes D=11 supergravity theory to strings D=10 type IIA supergravity [6].

Super-p-branes are p-branes with a global supersymmetry and a κ-symmetry, such that the worldvolume on-shell degrees of freedom for fermions and bosons are equal. They are identified with membrane solutions of the source free field equations of D=11 supergravity [7] and are considered analogous to superstrings in ultraviolet completion of D=10 supergravities. By considering particles sweeping out a worldline and strings with a worldsheet, the d-dimensional p-branes have a d-dimensional worldvolume where $d = p + 1$. Similar to strings and particles, branes are embedded in a target spacetime. There are restrictions on the possible super-p-branes given a spacetime dimension D and the number of supersymmetry charges $\mathcal{N}$ [8]. One requirement is that the dimension of the brane worldvolume d is smaller than or equal to the target spacetime dimension D. Another important requirement is that the number of on-shell degrees of freedom for bosons and fermions equal on the worldvolume, which is ensured by κ-symmetry which halves the number of degrees of freedom of the fermions. This leads to restrictions on the combinations of parameters $(d, D, \mathcal{N})$ for a super-p-brane to exist.

Superstring action in flat superspace is described by the Green-Schwarz action [9], which is a generalization of the superparticle action [10]. It is later shown in [11] and [8] that this action can be generalized to higher dimensional extended super-p-branes provided the condition on $(d, D, \mathcal{N})$ is satisfied. In addition to flat superspace actions, generalizations to curved space are also found [12]. The requirement of satisfying κ-symmetry for M2-brane actions in general curved target superspace results in a set of constraints on the superspace, which are shown to be equivalent to the constraints in the superspace formulation of D=11 supergravity. This suggests that M2-brane actions are consistent with D=11 supergravity. Given a target superspace satisfying D=11 supergravity constraints, one can construct M2-brane action in this background.

In Chapter 2, we will discuss the D=11 supergravity theory. The contents of this theory are three fields: a graviton field, a Majorana gravitino field, and a 3-form gauge field. D=11 supergravity is originally constructed using Noether's method, which is an iterative process for finding invariant actions. Its invariance under local supersymmetry and the closure of its super-Poincare algebra can be shown by applying Fierz identities. The M2-brane appearing as an exact solution of D=11 supergravity field equations is discussed in Chapter 3. A timelike singularity is found and leads to the proposition of a source term in the equations of motion. This corresponds to an additional term in the form of membrane action in the supergravity action. The actions of M2-branes are discussed in Chapter 4. We start from the action in flat superspace as a direct generalization of the Green-Schwarz superstring action in flat superspace. The choice of static gauge and light cone gauge is presented, as well as the semiclassical quantization of a toroidal membrane. The generalization of M2-brane action to curved superspace background is then discussed. Chapter 5 is about Kaluza-Klein dimension reduction from D=11 to D=10. The D=11 supergravity upon dimension reduction gives the D=10 type IIA supergravity, which is observed to have a string solution. By a similar argument as for D=11 supergravity and M2-brane, a source term in the form of superstring can be proposed. This leads us to suspect that the M2-brane action reduces to a superstring action in the D=10 type IIA background. This is confirmed by a double-dimension reduction, where the dimension to be reduced coincides with one of the spatial dimensions of the M2-brane.

CHAPTER 2

D=11 Supergravity

Supergravity is a gauge theory of supersymmetry and spacetime symmetries. While the vielbeins and the spin connections gauge the spacetime symmetry, the gravitino field gauges the supersymmetry. Supergravity theory is first proposed in the paper [1], where it is constructed using Noether's method. The superalgebras in different dimensions are classified by Nahm in the paper [2]. Under the assumption of considering no massless particles with spin larger than 2, D=11 is the highest number of dimensions allowed. This can be explained in D=4. The massless representation of a set of supersymmetry charges has a pair of lowering and raising operators. We define the vacuum by the state annihilated by the lowering operator, and the other operator creates a state that has raised helicity by $1/2$. Starting from helicity -2 to helicity 2, we can have a maximum of eight sets of these operators, which is $\mathcal{N} = 8$ supersymmetry. Since in D=4, each supersymmetry charge has 4 spinor components, there is a maximum of 32 spinor components in D=4. Since the number of supersymmetry charges is preserved in the torus reduction, the maximum number of supercharges in higher dimensions is also 32. The highest dimension with the number of spinor indices not exceeding 32 is D=11.

The D=11 supergravity action is proposed in [13]. Noether's method is used for constructing the action for D=11 supergravity. An action with a rigid symmetry has Noether's current, which is conserved on-shell since the transformation parameter ϵ is constant. For finding a corresponding local symmetry, ϵ is then promoted to be spacetime dependent, making the Noether's current term not vanishing under the variation. The action and the transformations are modified correspondingly to cancel out this non-vanishing term. This modification also introduces other non-vanishing terms, which require adding further terms in the action and the transformation. This process is repeated until the final action is invariant under the modified transformation rules. In this section, we would like to construct the D=11 supergravity action from a globally supersymmetric linearised theory. Following Noether's method, we describe how the D=11 supergravity action is constructed.

2.1 Global supersymmetry

To apply Noether's method to find D=11 supergravity action, we first would like to find an action that is invariant under a global supersymmetry. In this section, we start from the on-shell states and construct the linearized theory with a global supersymmetry. This follows closely the method given in chapter 7 of [14].

The irreducible representations of supersymmetry form multiplets containing both bosons and fermions.

The multiplets that contain a spin-2 particle also contain a spin-$3/2$ or a spin-$5/2$ particle. Under the assumption that we are not considering particles of spin higher than 2, we choose the multiplet with a spin-$3/2$ particle. The spin-2 graviton can be represented by a rank 2 symmetric tensor $h_{\mu\nu}$, and the spin-$3/2$ gravitino by a Majorana vector spinor $\psi_{\mu\alpha}$. The irreducible representation of the graviton is constructed from the irreducible representation of the little group of the Poincaré algebra. The massless little algebra corresponds to the $SO(D-2)$ group. The graviton is in the rank 2 representation of this group as a symmetric traceless matrix, thus having $D(D-3)/2 = 44$ degrees of freedom on-shell. Considering the Rarita-Schwinger equation, the gaugino has 128 degrees of freedom on-shell. Detailed arguments for this number can be found in chapter 5 of [15]. By the requirement of supersymmetry that the fermionic and bosonic degrees of freedom equal, we introduce another bosonic field with 84 degrees of freedom. This is satisfied by the three-form gauge field $A_{\mu\nu\rho}$ in rank 3 representation of $SO(D-2)$ group. Therefore D=11 supergravity contains graviton $h_{\mu\nu}$, gravitino ψ_μ, and a three-form field $A_{\mu\nu\rho}$.

Considering the free field equations of these fields, we propose the following action

$$S = \int d^{11}x \left[h^{\mu\nu} G_{\mu\nu} - \bar{\psi}_\mu \Gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - \frac{1}{48} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \right], \tag{2.1}$$

where $G_{\mu\nu}$ is the linearised Einstein tensor and $F_{\mu\nu\rho\sigma}$ is the field strength tensor

$$G_{\mu\nu} = \frac{1}{2} \left(h^{\rho}_{\mu,\rho\nu} + h^{\rho}_{\nu,\rho\mu} - h_{,\mu\nu} - \Box h_{\mu\nu} - \eta_{\mu\nu} h^{\rho\sigma}_{,\rho\sigma} + \eta_{\mu\nu} \Box h \right), \tag{2.2a}$$

$$F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]}. \tag{2.2b}$$

The equations of motion for $h_{\mu\nu}$, ψ_μ and $A_{\mu\nu\rho}$ are invariant under following gauge symmetries

$$\delta h_{\mu\nu} = 2\partial_{[\mu} \xi_{\nu]}, \tag{2.3a}$$

$$\delta \psi_{\mu\alpha} = \partial_\mu \eta_\alpha, \tag{2.3b}$$

$$\delta A_{\mu\nu\rho} = \partial_{[\mu} \theta_{\nu\rho]}. \tag{2.3c}$$

The global supersymmetry transformation of this action involves the variations of each of the component fields by a fermionic parameter ε. The transformation of the graviton $\delta h_{\mu\nu}$ and the three-form field $\delta A_{\mu\nu\rho}$ can be expressed in terms of ψ_μ, while the transformation of the gravitino $\delta\psi_\mu$ can be expressed in terms of $h_{\mu\nu}$ and $A_{\mu\nu\rho}$. We first propose the global supersymmetry transformation of the form

$$\delta h_{\mu\nu} = \frac{1}{2} (\bar{\varepsilon} \Gamma_\mu \psi_\nu + \bar{\varepsilon} \Gamma_\nu \psi_\mu), \tag{2.4a}$$

$$\delta \psi_\mu = a\, \Gamma^{ab} \partial_a h_{b\mu} \varepsilon + \left(b\, \Gamma_\mu{}^{\alpha\beta\gamma\delta} - c\, \delta^\alpha_\mu \Gamma^{\beta\gamma\delta} \right) F_{\alpha\beta\gamma\delta} \varepsilon, \tag{2.4b}$$

$$\delta A_{\mu\nu\rho} = d\, \bar{\varepsilon} \Gamma_{[\mu\nu} \psi_{\rho]}. \tag{2.4c}$$

The coefficients a, b, c and d are to be determined by requiring the action to be invariant under these transformations and that this fermionic transformation forms a closed algebra with the gauge transformations above. By looking at each term in variation, we expect the variation of $h^{\mu\nu} G_{\mu\nu}$ to cancel with the

$\delta\psi_\mu = a\Gamma^{ab}\partial_a h_{b\mu}\varepsilon + ...$ variation of the second term in the action. This gives the following requirement

$$\int d^{11}x \left[2\bar{\varepsilon}\Gamma^\mu\psi^\nu G_{\mu\nu} - 2a\,\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma^{ab}(\partial_\nu\partial_a h_{b\mu})\varepsilon\right] = 0. \tag{2.5}$$

By applying the properties of Gamma matrices as in equation (A.10), we found

$$\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma^{ab}(\partial_\nu\partial_a h_{b\mu}) = 2\bar{\psi}^\mu\Gamma^\nu G_{\mu\nu} \tag{2.6}$$

and thus require $a = 1/2$. The other coefficients are determined by the cancellation of the remaining terms. The $\delta\psi_\mu = ... + \left(b\,\Gamma_\mu{}^{\alpha\beta\gamma\delta} - c\,\delta^\alpha_\mu\Gamma^{\beta\gamma\delta}\right)F_{\alpha\beta\gamma\delta}\varepsilon$ variation of the second term in the action and the variation of the term $-\frac{1}{48}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}$ gives

$$2\int d^{11}x\bar{\varepsilon}\left[\left(b\,\Gamma_\mu{}^{\alpha\beta\gamma\delta} - c\,\Gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\Gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \frac{1}{12}d\,\Gamma_{\nu\rho}\psi_\sigma\partial_\mu F^{\mu\nu\rho\sigma}\right] = 0. \tag{2.7}$$

By looking at the product of gamma matrices

$$\begin{aligned}\Gamma^{\alpha\beta\gamma\delta}{}_\mu\Gamma^{\mu\nu\rho}F_{\alpha\beta\gamma\delta} =& (D-6)\Gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\delta} + 8(D-5)\Gamma^{\alpha\beta\gamma[\nu}\,F^{\rho]}{}_{\alpha\beta\gamma}\\ &- 12(D-4)\Gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho},\end{aligned} \tag{2.8a}$$

$$-\Gamma^{\beta\gamma\delta}\Gamma^{\mu\nu\rho}F_{\mu\beta\gamma\delta} = -\Gamma^{\nu\rho\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} - 6\Gamma^{\alpha\beta\gamma[\nu}\,F^{\rho]}{}_{\alpha\beta\gamma} + 6\Gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}, \tag{2.8b}$$

integrating by parts, and applying Bianchi identity $\partial_{[\mu}F_{\alpha\beta\gamma\delta]} = 0$, we obtain the relation between b, c and d:

$$48b - 6c = 0, \tag{2.9a}$$

$$84b - 6c = -\frac{1}{12}d. \tag{2.9b}$$

These are solved to give the relation $b = -d/432$ and $c = 8b$.

The coefficient c can then be determined by performing consecutive transformations on $A_{\mu\nu\rho}$

$$\begin{aligned}\left[\delta_1, \delta_2\right]A_{\mu\nu\rho} =& -\frac{1}{432}c^2\bar{\varepsilon}_2\Gamma_{[\mu\nu}\left(\Gamma^{\alpha\beta\gamma\delta}_{\rho]} - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_{\rho]}\right)\varepsilon_1 F_{\alpha\beta\gamma\delta}\\ &- \frac{1}{2}\bar{\varepsilon}_2\Gamma_{[\mu\nu}\Gamma^{\alpha\beta}h_{\rho]\beta,\alpha}\varepsilon_1 - (1\leftrightarrow 2)\end{aligned} \tag{2.10}$$

and requiring the gauge and the global supersymmetry transformations to form a closed algebra:

$$\left[\delta_{\epsilon_1}, \delta_{\epsilon_2}\right]A_{\mu\nu\rho} = -\frac{1}{2}\bar{\epsilon}_1\gamma^\sigma\epsilon_2 F_{\sigma\mu\nu\rho} + \text{gauge tranformations.} \tag{2.11}$$

Equation (2.10) has terms containing $F_{\sigma\mu\nu\rho}$ with gamma matrices of rank 7, 5, 3, and 1. By looking at the antisymmetrized ε_1 and ε_2, only the terms with gamma matrices of rank 1 and 5 are non-vanishing as the exchange of ε_1 and ε_2 gives a negative sign. We look for the rank 1 term in equation (2.10), which can be found by applying equation (A.10): $\left[\delta_1, \delta_2\right]A_{\mu\nu\rho} = -\frac{2}{9}d^2\bar{\epsilon}_1\gamma^\sigma\epsilon_2 F_{\sigma\mu\nu\rho} + ...$ Therefore, we fix the parameter $d^2 = 9/4$ and choose the negative root $d = -3/2$.

2.2 Local supersymmetry

To obtain the non-linear supergravity theory, we apply Noether's methods, as discussed at the beginning of this chapter, and observe that the linear theory in the last section possesses both local Abelian invariances and global supersymmetry. In this section, we follow the procedure given in chapter 13 of [16]. The first step is to make the parameter of transformation spacetime dependent $\varepsilon \to \varepsilon(x)$. The variation of the linear action with this spacetime dependent parameter is no longer zero. The non-vanishing term is found to be

$$\delta S = \int d^{11}x \; J^\nu \partial_\nu \varepsilon, \tag{2.12a}$$

$$J^\nu = \bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\partial_a h_{b\rho} + \frac{1}{48}\bar{\psi}_\mu \left(\Gamma^{\mu\nu\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} + 12\Gamma^{\alpha\beta}F_{\alpha\beta}{}^{\mu\nu}\right) \tag{2.12b}$$

This correspond to the Noether's current term for the global supersymmetry. This term can be cancelled if one adds $\frac{\kappa}{2}\int d^{11}x \; J^\nu \psi_\nu$ to the action and $\frac{1}{\kappa}\partial_\nu \varepsilon$ to $\delta\psi_\nu$. The full action becomes

$$\begin{aligned} S = \int d^{11}x \Big[& -h^{\mu\nu}G_{\mu\nu} - \bar{\psi}_\mu \Gamma^{\mu\nu\rho}\left(\partial_\nu - \frac{\kappa}{2}\Gamma^{ab}\partial_a h_{b\nu}\right)\psi_\rho - \frac{1}{48}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} \\ & -\frac{\kappa}{96}\bar{\psi}_\nu \left(\Gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\delta} + 12\Gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}\right)\psi_\rho \Big]. \end{aligned} \tag{2.13}$$

At this stage, the variations of zeroth order in κ cancel out.

This action also has variations of the first order in κ, which need to be canceled by further modifications to the action and the transformations. We will consider some of these first-order terms and describe how they lead to the addition of certain terms to the action and the transformation. Fine-tuning of the parameters for each term is needed for the first-order terms to cancel out. Here, we only consider how each possible new term can be found. The $\delta\psi_\mu = ... + \frac{1}{288}\left(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - 8\,\delta_\mu^\alpha \Gamma^{\beta\gamma\delta}\right)F_{\alpha\beta\gamma\delta}\varepsilon$ variation of the second part of the second term in the action cancels out with the $\delta\psi_\mu = ...\frac{1}{2}\,\Gamma^{ab}\partial_a h_{b\mu}\varepsilon$ variation of the last term. The $\delta\psi_\mu = ... + \frac{1}{288}\left(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - 8\,\delta_\mu^\alpha \Gamma^{\beta\gamma\delta}\right)F_{\alpha\beta\gamma\delta}\varepsilon$ variation of the last term in the action (2.13) contains rank 9, 7, 5, 3, and 1 contributions. While the rank 7, 5, and 3 terms canceled out within themselves, rank 9 and rank 1 terms remain. The rank 9 term has the form

$$\begin{aligned} \delta S = ... + \frac{\kappa}{16 \times 144}\int d^{11}x \;\; \bar{\psi}_\mu \Gamma^{\mu\alpha\beta\gamma\delta\rho\sigma\zeta\xi} \; \varepsilon \; F_{\alpha\beta\gamma\delta}F_{\rho\sigma\zeta\xi} & \\ = -\frac{\kappa}{3 \times 8 \times 144}\int d^{11}x \;\; (-\frac{3}{2}\bar{\psi}_\mu \Gamma_{\nu\theta}\varepsilon)\epsilon^{\alpha\beta\gamma\delta\rho\sigma\zeta\xi\mu\nu\theta}F_{\alpha\beta\gamma\delta}F_{\rho\sigma\zeta\xi}, & \end{aligned} \tag{2.14}$$

where we have used the property that a rank 9 gamma matrix is dual to a rank 2 gamma matrix in 11 dimensions. From the second line of this equation, we suspect a term

$$S = ... + \frac{2\kappa}{144^2}\int d^{11}x \;\; \epsilon^{\alpha\beta\gamma\delta\rho\sigma\zeta\xi\mu\nu\theta}F_{\alpha\beta\gamma\delta}F_{\rho\sigma\zeta\xi}A_{\mu\nu\theta} \tag{2.15}$$

in the action. The rank 1 in gamma term has the form

$$\delta S = ... + \frac{\kappa}{96}\int d^{11}x F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}\bar{\psi}_\mu \Gamma^\mu \varepsilon. \tag{2.16}$$

Since there are no other variations in the action that can give a term with two F's, we observe $-\bar{\psi}_\mu \Gamma^\mu \varepsilon$ as a δh transformation and add a new term into the action:

$$S = ... + \frac{1}{48}\int d^{11}x \; \frac{\kappa}{2} h \; F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}, \tag{2.17}$$

where $h = h^\mu_\mu$. The term containing two F's in the action then has a factor of $1 - \frac{\kappa}{2}h$, which can be recognised as $\det(e^a_\mu) = \det(\delta^a_\mu - \frac{\kappa}{2}h^a_\mu)$. This term has another variation with respect to $A_{\mu\nu\rho}$ of the form $\kappa h/2 \times F\psi$, which is not canceled by the variation of the terms we currently have in the action. However, we have seen in the cancellation of the zeroth order how the variation of FF term vanishes with the variations of other terms in the action. This leads us to suggest adding corresponding terms multiplied by $\kappa h/2$ into the action. This cancellation results in the terms multiplied by e in the final action.

There are also two terms containing $\partial_\mu \varepsilon$ not vanishing in the first order. These two terms are a part of the $\delta h_{\mu\nu}$ variation of $\int d^{11}x \; ... + \frac{\kappa}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\partial_a h_{b\nu}\psi_\rho + ...$ and the $\delta A_{\mu\nu\rho}$ variation of the last term in equation (2.13):

$$\begin{aligned} \delta S = \int d^{11}x \Big[& ... - \frac{1}{4}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho \left(\bar{\psi}_\nu \Gamma_b + \bar{\psi}_b \Gamma_\nu\right) \partial_a \varepsilon \\ & + \frac{3\kappa}{96}\left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\beta\gamma}\psi^\delta\right)\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\partial_{\delta]}\varepsilon + ... \Big] . \end{aligned} \tag{2.18}$$

These terms are to the third power of ψ, which is not found in the variations of other terms we already have. This suggests the addition of new terms for cancellation. We recognize these terms as the $\delta\psi_\mu$ variation of some new terms in the action with $\partial_\mu \varepsilon$ replaced by $\frac{\kappa}{2}\psi_\mu$. By noticing the antisymmetry of indices a and b in the first term in equation (2.18), we propose the following new term in the action

$$S = ... - \int d^{11}x \frac{\kappa^2}{8}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho \left(\bar{\psi}_\nu \Gamma_a \psi_b - \bar{\psi}_\nu \Gamma_b \psi_a + \bar{\psi}_a \Gamma_\nu \psi_b\right) . \tag{2.19}$$

Following similar argument for the second term (2.18), another new term can be added to the full action

$$S = ... - \int d^{11}x \frac{3\kappa^2}{96}\left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\beta\gamma}\psi^\delta\right)\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]} . \tag{2.20}$$

While the second term in equation (2.18) cancels half of the corresponding variation in this new term, the remaining half is canceled by other remaining terms of the same structure. After the introduction of these two terms in the action, one of the remaining terms is a part of the $\delta\psi_\mu = ... + \partial_\mu \varepsilon/\kappa$ variation of equation (2.19)

$$\delta S = ... - \frac{\kappa}{4}\int d^{11}x \; \bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\partial_\rho \varepsilon \;\; (\bar{\psi}_a \Gamma_b \psi_\nu + \bar{\psi}_a \Gamma_\nu \psi_b - \bar{\psi}_b \Gamma_a \psi_\nu), \tag{2.21}$$

which differs from other remaining first order terms. Since we have already introduced the term (2.19), another possible way of having this term in the first order is through the variation of the second term in equation (2.1) and introducing a new term in the $\delta\psi_\mu$ variation. We also notice similar contributions in second order of κ which suggests a new variation in $\delta\psi_\mu$ in the $\delta\psi_\mu = \ldots + \frac{1}{288}\left(\Gamma_\mu^{\alpha\beta\gamma\delta} - 8\delta^\alpha_\mu \Gamma^{\beta\gamma\delta}\right) F_{\alpha\beta\gamma\delta}\varepsilon +$

... variation of the $\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho$ part in the first term in equation (2.19)

$$\delta S = ...+\frac{\kappa}{96}\int d^{11}x\bar{\psi}_\mu\left(\Gamma^{\mu\nu\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}+12\Gamma^{\alpha\beta}F_{\alpha\beta}{}^{\mu\nu}\right)\frac{\kappa}{4}\Gamma^{ab}\left(\bar{\psi}_\nu\Gamma_a\psi_b-\bar{\psi}_\nu\Gamma_b\psi_a+\bar{\psi}_a\Gamma_\nu\psi_b\right). \quad (2.22)$$

Thus we have another possible term in the variation $\delta\psi$:

$$\delta\psi_\mu = ... + \frac{\kappa}{8}\Gamma^{ab}\left(\bar{\psi}_\nu\Gamma_a\psi_b-\bar{\psi}_\nu\Gamma_b\psi_a+\bar{\psi}_a\Gamma_\nu\psi_b\right) \quad (2.23)$$

and the algebra remains closed. This has also led to cancellations between the $\delta\psi_\mu$ variations of (2.19) with (2.23) variation of the second term in the action (2.13) to second order in κ. The remaining first-order terms are canceled by further modifications to the action and transformation. One may also need to modify the coefficients of the terms suggested above if remaining first-order terms of the same structure arise in the iterative process.

We obtain an invariant action order by order in terms of κ repeating the process described above. At each step, we also require the transformations to form a closed algebra, and ambiguities that arise in the process are eliminated by requiring the closure of this algebra.

2.3 D=11 supergravity action

In this section, we introduce the action of D=11 supergravity. We first look at the action in the form as it is constructed using Noether's method and how it shows order-by-order modifications to the action of global supersymmetry. Then, we would like to perform a rescaling of the fields to put it into its conventional form. The local supersymmetry invariant action constructed from Noether's method is

$$\begin{aligned}S = \int d^{11}x\Bigg\{&\frac{e}{\kappa^2}R[\omega]-\frac{e}{48}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}-e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\left[\frac{1}{2}(\omega+\hat{\omega})\right]\psi_\rho\\ &-\frac{e\kappa}{96}(\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu+12\bar{\psi}^\alpha\Gamma^{\beta\gamma}\psi^\delta)(F_{\alpha\beta\gamma\delta}+\hat{F}_{\alpha\beta\gamma\delta})\\ &+\frac{2\kappa}{12^4}\epsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'\mu\nu\rho}F_{\alpha\beta\gamma\delta}F_{\alpha'\beta'\gamma'\delta'}A_{\mu\nu\rho}\Bigg\}\end{aligned} \quad (2.24)$$

where D_ν is the covariant derivative in terms of spin connections ω

$$D_\mu(\omega) = \partial_\mu - \frac{1}{2}\Gamma^{ab}\omega_{\mu ab}. \quad (2.25)$$

The terms with hat are supercovariant terms defined by their variation containing on $\partial_\mu\epsilon$ terms. These are given by

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab}+\frac{\kappa^2}{4}\bar{\psi}_\nu\Gamma_{\mu ab}{}^{\nu\rho}\psi_\rho, \quad (2.26a)$$

$$\hat{F}_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]}+6\kappa\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]}, \quad (2.26b)$$

where

$$\omega_{\mu ab} = \omega_{\mu ab}(e)+\frac{\kappa^2}{2}\left(\bar{\psi}_\mu\Gamma_b\psi_a-\bar{\psi}_a\Gamma_\mu\psi_b+\bar{\psi}_b\Gamma_a\psi_\mu\right)-\frac{\kappa^2}{4}\bar{\psi}_\nu\Gamma^{\nu\rho}{}_{\mu ab}\psi_\rho. \quad (2.27)$$

$\omega_{\mu ab}(e)$ is the torsion-free spin connection, which can be written in terms of the vielbeins $\omega_\mu^{ab}(e) = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma^c$. From the action (2.24), we can see that the $h^{\mu\nu}G_{\mu\nu}$ term as discussed in the previous section has been replaced by the non-linear version $\frac{e}{\kappa^2}R$ using the expansion of e_μ^a in terms of h_μ^a . Similar replacements can be seen in the inclusion of e for the other terms. We can also see the additional terms in different powers of κ, which gives information of which order of cancellation it may have come from. In fact, our consideration of some of the terms in the first order has led to many of the terms in the final action. This action is invariant under the following local supersymmetry transformations

$$\delta e_m^\mu = \kappa\bar{\varepsilon}\Gamma^m\psi_\mu, \qquad \delta A_{\mu\nu\rho} = -\frac{3}{2}\bar{\varepsilon}\Gamma_{[\mu\nu}\psi_{\rho]} \tag{2.28a}$$

$$\delta\psi_\mu = \frac{1}{\kappa}D_\mu[\hat{\omega}]\varepsilon + \frac{1}{288}\left(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - 8\,\delta_\mu^\alpha\Gamma^{\beta\gamma\delta}\right)F_{\alpha\beta\gamma\delta}\varepsilon. \tag{2.28b}$$

While the above form of the action shows the order of each term by powers of κ as how they are obtained using Noether's method, this action also has a possible rescaling of fields such that each term in the action has the same power of κ. This rescaling can be useful later when we look for solutions to D=11 supergravity. We perform the rescalings $h \to -h/2$, $F \to F/2\kappa$ and $\psi \to \psi/2\kappa$ to obtain

$$\begin{aligned} S =& \frac{1}{2\kappa^2}\int d^{11}x\left\{eR[\omega] - \frac{e}{48}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta} - e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\left[\frac{1}{2}(\omega+\hat{\omega})\right]\psi_\rho\right. \\ &-\frac{e}{192}(\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha\Gamma^{\beta\gamma}\psi^\delta)(F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}) \\ &\left.+\frac{1}{12^4}\epsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'\mu\nu\rho}F_{\alpha\beta\gamma\delta}F_{\alpha'\beta'\gamma'\delta'}A_{\mu\nu\rho}\right\}. \end{aligned} \tag{2.29}$$

This is the action we will use in the following chapters. The invariance of this action under local supersymmetry transformations can be verified with the help of Fierz identities, and the vanishing of terms in each combination of ψ, F, and Γ is shown in [17].

One can also check that the superalgebra closes on-shell. The supersymmetry algebra of D=11 supergravity is given by

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_{\mathrm{C}}(\xi^\mu) + \delta_L\left(\lambda^{ab}\right) + \delta_Q(\varepsilon_3) + \delta_A(\theta_{\mu\nu}). \tag{2.30}$$

$\delta_C(\xi^\mu)$ is a general coordinate transformation $x'^\mu = x^\mu - \xi^\mu(x)$ by a parameter

$$\xi^\mu(x) = \frac{1}{2}\bar{\varepsilon}_2(x)\Gamma^\mu\varepsilon_1(x) \tag{2.31}$$

For example, this contributes a term $\delta_{\mathrm{C}}(\xi^\mu)\,e_\mu^a = \xi^\rho\partial_\rho e_\mu^a + \partial_\mu\xi^\rho e_\rho^a$ if it is acted on the vielbeins or $\delta_{\mathrm{C}}(\xi^\mu)\,A_{\mu\nu\sigma} = \xi^\rho\partial_\rho A_{\mu\nu\sigma} + 3(\partial_{[\mu}\xi^\rho)A_{\rho\nu\sigma]}$ if it is acted on $A_{\mu\nu\sigma}$. The second term $\delta_L\left(\lambda^{ab}\right)$ is a local Lorentz transformation by a field dependent parameter

$$\lambda^{ab} = -\xi^\rho\omega_\rho^{ab} + \frac{1}{288}\bar{\varepsilon}_2\left(\Gamma^{\alpha\beta\gamma\delta ab}\hat{F}_{\alpha\beta\gamma\delta} + 24\Gamma^{\alpha\beta}\hat{F}_{\alpha\beta}{}^{ab}\right)\varepsilon_1. \tag{2.32}$$

The terms with rank 2 and rank 6 terms in gamma in this parameter do not vanish in the classical M2-

and M5- brane solutions [15]. The rank 6 term can be written as a rank 5 term applying the properties of gamma matrices in 11 dimensions. These contribute to the central charges of the global supersymmetry algebra of the branes. $\delta_Q(\varepsilon_3)$ is a local supersymmetry transformation. The parameter ε_3 is given by

$$\varepsilon_3 = -\xi^\sigma \psi_\sigma. \tag{2.33}$$

$\delta_A(\theta_{\mu\nu})$ is the gauge symmetry of the 3-form field $A_{\mu\nu\rho}$. This term is found in the commutator of the transformations of $A_{\mu\nu\rho}$. The parameter $\theta_{\mu\nu}$ is given by

$$\theta_{\mu\nu} = -\xi^\rho A_{\rho\mu\nu} + \frac{1}{2}\bar{\varepsilon}_1 \Gamma_{\mu\nu}\varepsilon_2. \tag{2.34}$$

The action is separately invariant under these transformations. For the variation of gravitino ψ_μ, the equation of motion is applied to eliminate extra terms that are not included in the equation (2.30) to close the algebra. Thus, the superalgebra by the transformations closes on-shell. Apart from the local Lorentz transformation term, these extra terms do not affect the algebra of the physical states.

2.4 D=11 supergravity fields in superspace

The on-shell D=11 supergravity can be formulated in superspace with its field equations of motion expressed in terms of geometrical quantities [18]. This is done by comparing the supersymmetry transformation shown in equation (2.28) and the general coordinate transformations of the three component fields to a general coordinate transformation in a corresponding superspace of 11 spacetime and 32 fermionic coordinates [19, 20]. This comparison leads to the parametrization of the superfields and their transformation parameters in terms of the component fields of D=11 supergravity order-by-order in θ-coordinates.

We start with defining the superspace coordinates, vielbeins, torsion, and curvature. Going to the superspace, we make a new definition of the coordinates $Z^M = (X^m, \theta^\mu)$ and the tangent coordinates $\Pi^A = (X^a, \theta^\alpha)$, where $m, a = 0, ..., 10$ and $\mu, \alpha = 0, ..., 31$. The vielbeins for these superspace coordinates are defined by $\Pi^A = dZ^M \Pi_M^A$ where $\Pi_M^A \Pi_B^M = \delta_B^A$. The torsion and curvature tensors are defined in a similar way as in spacetime coordinates and are given by

$$\mathcal{T}^A = d\Pi^A + \Pi^B \Omega_B{}^A, \quad \mathcal{R}^{AB} = d\Omega^{AB} + \Omega^A{}_C \wedge \Omega^{CB}. \tag{2.35}$$

where Ω_{AB} is the superspace spin connection one form. We also define a tensor field A_{MNP} with gauge transformations

$$\delta A_{MNP} = 3\partial_{[M}\Sigma_{NP\}}, \quad \Sigma_{MN} = -(-)^{MN}\Sigma_{NM}, \tag{2.36}$$

where $[X, Y\}$ denotes the graded Poisson bracket.

Consider the following transformations of the superspace by superspace parameters Ξ^N, Λ^{ab}, Σ_{MN}:

$$\delta\Pi_M^A = \Xi^N\partial_N\Pi_M^A + \partial_M\Xi^N\Pi_N^A + (\Lambda^{ab}L_{ab})_B^A\Pi_M^B, \tag{2.37a}$$

$$\delta\Omega_M^{ab} = \Xi^N\partial_N\Omega_M^{ab} + \partial_M\Xi^N\Omega_N^{ab} - \partial_M\Lambda^{ab} - \Omega_M^{ac}\Lambda_c{}^b + \Omega_M^{bc}\Lambda_c{}^a, \tag{2.37b}$$

$$\delta A_{NMP} = \Xi^Q\partial_Q A_{MNP} + \partial_{[M}\Xi^Q A_{QNP]} + 3\partial_{[M}\Sigma_{NP\}}. \tag{2.37c}$$

where L_{ab} are the Lorentz generators. We match the components of these superspace vielbeins, spin connections, and the 3-form field with the fields of D=11 supergravity order-by-order in θ-coordinate by requiring these transformations to be consistent with equation (2.28). The transformation parameters in the superspace Ξ^N, Λ^{ab}, Σ_{MN} will be dependent on the component fields of D=11 supergravity and their transformation parameter ε. Ambiguities that arise in the process are resolved by requiring the supersymmetry algebra to be consistent with the D=11 supergravity case. The result is the expression of superspace transformation parameters and superfields in terms of the component fields of D=11 supergravity.

The first step is the identification of the component field at zeroth order in θ. This can be done by a gauge choice such that

$$\Pi_m^a(X,\theta=0) = e_m^a(X), \quad \Pi_\mu^\alpha(X,\theta=0) = \psi_\mu^\alpha, \tag{2.38a}$$

$$\Xi^m(X,\theta=0) = \xi^m(X), \quad \Xi^\mu(X,\theta=0) = \varepsilon^\mu(X), \tag{2.38b}$$

$$A_{mnp}(X,\theta=0) = A_{mnp}(X), \quad \Lambda^{ab}(X,\theta=0) = \lambda^{ab}(X), \quad \Sigma_{mn}(X,\theta=0) = \xi_{mn}(X). \tag{2.38c}$$

The expression of Ξ^M in terms of component field can be found by considering the supersymmetry algebra by the commutator of two consecutive transformations acting on a scalar superfield. By requiring this to be consistent with the general coordinate transformation generated by (2.31) to zeroth order in θ gives

$$\Xi^m(X,\theta) = \frac{1}{4}\bar{\theta}\Gamma^m\varepsilon + \mathcal{O}(\theta), \quad \Xi^\mu(X,\theta) = \varepsilon^\mu - \frac{1}{4}\bar{\theta}\Gamma^n\varepsilon\psi_n^\mu + \mathcal{O}(\theta). \tag{2.39}$$

With these parameters in terms of component fields, we then look at the transformations of the superspace vielbeins by Ξ^M and compare to the spacetime vielbein transformation and the supersymmetry transformation of ψ_μ^a according to the $\theta = 0$ condition by (2.38a). The zeroth order term in $\delta\Pi_m^a$ is compared to the supersymmetry transformation of the spacetime vielbein

$$\varepsilon^\mu\partial_\mu(\theta^\nu)K_{m;\nu}^a + \mathcal{O}(\theta) = \frac{1}{2}\bar{\varepsilon}\Gamma^a\psi_m, \tag{2.40}$$

where $\bar{\theta}K_m^a = \theta^\nu K_{m;\nu}^a$ is the first order term in the expansion of Π_m^a. This suggests

$$\Pi_m^a(X,\theta) = e_m^a + \frac{1}{2}\bar{\theta}\Gamma^a\psi_m + \mathcal{O}(\theta^2). \tag{2.41}$$

Following similar procedure of comparing different components of Π_M^A, one finds

$$\Pi_m^\alpha(X,\theta) = \psi_m^\alpha + \frac{1}{4}\hat{\omega}_m^{ab}(\Gamma_{ab}\theta)^\alpha + \frac{1}{288}\hat{F}_{npqr}[(\Gamma_m^{npqr} - 8\delta_m^n\Gamma^{pqr})\theta]^\alpha + \mathcal{O}(\theta^2), \tag{2.42a}$$

$$\Pi_\mu^a(X,\theta) = -\frac{1}{4}(\bar{\theta}\Gamma^a)_\mu + \mathcal{O}(\theta^2), \quad \Pi_\mu^\alpha(X,\theta) = \delta_\mu^\alpha + \mathcal{O}(\theta^2). \tag{2.42b}$$

The same process also applies to the gauge transformation by Σ_{MN} of the three-form field. The transformation parameter Σ_{MN} is first determined by the consistency with the transformation by equation (2.34):

$$\Sigma_{mn}(X,\theta) = \frac{1}{4}\bar{\theta}\left(\Gamma_{mn} + \Gamma^p A_{pmn}\right)\varepsilon + \mathcal{O}(\theta^2), \quad \Sigma_{m\mu}(X,\theta) = \Sigma_{\mu\nu}(X,\theta) = 0 + \mathcal{O}(\theta^2). \tag{2.43}$$

Applying these parameters to equation (2.37c) and requiring the result to be consistent with (2.28a), we found

$$A_{mnp}(X,\theta) = A_{mnp}(X) - \frac{3}{2}\bar{\theta}\Gamma_{[mn}\psi_{p]} + \mathcal{O}(\theta^2), \tag{2.44a}$$

$$A_{mn\mu}(X,\theta) = \frac{1}{4}(\bar{\theta}\Gamma_{mn})_\mu + \mathcal{O}(\theta^2), \quad A_{m\mu\nu} = A_{\mu\nu\rho} = 0 + \mathcal{O}(\theta^2). \tag{2.44b}$$

Following the same process, the superspace spin connection can also be determined, and thus, the superspace torsion and curvature tensors can be obtained by applying the equation (2.35). Detailed computation of the superspace fields and transformation parameters up to second order in θ is discussed in [21].

The final result is a single superfield $W_{abcd}(X,\theta=0) = \hat{F}_{abcd}(X)$, satisfying the equation

$$(\Gamma^{bcd}D)_\alpha W_{abcd}(X,\theta) = 0. \tag{2.45}$$

All the components of the torsion and curvature in superspace can be expressed in terms of W_{abcd} [18]. The Bianchi identities for torsion and curvature in the superspace are equivalent to the D=11 supergravity field equations of motion and their Bianchi identities.

CHAPTER 3

M2-brane solution of D=11 supergravity

M2-brane emerges as a brane solution to the equations of motion of D=11 supergravity [22]. The brane solutions to supergravity theories are found by solving the equations of motion of the fields with proposed ansatz possessing certain symmetries. In this chapter, we would like to consider the bosonic part of the supergravity action (2.29), which has the following equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \tag{3.1a}$$

$$\partial_\mu \left(\sqrt{-g} F^{\mu\sigma\zeta\rho} \right) + \frac{1}{8 \times 12^2} \epsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'\sigma\zeta\rho} F_{\alpha\beta\gamma\delta} F_{\alpha'\beta'\gamma'\delta'} = 0, \tag{3.1b}$$

where

$$T_{\mu\nu} = \frac{1}{12} F_\nu{}^{\beta\gamma\delta} F_{\mu\beta\gamma\delta} - \frac{1}{96} g_{\mu\nu} F^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}. \tag{3.2}$$

Since only the bosonic part is considered, the supersymmetry transformation for bosons vanishes as the fermions are set to zero. For a supersymmetric solution, we also check the vanishing of the fermionic transformation:

$$\delta\psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \Gamma_{\alpha\beta} \epsilon + \frac{1}{288} \left(\Gamma_\mu^{\alpha\beta\gamma\delta} - 8 \Gamma^{\beta\gamma\delta} \delta_\mu^\alpha \right) F_{\alpha\beta\gamma\delta} \epsilon = 0. \tag{3.3}$$

A spinor ϵ satisfying this condition is called a Killing spinor. We look for solutions to these equations of motion that also satisfy the Killing spinor condition.

3.1 Brane solutions ansatz

As can be seen in equations (3.1), the bosonic part of D=11 supergravity involves two gauge fields $g_{\mu\nu}$ and $A_{\mu\nu\rho}$. The brane solution ansatz for supergravity thus involves ansatz for both of these fields. The $A_{\mu\nu\rho}$ ansatz determines the dimensions of branes that it can couple to.

3.1.1 $\mathrm{ISO}(1, \mathrm{d}-1) \times \mathrm{SO}(\mathrm{D}-\mathrm{d})$ symmetric ansatz for metric

In solving for the brane solutions, we look for solutions that preserve some of the supersymmetry and translation symmetries. We also require unbroken isotropy in the directions transverse to the translation symmetry, which can be relaxed when we look for more general solutions. We make ansatz according to these requirements with symmetry $ISO(1, d-1) \times SO(D-d)$. This symmetry splits the coordinates into two parts, one satisfying the Poincaré symmetry $ISO(1, d-1)$ and the other satisfying the isotropy $SO(D-d)$. It is important that we redefine our labeling of coordinates here to show this splitting

$$X^M = (x^\mu, y^m), \quad \mu = 0, 1, ..., d; \; m = d, d+1, ..., 10 \tag{3.4}$$

Then, the ansatz for the metric in accordance to the symmetry requirement is

$$ds^2 = g_{MN} dX^M dX^N = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} \delta_{mn} dy^m dy^n \tag{3.5}$$

where $r = \sqrt{y^m y^m}$. This describes a flat hyperplane of d-dimensions embedded in the 11-dimensional spacetime. We substitute this ansatz into the equation of motion to solve for the undetermined functions $A(r)$ and $B(r)$.

This would require the calculation of the Ricci tensor and Ricci scalar corresponding to the ansatz metric. These can be computed using vielbeins defined by

$$\theta^\alpha = e^{A(r)} \delta^\alpha_\mu dx^\mu, \qquad \theta^a = e^{B(r)} \delta^A_m dy^m, \tag{3.6}$$

where the tangent coordinates are defined by $X^{\underline{A}} = (x^\alpha, y^a)$. The torsion-free spin connections are found from these vielbeins as

$$\omega^\alpha{}_\beta = 0, \tag{3.7a}$$

$$\omega^\alpha{}_a = \frac{y_a}{r} A' e^{-B} \theta^\alpha, \tag{3.7b}$$

$$\omega^a{}_b = \frac{y_b}{r} B' e^{-B} \theta^a - \frac{y^a}{r} B' e^{-B} \theta^c \eta_{cb}. \tag{3.7c}$$

The Ricci tensor components are found from the torsion free spin connection applying Cartan's second equation

$$R_{\mu\nu} = -\eta_{\mu\nu} e^{2(A-B)} \left(A'' + d\,(A')^2 + \tilde{d} A' B' + \frac{(\tilde{d}+1)}{r} A' \right), \tag{3.8a}$$

$$\begin{aligned} R_{mn} = & - \delta_{mn} \left(B'' + dA'B' + \tilde{d}\,(B')^2 + \frac{(2\tilde{d}+1)}{r} B' + \frac{d}{r} A' \right) \\ & - \frac{y^m y^n}{r^2} \left(\tilde{d} B'' + dA'' - 2dA'B' + d\,(A')^2 - \tilde{d}\,(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' \right), \end{aligned} \tag{3.8b}$$

where $A' = \partial_r A(r)$.

3.1.2 The ansatz for the 3-form field

For 11-dimensional supergravity, there are two possible ansatz for the three-form field, giving the elementary (electric) and the solitonic (magnetic) solutions. The three-form coupling to a brane of world-volume of three gives the electric ansatz . A four-form field strength F is constructed from a three-form

A by the relation $F = dA$. According to the symmetries required, the electric ansatz is defined as

$$A_{\mu_1\mu_2\mu_3} = \epsilon_{\mu_1\mu_2\mu_3}\ e^{C(r)}, \quad F_{\mu_1\mu_2\mu_3 m} = \epsilon_{\mu_1\mu_2\mu_3}\ \partial_m e^{C(r)}, \tag{3.9a}$$

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\delta_{mn}dy^m dy^n, \quad \mu,\nu = 0,1,2; \ \ m,\ n = 3,...,11. \tag{3.9b}$$

where $A(r)$, $B(r)$, and $C(r)$ are arbitrary functions depending on only the radial coordinate r. This ansatz gives the M2-brane solution. The ansatz of the 3-form $A_{\mu\nu\rho}$ only has a dependence on indices corresponding to coordinates having the Poincaré symmetry, thus coupling to a world volume with this symmetry. The 11-dimensional spacetime in this case has $ISO(1,2) \times SO(8)$ symmetry. This ansatz satisfies the Bianchi identity $dF = 0$ and $F \wedge F \wedge A = 0$ due to the antisymmetry of the indices:

$$\epsilon_{[\mu_1\mu_2\mu_3}\partial_{m_1}\partial_{m_2]}e^C = 0, \tag{3.10a}$$

$$\epsilon^{\mu_1\mu_2\mu_3 m_1\ \nu_1\nu_2\nu_3 m_2\ \rho_1\rho_2\rho_3}\epsilon_{\mu_1\mu_2\mu_3}\epsilon_{\nu_1\nu_2\nu_3}A_{\rho_1\rho_2\rho_3}\partial_{m_1}e^C\partial_{m_2}e^C = 0. \tag{3.10b}$$

Referring to the bosonic part of the action, we have only the kinetic term of the three-form field that contains $A_{\mu\nu\rho}$ and is non-vanishing with the ansatz applied

$$-\int d^{11}x\ \ \frac{e}{2\kappa^2}F \wedge *F. \tag{3.11}$$

The equation of motion for this action is unchanged if one makes the change $F \to *F$. If F is a four-form solution to the equation of motion, $*F$ is a seven-form solution. One can also find a seven-form in a similar expression as (3.9a) whose dual gives another solution. We apply this argument to see the other ansatz in the form $*F = \epsilon_{\mu_1\mu_2...\mu_6}\ \partial_m e^C(r)dx^{\mu_1} \wedge ... \wedge dx^{\mu_6} \wedge dy^m$, $\alpha, ..., \rho = 0,1,...,5$. Performing the Hodge dual of this to find F up to a difference in sign and ignore the constant coefficients, we have

$$F \propto \epsilon_{m_1...m_4 n}\partial^n e^C(r)dy^{m_1} \wedge ... \wedge dy^{m_4}. \tag{3.12}$$

The requirement that this field strength satisfies Bianchi identity leads to

$$dF \propto \partial_p\partial^n e^C dy^p \wedge dy^{m_1} \wedge ... \wedge dy^{m_4} \tag{3.13a}$$

$$= \partial_p[\partial_r e^C \frac{y^n}{r}]dy^p \wedge dy^{m_1} \wedge ... \wedge dy^{m_4} \tag{3.13b}$$

$$= [\partial_r^2 e^C \frac{y^n y_p}{r^2} + \partial_r e^C \frac{\delta^n_p}{r} - \partial_r e^C \frac{y^n y_p}{r^3}]dy^p \wedge dy^{m_1} \wedge ... \wedge dy^{m_4} \tag{3.13c}$$

We notice when performing the Hodge dual, $n \neq m_1, ..., m_4$ and $n, m_1, ..., m_4 = 6, 7, .., 10$. From equation (3.13), $p \neq m_1, ..., m_4$. Thus, we see only terms which has $m = p$ are non-zero. The Bianchi identity condition on F becomes

$$\partial_r^2 e^C + \partial_r e^C \frac{4}{r} = 0. \tag{3.14}$$

This is solved by $e^C \propto 1/r^3$, which gives $\partial^n e^C \propto y^n/r^5$. Therefore, we suggest the magnetic ansatz

$$F_{m_1...m_4} = \lambda\epsilon_{m_1...m_4\,n}\,\frac{y^n}{r^5}, \tag{3.15a}$$

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\delta_{mn}dy^m dy^n, \mu, \nu = 0, ..., 5;\;\; m,\; n = 6, ..., 10. \tag{3.15b}$$

The 11-dimensional spacetime here has $ISO(1,5)\times SO(5)$ symmetry. This gives the M5-solution of D=11 supergravity, which we may not consider further in this review.

3.2 M2-brane solution

The M2-brane solution of D=11 supergravity is the solution obtained by substituting in the electric ansatz given in equation (3.9) and requires it to satisfy the equations of motion (3.1) and the Killing spinor condition (3.3). We make a three-eight splitting of the gamma matrices consistent with the $ISO(1,2)\times SO(8)$ symmetry

$$\Gamma_A = (\gamma_\alpha \otimes \Gamma_9, \mathbb{1}\otimes\Sigma_a), \tag{3.16}$$

where γ_α are the gamma matrices with tangent indices for $SO(1,2)$, Σ_a are the gamma matrices with tangent indices for $SO(8)$, $\Lambda_9 = \Sigma_3\Sigma_4...\Sigma_{10}$. The spinor ϵ can also be split into three and 8-dimensional components consistent with the $ISO(1,2)\times SO(8)$ symmetry.

$$\epsilon(x,y) = \zeta\otimes\eta(r), \tag{3.17}$$

where ζ is a constant spinor of $SO(1,2)$ and $\eta(r)$ is a transverse radially dependent spinor of $SO(8)$.

We would like to first look at the Killing spinor equations given by equation (3.3). From equations (3.7), one obtain the expressions

$$\omega_\mu{}^{\alpha a} = -\omega_\mu{}^{a\alpha} = -e^{-B}\partial_n e^A\delta_{na}\delta_{\alpha\mu}, \quad \omega_m{}^{\alpha a} = 0, \tag{3.18a}$$

$$\omega_\mu{}^{ab} = -\omega_\mu{}^{ab} = 0, \quad \omega_m{}^{ab} = -\omega_m{}^{ba} = e^{-B}\partial_n e^B\delta_{nb}\delta_{ma}. \tag{3.18b}$$

Evaluating each term in Killing spinor condition (3.3) gives

$$\omega_\mu{}^{AB}\Gamma_{AB} = 2\omega_\mu{}^{\alpha a}\Gamma_{\alpha a} = -2e^{-A}\partial_n e^A\gamma_\mu\Sigma_n\Lambda_9, \tag{3.19a}$$

$$\Gamma^{PQRS}{}_\mu F_{SPQR} = 0, \tag{3.19b}$$

$$\Gamma^{PQR}\delta^S_\mu F_{SPQR} = 3\Gamma^{m\alpha\beta}\epsilon_{\alpha\beta\mu}\partial_m e^C = 6e^{-3A}\gamma_\mu\Sigma^m\partial_m e^C, \tag{3.19c}$$

$$\omega_m{}^{AB}\Gamma_{AB} = 2\omega_m^{ab}\Gamma_{ab} = e^{-B}[\Sigma_m,\Sigma^n]\partial_n e^B, \tag{3.19d}$$

$$\Gamma^{PQRS}{}_m F_{SPQR} = -12e^{-3A}[\Sigma_m,\Sigma^n]\partial_n e^C\Lambda_9, \tag{3.19e}$$

$$\Gamma^{PQR}\delta^S_m F_{SPQR} = -6e^{-3A}\partial_m e^C\Lambda_9. \tag{3.19f}$$

Putting together one has two Killing spinor equations

$$\tilde{D}_\mu \epsilon = -\frac{1}{2}e^{-A}\partial_n e^A \left(\gamma_\mu \zeta\right)\Sigma^n \Lambda_9 \eta - \frac{1}{6}e^{-3A}\partial_n e^c \left(\gamma_\mu \zeta\right)\Sigma^n \eta = 0, \tag{3.20a}$$

$$\begin{aligned}\tilde{D}_m \epsilon =& \zeta \partial_m \eta + \frac{1}{6}e^{-3A}\partial_m e^c \zeta(\Lambda_9 \eta) \\ &+ \frac{1}{4}e^{-B}\partial_n e^B \left[\Sigma_m, \Sigma^n\right]\zeta\eta - \frac{1}{24}e^{-3A}\partial_n e^c \left[\Sigma_m, \Sigma^n\right]\zeta \Lambda_9 \eta = 0.\end{aligned} \tag{3.20b}$$

Observing the first Killing spinor equation, one can make the ansatz $C = 3A$. Equation (3.20a) then reduces to

$$\tilde{D}_\mu \epsilon = \frac{1}{2}\partial_n A(\gamma_\mu \zeta)\Sigma^n (1 + \Lambda_9)\eta = 0. \tag{3.21}$$

Thus also requires the projection $(1+\Lambda_9)\eta = 0$ to satisfy this Killing spinor equation. With these ansatz made, we look at the second Killing spinor equation (3.20b). The second line of this Killing spinor equation suggests $2\partial_n B = -\partial_n A$, thus $A = -2B + \text{const}$. The first line gives the relation

$$\partial_m \eta = \frac{1}{2}(\partial_m A)\eta, \tag{3.22}$$

therefore $\eta = \eta_0 e^{A/2} = \eta_0 e^{C/6}$. In solving Killing spinor equations, we have found relations between $A(r)$, $B(r)$, and $C(r)$ and a projection relation:

$$3A = -6B = C, \quad (1 + \Lambda_9)\eta = 0, \quad \eta = \eta_0 e^{C/6}. \tag{3.23}$$

One only has to determine one of $A(r)$, $B(r)$, and $C(r)$ and use the above relation to determine the others. This can be done by looking at the equation of motion. By substituting equations (3.23) into equation (3.1b), one has

$$\partial_m \partial^m e^{-C} = 0. \tag{3.24}$$

Solving this differential equation for e^{-C} gives

$$e^{-C} = 1 + \frac{L^6}{r^6}, \tag{3.25}$$

where L is a constant. Applying the relations in equations (3.25) and (3.23) to the line element ansatz and the three-form ansatz, we have

$$ds^2 = \left(1 + \frac{L^6}{r^6}\right)^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^6}{r^6}\right)^{1/3} \delta_{mn} dy^m dy^n, \tag{3.26a}$$

$$A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}\left(1 + \frac{L^6}{r^6}\right)^{-1}. \tag{3.26b}$$

One can check that these satisfies also equation (3.1a). Here, we also notice that by defining ansatz for the three-form with an extra negative sign or using the positive convention for gamma matrices as shown in equation (A.3), we obtain the projection equation for the $SO(8)$ spinor as $(1 \pm \Lambda_9)\eta = 0$. Therefore, the Killing spinor equation admits two solutions

$$A_{\mu\nu\rho} = \pm\epsilon_{\mu\nu\rho}\, e^{C}(r) \quad \to \quad (1 \pm \Lambda_9)\eta = 0. \tag{3.27}$$

By the projection $(1 \pm \Lambda_9)$, the number of independent spinor components is halved. This suggests that half of the rigid supersymmetry is preserved in this solution. This describes supersymmetric extended objects.

3.3 Interpolation between Minkowski spacetime and $\mathrm{AdS}_4 \times \mathrm{S}^7$

Many of the brane solutions to supergravity theories interpolate between Minkowski spacetime and a compact spacetime [23], so does the M2-solution to D=11 supergravity. In the limit $r \to \infty$, the metric as given in equation (3.26a) tends to

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \delta_{mn} dy^m dy^n. \tag{3.28}$$

This is the Minkowski limit of the metric at the spatial infinity of the transverse coordinates. The other limit is when $r \to 0$; at this limit we can first expand the metric and write the transverse coordinates into polar coordinates to obtain

$$ds^2 = \frac{r^4}{L^4}\left(\eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^6}{r^6} dr^2\right) + L^2 d\Omega_7^2. \tag{3.29}$$

This is followed by a recaling performed on D=11 coordinate $dX^M \to (1/L) dX^M$ and a redefinition $R = L^2/2r^2$, we find

$$ds^2 = \frac{1}{4L^2R^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dR^2\right) + d\Omega_7^2. \tag{3.30}$$

This suggests at $r \to 0$, the spacetime tends to $AdS_4 \times S^7$. The M2-brane solution interpolates between Minkowski spacetime and $AdS_4 \times S^7$. The latter is also known as the maximally supersymmetric vacuum.

We also notice at $r = 0$, the metric is singular. However, the curvature tensor R_{MNPQ} and $F_{m\mu\nu\rho}$ are non-singular at this point. This is a coordinate singularity which can be removed by redefinition of the coordinates. We can also perform a redefinition of the coordinate by $r = (\tilde{r}^6 - L^6)^{1/6}$, and consider analytic continuation of the spacetime. In this coordinate, the solution becomes

$$ds^2 = \left(1 - \frac{L^6}{\tilde{r}^6}\right)^{2/3} \left(-dt^2 + d\sigma^2 + d\rho^2\right) + \left(1 - \frac{L^6}{\tilde{r}^6}\right)^{-2} d\tilde{r}^2 + \tilde{r}^2 d\Omega_7^2 \tag{3.31a}$$

$$A_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda}\left(1 - \frac{L^6}{\tilde{r}^6}\right) \tag{3.31b}$$

Similar to Schwarzchild metric, $r = 0$ behaves like a horizon. Thus, the $AdS_4 \times S^7$ limit is also known as the near horizon limit. The normal to this surface is a null vector. However, the spacelike and timelike regions do not exchange crossing the horizon due to the $2/3$ exponent. $\tilde{r} = 0$ is a true singularity and is timelike [24].

3.4 D=11 supergravity with M2-brane term

The coordinate singularity suggests that the solution we have found does not solve the equations of motion everywhere. This invites one to consider the coupling of a source to the solution at the position of this singularity and thus adding a source term to the equations of motion [22]. This source term corresponds to a new term added into the action in the form of a D=11 supermembrane action. To see how a supermembrane action leads to a source term in the equations of motion, we consider the combined supergravity-supermembrane action

$$S = S_G + S_M \tag{3.32}$$

where S_G is the bosonic sector of D=11 supergravity and the supermembrane action S_M is given by

$$S_M = T \int d^3\xi \left(-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i X^M \partial_j X^N g_{MN} + \frac{1}{2}\sqrt{-h} + \frac{1}{3!}\varepsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P A_{MNP} \right). \tag{3.33}$$

T is the tension of the membrane. The last term in the supermembrane action is known as the Wess-Zumino term. The variation of g_{MN} and A_{MNP} gives equations of motion

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \tag{3.34a}$$

$$\begin{aligned} &\partial_M \left(\sqrt{-g}F^{MUVW}\right) + \frac{1}{1152}\varepsilon^{UVWMNOPQRST}F_{MNOP}F_{QRST} \\ &\qquad = -2\kappa^2 T \int \mathrm{d}^3\xi \; \varepsilon^{ijk}\partial_i X^U \partial_j X^V \partial_k X^W \delta^{11}(x - X), \end{aligned} \tag{3.34b}$$

where the stress-energy tensor T_{MN} is

$$\begin{aligned} \kappa^2 T^{MN} = &\frac{1}{12}\left(F^M_{PQR}F^{NPQR} - \frac{1}{8}g^{MN}F_{PQRS}F^{PQRS}\right) \\ &- \kappa^2 T \int \mathrm{d}^3\xi \sqrt{-h}h^{ij}\partial_i X^M \partial_j X^N \frac{\delta^{11}(x-X)}{\sqrt{-g}}. \end{aligned} \tag{3.35}$$

The inclusion of a supermembrane also has introduced new variables and corresponding equations of motion:

$$\begin{aligned} &\partial_i \left(\sqrt{-h}h^{ij}\partial_j X^N g_{MN}\right) + \frac{1}{2}\sqrt{-h}h^{ij}\partial_i X^N \partial_j X^P \partial_M g_{NP} \\ &\qquad \pm \frac{1}{3!}\varepsilon^{ijk}\partial_i X^N \partial_j X^P \partial_k X^Q F_{MNPQ} = 0, \end{aligned} \tag{3.36a}$$

$$h_{ij} = \partial_i X^M \partial_j X^N g_{MN}. \tag{3.36b}$$

The delta function comes from the fact that the S_G terms are integrated over the D=11 spacetime $\int d^{11}x$ while the S_M terms are integrated over the worldvolume $\int d^3\xi$. One can substitute in $\int d^{11}x \; \delta^{11}(X - x) = 1$ and obtain the above expressions. In obtaining equations (3.36a), one also remember that

$g_{MN} = g_{MN}(X)$ and $A_{MNP} = A_{MNP}(X)$.

Following a similar procedure as before, we substitute the electric ansatz into the equation of motion from the variation of A_{MNP} (3.34b) to obtain

$$\partial_m \partial^m e^{-C} = -\frac{2}{3!}\kappa^2 T \int d^3\xi \; \varepsilon_{\mu\nu\rho}\varepsilon^{ijk}\partial_i x^\mu \partial_j x^\nu \partial_k x^\rho \delta^{11}(x - X). \tag{3.37}$$

Thus, the equation of motion now has a source term. At $x^\mu = \xi^\mu$, the membrane coordinates are aligned with spacetime x^μ coordinates. The equation of motion reduces to

$$\partial_m \partial^m e^{-C} = -2\kappa^2 T \delta^8(y). \tag{3.38}$$

This equation can be solved by constructing Green's function using the homogeneous solution. The generalization to multi-membrane configuration can be achieved by linear combinations of the solution.

3.5 Charges, mass density and saturation of BPS bound

The existence of the supermembrane source term contributes to non-vanishing central charges for the supersymmetry algebra. This can also be seen from the Wess-Zumino term in the supermembrane Lagrangian [25]. The Wess-Zumino term contains a 3-form field A with a closed-form field strength F, which can give a variation that changes the Lagrangian by a total derivative while keeping the action unchanged. As a result, an extra term is added to the expression of the conserved current and thus modifies the algebra of the conserved charge. The supersymmetry algebra of D=11 supergravity is given by

$$\{Q_\alpha, Q_\beta\} = \left(\Gamma^M C^{-1}\right)_{\alpha\beta} P_M + \left(\Gamma^{MN} C^{-1}\right)_{\alpha\beta} Q^{(e)}_{MN} + \left(\Gamma^{MNPQR} C^{-1}\right)_{\alpha\beta} Q^{(m)}_{MNPQR}, \tag{3.39}$$

where P_M is the generator of translation, $Q^{(e)}$ and $Q^{(m)}$ are 2- and 5-form central charges related to the electric charge Q_e of M2-brane and magnetic charge Q_m of M5-brane. In the rest frame, $P_m = (M, 0, 0, 0)$ where M is the mass density of the supermembrane. The central charges in the algebra lead to lower limit of the soliton mass by the positivity of Q^2; this is the BPS bound [26]. In 11 dimensions, the electric bound is given by $M \geq Q_e$. The equation of motion (3.34b) can be rewritten in terms of differential forms

$$d * F + \frac{1}{2} A \wedge F = - * J. \tag{3.40}$$

From this, we identify the electric charge as

$$Q_e = \frac{1}{2\kappa^2} \int_{\mathcal{M}_8} *J = -\frac{1}{2\kappa^2} \int_{\partial\mathcal{M}_8} (*F + \frac{1}{2} A \wedge F). \tag{3.41}$$

Since the Hodge dual of the field strength 4-form in 11 dimensions is a 7-form, the integral is taken over a 7-sphere. The dimension of the object enclosed by a d-dimensional surface in D-dimensional spacetime is generally given by $D - d - 1$. Thus, this enclosed an object of 3 spacetime dimensions, which is just the M2-brane. By a similar argument, and noticing magnetic charge from the Bianchi identity $dF = 0$:

$$Q_m = \frac{1}{2\kappa^2} \int_{S^4} F, \tag{3.42}$$

one expects to have a magnetic-charged object with 6 spacetime dimensions. These expected solutions correspond to the solitonic interpretations of M2 and M5 branes. With out a source, these charges are evaluated to zero by the equation of motion.

For the M2-brane ansatz we calculate the electric charge Q_e applying equation (3.41). The $A \wedge F$ term vanishes, thus we only evaluate the integration of $*F$. At the asymptotic limit,

$$F = dA = \epsilon_{\mu\nu\rho} L^6 r^{-8} y_n dy^n \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho \tag{3.43a}$$

$$\begin{aligned} Q_e = -\frac{1}{2\kappa^2} \int *F &= \frac{6L^6}{2\kappa^2} \int \sum_{n=1}^{n=8} (-1)^{n-1} r^{-8} y_n dy^1 \wedge ... \wedge dx_{n-1} \wedge dx_{n+1} \wedge ... \wedge dy^8 \\ &= \frac{6L^6}{2\kappa^2} \int d\Omega_7 = \frac{6L^6}{2\kappa^2} \Omega_7. \end{aligned} \tag{3.43b}$$

This ansatz gives a non-vanishing electric charge, which gives further evidence for the existence of a source at the singular point.

The mass density at the asymptotic limit can be found by the ADM formula for the energy density:

$$\mathcal{E} = \int dx^8 T_{00} = \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}_8} r^6 y_m d\Omega_7 \left(-\partial^m h_{00} + \partial_n h^{mn} - \partial^m h^\alpha_\alpha\right) \tag{3.44a}$$

$$= \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}_8} d\Omega_7 \left(6L^6\right) = \frac{6L^6}{2\kappa^2} \Omega_7, \tag{3.44b}$$

where we have used the expression of stress-energy tensor in terms of $h_{\mu\nu}$,

$$T_{\mu\nu} = \frac{1}{2\kappa^2} \left(-\Box h_{\mu\nu} - \eta_{\mu\nu} h^{mn}{}_{,mn} + \eta_{\mu\nu} \Box h\right). \tag{3.45}$$

In the rest frame, this energy density equals to the mass density. We thus found the mass density of the M2-brane having the same expression as the electric charge. The M2-brane solution of D=11 supergravity saturates the BPS bound.

Finally, by comparing the two expressions given in equation (3.41), one can find the expression of the supermembrane tension. The electric charge found by integrating the source term is given by

$$Q_e = \frac{1}{2\kappa^2} \int 2\kappa^2 T \delta^8(y) = T. \tag{3.46}$$

Compared to equation (3.43b), we find the brane tension to have the expression

$$T = \frac{3L^6 \Omega_7}{\kappa^2}. \tag{3.47}$$

CHAPTER 4

M2-brane action

The electric solution to the bosonic sector of D=11 supergravity discussed in the last chapter suggests the supermembrane action as a source term. In this chapter, we look at the super-p-brane action more generally, considering both the bosonic and fermionic parts in a superspace description.

Super-p-branes are p-branes that satisfy supersymmetry on worldvolume. It was first shown in [11] that the effective action of the worldvolume theory of a 3-brane solution in flat D=6 N=1 supersymmetry theory can be written in the form of a generalized Green-Schwarz action [9]. The Green-Schwarz superstring action is a generalization of the superparticle action together with its global supersymmetry by parameter ϵ and local fermionic symmetry by parameter κ. Upon the generalization to superstring, one also expects these symmetries to be generalized correspondingly. The local fermionic symmetry by κ is known as the kappa symmetry. The generalization of superstrings to supermembranes in flat target space is similar to that from superparticles to superstrings. One requirement for this generalization is that the form-field coupling to the worldvolume of the brane has a closed field strength tensor. This requirement is shown by [8] to be equivalent to requiring the number of on-shell worldvolume degrees of freedom for fermions and bosons to be equal. This places conditions on the dimension d of the possible branes that can exist given the target spacetime dimension D and the number for extended supersymmetry $\mathcal{N}$. The construction of a supermembrane action in curved background is described by [12]. The action satisfies a kappa symmetry with constraints that are equivalent to the constraints for D=11 supergravity in the superspace formulation [18].

In this chapter, we start from the Green-Schwarz superstring action in section 4.1 and discuss how this action can be generalized to extended objects of higher dimensions in a flat background. The conditions for the existence of super-p-branes in a D-dimensional background with $\mathcal{N}$ sets of supercharges will be discussed. The example of generalization to M2-branes in D=11 flat superspace is then shown in section 4.2, before we look at the generalization to a curved superspace in section 4.3. The M2-brane action in $AdS_4 \times S^7$ background will be discussed in section 4.4. Some aspects of gauge fixing and semiclassical quantization is also presented for the flat superspace case.

4.1 Green-Schwarz action for superstrings and super-p-branes

We make the following definition of the coordinates to incorporate the fermionic coordinates for the superspace description: the coordinates $Z^M = (X^m, \theta^{\mu\mathcal{I}})$ and the tangent space coordinates $Z^A = (X^a, \theta^{\alpha\mathcal{I}})$, where m and a label spacetime dimensions, μ and α label the spinor components and $\mathcal{I}$

labels the sets of supersymetries. The worldvolume is labelled by indices i, j. The vielbeins for these coordinates are Π_M^A, and $\Pi_A^M \Pi_M^B = \delta_A^B$. Π_i^A is the pullback of Π_M^A on the worldvolume $\Pi_i^A = \partial_i Z^M \Pi_M^A$.

Supermembranes are described by Green-Schwarz type action, which is generalized from the Green-Schwarz covariant action for superstrings in flat background [9]

$$S_{GS} = \int \mathrm{d}\xi^2 \left(L_1 + L_2\right), \tag{4.1a}$$

$$L_1 = -\frac{1}{2}\sqrt{-h} h^{ij} \Pi_i^m \Pi_{mj}, \tag{4.1b}$$

$$L_2 = -\mathrm{i}\epsilon^{ij}\partial_i X^m \left[\bar{\theta}^1 \Gamma_m \partial_j \theta^1 - \bar{\theta}^2 \Gamma_m \partial_j \theta^2\right] + \epsilon^{ij}\bar{\theta}^1 \Gamma^m \partial_i \theta^1 \bar{\theta}^2 \Gamma_m \partial_j \theta^2, \tag{4.1c}$$

where

$$\Pi_i^m = \partial_i X^m - i\bar{\theta}^{\mathcal{I}} \Gamma^m \partial_i \theta^{\mathcal{I}}, \quad \Pi_i^\mu = \partial_i \theta^\mu. \tag{4.2}$$

The action by L_1 has reparametrization and target space Poincaré symmetries and is also invariant under a global supersymmetry that leaves Π_i^m invariant

$$\delta_\varepsilon \theta^{\mathcal{I}} = \varepsilon^{\mathcal{I}}, \quad \delta_\varepsilon x^m = i\bar{\varepsilon}\Gamma^m \theta^{\mathcal{I}}. \tag{4.3}$$

However, the L_1 kinetic term alone does not guarantee worldvolume supersymmetry. This covariant action for superstring, in turn, is a generalization of the superparticle action, which has a local supersymmetry that allows half of the fermionic degrees of freedom to be gauged away [27]. This suggests a similar approach and a similar local fermionic symmetry for Lorentz-covariant string action. This fermionic symmetry is known as the kappa-symmetry (κ-symmetry). The L_1 term in the action does not possess a local fermionic symmetry. The second term L_2, also known as the Wess-Zumino term, plays an important part in ensuring this. The L_2 term in the action can also be viewed as an integral of a two-form $\int_{\partial\Sigma} \Omega_2$ over the worldvolume. The requirement on this two-form from κ-symmetry is that its field strength three-form $\Omega_3 = d\Omega_2$ is closed. The κ-symmetry transformations for the Green-Schwarz superstring action are

$$\delta_\kappa \theta^1 = (1+\Gamma)\kappa^1, \quad \delta_\kappa \theta^2 = (1-\Gamma)\kappa^2, \quad \delta_\kappa X^m = i\bar{\theta}^{\mathcal{I}} \Gamma^m \delta_\kappa \theta^{\mathcal{I}}. \tag{4.4}$$

In terms of tangent space coordinates by performing the transformation $\Pi^A = \Pi_M^A dZ^M$ with

$$\Pi_m^a = \delta_m^a, \quad \Pi_\mu^a = -i(\bar{\theta}\Gamma^a)_\mu, \quad \Pi_m^\alpha = 0, \quad \Pi_\mu^\alpha = \delta_\mu^\alpha, \tag{4.5}$$

these transformations become

$$\delta\Pi^a = \Pi_M^a \delta Z^M = 0, \quad \delta\Pi^\alpha = (1+\Gamma)_\beta^\alpha \kappa^\beta, \tag{4.6}$$

where

$$\Gamma = \frac{1}{2!\sqrt{-h}} \epsilon^{ij} \not{\Pi}_i \not{\Pi}_j. \tag{4.7}$$

This expression in terms of tangent space coordinates is useful when we look at the generalization to curved backgrounds. Using this expression, one can show that $\Gamma^2 = 1$ and that $(1 \mp \Gamma)$ acts as a projection operator. Only half of the components of κ are in effect in the transformation, and this can be used to gauge away half of the degrees of freedom of θ.

Green-Schwarz type action can be constructed not only for strings, but it can also be generalized to higher dimensional extended objects upon requiring some conditions relating to the spacetime dimension, worldvolume dimension, and number of sets of supersymmetries. The action in equation (4.1) describes a flat superspace background. This can also be generalized to curved backgrounds. Examples of these generalizations include other string theories in different backgrounds [28, 29], and M2-brane in D=11 [12].

The existence of κ-summetry invariant super-p-brane action in worldvolume of dimension d, target spacetime dimension D, and number of sets of supersymmetry charges $\mathcal{N}$ is classified by [8]. The requirement of on-shell supersymmetry on the worldvolume places the constraint that the number of on-shell degrees of freedom for worldvolume bosons and fermions equal. The bosonic degree of freedom for a d-dimensional extended object in D-dimensional target space is $D - d$. The fermionic degrees of freedom are found by the number of sets of supersymmetry charges $\mathcal{N}$ multiplying the minimum number of spinor components in D-dimensions n_{min}. This number is halved by the requirement of kappa symmetry and is further halved on-shell. Thus one requires

$$D - d = n_{\text{min}}\mathcal{N}/4 \tag{4.8}$$

for a super-p-brane to exist in D dimensions, where $d = p + 1$. This requirement is equivalent to requiring the existence of a closed (d+1)-form. Similar to the integral over the two-form $\int_{\partial\Sigma} \Omega_2$ and its closed field strength $d\Omega_3 = 0$ in the superstring case, we consider a closed Lorentz invariant (d+1)-form in the generalization to higher dimensional extended objects in the flat background. This takes the general form

$$F_{[d+1]} = \zeta\ \Pi^{a_1}...\Pi^{a_p} d\bar{\theta}\Gamma_{a_1...a_p} d\theta \tag{4.9}$$

where $\Pi^a = \delta^a_m(dX^m - i\bar{\theta}\Gamma^m d\theta)$ and ζ is a constant. The closure of this field strength is equivalent to the requirement

$$(d\bar{\theta}\Gamma^{a_1} d\theta)(d\bar{\theta}\Gamma_{a_1...a_p} d\theta) = 0. \tag{4.10}$$

This can be shown using identities of gamma matrices to be equivalent to the requirement $D - d = n_{\text{min}}\mathcal{N}/4$. Thus, the closure of this (d+1)-form is equivalent to the requirement of κ-symmetry. When the condition by equation (4.8) is satisfied, a super-p-brane action can be found with the corresponding L_1 term being a direct generalization of Polyakov type action and the L_2 Wess-Zumino term being the integral of the (d+1)-form given by equation (4.9). The action obtained satisfies a κ-symmetry in the form of a generalization of equations (4.4) and (4.7). The generalization to M2-brane action in a flat background will be shown in the following section. Following a similar procedure, one can also obtain the action of higher dimensional extended branes with flat background.

4.2 M2-brane in flat background

4.2.1 Construction of the action

We perform the generalization of the Green-Schwarz action to the M2-brane action in a flat D=11 background with $\mathcal{N} = 1$. For D=11, $n_{\text{min}}\mathcal{N} = 32$. A 3-dimensional supersymmetric extended brane can exist, which is the M2-brane. While the kinetic term in the action generalizes directly as a supersymmetrized version of Polyakov action, the procedure of finding the corresponding Wess-Zumino term is described in [30]. The kinetic term in this action is a supersymmetrized version of Polyakov type action for a 2-brane and is given as

$$\tilde{L}_1 = -\frac{1}{2}\sqrt{-h}\left(h^{ij}\Pi_i^m\Pi_{jm} - 1\right). \tag{4.11}$$

The Wess-Zumino term is found by constructing the four-form field strength $F = dA$ corresponding to the three-form A that couples to the membrane worldvolume:$F = F_{MNPQ}dZ^M \wedge dZ^N \wedge dZ^P \wedge dZ^Q$. The coefficients F_{MNPQ} need to be the components of a Lorentz covariant quantity for the four-form to be Lorentz invariant. Another requirement is that F is closed. Considering these conditions, the only form F can take is

$$F = i\,\zeta\Pi^m\Pi^n\; d\bar{\theta}\;\Gamma_{mn}\; d\theta. \tag{4.12}$$

This satisfies $dF = 0$ by the antisymmetric properties of the wedged product and the symmetric indices in second-order differentiation on θ. Its non-vanishing components are $F_{\mu\nu\alpha\beta} = 2i\zeta(\Gamma_{\mu\nu})_{\alpha\beta}$. We can also easily find the expression of the three-form field by expanding out Π^m:

$$\begin{aligned} &i\int_\Sigma \zeta\left(\Pi_A^m\Pi_B^n + 2i\bar{\theta}\Gamma^m\partial_A\theta\Pi_B^n - (\bar{\theta}\Gamma^m\partial_A\theta)(\bar{\theta}\Gamma^n\partial_B\theta)\right)\partial_C\bar{\theta}\Gamma_{mn}\partial_D\theta\;\; dZ^A\wedge dZ^B\wedge dZ^C\wedge dZ^D \\ &\quad = i\int_{\partial\Sigma} d^3\xi\; 3!\zeta\;\epsilon^{ijk}\bar{\theta}\Gamma_{mn}\partial_k\theta\left(\Pi_i^m\Pi_j^n + i\bar{\theta}\Gamma^m\partial_i\theta\Pi_j^n - \frac{1}{3}(\bar{\theta}\Gamma^m\partial_i\theta)(\bar{\theta}\Gamma^n\partial_j\theta)\right) \end{aligned} \tag{4.13}$$

Choosing $3!\;\zeta = -1/2$, the full M2-brane action in flat background is

$$\begin{aligned} S_{M2}^{flat} = -\frac{1}{2}\int d^3\xi\,\Big[&\sqrt{-h}\left(h^{ij}\Pi_i^m\Pi_{jm} - 1\right) \\ &+ i\epsilon^{ijk}\bar{\theta}\Gamma_{mn}\partial_k\theta\left(\Pi_i^m\Pi_j^n + i\bar{\theta}\Gamma^m\partial_i\theta\Pi_j^n - \frac{1}{3}(\bar{\theta}\Gamma^m\partial_i\theta)(\bar{\theta}\Gamma^n\partial_j\theta)\right)\Big]. \end{aligned} \tag{4.14}$$

Similar to the action for superstring, this action has worldvolume reparametrization, target space Poincaré symmetry, and global supersymmetry by equation (4.3). It is invariant under a κ-symmetry with $\mathcal{N} = 1$ in a similar form as equation (4.4)

$$\delta_\kappa\theta = (1 \mp \Gamma)\kappa, \quad \delta_\kappa X^m = i\bar{\theta}\Gamma^m\delta\theta, \tag{4.15a}$$

$$\begin{aligned} \delta_\kappa(h^{ij}\sqrt{h}) =& ig^{l_1(i}\epsilon^{j)l_2l_3}\bar{\kappa}(1+\Gamma)\partial_{l_1}\theta\not{\Pi}_{l_2}\not{\Pi}_{l_3} \\ &+ \frac{2i}{3\sqrt{-h}}\epsilon^{l_1l_2(i}\epsilon^{j)l_3l_4}\bar{\kappa}\not{\Pi}^{l_5}\partial_{l_5}\theta\left(\Pi_{l_1}^M\Pi_{l_3M}\Pi_{l_2}^N\Pi_{l_4N} + \Pi_{l_1}^M\Pi_{l_3M}g_{l_2l_4} + g_{l_1l_3}g_{l_2l_4}\right), \end{aligned} \tag{4.15b}$$

where Γ is generalised to

$$\Gamma = \frac{1}{3!\sqrt{-g}}\epsilon^{ijk}\not{\Pi}_i\not{\Pi}_j\not{\Pi}_k. \tag{4.16}$$

In addition to these fermionic symmetries, the action also has worldvolume reparametrisation and target spacetime Poincaré symmetry, which has the transformations

$$\delta X^m = \eta^i\partial_i X^m + L^m_n X^n, \quad \delta\theta = \eta^i\partial_i\theta + \frac{1}{4}L_{mn}\Gamma^{mn}\theta, \tag{4.17a}$$

$$\delta(\sqrt{-h}h^{ij}) = \partial_k(\sqrt{-h}h^{ij}\eta^k) - 2\sqrt{-h}h^{k(i}\partial_k\eta^{j)}, \tag{4.17b}$$

where η^i is the parameter of general coordinate transformation on worldvolume (reparametrization), and L^m_n are the parameters of target space Poincaré transformation.

Since the Wess-Zumino term does not depend on the worldvolume metric, it does not contribute to the equation of motion of h^{ij}. This equation of motion, similar to the result by only considering the bosonic part of the action, gives the embedding equation

$$h_{ij} = \Pi^m_i\Pi^n_j\eta_{mn}. \tag{4.18}$$

The equations of motion of the bosonic coordinates X^m is

$$\partial_i\left(\sqrt{-h}h^{ij}\Pi^m_j - \frac{i}{2}\epsilon^{kij}(\bar{\theta}\Gamma_{mn}\partial_i\theta)(2\Pi^n_j + i\bar{\theta}\Gamma_n\partial_j\theta)\right) = 0, \tag{4.19}$$

and for the fermionic coordinates we have

$$(1-\Gamma)h^{ij}\Pi^m_i\Gamma_m\partial_j\theta = 0. \tag{4.20}$$

4.2.2 Static gauge

A static gauge can be chosen to fix the worldvolume reparametrization and κ-symmetry. We start with a splitting of the spacetime coordinates

$$X^m = (X^i, X^I), \quad i = 0,1,2, \quad I = 3,4,...,10, \tag{4.21}$$

and choosing the gauge

$$X^i = \xi^i. \tag{4.22}$$

This choice of gauge can be done locally if the brane is closed and can be done globally if the brane has an infinite extent. By fixing the gauge, the symmetry properties change. The remaining symmetries are those satisfying the gauge condition and leave the action invariant. While the reparametrization or the Poincaré transformation alone violates the gauge condition, we may find a combination of them that gives the residual bosonic symmetry. After fixing the gauge, the bosonic transformation, as given in equations (4.17a), becomes

$$\delta X^i = \eta^i + L^i_j \xi^j + L^i_I X^I, \quad \delta X^I = \eta^i \partial_i X^I + L^I_i \xi^i + L^I_J X^J. \tag{4.23}$$

For the transformation to satisfy the gauge condition, we require $\delta X^i = 0$. This leads to an expression of η, which can be substituted into δX^I transformation to give a symmetry transformation that satisfies the gauge condition

$$\delta X^I = -(L^i_j \xi^j + L^i_I X^I)\partial_i X^I + L^I_J X^J + (L^I_i \xi^i). \tag{4.24}$$

The first term in this transformation can be seen as a worldvolume Lorentz transformation $\xi^i \to \xi^i + l^i_j \xi^j$ on $X^I(\xi)$, and is identified with the $SO(1,2)$ subgroup of the spacetime Poincaré symmetry $SO(1,10)$. The third term is identified with the $SO(8)$ rotations in the transverse directions. The remaining terms give the transformations corresponding to $SO(1,10)/SO(1,2) \times SO(8)$. Considering only the $SO(1,2)$ and $SO(8)$ transformations, the transformation of θ is written as

$$\delta\theta = -L^i_j \xi^j \partial_i \theta + \frac{1}{4} L_{ij} \Gamma^{ij} \theta + \frac{1}{4} L_{IJ} \Gamma^{IJ} \theta, \tag{4.25}$$

where the second term is a transformation of worldvolume spinors and the third term is an $SO(8)$ rotation of spacetime spinors.

To fix the κ-symmetry, we make the choice of projecting the spinors to a $SO(8)$ chirality by satisfying

$$(1 + \Gamma^*)\theta = 0, \tag{4.26}$$

where

$$\Gamma^* = \Gamma^1 \Gamma^2 ... \Gamma^8. \tag{4.27}$$

Another convenient choice could be setting $(1 + \Gamma)\theta = 0$, which corresponds to making a κ-symmetry transformation by $\kappa = -\theta$.

The bosonic part of the action under static gauge becomes:

$$S^{static\ gauge}_{bosonic} = \int d^3\xi \sqrt{-\text{det}(\eta_{ij} + \partial_i X^I \partial_j X^I)}. \tag{4.28}$$

The gauge fixed action has not the rigid supersymmetry but also compensating κ-symmetry transformations. The static gauge fixed theory has $\mathcal{N} = 8$ supersymmetry [31, 32].

4.2.3 Semiclassical quantisation

Since the membrane theory is non-linear, we approach the quantization by a semiclassical method. This is done by choosing a stable classical solution under a certain gauge choice and quantizing small fluctuations around it. In this subsection, we discuss an example of semiclassical quantization of the supermembrane in a spacetime with topology $R^9 \times S^1 \times S^1$.

Similar to superstrings, we can impose an analog of lightcone gauge on the supermembrane by proposing the following conditions

$$X^+ = p^+\tau, \quad h_{0v} = 0, \quad h_{00} = -l, \quad \Gamma^+\theta = 0, \tag{4.29}$$

where $v = 1, 2$,

$$l = \det(l_{vw}) = \det(\partial_v X^m \partial_w X_m) \tag{4.30}$$

and

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{10}), \quad \Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^{10}). \tag{4.31}$$

By fixing the coordinate $X^+ = p^+\tau$, X^+ is no longer counted as an independent degree of freedom. Applying the embedding equation given in (4.18) with the gauge choices of h_{ij}, X^- can be expressed in terms of the X^I coordinates where we have defined $I = 1, ..., 9$. Considering only the bosonic sector, this gives the following equations

$$\dot{X}^- = \frac{1}{2p^+}(\dot{X}^I\dot{X}^I + l), \quad \partial_v X^- = \frac{1}{p^+}(\partial_v X^I)\dot{X}_I. \tag{4.32}$$

The curl of the second equation in (4.32) gives an important constraint on X^1 and X^2:

$$\partial_v X^I \partial_w \dot{X}_I - \partial_w X^I \partial_v \dot{X}_I = 0. \tag{4.33}$$

Under this gauge choice, the equations of motion (4.18), (4.19) and (4.20) become

$$l_{vw} = \partial_w X^I \partial_v X_I, \tag{4.34a}$$

$$-\ddot{X}^I + \partial_v(l\, l^{vw}\partial_w X^I) - \sqrt{2}ip^+\epsilon^{vw}(\partial_v\bar{S}\gamma^I\partial_w S) = 0, \tag{4.34b}$$

$$\dot{S} + \epsilon^{vw}\partial_v X^I \gamma_I \partial_w S = 0, \tag{4.34c}$$

where S is defined by the solution $\theta = (0, S)$ to $\Gamma^+\theta = 0$, and γ is the corresponding gamma matrices.

We look to find specific classical solutions by choosing a background that satisfies D=11 supergravity before solving the equations of motion of the membrane. The fermionic equations can be solved by $\theta =$ const. [32]. Thus, we focus on solving the bosonic equations. One classical solution under a light cone gauge is the static toroidal membrane. This is given by the ansatz

$$\theta = 0, \quad X^1 = L_1 R_1 \sigma, \quad X^2 = L_2 R_2 \rho, \quad \partial_\rho X^{\tilde{I}} = \partial_\sigma X^{\tilde{I}} = 0, \tag{4.35}$$

where $\tilde{I} = 3, ..., 9$. This is a solution by two consecutive double dimension reductions reducing the membrane action to a point particle action, which admits the solution

$$X^{\tilde{I}} = p^{\tilde{I}}\tau + a^{\tilde{I}}, \tag{4.36}$$

together with $p^2 = m^2$ and conditions in (4.35). From the above ansatz, we have

$$l_{vw} = \mathrm{diag}((L_1R_1)^2, (L_2R_2)^2), \quad \text{and} \quad l = (L_1L_2R_1R_2)^2. \tag{4.37}$$

Noticing the light cone gauge condition $X^+ = p^+\tau$, and the equations (4.32) satisfied by X^-, we find

$$X^- = \frac{1}{2p^+}(L_1L_2R_1R_2)^2\tau. \tag{4.38}$$

With a classical solution found, we look at fluctuations around this solution by proposing

$$X^1 = L_1R_1\sigma + Y^1, \quad X^2 = L_2R_2\rho + Y^2, \quad X^{\tilde{I}} = Y^{\tilde{I}}, \quad \theta = \theta_Y = (0, S_Y), \tag{4.39}$$

where Y and θ_Y are small fluctuations. We then have

$$\ddot{Y}^I = (L_1L_2R_1R_2)^{-2}\left[(L_1R_1)^{-2}\partial_\sigma\partial_\sigma Y^I + (L_2R_2)^2\partial_\rho\partial_\rho Y^I\right] \tag{4.40}$$

for the bosonic equation of motion (4.34b). It is useful to make the redefinition here for $\theta_Y = (0, S_Y)$ as

$$\theta_Y = (16\sqrt{2}p^+)^{-1/2}\left(\chi, -i\chi^*, -\sqrt{2}p^+\chi, -i\sqrt{2}p^+\chi^*\right)^T, \tag{4.41}$$

such that the fermionic equation (4.34c) can be written as

$$\dot{\chi} = w_2R_2\partial_\sigma\chi^* - iw_1R_1\partial_\rho\chi^*. \tag{4.42}$$

Detailed procedure for obtaining this can be found in [32] and [33]. With the ansatz chosen, the constraint given in equation (4.33) reduces to

$$L_1R_1\partial_\rho\dot{Y}^1 = L_2R_2\partial_\sigma\dot{Y}^2, \tag{4.43}$$

which can be integrated to $L_1R_1\partial_\rho Y^1 = L_2R_2\partial_\sigma Y^2 + k(\sigma,\rho)$, showing the residual gauge symmetry. We fix this residual symmetry by introducing another constraint

$$L_1R_1\partial_\rho Y^1 = L_2R_2\partial_\sigma Y^2. \tag{4.44}$$

The general solutions to equations (4.40) and (4.42) are

$$Y^I = y_0^I + p^I\tau + \frac{1}{\sqrt{2}}\sum_{m^2+n^2\neq 0}\frac{1}{\omega_{mn}}e^{i(m\sigma+n\rho)}\left[\alpha_{mn}^{I\,\dagger}e^{i\omega_{mn}\tau} + \alpha_{-m-n}^{I}e^{-i\omega_{mn}\tau}\right], \tag{4.45a}$$

$$\chi = \sqrt{2}S_{00} + \sum_{m^2+n^2\neq 0}e^{i(m\sigma+n\rho)}\left[\frac{m-in}{\omega_{mn}}S_{mn}^\dagger e^{i\omega_{mn}\tau} + S_{-m-n}e^{-i\omega_{mn}\tau}\right], \tag{4.45b}$$

where $\omega_{mn} = \sqrt{(L_1R_1n)^2 + (L_2R_2m)^2}$. In proceeding to the quantisation, it is important to notice the constrained variables Y^1 and Y^2 and the unconstrained $Y^{\tilde{I}}$. The constrained variables are quantized using Dirac's procedure of quantizing constrained variables [34] noticing (4.43) and (4.44) . The following canonical quantizations are proposed

$$[\dot{Y}^{\tilde{I}}, Y^{\tilde{J}}] = -(2\pi)^2 i\delta^{\tilde{I}\tilde{J}}\delta(\sigma-\sigma')\delta(\rho-\rho'), \tag{4.46a}$$

$$[\dot{Y}^{1}, Y^{1}] = -(2\pi)^2 i\left(1-\frac{1}{(L_1R_1)^2}\frac{\partial_\sigma^2}{\nabla^2}\right)\delta(\sigma-\sigma')\delta(\rho-\rho'), \tag{4.46b}$$

$$[\dot{Y}^{2}, Y^{2}] = -(2\pi)^2 i\left(1-\frac{1}{(L_2R_2)^2}\frac{\partial_\rho^2}{\nabla^2}\right)\delta(\sigma-\sigma')\delta(\rho-\rho'), \tag{4.46c}$$

$$\{\chi^{*A}, \chi^{B}\} = 2(2\pi)^2\delta^{AB}\delta(\sigma-\sigma')\delta(\rho-\rho'). \tag{4.46d}$$

By substituting the solutions by equation (4.45) into these commutator relations, we find the commutator relations between α_{mn} and S_{mn}. These are found to be

$$[\alpha^1_{mn}, \alpha^{1\dagger}_{m'n'}] = \frac{(L_2R_2m)^2}{\omega_{mn}}\delta_{mm'}\delta_{nn'}, \quad [\alpha^2_{mn}, \alpha^{2\dagger}_{m'n'}] = \frac{(L_1R_1n)^2}{\omega_{mn}}\delta_{mm'}\delta_{nn'} \tag{4.47a}$$

$$[\alpha^{\tilde{I}}_{mn}, \alpha^{\tilde{J}\dagger}_{m'n'}] = \omega_{mn}\delta^{\tilde{I}\tilde{J}}\delta_{mm'}\delta_{nn'}, \quad \{S^A_{mn}, S^{B\dagger}_{m'n'}\} = \delta^{AB}\delta_{mm'}\delta_{nn'} \tag{4.47b}$$

$$[p^I, y^J_0] = -i\delta^{IJ}, \quad \{S^A_{00}, S^{B\dagger}_{00}\} = \delta^{AB}. \tag{4.47c}$$

With the commutation relations found, the next step is to look at the mass formula. This can be found if p^- is known. $P^- = \dot{X}^-$ is found by applying the ansatz in equation (4.35) and (4.41) to the light cone gauge constraint on h_{ij} in equation (4.29). This gives an expression of $\dot{X}^-$ in terms of Y^I and χ similar to equation (4.32) but now with fermionic terms. Substituting in the expressions for these variables as given in equation (4.45), the result is the following expression for P^-:

$$P^- = \frac{1}{2p^+}\left\{l + p^Ip_I + \sum_{m^2+n^2\neq 0}\left[(\alpha_{mn}\alpha^\dagger_{mn} + \alpha^\dagger_{mn}\alpha_{mn}) + \omega_{mn}(-S^A_{mn}S^{A\dagger}_{mn} + S^{A\dagger}_{mn}S^A_{mn})\right]\right\}. \tag{4.48}$$

From the light cone gauge constraint (4.29) and the solutions (4.45), $P^+ = \dot{X}^+ = p^+$ and $P^I = \dot{X}^I = \dot{Y}^I = p^I$. This gives the mass formula

$$m^2 = 2P^+P^- - P^IP_I = l + H, \quad H = 2\sum_{m^2+n^2\neq 0} :(\alpha^\dagger_{mn}\alpha_{mn} + \omega_{mn}S^{A\dagger}_{mn}S^A_{mn}): \tag{4.49}$$

For strings, the spectrum of states is determined by the constraints from relating the left and right hamiltonians. Applying similar argument for membrane, these constraint becomes [12]

$$L_1R_1p^1 + N^1_b + N^1_f = 0, \quad L_2R_2p^2 + N^2_b + N^2_f = 0, \tag{4.50}$$

where the number operators N_b and N_f are

$$N^k_b = \sum_{m^2+n^2\neq 0}\frac{m^k}{\omega_{mn}}\alpha^\dagger_{mn}\alpha_{mn}, \quad N^k_f = \sum_{m^2+n^2\neq 0} m^kS^{A\dagger}_{mn}S^A_{mn}, \tag{4.51}$$

where we have defined for simplcity that $m^k = (m, n)$ and $k = 1, 2$. From equation (4.47), we cam compute the commutation of the number operators N and the H operator with α_{mn} and S_{mn}. The results are

$$[N_b^k, \alpha_{mn}^\dagger] = m^k \alpha_{mn}^\dagger, \quad [N_b^k, \alpha_{mn}] = -m^k \alpha_{mn}, \tag{4.52a}$$
$$[N_b^k, S_{mn}^{A\dagger}] = m^k S_{mn}^{A\dagger}, \quad [N_b^k, S_{mn}^A] = -m^k S_{mn}^A, \tag{4.52b}$$
$$[H, \alpha_{mn}^\dagger] = 2\omega_{mn} \alpha_{mn}^\dagger, \quad [H, \alpha_{mn}] = -2\omega_{mn} \alpha_{mn}, \tag{4.52c}$$
$$[H, S_{mn}^{A\dagger}] = 2\omega_{mn} S_{mn}^{A\dagger}, \quad [H, S_{mn}^A] = -2\omega_{mn} S_{mn}^A \tag{4.52d}$$

By defining the vacuum to satisfy [35]

$$\alpha_{mn} |\text{vac}\rangle, \quad S_{mn}^A |\text{vac}\rangle, \quad m^2 + n^2 \neq 0, \tag{4.53}$$

we can construct particle states by acting creation operator on the vacuum

$$\alpha_{m_1 n_1}^\dagger ... \alpha_{m_p n_p}^\dagger S_{m_{p+1} n_{p+1}}^\dagger ... S_{m_{p+q} n_{p+q}}^\dagger |\text{vac}\rangle . \tag{4.54}$$

Applying equations (4.52), (4.53) and (4.49), we have the expression for the state mass

$$m^2 = (L_1 L_2 R_1 R_2)^2 + 2\omega_{m_1 n_1} + ... + 2\omega_{m_p n_p} + 2\omega_{m_{p+1} n_{p+1}} + ... + 2\omega_{m_{p+q} n_{p+q}}. \tag{4.55}$$

The constraints in equation (4.50) determines the allowed combinations of the creation operators

$$L_1 R_1 p^1 + m_1 + ... + m_{p+q} = 0, \quad L_2 R_2 p^2 + n_1 + ... + n_{p+q} = 0. \tag{4.56}$$

From the mass term (4.55), the vacuum state may have a non-vanishing mass. By taking the limit that R_2 tends stepwise to zero while keeping the product $L_2 R_2 = 1$, the dimension ρ is shrunk, and the membrane is rolled up to give a closed string mass formula and constraints.

4.3 M2-brane in curved background

In addition to the generalization to higher dimensional extended objects, the Green-Schwarz superstring action can also be generalized to curved backgrounds. The generalization to superstrings in a curved background is discussed by [36, 37]. Having looked at the M2-brane in a flat background via direct generalization of the Green-Schwarz action, we would like to see how this generalization applies to find the M2-brane action in a curved background. This is first described in [12].

Following similar arguments as before, we expect the kinetic term to be a direct generalization of the Polyakov action. The Wess-Zumino term in the action is expected to be the integral of a closed form as a generalization of equation (4.9) in curved superspace background. We start by proposing an action of the form

$$S_{M2}^{curved} = -\frac{1}{2} \int d^3\xi \left[\sqrt{-h} \left(h^{ij} \Pi_i^a \Pi_j^b \eta_{ab} - 1 \right) + \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C A_{CBA} \right]. \tag{4.57}$$

This action is required to satisfy a κ-symmetry given in similar form as equation (4.15a). Since we are now considering curved superspace, it is useful to express these transformations in tangent space coordinates

$$\delta\Pi^a = \Pi^a_M \delta Z^M = 0, \quad \delta\Pi^\alpha = (1+\Gamma)^\alpha_\beta \kappa^\beta. \tag{4.58}$$

We perform the variation of the action by this κ-symmetry keeping in mind $\delta\Pi^a = 0$ to give

$$\begin{aligned}\delta S = \frac{1}{4}\int d^3\xi \Big[&-\sqrt{-h}\,\delta h^{ij}\left(h_{ij}h^{kl}\Pi^a_k\Pi^b_l\eta_{ab} - h_{ij} - 2\Pi^a_i\Pi^b_j\eta_{ab}\right) \\ &+4\sqrt{-h}h^{ij}(-\delta\Pi^\beta\Pi^A_i T^a_{A\beta})\Pi_{aj} + 4\epsilon^{ijk}\Pi^A_i\Pi^B_j\Pi^C_k\delta\Pi^\alpha F_{\alpha CBA}\Big],\end{aligned} \tag{4.59}$$

where $T^a_{A\beta}$ is the torsion tensor. To obtain this expression, we have used the property

$$\begin{aligned}\delta\Pi^A_i &= \partial_i(\delta z^M)\Pi^A_M + \partial_i z^M \delta\Pi^A_M, \\ &= \partial_i(\delta z^M \Pi^A_M) + 2\delta z^M \partial_i z^N \partial_{[M}\Pi^A_{N]}, \\ &= D_i(\delta z^M \Pi^A_M) - \delta z^M \partial_i z^N T_{NM}{}^A - \Pi^B_i \delta z^M \Omega_{MB}{}^A,\end{aligned} \tag{4.60}$$

where we used the Cartan's equation $T^A = d\Pi^A + \Pi^B\Omega_B{}^A$ from the second line to the third line. The field strength $F_{\alpha CBA}$ is defined as $4\partial_{[\alpha}A_{CBA]}$. By requiring $\delta S = 0$ under the proposed κ-symmetry transformation, we find constraints on the supertorsion and the four-form field strength order by order in terms of Π^α. The terms to the second and third order in Π^α only occur from the Wess-Zumino term; thus, this requires

$$F_{\alpha\beta\gamma\delta} = 0, \quad F_{\alpha\beta\gamma d} = 0. \tag{4.61}$$

The terms to the first order in Π^α come from the terms in the second line in equation (4.59). However, the order of Π^M_i does not equal on both sides. One useful identity in this case is $\delta\Pi^\alpha = \Gamma^\alpha_\beta\Pi^\beta + (1-\Gamma^2)^\alpha_\beta\kappa^\beta$. Substituting this expression into the term from the variation of kinetic term, we obtain

$$-\frac{1}{3!}h^{i'j'}\epsilon^{ijk}\Pi^a_i\Pi^b_j\Pi^c_k\delta\Pi^\alpha\Pi^{\alpha'}_{i'}T^{a'}_{\alpha'\beta'}(\Gamma_{abc})^{\beta'}_\alpha\Pi_{a'j'} + \epsilon^{ijk}\Pi^a_i\Pi^b_j\Pi^\delta_k\delta\Pi^\alpha F_{\alpha\delta ba} + \text{residual terms} \tag{4.62}$$

If we set

$$T^a_{\beta\gamma} = -2(\Gamma^a)_{\beta\gamma}, \tag{4.63}$$

the first order terms become $F_{\alpha\beta ab} + \frac{1}{2}(\Gamma_{ab})_{\alpha\beta}$ + residual terms, applying $\Gamma^a\Gamma_{bcd} = 3\delta^a_{[b}\Gamma_{cd]} + ...$ The residual terms have the same factor of $\sqrt{-h}h^{ij}$ in front and can be thought of as modifications to δh^{ij} variations. Then, the cancellation of the term F from the variation of the Wess-Zumino term requires

$$F_{\alpha\beta ab} = \frac{1}{3}(\Gamma_{ab})_{\alpha\beta} \tag{4.64}$$

Following a similar argument, the cancellation between the zeroth order term gives the relation $F_{\alpha abc} = -\frac{1}{3!}(\Gamma_{abc})^\beta_\alpha T^d_{d\beta}$. We choose

$$\eta_{c(a}T^c{}_{b)\beta} = \eta_{ab}\Psi_\alpha, \quad F_{\alpha abc} = -\frac{1}{2}\Psi_\beta(\Gamma_{abc})^\beta_\alpha, \tag{4.65}$$

where Ψ_α is an arbitrary spinor field.The remaining terms contributes to the variation of δh_{ij}, and this is found to be [32]

$$\delta(\sqrt{-h}h^{ij}) = -2i(1+\Gamma)^{\alpha}{}_{\beta}\kappa^{\beta}(\Gamma_{ab})_{\alpha\gamma}\Pi^{\gamma}_{i'}h^{i'(i}\epsilon^{j)kl}\Pi^{a}_{k}\Pi^{b}_{l}$$
$$-\frac{2i}{3\sqrt{-g}}\kappa^{\alpha}(\Gamma_{c})_{\alpha\beta}\Pi^{\beta}_{\kappa}\Pi^{c}_{l}h^{kl}\epsilon^{i'j'(i}\epsilon^{j)k'l'} \tag{4.66}$$
$$\times(\Pi^{a}_{i'}\Pi_{k'a}\Pi^{b}_{j'}\Pi_{l'b}+\Pi^{a}_{i'}\Pi_{k'a}h_{j'l'}+h_{i'k'}h_{j'l'}).$$

The constraints (4.61), (4.63), (4.64) and (4.65) from satisfying the κ-symmtry are equivalent to some of the constraints of D=11 supergravity in superspace formulation [18]. This means the superfields satisfying constraints for superspace formulation of D=11 supergravity also satisfy the κ-symmetry constraints here. The additional freedom allowed by the κ-symmetry constraints corresponds to the freedom allowed in the choice of the constraints of D=11 supergravity. When $\Psi_{\alpha}=0$ and $T^{\alpha}_{\beta\gamma}=T^{a}_{bc}=0$, the κ-symmetry constraints coincide with the D=11 supergravity constraints. The field strength tensor F has non-zero components F_{abcd} and $F_{\alpha\beta ab}$. These constraints allow the construction of the closed form F

$$F=\frac{1}{4!}\Pi^{A}\Pi^{B}\Pi^{C}\Pi^{D}F_{ABCD}=\frac{1}{12}\Pi^{\alpha}\Pi^{\beta}\Pi^{a}\Pi^{b}(\Gamma_{ab})_{\alpha\beta}+\frac{1}{4!}\Pi^{a}\Pi^{b}\Pi^{c}\Pi^{d}F_{abcd}, \tag{4.67}$$

which determines the Wess-Zumino term of the action. The superspace vielbeins solved from these constraints are equivalent to those solved from the formulation of D=11 supergravity in superspace. Upon using the results from superspace formulation of supergravity, the expressions for the supervielbeins up to first order in θ are given in equations (2.41) and (2.42). This gives the pullback on the world volume

$$\Pi^{a}_{i}=\partial_{i}X^{a}+\frac{1}{2}\bar{\theta}\Gamma^{a}(\psi_{m}\partial_{i}X^{m}+\frac{1}{2}\partial_{i}\theta), \tag{4.68a}$$
$$\Pi^{\alpha}_{i}=\partial_{i}\theta^{\alpha}+\partial_{i}X^{m}\left(\psi^{\alpha}_{m}+\frac{1}{4}\omega^{ab}_{m}\left(\Gamma_{ab}\theta\right)^{\alpha}+\frac{1}{288}F_{npqr}\left[\left(\Gamma^{npqr}_{m}-8\delta^{n}_{m}\Gamma^{pqr}\right)\theta\right]^{\alpha}\right). \tag{4.68b}$$

We have thus found D=11 supergravity as the allowed curved background of M2-brane action. To summarise, the action of M2-brane in curved superspace has the form (4.57), where the Wess-Zumino term is constructed using (4.67) and the supervielbeins are given by (4.68).

4.4 M2-brane in $\mathrm{AdS}_4\times\mathrm{S}^7$ background

In the last section, it is shown that the background of the M2-brane satisfies the superspace torsion and curvature constraints of D=11 supergravity. Considering only the bosonic sector of supergravity, the M2-brane solution interpolates between a flat and an $AdS_4\times S^7$ background. In this section, we would like to derive the M2-brane action in an $AdS_4\times S^7$ background by finding the corresponding solution to the superspace constraints.

The near horizon superspace is considered in [38], where the supergravity constraints on superspace are solved to all orders of θ-coordinate. The derivation of the geometric superfields is considered in [39] for $AdS_4\times S^7$ and $AdS_7\times S^7$ backgrounds using coset representations. These can also be derived using the constraints on the supergravity torsion and curvature [40], obtaining the same results. We will focus on the latter method in this section.

Considering only the bosonic sector, the on-shell constraints on torsion for D=11 supergravity are [18]

$$\mathcal{T}^a = -\Pi^\alpha \Gamma^a_{\alpha\beta}\Pi^\beta, \quad \mathcal{T}^\alpha = \Pi^a F_{bcde}(T_a^{bcde})_{\alpha\beta}\Pi^\beta, \tag{4.69}$$

where

$$T^{abdce} = \frac{1}{12^2}(\Gamma^{abcde} - 8\Gamma^{[bcd}\eta^{e]a}). \tag{4.70}$$

To find the constraints on the curvature two-form, we first look at the spin connections in the superspace. With the fermionic part set to zero, the superspace spin connection one-forms are given by

$$\Omega^{ab}_m = \omega^{ab}_m, \quad \Omega^{ab}_\alpha = \frac{1}{2}(\bar{\theta}S^{abcdef})_\alpha F_{cdef}, \tag{4.71}$$

where

$$S_{ab}{}^{cdef} = \frac{1}{72}(\Gamma_{ab}{}^{cdef} + 24\delta_a^{[c}\delta_b^d\Gamma^{ef]}). \tag{4.72}$$

The constraints on the curvature are found by applying Cartan's equations using these constraints on the spin connections. The result is a constraint in the form

$$\mathcal{R}^{ab} = \frac{1}{2}\Pi^c\Pi^d R_{cd}{}^{ab} + \frac{1}{2}\Pi^\alpha(S^{abcdef}F_{cdef})_{\alpha\beta}\Pi^\beta. \tag{4.73}$$

From the constraints by equations (4.67), (4.69) and (4.73), we look to solve for the expression for the vielbeins and the spin connection in $AdS_4 \times S^7$ background. A useful trick to find superspace vielbeins and spin connections is to rescale the θ coordinates by a parameter t, which is set to 1 at the end of the calculation. Considering the redefinition of the coordinates

$$X^m \to X^m, \quad \theta^\alpha \to t\theta^\alpha, \tag{4.74}$$

we compute the derivative of the vielbeins and the spin connections with respect to t:

$$\frac{d}{dt}\Pi^a = d(\theta^\mu\Pi^a_\mu) + \theta^\nu\Pi^B_\nu T^a{}_{BC}\Pi^C - \theta^\nu\Pi^B_\nu\Omega_B{}^a + \theta^\nu\Pi^B\Omega_{\nu B}{}^a, \tag{4.75a}$$

$$\frac{d}{dt}\Pi^\alpha = d(\Pi^\alpha_\mu\theta^\mu) + \theta^\nu\Pi^A_\nu T^\alpha{}_{AB}\Pi^B + \frac{1}{4}(\Gamma_{ab}\Omega^{ab})^\alpha_\beta\Pi^\beta_\nu\theta^\nu - \frac{1}{4}(\Gamma_{ab}\Omega_\nu{}^{ab})^\alpha_\beta\Pi^\beta_\nu\theta^\nu \tag{4.75b}$$

$$\frac{d}{dt}\Omega^{ab} = d(\Omega_\mu{}^{ab}\theta^\mu) - \theta^\nu\Pi^A_\nu\Pi^B R^{ab}{}_{AB} - \Omega^a{}_c\theta^\nu\Omega_\nu{}^{cb} + \theta^\nu\Omega_\nu{}^a{}_c\Omega^{cb}. \tag{4.75c}$$

By making the assumption $\theta^\mu E^a_\mu = \theta^\mu\Omega^{ab}_\mu = 0$ and substituting in equations (4.69) and (4.73), we obtain a set of equations to be solved:

$$\frac{d}{dt}\Pi^a = 2\bar{\Theta}^\alpha(\Gamma^a)_{\alpha\beta}\Pi^\beta, \quad \frac{d}{dt}\Pi^\alpha = (d - \frac{1}{4}\Omega^{ab}\Gamma_{ab} + \Pi^a T_a^{bcde}F_{bcde})^\alpha_\beta\Theta^\beta, \tag{4.76a}$$

$$\frac{d}{dt}\Omega^{ab} = -\Theta^\alpha(F_{bcde}T_a^{bcde})_{\alpha\beta}\Pi^\beta, \tag{4.76b}$$

where $\Theta^\alpha = \Pi^\alpha_\mu\theta^\mu$. These equations are the same as those derived in [39] using coset space representations. These equations are solved considering the initial conditions

$$\Pi^\alpha|_{t=0} = 0, \quad \Pi^a|_{t=0} = e^a, \quad \Omega^{ab}|_{t=0} = \omega^{ab}. \tag{4.77}$$

The solution is found to all orders of θ and is given by [38]

$$\Pi^\alpha = \sum_{n=0}^{16} \frac{1}{(2n+1)!} (\mathcal{M}^n D\Theta_f)^\alpha, \quad \Pi^a = dX^m e_m^a + 2\sum_{n=0}^{15} \frac{1}{(2n+2)!} \bar{\Theta}_f \Gamma^a \mathcal{M}^n D\Theta_f, \tag{4.78a}$$

$$\Omega^{ab} = dX^m \omega_m{}^{ab} + \sum_{n=0}^{15} \frac{1}{(2n+2)!} \bar{\Theta}_f S^{abcdef} F_{cdef} \mathcal{M}^n D\Theta_f, \tag{4.78b}$$

where

$$(\mathcal{M})_\alpha^\beta = 2\left(T_a^{bcde} F_{bcde}\Theta_f\right)_\alpha \left(\bar{\Theta}_f \Gamma^a\right)^\beta - \frac{1}{4}\left(\Gamma_{ab}\Theta_f\right)_\alpha \left(\bar{\Theta}_f S^{abcdef} F_{cdef}\right)^\beta, \tag{4.79a}$$

$$\Theta_f^\alpha = \Theta^\mu \Pi_\mu{}^\alpha|_{t=0} \equiv \Theta^\mu e_\mu{}^\alpha(X), \quad D\Theta_f^\alpha = \left(d - \frac{1}{4}\omega^{ab}\Gamma_{ab} + e^a T_a^{bcde} F_{bcde}\right)_\beta^\alpha \Theta_f^\beta. \tag{4.79b}$$

This result for superspace vielbeins can be expanded in low orders of θ to give

$$\Pi^a = e^a + \bar{\theta}\Gamma^a d\theta + \bar{\theta}\Gamma^a (e^b T_b^{cdef} F_{cdef} - \frac{1}{4}\omega^{cd}\Gamma_{cd})\theta + \mathcal{O}(\theta^4) \tag{4.80a}$$

$$\Pi^\alpha = d\theta^\alpha + \left[(e^b T_b^{cdef} F_{cdef} - \frac{1}{4}\omega^{ab}\Gamma_{ab})\theta\right]^\alpha + \mathcal{O}(\theta^3). \tag{4.80b}$$

This agrees with our results in equations (2.41) and (2.42) with the fermionic sector set to zero. This is also in agreement with the result in [21] where the superspace vielbeins are computed to second order in θ in a general curved D=11 supergravity background.

The Wess-Zumino part of the action takes the form given by equation (4.67). We apply the same procedure as we did for Π and Ω to the four-form F to find the differential equation of F with respect to t. In the process, we also apply equation (4.76) when encountering the t derivative of tangent space coordinates and superspace spin connections. Noticing the properties of gamma matrices, we obtain the following equation satisfied by F

$$\frac{d}{dt} F = -d(\Theta_f^\alpha (\Gamma_{ab})_{\alpha\beta} \Pi^\beta \Pi^a \Pi^b). \tag{4.81}$$

This can be solved by directly integrating both sides with respect to t to find the three-form field A

$$A = \frac{1}{6} A_{abc} e^a e^b e^c - \int_0^1 dt (\Theta_f^\alpha (\Gamma_{ab})_{\alpha\beta} \Pi^\beta \Pi^a \Pi^b). \tag{4.82}$$

This recovers the flat superspace result by substituting in the corresponding expressions for flat superspace vielbeins. The supermembrane action can then be written out as

$$S_{M2}^{AdS_4 \times S^7} = -\frac{1}{2} \int d^3\xi \sqrt{-h} \left(h^{ij} \Pi_i^a \Pi_j^b \eta_{ab} - 1\right) + \int_{M2} A, \tag{4.83}$$

where Π_i^a is the pullback of the vielbeins onto the worldvolume and the three-form field A is integrated over the worldvolume. The invariance under κ-symmetry of this action is ensured, as discussed in the last section.

CHAPTER 5

Dimension reduction from D=11 to D=10 and the relation of M2-brane action to superstring action

In this section, we look at the relation between M2-brane action and superstring action by double-dimension reduction. Dimension reductions are performed by expressing the higher dimensional fields into lower dimensional components and introducing new fields for the extra components. In this section, we consider the D=11 coordinates $\hat{Z}^{\hat{m}} = (\hat{X}^{\hat{m}}, \hat{\theta}^{\hat{\mu}})$ and the D=10 coordinates $Z^M = (X^m, \theta^\mu)$, where $\hat{m} = 0, ..., 10$ and $m = 0, ..., 9$. 11-dimensional variables can be expressed in terms of 10-dimensional variables with the splitting of coordinates $\hat{X}^{\hat{m}} = (\hat{X}^m, z)$. The D=11 supergravity can be reduced to D=10 type IIA supergravity. The metric in D=11 $\hat{g}_{\hat{m}\hat{n}}$ expressed in terms of the D=10 metric g_{mn} and a one-form field $\mathcal{A}_{\Updownarrow}$. The field strength $F^{(11)}_{\hat{m}\hat{n}\hat{p}\hat{q}}$ is expressed in terms of the 10-dimensional four-form with compoenents $\tilde{F}^{(10)}_{mnpq}$ and three-form with components $H^{(10)}_{mpq}$. We show the dimension reduction of the bosonic part of D=11 supergravity to the bosonic part of the D=10 type IIA supergravity in the first subsection. The D=10 type IIA supergravity action is observed to admit a fundamental string solution. This leads to the suggestion of a source term by similar arguments as what we discussed in Chapter 3. We also raise the question of whether the M2-brane action reduces to a superstring action in D=10 upon a dimensional reduction. This is shown to be true by a double dimension reduction, where the dimension to be reduced coincides with one of the spatial worldvolme coordinates of the M2-brane.

5.1 D=10 type IIA supergravity from D=11 supergravity

The 11-dimensional metric in the form of 11×11 matrix can be written into a 10×10 matrix, a vector of 10 components, and a scalar. The line element can be written as

$$d\hat{s}^2 = \left(\hat{g}_{mn} - \frac{\hat{g}_{\hat{m}z}\hat{g}_{\hat{n}z}}{\hat{g}_{zz}}\right) d\hat{X}^m d\hat{X}^n + \hat{g}_{zz}\left(\frac{\hat{g}_{\hat{m}z}}{\hat{g}_{zz}} dX^m + dz\right)^2 . \tag{5.1}$$

With this, one can perform a redefinition of variables

$$g_{mn} = \hat{g}_{mn} - \frac{\hat{g}_{\hat{m}z}\hat{g}_{\hat{n}z}}{\hat{g}_{zz}}, \quad \Phi^2 = \hat{g}_{zz}, \quad \mathcal{A}_m = \frac{\hat{g}_{\hat{m}z}}{\hat{g}_{zz}}. \tag{5.2}$$

Then, the line element can be written as

$$d\hat{s}^2 = g_{mn} dX^m dX^n + \Phi^2 \left(\mathcal{A}_m dX^m + dz\right)^2 . \tag{5.3}$$

We can also consider a possible rescaling of the metric to give

$$d\hat{s}^2 = e^{2\alpha\phi}ds^2 + e^{2\beta\phi}(dz + \mathcal{A}_m dz^m)^2, \tag{5.4}$$

where z is the dimension to be reduced. $z^{\hat{m}}$ is the 11-dimensional coordinate, while z^m is the 10-dimensional coordinate. An essential step of Kaluza-Klein reduction is the consistent truncation of the field variables, made by choosing ϕ and A_M to be independent of the reduction coordinate z. We begin with the dimensional reduction of the Einstein-Hilbert action by expressing it in terms of D=10 coordinates via the relation by the metric given above.

The first step is to compute the Ricci scalar. This can be obtained using vielbeins defined by

$$\Theta^a = e^{\alpha\phi}\theta^a, \quad \Theta^z = e^{\beta\phi}(dz + \mathcal{A}_m dX^m), \tag{5.5}$$

where Θ labels the vielbeins in $D+1$ dimensions while θ labels vieldbeins in D dimesions. The spin connections are

$$\hat{\omega}^z{}_a = e^{-\alpha\phi}\left(\beta\phi,_a \Theta^z + \frac{1}{2}e^{\beta\phi}\mathcal{F}_{ab}\theta^b\right), \tag{5.6a}$$

$$\hat{\omega}^a{}_b = \omega^a{}_b + \alpha e^{-\alpha\phi}\phi,_b \Theta^a - \alpha e^{-\alpha\phi}\phi,^a \Theta^c\eta_{bc} - \frac{1}{2}e^{(\beta-2\alpha)\phi}\mathcal{F}^a{}_b\Theta^z, \tag{5.6b}$$

where $\phi,_a = \partial_a\phi$. The curvature two-form can be found from these spin connections applying Cartan's second equation. The curvature tensor can subsequently be found by comparing it to the components of the curvature two-form. Noticing the relation $\sqrt{-\hat{g}} = e^{(D\alpha+\beta)\phi}\sqrt{-g}$, the Lagrangian takes the form $e^{((D-2)\alpha+\beta)\phi}\sqrt{-g}R + ...$ without defining the relation between α and β. To obtain a resulting Lagrangian with no ϕ dependent scaling in front of the Einstein term, we make a choice $\beta = -(D-2)\alpha$. To fix a coefficient of $-1/2$ in front of the dilaton kinetic term, we also fix $\alpha = -1/12$. The Einstein-Hilbert action is then expressed as

$$\frac{1}{2\kappa^2}\int d^{11}\hat{X}\sqrt{-\hat{g}}\hat{R}(\hat{g}) = \frac{1}{2\kappa'^2}\int d^{10}X\sqrt{-g}\left\{R - \frac{1}{4}e^{\frac{3}{2}\phi}F_{ab}F^{ab} - \frac{1}{2}\phi,_a \phi,^a\right\}, \tag{5.7}$$

where $2\pi\mathcal{R}\kappa'^2 = \kappa^2$.

For the kinetic term for the 3-form gauge field A_{MNP}, the field strength $F^{(11)}_{[4]}$ can be expressed in terms of D=10 field strengths $F^{(10)}_{[4]}$ and $H^{(10)}_{[3]}$

$$\begin{aligned} F^{(11)}_{\hat{a}_1\hat{a}_2...\hat{a}_4}&\Theta^{\hat{a}_1}\wedge\Theta^{\hat{a}_2}\wedge ... \wedge \Theta^{\hat{a}_4} \\ &= F^{(10)}_{a_1a_2...a_4}\theta^{a_1}\wedge\theta^{a_2}\wedge...\wedge\theta^{a_4} + nH^{(10)}_{a_1a_2a_3}\theta^{a_1}\wedge\theta^{a_2}\wedge...\wedge dz, \end{aligned} \tag{5.8}$$

where

$$F^{(11)}_{a_1a_2a_3z} = e^{-\frac{5}{12}\phi}H^{(10)}_{a_1a_2a_3}, \tag{5.9a}$$

$$F^{(11)}_{a_1a_2...a_4} = e^{\frac{1}{3}\phi}\left(F^{(10)}_{a_1a_2...a_4} - 4H^{(10)}_{[a_1a_2a_3}\mathcal{A}_{a_4]}\right) = e^{\frac{1}{3}\phi}\tilde{F}^{(10)}_{a_1a_2...a_4}. \tag{5.9b}$$

The kinetic term of gauge field written in terms of D=10 field strengths is

$$\begin{aligned}\frac{1}{2\kappa^2}\int d^{11}\hat{X}\sqrt{-\hat{g}}F^{(11)}_{abcd}F^{(11)abcd}\\ =\frac{1}{2\kappa'^2}\int d^{10}Xe^{-\frac{1}{6}\phi}\sqrt{-g}\left(e^{\frac{2}{3}\phi}\tilde{F}^{(10)}_{abcd}\tilde{F}^{(10)abcd}+4e^{-\frac{5}{6}\phi}H^{(10)}_{abc}H^{(10)abc}\right)\end{aligned} \tag{5.10}$$

The term in the action is topological, and is expressed in terms of $F^{(10)}$ as

$$\begin{aligned}\frac{1}{2\kappa^2}\int\ \epsilon^{mnpqm'n'p'q'rst}F^{(11)}_{mnpq}F^{(11)}_{m'n'p'q'}A^{(11)}_{rst}d\hat{x}^{11}\\ =\frac{1}{2\kappa'^2}\int\left(\epsilon^{mnpqm'n'p'q'rs}F^{(10)}_{mnpq}F^{(10)}_{m'n'p'q'}B^{(10)}_{rs}dx^{10}\right)\wedge dz,\end{aligned} \tag{5.11}$$

where $dB=H$. Thus the action becomes

$$\begin{aligned}S=\frac{1}{2\kappa'^2}\int d^{10}X\left\{\sqrt{-g}\left[R-\frac{1}{2}\phi_{,n}\phi_{,}{}^{n}-\frac{1}{12}e^{-\phi}H^{(10)}_{mpq}H^{(10)mpq}-\frac{1}{4}e^{\frac{3}{2}\phi}\mathcal{F}_{mn}\mathcal{F}^{mn}\right.\right.\\ \left.\left.-\frac{1}{48}e^{\frac{1}{2}\phi}\tilde{F}^{(10)}_{mnpq}\tilde{F}^{(10)mnpq}\right]+\frac{1}{12^4}\epsilon^{mnpqm'n'p'q'rs}F^{(10)}_{mnpq}F^{(10)}_{m'n'p'q'}B^{(10)}_{rs}\right\}.\end{aligned} \tag{5.12}$$

This is the type IIA bosonic action in Einstein frame. The dimensional reduction around a circle of D=11 supergravity theory gives D=10 type IIA supergravity. One can also perform a Weyl rescaling of the metric

$$g_{mn}\rightarrow e^{-\phi/2}g_{mn} \tag{5.13}$$

to obtain this action in the string frame

$$\begin{aligned}S=\frac{1}{2\kappa'^2}\int d^{10}X\left\{\sqrt{-g}\left[e^{-2\phi}\left(R+4\phi_{,n}\phi_{,}{}^{n}-\frac{1}{12}H^{(10)}_{mpq}H^{(10)mpq}\right)-\frac{1}{4}\mathcal{F}_{mn}\mathcal{F}^{mn}\right.\right.\\ \left.\left.-\frac{1}{48}\tilde{F}^{(10)}_{mnpq}\tilde{F}^{(10)mnpq}\right]+\frac{1}{12^4}\epsilon^{mnpqm'n'p'q'rs}F^{(10)}_{mnpq}F^{(10)}_{m'n'p'q'}B^{(10)}_{rs}\right\},\end{aligned} \tag{5.14}$$

where in this case $\alpha=-1/3$ and $\beta=2/3$.

Similar to D=11, this action admits brane solutions of the form given by ansatz (3.5). One of the solutions is the electric 1-brane solution coupled to the two-form field B_{mn}. This is the fundamental string solution [41] and is given by

$$ds^2=\left(1+\frac{L^6}{r^6}\right)^{-2}\eta_{\mu\nu}dx^\mu dx^\nu+\left(1+\frac{L^6}{r^6}\right)^{\frac{2}{3}}\delta_{IJ}dy^Idy^J, \tag{5.15a}$$

$$e^\phi=\left(1+\frac{L^6}{r^6}\right)^{-1},\quad B_{\mu\nu}=\sqrt{2}\epsilon_{\mu\nu}\left(1+\frac{L^6}{r^6}\right)^{-1}. \tag{5.15b}$$

Here for this equation we used the splitting of coordinate $X^m=(x^\mu,y^I)$ where $\mu=0,1$ and $y=2,...,0$. Following similar arguments we made for the M2-brane solution in D=11, we found $r=0$ is a coordinate singularity corresponding to an event horizon. Approaching this event horizon, the spacetime tends to $AdS_3\times S^7$. The spacetime approaches Minkowski at spatial infinity of the transverse coordinates. This metric is related to the metric appearing in the string sigma model by the same rescaling as in equation (5.13) from Einstein frame to string frame. By a similar argument as for the M2-brane case, the analytic continuation of the coordinates into the event horizon encounters a curvature singularity, suggesting a possible source term in the form of a string sigma model. This leads us to the question of whether the dimensional reduction of the M2-brane source term in D=11 supergravity gives string sigma source term in D=10.

5.2 From M2-brane action to superstring action

Corresponding to the suspected superstring source term in the last section, the M2-brane action under a double dimension reduction gives type IIA superstring in D=10 [6]. In this section, we show the double-dimension reduction performed on the bosonic part of the M2-brane action and consider the generalization to the reduction of the full action. The bosonic sector of the M2-brane action is

$$S = \frac{1}{2}\int \mathrm{d}^3\hat{\xi}\left(\sqrt{-\hat{h}}\hat{h}^{\hat{i}\hat{j}}\partial_{\hat{i}}\hat{X}^{\hat{m}}\partial_{\hat{j}}\hat{X}^{\hat{n}}\hat{g}_{\hat{m}\hat{n}}(\hat{X}) - \sqrt{-\hat{h}} + \frac{1}{3}\epsilon^{\hat{i}\hat{j}\hat{k}}\partial_{\hat{i}}\hat{X}^{\hat{m}}\partial_{\hat{j}}\hat{X}^{\hat{n}}\partial_{\hat{k}}\hat{X}^{\hat{p}}\hat{A}_{\hat{m}\hat{n}\hat{p}}(\hat{X})\right). \tag{5.16}$$

A double-dimension reduction can be performed to reduce the 11-dimensional action and equation of motion into 10-dimensional ones while also reducing the dimension of the membrane by one. The first step is to split the coordinates. We make the choice of a two-one split of the world volume coordinates and a ten-one split of the spacetime coordinates, which are defined as

$$\hat{\xi}^{\hat{i}} = (\xi^i, \rho), \quad i = 1, 2, \tag{5.17a}$$
$$\hat{X}^{\hat{m}} = (X^m, z), \quad m = 1, ..., 10, \tag{5.17b}$$

where both ρ and z are spatial coordinates to be compactified around a circle. We make a partial static gauge for dimension reduction by demanding $\rho = z$ and assume the metric $\hat{g}_{\hat{m}\hat{n}}$ and the three-form field $\hat{A}_{\hat{m}\hat{n}\hat{p}}$ are independent of the z-coordinate:

$$\partial_z\hat{g}_{\hat{m}\hat{n}}(X^m) = 0\ , \quad \partial_z\hat{A}_{\hat{m}\hat{n}\hat{p}}(X^m) = 0. \tag{5.18}$$

The expression of the D=11 metric in terms of D=10 coordinates in the string frame is

$$d\hat{s}^2 = \Phi^{-2/3}g_{mn}dX^m dX^n + \Phi^{4/3}\left(\mathcal{A}_m dX^m + dz\right)^2, \tag{5.19}$$

where $\Phi = e^{\phi}$. The metric can be expressed as

$$\hat{g}_{\hat{m}\hat{n}} = \Phi^{-2/3}\begin{pmatrix} g_{mn} + \Phi^2\mathcal{A}_m\mathcal{A}_n & \Phi^2\mathcal{A}_m \\ \Phi^2\mathcal{A}_n & \Phi^2 \end{pmatrix}. \tag{5.20}$$

This can adjust the values of det $\hat{g}$. By a similar method, the metric on the worldvolume in terms of worldvolume coordinates is given in the form

$$\hat{h}_{\hat{i}\hat{j}} = \varphi^{-2/3} \begin{pmatrix} h_{ij} + \varphi^2 V_i V_j & \varphi^2 V_i \\ \varphi^2 V_j & \varphi^2 \end{pmatrix}, \quad \hat{h}^{\hat{i}\hat{j}} = \varphi^{2/3} \begin{pmatrix} h^{ij} & -V_i \\ -V_j & \frac{1}{\varphi^2} + V_j V^j \end{pmatrix}. \tag{5.21}$$

This definition of the worldvolume metric gives det $\hat{h} =$ det h, and thus

$$\sqrt{-\hat{h}} = \sqrt{-h}. \tag{5.22}$$

By substituting equations (5.20), (5.22), and (5.21) into the action given in equation (5.16), one can rewrite the action in terms of 10-dimensional spacetime metric g_{mn}, 2-dimensional world volume metric h_{ij}, 10-dimensional vectors $\mathcal{A}_i$ and V_i, and a scalars ϕ and Φ. The first term in the action can be written as

$$\begin{aligned} &\sqrt{-\hat{h}}\hat{h}^{\hat{i}\hat{j}}\partial_{\hat{i}}\hat{X}^{\hat{m}}\partial_{\hat{j}}\hat{X}^{\hat{n}}\hat{g}_{\hat{m}\hat{n}} \\ &= \varphi^{2/3}\Phi^{-2/3}\sqrt{-h}\left[h^{ij}g_{ij} + h^{ij}(\mathcal{A}_i - V_i)(\mathcal{A}_j - V_j)\Phi^2 + \varphi^{-2}\Phi^2\right], \end{aligned} \tag{5.23}$$

where $g_{ij} = \partial_i X^m \partial_j X^n g_{mn}$ is the pullback of the spacetime metric onto the world volume and $\mathcal{A}_i = \partial_i X^m \mathcal{A}_m$ is the pullback of the vector $\mathcal{A}_m$ on to the world volume. The second term is given by equation (5.22), and the third term in the action is

$$\frac{1}{3}\epsilon^{\hat{i}\hat{j}\hat{k}}\partial_{\hat{i}}\hat{X}^{\hat{m}}\partial_{\hat{j}}\hat{X}^{\hat{n}}\partial_{\hat{k}}\hat{X}^{\hat{p}}\hat{A}_{\hat{m}\hat{n}\hat{p}} = \epsilon^{ij}\partial_i X^m \partial_j X^n A_{mn}, \tag{5.24}$$

where $\epsilon^{ijk} = 0$ because $i, j, k = 1, 2$. Finally, the action reduces to

$$\begin{aligned} S_{10} = \frac{1}{2}\int d^2\xi \Big(&\varphi^{2/3}\Phi^{-2/3}\sqrt{-h}\left[h^{ij}g_{ij} + h^{ij}(\mathcal{A}_i - V_i)(\mathcal{A}_j - V_j)\Phi^2 + \varphi^{-2}\Phi^2\right] \\ &-\sqrt{-h} + \epsilon^{ij}\partial_i x^m \partial_j x^n A_{mn}\Big). \end{aligned} \tag{5.25}$$

One can eliminate h^{ij}, V_i, $\mathcal{A}_i$, Φ and ϕ by substituting in the equations of motions of h^{ij}, V_i and ϕ:

$$\delta\varphi: \quad \phi^{-2}\Phi^2 = \frac{1}{2}h^{ij}g_{ij}, \tag{5.26a}$$

$$\delta V_i: \quad \mathcal{A}_i = V_i \tag{5.26b}$$

$$\begin{aligned} \delta h^{ij}: \quad &-\frac{1}{2}\sqrt{-h}h_{ij}\left[h^{kl}g_{kl} + h^{kl}\left(\mathcal{A}_k - V_k\right)\left(\mathcal{A}_l - V_l\right)\Phi^2 + \varphi^{-2}\Phi^2\right] \\ &+\sqrt{-h}\left[g_{ij} + \left(\mathcal{A}_i - V_i\right)\left(\mathcal{A}_j - V_j\right)\Phi^2\right] + \frac{1}{2}\varphi^{-2/3}\Phi^{2/3}\sqrt{-h}h_{ij} = 0 \end{aligned} \tag{5.26c}$$

Substitute equations of motion (5.26a) into equation (5.26c) and contract the equation by h^{ij}, one obtain

$$h^{ij}g_{ij} + \varphi^{-2}\Phi^2 = h^{ij}g_{ij} + \varphi^{-2/3}\Phi^{2/3}. \tag{5.27}$$

This implies $\varphi\Phi = 1$, and thus using equation (5.26a) one has

$$h^{ij}h_{ij} = h^{ij}g_{ij} = g^{ij}g_{ij} = 2, \quad \text{and} \quad h_{ij} = g_{ij} = \partial_i X^m \partial_j X^n g_{mn}. \tag{5.28}$$

By substituting equations (5.26a) and (5.28) into (5.25), the action reduce to

$$S_{10} = \int d^2\xi \left\{ \sqrt{-\det\left(\partial_i X^m \partial_j X^n g_{mn}\right)} + \frac{1}{2}\epsilon^{ij}\partial_i X^m \partial_j X^n A_{mn} \right\}, \tag{5.29}$$

which is a Nambu-Goto action for string coupling to a two-form field A_{mn}.

Following similar procedure, we can apply the double dimension reduction in the superspace setting. The Kaluza-klein ansatz for reduction for the supervielbein is

$$\hat{\Pi}_{\hat{M}}^{\hat{A}} = \begin{pmatrix} \hat{\Pi}_M{}^a & \hat{\Pi}_M{}^\alpha & \hat{\Pi}_M{}^{11} \\ \hat{\Pi}_z{}^a & \hat{\Pi}_z^\alpha & \hat{\Pi}_z{}^{11} \end{pmatrix} = \Phi^{-\frac{1}{3}} \begin{pmatrix} \Pi_M{}^a & \Pi_M{}^\alpha + \mathcal{A}_M \chi^\alpha & \Phi \mathcal{A}_M \\ 0 & \chi^\alpha & \Phi \end{pmatrix}, \tag{5.30}$$

where $\Pi_M^A = (\Pi_M^a, \Pi_M^\alpha)$ is the vielbein in D=10. Similar to the bosonic case, A_M is a one-form gauge field, and Φ and χ^α are superfields corresponding to dilaton and dilatino in the leading terms. For the reduction of the three-form gauge field $\hat{A}_{\hat{M}\hat{N}\hat{P}}$, define corresponding components in terms of reduced indices

$$\hat{A}_{MNP} = A_{MNP}, \quad \hat{A}_{MNz} = A_{MN}. \tag{5.31}$$

Upon dimension reduction, we assume all the D=10 superfields to be independent of the dimension that is to be reduced z:

$$\partial_z E_m^A = \partial_z \chi^\alpha = \partial_z A_m = \partial_z \Phi = \partial_z A_{MN} = \partial_z A_{MNP} = 0. \tag{5.32}$$

By double dimensional reduction, we also have the condition of a partial static gauge that the dimension to be reduced z coincide with one of the spatial dimension of the world volume

$$z = \rho, \quad \text{and} \quad \partial_z Z^M = 0. \tag{5.33}$$

For the M2-brane action in flat space given in equation (4.14), we expect it to reduce to the Green-Schwarz action (4.1) upon this dimension reduction. For the full M2-brane action as given in equation (4.57), it is simpler to take the kinetic term in the Nambu-Goto form

$$S = \int d^3\hat{\xi} \left(\sqrt{-\det \hat{\Pi}_{\hat{i}}^{\hat{a}} \hat{\Pi}_{\hat{j}}^{\hat{b}} \hat{\eta}_{\hat{a}\hat{b}}} - \frac{1}{6}\epsilon^{\hat{i}\hat{j}\hat{k}} \hat{\Pi}_{\hat{i}}^{\hat{A}} \hat{\Pi}_{\hat{j}}^{\hat{B}} \hat{\Pi}_{\hat{k}}^{\hat{C}} \hat{A}_{\hat{C}\hat{B}\hat{A}} \right). \tag{5.34}$$

Substituting equations (5.30), (5.31) into the action (5.34) and noting the properties (5.32) and (5.33), the action reduce to [6]

$$S = \int d^2\xi \left(\sqrt{-\det \Pi_i^a \Pi_j^b \eta_{ab}} - \frac{1}{2}\epsilon^{ij} \Pi_i^A \Pi_j^B A_{AB} \right). \tag{5.35}$$

The κ-symmetry transformations also undergo the dimension reductions. Since the M2-brane action in curved superspace is invariant under κ-symmetry if the D=11 supergravity constraints are satisfied, which are equivalent to the field equations of D=11 supergravity. It follows by dimension reduction as in the last section that these field equations reduce to D=10 type IIA supergravity field equations. Thus the κ-symmetry requirement on the action (5.35) is equivalent to the superspace constraints on D=10 type IIA supergravity.

CHAPTER 6

Conclusion

In this review, we looked at the construction of D=11 supergravity from Noether's method. We started with the supermultiplet containing graviton, gravitino, and the three-form field and constructed a global supersymmetry. The supergravity theory is found by gauging this supersymmetry, applying iterative modifications to the action and the supersymmetry transformations. We obtained the supergravity action up to a rescaling compared to its conventional form. Then, we discussed briefly how D=11 supergravity can be formulated in superspace by requiring the general coordinate transformation and the supersymmetry algebra of the superfields to match with the supersymmetry and the algebra of the D=11 supergravity component fields. We found the supervielbein and superspace version of the three-form field A up to the first order in θ.

Following this, we looked for brane solutions to the bosonic sector of D=11 supergravity by substituting in ansatz of $ISO(1, d-1) \times SO(D-d)$ symmetry. The ansatz also considers how the three-form is coupled to the worldvolume of the brane, giving electric and magnetic ansatz. While the electric ansatz solves for the M2-brane solution, the magnetic ansatz solves for the M5-brane. We found the M2-brane solution by substituting the ansatz into the equations of motion and the Killing spinor condition. This solves the expression for the spacetime metric and the three-form field. The metric shows interpolation between an asymptotic Minkowski spacetime and a near horizon $AdS_4 \times S^7$ spacetime. The analytic continuation of the coordinates into the event horizon would eventually encounter a timelike singularity, which suggests a possible source term for the supergravity field equations. This source term turns out to correspond to a supermembrane action. We then briefly discussed the electric charge and the mass density from the M2-brane solution, which are found to be equal. Thus, the M2-brane solution saturates the BPS-bound condition.

The construction supermembrane action is then discussed in the following chapter. We started from the Green-Schwarz action for superstrings in flat superspace. This action contains two parts. The first part is similar to a Polyakov action generalized to superspace, and the second is a term added to ensure a local supersymmetry known as the κ-symmetry. This action is readily generalized not only to higher dimensional extended objects but also to curved backgrounds. We first looked at the generalization of higher dimensional extended objects in flat superspace. We note that the condition for κ-symmetry requires the fermionic and bosonic on-shell degrees of freedom to equal on the worldvolume, leading to a condition on the spacetime dimension D, membrane worldvolume dimension d and the supersymmetry charges $\mathcal{N}$. With this in mind, the Green-Schwarz action can be directly generalized to higher dimensional super-p-branes. Following the generalization to supermembranes in flat superspace, we discussed the static gauge and light cone gauge choices, under which conditions we solve for classical solutions. We then performed the semiclassical quantization in a similar method as those with strings for the supermembrane. Then, we moved on to look at the supermembrane action in a general curved background. The requirement from satisfying κ-symmetry results in constraints in the superspace, which we find to be consistent with the superspace formulation constraints for supergravity. This suggests the M2-brane action is consistent with a supergravity background. Thus, we can look at the M2-brane action in an $AdS_4 \times S^7$ background, which is then discussed at the end of the chapter. In the last chapter, we looked at the relation between the M2-brane action and the superstring action by double-dimensional reduction. While the D=11 supergravity is reduced to D=10 type IIA supergravity, the supermembrane action reduces to superstring action.

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APPENDIX A

Clifford algebra in general dimensions

We construct the representation of the gamma matrices from the 2-dimensional Pauli matrices such that the Clifford algebra is satisfied. We follow a general approach by first constructing gamma matrices that satisfy an Euclidean Clifford algebra by observing

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ... \\
\gamma^6 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes ... \\
\gamma^7 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \\
&\vdots
\end{aligned} \tag{A.1}$$

which satisfy the Euclidean Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$. From the property of the Pauli matrices $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, we observe that the above construction of gamma matrices satisfies the Euclidean Clifford algebra. These gamma matrices are also hermitian and square to $\mathbb{1}$. Each of the Pauli matrices in the direct product is 2-dimensional and contributes two dimensions to the representation.

For even dimension D, one has D gamma matrices corresponding to a representation of direct product D/2 Pauli matrices together. This corresponds to a $2^{D/2}$ dimensional representation. For odd dimensions, it is sufficient to use instead of $\gamma^D = \sigma_3 \otimes ... \otimes \sigma_3 \otimes \sigma_1$ but $\gamma^D = \sigma_3 \otimes ... \otimes \sigma_3$, dropping the last σ_1 while still giving a good representation of Clifford algebra. This gives $2^{(D-1)/2}$ dimensional representation. One can also see for odd dimensions, the higher dimensional equivalent for the γ^5 in 4-dimensions becomes one of the bases in one dimension higher. The change from Euclidean signature to Lorentzian signature is done by a Wick rotation. This is done by picking one of the gamma matrices to be multiplied by i and relabelled as γ^0 for the time-like direction. Thus one obtain $(\gamma^0)^2 = -\mathbb{1}$. For example, in the 11-dimensional example above, one can redefine

$$\gamma^0 = i\gamma^{11} = i\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \tag{A.2}$$

to obtain the Clifford algebra in Lorentzian signature. There are two conventions of signs for gamma matrices γ_α for odd dimensions such as d=3:

$$\epsilon_{\alpha\beta\delta}\gamma^{\alpha\beta} = \pm 2!\gamma_\delta, \tag{A.3}$$

$$\epsilon_{\alpha\beta\delta}\gamma^{\alpha\beta\delta} = \pm 3!\mathbb{1}. \tag{A.4}$$

This corresponds to defining $i\sigma_1$ or $-i\sigma_1$ as γ^0. We chose the negative sign convention for our calculations.

It is worth noticing that the Clifford algebra is not an algebra satisfying Lie bracket, bilinearity, and Jacobi identity altogether. Thus, the Clifford algebra is not a Lie algebra. Clifford algebra is an unital associative algebra. Since the symmetric product of the gamma matrices is the anticommutators, which reduces to $\eta_{\mu\nu}$ in the γ^μ case, we look to construct new elements by antisymmetric product and define

$$\gamma^{\mu_1\mu_2\ldots\mu_r} = \gamma^{[\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_r]} = \frac{1}{r!}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_r} + \text{permuations}) \tag{A.5}$$

where r is the rank of the gamma matrix. The antisymmetrization of the indices comes with a factor of $1/r!$. In the above expression for gamma, if there is $\mu_i = \mu_j$, the antisymmetrisation of the indices would give $\gamma^{\mu_1\mu_2\ldots\mu_r} = 0$. Thus we can also write

$$\gamma^{\mu_1\mu_2\ldots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_r} \quad \text{for} \quad \mu_1 \neq \mu_2 \neq \ldots \neq \mu_r. \tag{A.6}$$

One can raise or lower the Lorentz indices on gamma matrices using the metric:

$$\begin{aligned} \gamma^\mu &= \eta^{\mu\nu}\gamma_\nu && \text{in Minkowski spacetime} \\ \gamma^\mu(x) &= g^{\mu\nu}\gamma_\nu(x) && \text{in general} \end{aligned} \tag{A.7}$$

One can also write gamma matrices with tangent space indices

$$\gamma^\mu(x) = e^\mu{}_{\underline{\mu}}(x)\gamma^{\underline{\mu}}, \tag{A.8}$$

where the x-dependent is only on the transformation matrix $e^\mu{}_a(x)$. Thus, contraction between gamma matrices can be defined from the Clifford algebra:

$$\begin{aligned} &\gamma^\mu\gamma_\mu = D \\ &\gamma^{\mu\nu}\gamma_\nu = (D-1)\gamma^\mu \\ &\gamma^{\mu\nu\rho}\gamma_{\nu\rho} = (D-2)\gamma^\mu \\ &\vdots \\ &\gamma^{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\gamma_{v_s\ldots\nu_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r} \end{aligned} \tag{A.9}$$

Another useful identity is

$$\Gamma^{a_j \dots a_2 a_1} \Gamma_{b_1 b_2 \dots b_k} = \sum_{l=0}^{\min(j,k)} l! \begin{pmatrix} j \\ l \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \delta^{[a_1}_{[b_1} \cdots \delta^{a_l}_{b_l} \Gamma^{a_j \cdots a_{l+1}]}{}_{b_{l+1} \cdots b_k]} . \tag{A.10}$$

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