On the Capacity Region of a Multiple Access Channel with Common Messages

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***Abstract*—The capacity region for a multiple access channel (MAC) with arbitrary sets of common messages was derived by Han in 1979, extending a result by Slepian and Wolf from 1973. The general characterization by Han involves one auxiliary random variable per message and one inequality per subset of messages. In this paper, at first, a special hierarchy of common messages is identified for which the capacity region is characterized with generally fewer auxiliary random variables and inequalities. It is also shown that this characterization requires no auxiliary random variable for certain message structures. A procedure is then proposed to transform any common message structure to this special hierarchy, leading to a general capacity characterization which generally requires fewer auxiliary random variables than the one given by Han.**

I. Introduction

A multiple access channel (MAC) consists of multiple users transmitting to a common receiver. The capacity region of the MAC was first characterized by Ahlswede [1] and Liao [2], assuming independent messages at the users. In this model, the independence of the transmitted messages prevents the channel inputs of different users to be chosen as correlated (apart from possible time-sharing), and the capacity region is characterized via the set of probability distributions that factorize as the product of the input distributions of the users [1][2].

A more general model encompasses arbitrarily correlated sources at the users. This leads to a significantly harder problem, which remains open in the most general setup, as the source correlation allows the users to generate correlated channel inputs [3]. A few special cases have been solved in [4]-[10]. Among such special cases is the model first considered by Slepian and Wolf in [7], in which users have access to a given subset of a number of independent messages. This scenario generalizes the classical MAC of [1], [2] in which each independent message is available to only a single user.

Within this class of channels, in [7], the capacity region is found for a two-user three-message MAC with two private and one common messages. An "educated guess" on the extension of this result to three users (and thus seven messages, including the three pair-wise common messages and one common message to all users) is also provided. Later, it was shown by Pinsker with a simple example that such guess does not provide the correct capacity region (see [9]). The correct generalization to a MAC with an arbitrary number of users and messages is derived by Han in [8]. Han's characterization requires one auxiliary random variable for each of the independent messages. However, it is known that, in certain special cases, it is possible to describe the capacity region without resorting to auxiliary random variables which significantly simplifies the numerical evaluation of the capacity region. The classical case of a single independent message at each user studied in [1] and [2], and the case of degraded message sets studied in [10] are two examples. Our goal in this paper is to give a generalization of such special cases and also to provide an alternative capacity characterization for the general case which would reduce the number of auxiliary random variables involved.

We identify a special message hierarchy for MACs with common messages such that the capacity region is described with a limited number of auxiliary random variables. We show for a special subset, generalizing [1], [2], [10], no auxiliaries are needed, unlike the characterization of [8]. Then, we propose a procedure to characterize the capacity region of any common message structure with a number of auxiliary variables smaller than (or, in the worst case, equal to) the general description of [8]. The procedure works by transforming the message set into the identified special message hierarchy through the definition of additional virtual users.

II. System Model

We consider M independent messages and K non-cooperating users. Each user has access to a non-empty subset of the messages, and all messages are to be conveyed to a destination over a MAC. We denote the messages by $W_m \in \mathcal{W}_m$ for $m \in [1, M]$, and the channel input of user k by $X_k \in \mathcal{X}_k$ for $k \in [1, K]$. The discrete memoryless channel is characterized by a probability mass function (pmf) $p(y|x_1, \ldots, x_K)$ where $Y \in \mathcal{Y}$ is the channel output available at the destination. We denote the set of the messages available to user k by $\mathcal{I}_k$:

$$\mathcal{I}_k = \{W_{m_1^k}, \ldots, W_{m_{i_k}^k}\}, \quad k \in [1, K], \tag{1}$$

where i_k is the number of messages at user k and $m_j^k \in [1, M]$ is the index of the jth message ($j \in [1, i_k]$) at user k. Without loss of generality, we assume that $\mathcal{I}_i \neq \mathcal{I}_j$ for $i \neq j$, since, otherwise, two users with the exact set of messages can be combined into a single super-user. Similarly, we assume that no two messages are available to the same set of users.

Definition 2.1: A $(2^{nR_1}, \ldots, 2^{nR_M}, n)$ code for a K-user MAC consists of M sets $\mathcal{W}_m = \{1, \ldots, 2^{nR_m}\}$, $m \in [1, M]$, K encoding functions: $f_k : \mathcal{W}_{m_1^k} \times \cdots \times \mathcal{W}_{m_{i_k}^k} \to \mathcal{X}_k^n, \quad k \in$

$[1, K]$, and a decoding function: $h : \mathcal{Y}^n \to \mathcal{W}_1 \times \ldots \times \mathcal{W}_M$, with $(\hat{W}_1, \ldots, \hat{W}_M) = h(\mathcal{Y}^n)$.

The average probability of error for a code is defined as

$$P_e^n \triangleq \Pr\{(\hat{W}_1, \ldots, \hat{W}_M) \neq (W_1, \ldots, W_M)\},$$

where W_j are independent and uniformly distributed in their domains $\mathcal{W}_j$, $j \in [1, M]$.

Definition 2.2: A rate tuple $(R_1, \ldots, R_M)$ is said to be *achievable* if there exists a sequence of $(2^{nR_1}, \ldots, 2^{nR_M}, n)$ codes with $P_e^n \to 0$ as $n \to \infty$. The *capacity region* is the closure of the set of all achievable rate tuples.

Theorem 2.1: [8] The capacity region of the MAC at hand is given by the closure of the convex hull of the set of all rate tuples $(R_1, \ldots, R_M)$ satisfying[1]

$$\sum_{j \in \mathcal{M}} R_j \leq I(U_{\mathcal{M}}; Y | U_{\mathcal{M}^c}) \qquad (2)$$

for all $\mathcal{M} \subseteq [1, M]$, for some auxiliary variables $U_j \in \mathcal{U}_j$, $j \in [1, M]$ and pmf $\prod_{j=1}^{M} p(u_j) \ \prod_{k=1}^{K} p(x_k | u_{m_1^k}, ..., u_{m_{i_k}^k})$ $p(y | x_1, \ldots, x_K)$. Moreover, the conditional pmfs $p(x_k | u_{m_1^k}, ..., u_{m_{i_k}^k})$ can be restricted to take values only in $\{0, 1\}$ and $|\mathcal{U}_j| \leq \prod_{k:\ W_j \in \mathcal{I}_k} |\mathcal{X}_k| + M$.

III. MAC WITH A SPECIAL MESSAGE HIERARCHY

A. Definitions and Associated Message Graph

Our focus here is on a MAC with a special message hierarchy, which is specified by the following definition.

Definition 3.1: A given MAC with K users and M messages, where the kth user has access to messages $\mathcal{I}_k$ (1) for $k \in [1, K]$, is said to have the *special message hierarchy* if, for any $i \neq j \in [1, K]$, the set $\mathcal{I}_i \cap \mathcal{I}_j$ is either an empty set or is equal to $\mathcal{I}_k$ for some $k \in [1, K]$.

This special message hierarchy induces a particular structure on the message subsets $\mathcal{I}_k$s. It is useful to describe this structure as a graph, in order to ease the description of the capacity region (see Fig. 1 and 2). We first give some necessary definitions. A graph G is denoted by $G = (V, E)$ with vertex set V and edge set E, $E \subseteq [V]^2$. G is a *directed graph* if we have two maps, $\mathrm{init} : E \to V$ and $\mathrm{ter} : E \to V$, assigning an initial vertex $\mathrm{init}(e)$ and a terminal vertex $\mathrm{ter}(e)$ to each edge $e \in E$, where $\mathrm{init}(e) \neq \mathrm{ter}(e)$. Then, edge e is said to be directed from $\mathrm{init}(e)$ to $\mathrm{ter}(e)$, and is denoted by the ordered pair $e = (\mathrm{init}(e), \mathrm{ter}(e))$. A subgraph $P = (V', E')$, with $V' = \{x_0, \ldots, x_k\} \subseteq V$ and $E' = \{e_0, \ldots, e_{k-1}\} \subseteq E$, is a directed path on G if e_i is an edge directed from x_i to x_{i+1} for all $i < k$. A directed graph is called a *rooted directed graph* if there exists a directed path between a vertex designated as the root and every vertex in the graph. A directed cycle is a directed path that starts and ends at the same vertex. A directed graph is *acyclic* of if it contains no directed cycles.

Definition 3.2: A rooted, directed, acyclic graph $G = (V, E)$ is referred to as a *message graph* if for any $e \in E$ it has no directed path from $\mathrm{init}(e)$ to $\mathrm{ter}(e)$ other than e.

It is noted that a message graph is not in general a (directed) tree, since more than one directed path may exist between two vertices not on the same edge. However, we can define some terminology similar to trees, which will be useful in the following. The *parents* of a node x_i are defined as the nodes that are directly connected to x_i and are on a path from x_i to the root. A *child* of a vertex x_i is a vertex of which x_i is the parent. The set D_k of *descendants* of a vertex v_k contains all its children, the children of its children and so forth. A *leaf* is a vertex that does not have a child so its descendant set is empty. The *ancestors* of a vertex are all its parents, the parents of its parents and so forth.

Definition 3.3: Given a MAC with a message structure as in (1), the *associated graph* $\Gamma = (V, E)$ is defined as follows. We have $V = \{v_0, v_1, \ldots, v_K\}$, where vertex v_k corresponds to user k in the MAC for $k = 1, \ldots, K$. We add a directed edge from v_k to v_l if $\mathcal{I}_k \subset \mathcal{I}_l$ and if there exists no $\mathcal{I}_j$, $j \neq k, l$, such that $\mathcal{I}_k \subset \mathcal{I}_j \subset \mathcal{I}_l$. Finally, we add the edges (v_k, v_0) if v_k has no parent. Root vertex v_0 does not correspond to a user in the network, but is added to the graph to make it a rooted graph[2].

The associated graph for a MAC with any message structure is a message graph. As an example, consider the 6-user MAC with 4 messages in Fig. 1 and its associated graph in Fig. 2. The graph has a total of 7 vertices, one for each user and a root v_0. Note, for example that, vertex v_4 is the parent of v_1 and v_2 since the messages W_1 and W_2 are also available at user 4. Vertex v_1 has two parents as W_1 is also available to user 3. Moreover, examples of descendant sets are $D_4 = \{v_1, v_2\}$ and $D_5 = \{v_1, v_2, v_3, v_4\}$. Note that the MAC in this example has the special message hierarchy.

Definition 3.4: In a message graph $\Gamma = (V, E)$, we call a vertex $v \in V$ *multiple parent* (mp) if it has more than one parent. We denote the set of all mp-vertices in V by $\mathcal{M}$.

Note that a message graph is a tree if and only if there are no mp-vertices. In the example of Fig. 2 we have $\mathcal{M} = \{v_1\}$ and the graph is not a tree. As seen below, in the characterization of the capacity region, users corresponding to mp-vertices need to be handled in a special way.

B. Capacity Region

We define the *private messages* of user k as the messages that are in $\mathcal{I}_k$ but are not available to any of the users in the descendant set D_k of user k. If D_k is empty, then all the messages available to user k are its private messages.

Lemma 3.1: If the underlying MAC has the special source hierarchy, there can be at most one private message for each user k (and associated vertex $v_k \neq v_0$). This private message is denoted as $W(k)$. Moreover, each message W_j in the system is a private message for exactly one user, i.e., $W_j = W(k)$ for only one $k \in [1, K]$.

Proof: Assume that W_1 and W_2 are both in $\mathcal{I}_k$ but not available to any of the users in D_k. Then, there needs to be at least one user in the MAC such that $W_1 \in \mathcal{I}_l$ but $W_2 \notin \mathcal{I}_l$, as otherwise one can combine W_1 and W_2 into a single message. But now $W_1 \in \mathcal{I}_k \cap \mathcal{I}_l$ and thus, by Definition 3.1, we need

[1] $X_{\mathcal{M}} \triangleq \{X_j \colon j \in \mathcal{M}\}$ for any set of indices $\mathcal{M}$ and $\mathcal{M}^c = [1, M] \backslash \mathcal{M}$.

[2] This step may be skipped if there exists a user observing all messages.

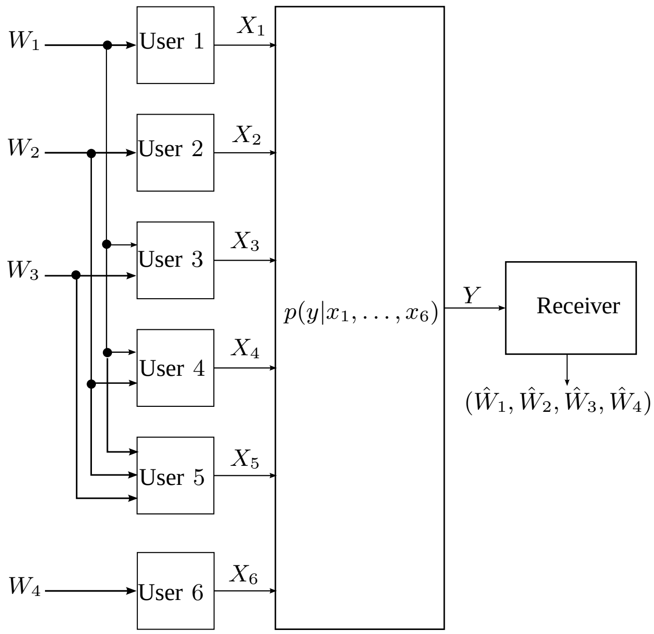

Figure 1. Example of a MAC with a special message hierarchy.

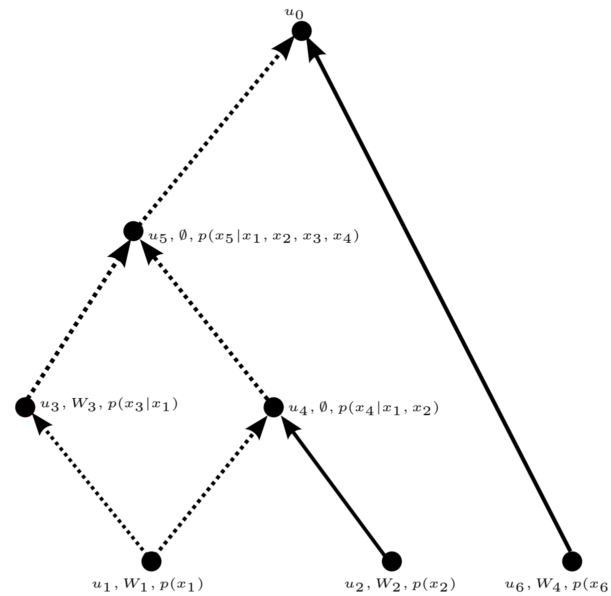

Figure 2. The message graph Γ corresponding to the MAC in Fig. 1.

to have $\mathcal{I}_m = \mathcal{I}_k \cap \mathcal{I}_l$ for some user $m \in [1, M]$. But, since $W_2 \notin \mathcal{I}_m$, we have $m \neq k$ and $\mathcal{I}_m \in D_k$. However, this contradicts the initial assumption. ∎

Private messages $W(k)$ for the MAC in Fig. 1 are also included in Fig. 2. For $v_k \neq v_0$ let $R(v_k)$ denote the rate of message $W(k)$. Note that, nodes v_4 and v_5 having no private messages have $W(4) = W(5) = \emptyset$ and $R(v_4) = R(v_5) = 0$.

Definition 3.5: A graph $F = (V', E')$ is a *proper rooted subgraph* of the message graph $\Gamma = (V, E)$ if $V' \subset V$, $E' \subset E$, $v_0 \in V'$ and for any $v \in V'$ all the edges in E that originate from v belong to E', i.e., $e \in E'$ if $e \in E$ and $\text{init}(e) \in V'$.

Each rooted subgraph corresponds to a set of users, and this set is considered to be proper if for each message that is available to a user in the chosen set, all the users that have access to this message are also included in the set.

For example, in the message graph Γ of Fig. 2, the rooted subgraph (V', E') with $V' = \{v_0, v_5, v_3, v_4, v_1\}$ and $E' = \{(v_5, v_0), (v_3 v_5), (v_4, v_5), (v_1, v_3), (v_2, v_3)\}$ is a proper rooted subgraph. This subgraph is illustrated with the dotted edges in Fig. 2. Note that the subgraph with $V' = \{v_0, v_5, v_3, v_1\}$ and $E' = \{(v_5, v_0), (v_3 v_5), (v_1, v_3)\}$ is not a proper one since $(v_1, v_5) \notin E'$ even though $v_1 \in V'$.

Since each proper rooted subgraph F is defined by its vertices, with abuse of notation, we will denote F by its vertex set. In Fig. 2, we have 18 proper rooted subgraphs: $v_0, \{v_0, v_5\}$, $\{v_0, v_6\}$, $\{v_0, v_5, v_3\}$, $\{v_0, v_5, v_4\}$, $\{v_0, v_5, v_6\}$, $\{v_0, v_5, v_3, v_4\}$, $\{v_0, v_5, v_4, v_2\}$, $\{v_0, v_5, v_4, v_6\}$, $\{v_0, v_5, v_3, v_6\}$, $\{v_0, v_5, v_3, v_4, v_1\}$, $\{v_0, v_5, v_3, v_4, v_2\}$, $\{v_0, v_5, v_3, v_4, v_6\}$, $\{v_0, v_5, v_3, v_4, v_1, v_2\}$, $\{v_0, v_5, v_3, v_4, v_1, v_6\}$, $\{v_0, v_5, v_4, v_2, v_6\}$, $\{v_0, v_5, v_3, v_4, v_2, v_6\}$, $\{v_0, v_5, v_3, v_4, v_1, v_2, v_6\}$.

Theorem 3.2: For a MAC with a special message hierarchy as in Defn. 3.1, the capacity region is the closure of the convex hull of the set of all rate tuples $(R_1, \ldots, R_M)$ satisfying[3]

$$0 \leq \textstyle\sum_{v \in F} R(v) \leq I(X_F; Y | X_{F^c}, U_{F^c \cap \mathcal{M}}) \qquad (3)$$

for all the proper rooted subgraphs F of the associated message graph Γ and some pmf $\prod_{k=1}^{K} p(\tilde{x}_k | x_{D_k}, u_{D_k}) p(y | x_1, \ldots, x_K)$, where $\tilde{x}_k = (x_k, u_k)$ if $k \in \mathcal{M}$and $\tilde{x}_k = x_k$ otherwise.

Proof: A sketch of the proof is given in the Appendix. ∎

The characterization in Theorem 3.2 involves one inequality per proper rooted subgraph F, each concerning the rates of the private messages in F. This is unlike Theorem 2.1 by Han which involves one inequality for each subset of messages. Moreover, Theorem 3.2 requires one auxiliary variable for each mp-vertex only, whereas Theorem 2.1 uses one auxiliary variable per message. It should be noted, however, that it appears difficult to give general bounds on the cardinality of the auxiliary variables in Theorem 3.2, unlike for Theorem 2.1. Finally, we remark that Theorem 3.2 generalizes the capacity region in [7][4] while the following corollary provides a generalization of the capacity regions in [1], [2] and [10].

Corollary 3.3: The capacity characterization (3) for a MAC with a special message structure that does not involve any mp-vertex (i.e., whose message graph Γ is a tree) does not require any auxiliary random variables.

Some of the inequalities obtained from Theorem 3.2 may be redundant or trivial. For instance, in Fig. 2, since no private message is assigned to v_0, (v_0, v_5) or (v_0, v_5, v_4), they can be ignored (corresponding inequalities are trivial). Also,

[3] $X_F \triangleq \{X_j \colon v_j \in V'\}$ for a proper rooted subgraph $F = (V', E')$, while $X_{F^c} \triangleq \{X_j \colon v_j \in V \backslash V'\}$.

[4] Notice that in this case the cardinality of the only auxiliary random variable can be bounded as shown in [11].

(v_0, v_5, v_3, v_4) can be ignored as we already have (v_0, v_5, v_3) and v_4 has no private message (corresponding inequality is redundant). In general, for all the combinations that include the same subset of vertices with private messages all but one can be ignored. Following these arguments, it can be seen from Theorem 3.2 that the capacity region is the closure of the convex hull of the rate tuples satisfying

$$
\begin{aligned}
0 \le R_3 &\le I(X_3, X_5; Y|U_1, X_1, X_2, X_4, X_6)\\
0 \le R_1 + R_3 &\le I(X_1, X_3, X_4, X_5; Y|X_2, X_6)\\
0 \le R_2 &\le I(X_2, X_4, X_5; Y|U_1, X_1, X_3, X_6)\\
0 \le R_1 + R_2 + R_3 &\le I(X_1, X_2, X_3, X_4, X_5; Y|X_6)\\
0 \le R_2 + R_3 &\le I(X_2, X_3, X_4, X_5; Y|U_1, X_1, X_6)\\
0 \le R_4 &\le I(X_6; Y|U_1, X_1, X_2, X_3, X_4, X_5)\\
0 \le R_3 + R_4 &\le I(X_3, X_5, X_6; Y|U_1, X_1, X_2, X_4)\\
0 \le R_1 + R_3 + R_4 &\le I(X_1, X_3, X_4, X_5, X_6; Y|X_2)\\
0 \le R_2 + R_4 &\le I(X_2, X_4, X_5, X_6; Y|U_1, X_1, X_3)\\
0 \le R_1 + R_2 + R_3 + R_4 &\le I(X_1, X_2, X_3, X_4, X_5, X_6; Y)
\end{aligned}
$$

for some joint pmf of the form $p(u_1, x_1)$ $p(x_2)$ $p(x_3|u_1, x_1)$ $p(x_4|u_1, x_1, x_2)$ $p(x_5|u_1, x_1, x_2, x_3, x_4)$ $p(x_6)$ $p(y|x_1, x_2, x_3, x_4, x_5, x_6)$. We would like to remark here that using the formulation given by Han in [8], for the same MAC, we would obtain a capacity region characterization defined by 16 inequalities rather than 11, which would involve 4 auxiliary random variables.

IV. MAC WITH GENERAL MESSAGE HIERARCHY

Now, we show that the characterization for the special message hierarchy given in Section III can be used to obtain the capacity region in general. Given a MAC with any message structure, consider all possible pairs of sets $\mathcal{I}_i$ and $\mathcal{I}_j$ with $i \neq j$. If $\mathcal{I}_i \cap \mathcal{I}_j$ is neither empty nor equal to the message set of any of the existing users, create a "virtual user" that has access to messages in $\mathcal{I}_i \cap \mathcal{I}_j$ but no channel input. After going through all pairs of users, apply the same procedure in the new MAC including the virtual users, and repeat until there is no further need to create virtual users. Since the cardinality of the message sets of the virtual users will be decreasing at each stage, this process will stop after finite number of steps. At the end, we obtain a MAC that satisfies the special message hierarchy. We can then characterize the capacity region of this MAC using the technique in Section III. Note that, although the virtual users have no channel input, since they are all mp-vertices, there will be one auxiliary random variable associated with each virtual user.

As an example, consider the same MAC as in Fig. 1 without user 1. The new MAC does not satisfy the special message hierarchy property since the intersection of the message sets of user 3 and user 4 is $\{W_1\}$, which neither is empty, nor corresponds to another user's message set. We then create a "virtual" user 1, without a channel input, which has access to W_1. This new MAC satisfies the special message hierarchy and its message graph is as given in Fig. 2. The capacity region for the MAC in Fig. 1 without user 1 is then obtained as the closure of the convex hull of the rate tuples satisfying

$$
\begin{aligned}
0 \le R_3 &\le I(X_3, X_5; Y|U_1, X_2, X_4, X_6)\\
0 \le R_1 + R_3 &\le I(X_3, X_4, X_5; Y|X_2, X_6)\\
0 \le R_2 &\le I(X_2, X_4, X_5; Y|U_1, X_3, X_6)\\
0 \le R_1 + R_2 + R_3 &\le I(X_2, X_3, X_4, X_5; Y|X_6)\\
0 \le R_2 + R_3 &\le I(X_2, X_3, X_4, X_5; Y|U_1, X_6)\\
0 \le R_4 &\le I(X_6; Y|U_1, X_2, X_3, X_4, X_5)\\
0 \le R_3 + R_4 &\le I(X_3, X_5, X_6; Y|U_1, X_2, X_4)\\
0 \le R_1 + R_3 + R_4 &\le I(X_3, X_4, X_5, X_6; Y|X_2)\\
0 \le R_2 + R_4 &\le I(X_2, X_4, X_5, X_6; Y|U_1, X_3)\\
0 \le R_1 + R_2 + R_3 + R_4 &\le I(X_2, X_3, X_4, X_5, X_6; Y)
\end{aligned}
$$

for some joint pmf $p(u_1)$ $p(x_2)$ $p(x_3|u_1)$ $p(x_4|u_1, x_2)$ $p(x_5|u_1, x_2, x_3, x_4)$ $p(x_6)$ $p(y|x_2, x_3, x_4, x_5, x_6)$.

After having included all the virtual users as above, one can potentially reduce the number of auxiliary variables required. For any virtual user that does not have a private message, we can assign its auxiliary random variable to be equivalent to the auxiliaries of its descendants, hence we do not need an additional auxiliary variable for these users. Consider, for example, the 4 user MAC with $\mathcal{I}_1 = \{W_1, W_2, W_3\}$, $\mathcal{I}_2 = \{W_2, W_3, W_4\}$, $\mathcal{I}_3 = \{W_2, W_5\}$, $\mathcal{I}_4 = \{W_3, W_6\}$. Now following the above algorithm, we end up adding the following virtual users: $\mathcal{I}_5 = \{W_2, W_3\}$, $\mathcal{I}_6 = \{W_2\}$ and $\mathcal{I}_7 = \{W_3\}$. Note that, we will have auxiliary random variables, say U_6 and U_7, assigned to the two virtual users which are the leaves of the message graph, but we do not need an additional variable for the virtual user without a private message, and simply assign it as (U_6, U_7). Hence, for this example, we can define the capacity region with only 2 auxiliaries and 8 inequalities as opposed to 6 auxiliaries and 63 inequalities of Thm. 2.1.

In general, the number of auxiliary variables involved will be less than Theorem 2.1. This follows from the fact that, in the worst case, we will create one virtual user for each message in the system. In this case Theorem 3.2 will give us the same capacity characterization as the one given by Han in Theorem 2.1. Moreover, even in the case when the numbers of auxiliaries involved in both characterizations are equal, the number of inequalities in Theorem 3.2 will be less than or equal to the ones in Theorem 2.1.

V. CONCLUSIONS

We have considered a MAC with multiple users and messages, in which each user has access to a certain subset of the messages. We have provided the corresponding capacity region under the assumption of a special message hierarchy. This single-letter capacity region characterization involves less auxiliary random variables and inequalities than the general characterization given by [8] and generalizes the results in [1], [2][10]. We then used this result to give the capacity region for the general MAC with common messages that again requires in general less auxiliary random variables and inequalities than the capacity region characterization given in [8].

APPENDIX

A. Achievability

Code Construction: We fix a joint distribution $\prod_{k=1}^{K} p(x_k, u_k | x_{D_k}, u_{D_k})$. We start generating the code from the leaves of the corresponding message graph. For each leaf v_k whose private message is $W(k)$, generate $2^{nR(v_k)}$ codeword pairs $u_k^n(w_k)$ and $x_k^n(w_k)$ $w_k = 1, 2, ..., 2^{nR(v_k)}$ independent and identically distributed (i.i.d.) with $\prod_{i=1}^{n} p(u_{ki})$ and $\prod_{i=1}^{n} p(x_{ki})$, respectively. For each of the parents of the leaves, generate a separate codebook for each combination of the messages of its descendants. For example, consider a parent v_k. For each combination of the messages of its descendants, denoted by w_{D_k}, generate $2^{nR(k)}$ codeword pairs $u_k^n(w_k)$ and $x_k^n(w_k)$ i.i.d. according to the probability distribution $\prod_{i=1}^{n} p(x_{ki}, u_{ki} | u_{D_k i}(w_{D_k}), x_{D_k i}(w_{D_k}))$, where $x_{D_k i}(w_{D_k})$ and $u_{D_k i}(w_{D_k})$ correspond to the codewords of the descendants D_k. Label these codewords as $u_k^n(w_k, w_{D_k})$ and $x_k^n(w_k, w_{D_k})$ for $w_k \in [1, 2^{nR(v_k)}]$. We continue similarly until we reach v_0. Since v_0 does not correspond to a user in the system, it does not have a channel input.

Encoders: Given $(W(1), ..., W(M)) = (w_1, \ldots, w_M) = \mathbf{w}$, encoder k transmits $x_k^n(\mathbf{w}) \triangleq x_k^n(w_{m_1^k}, \ldots, w_{m_{i_k}^k})$.

Decoders: The decoder uses a joint typicality decoder and looks for a tuple $\mathbf{w}$ such that $(x_1^n(\mathbf{w}), \ldots, x_K^n(\mathbf{w}), u_1^n(\mathbf{w}), \ldots, u_K^n(\mathbf{w}), y^n) \in T_n^\epsilon$, where T_n^ϵ denotes the set of jointly typical sequences.

Error analysis: Assume $(w_1, \ldots, w_M) = \mathbf{1}$ was sent, where $\mathbf{1}$ is a length M vector of 1's. We define the following set of events: $\mathcal{E}_\mathbf{1} = \{(X_1^n(\mathbf{1}), \ldots, X_K^n(\mathbf{1}), U_1^n(\mathbf{1}), \ldots, U_K^n(\mathbf{1}), Y^n) \notin T_n^\epsilon\}$, and for $\mathbf{b} = b_1 \cdots b_M$ with $b_i \in \{0, 1\}$ and $\mathbf{b} \neq \mathbf{1}$,

$$\mathcal{E}_\mathbf{b} = \{(X_1^n(\mathbf{b}), \ldots, X_K^n(\mathbf{b}), U_1^n(\mathbf{b}), \ldots, U_K^n(\mathbf{b}), Y^n) \in T_n^\epsilon \text{ for some } w_i \neq 1 \text{ if } b_i = 0, i = 1, \ldots, M\}$$

where we define $X_1^n(\mathbf{b}) = X_1^n(w_1, \ldots, w_M)$ such that $w_i = 1$ if $b_i = 1$. We have a total of 2^M error events. However, we observe that all the error events that have a message $w_k \neq 1$, i.e., $b_k = 0$, correspond to the same induced distribution, irrespective of the private messages of the ancestors of v_k. Therefore, we can combine these into a single error event corresponding to one proper rooted subgraph of Γ. We can use the union bound to bound the probability of the union of the error events, and the proof follows from standard arguments. Finally, we set $U_k = X_k$ if $v_k \notin \mathcal{M}$.

B. Converse

Assume that a sequence of $(2^{nR_1}, \ldots, 2^{nR_M}, n)$ codes exists such that $P_e^n \to 0$ as $n \to \infty$. From Fano's inequality, we have $H(W_\Gamma | Y^n) \leq n\delta_n$ with $\delta_n \to 0$ as $n \to \infty$[5]. This also leads to $H(W_F | Y^n, W_{F^c}) \leq n\delta_n$ for any proper rooted subgraph F of Γ. Then, for any F, we have

$$n \textstyle\sum_{v \in F} R(v) = H(W_F) = H(W_F | W_{F^c})$$

$$\begin{aligned}
&\leq I(W_F; Y^n | W_{F^c}) + n\delta_n \\
&= \sum_{i=1}^{n} I(W_F; Y_i | W_{F^c}, Y^{i-1}) + n\delta_n \\
&= \sum_{i=1}^{n} I(W_F, X_{F,i}; Y_i | W_{F^c}, X_{F^c,i}, Y^{i-1}) + n\delta_n \quad (4) \\
&\leq \sum_{i=1}^{n} I(X_{F,i}; Y_i | W_{F^c}, X_{F^c,i}) + n\delta_n \quad (5) \\
&\leq \sum_{i=1}^{n} I(X_{F,i}; Y_i | W_{F^c \cap \mathcal{M}}^D, X_{F^c,i}) + n\delta_n \quad (6) \\
&= \sum_{i=1}^{n} I(X_{F,i}; Y_i | U_{F^c,i}, X_{F^c,i}) + n\delta_n \quad (7)
\end{aligned}$$

where (4) follows from the fact that the codewords are functions of the messages and the definition of a proper rooted subgraph; (5) follows from the facts that conditioning reduces entropy and $(W_\Gamma, Y^{i-1}) - X_{\Gamma,i} - Y_i$ forms a Markov chain; in (6) we define $W_S^D \triangleq \{W_{D_k} : v_k \in S\}$ and use the fact that $D_k \subseteq F^c$ for $v_k \in F^c$; and finally in (7) we define $U_{k,i} \triangleq W_{D_k}$ for $v_k \in \mathcal{M}$. We then introduce a time-sharing random variable Q, uniformly distributed over $\{1, .., n\}$, independent from everything else and define $X_k \triangleq X_{kQ}$ and $U_k \triangleq U_{kQ}$ for $k \in [1, K]$ and $Y \triangleq Y_Q$. This results in a rate region with all the mutual information terms conditioned on Q. The channel input distributions are also conditioned on q.

It can also be shown that the joint distribution of the chosen auxiliary random variables and the channel inputs is of the form given in Thm. 3.2. This can be done rigorously by induction on the vertices of the message graph. It can be shown for the mp vertices that the input distributions of nodes following an mp are conditionally independent given the set of messages of the descendants of the mp itself.

[5] We define $W_S \triangleq \{W(k) : v_k \in S\}$.

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